

CONWAY POLYNOMIAL AND MAGNUS EXPANSION

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ABSTRACT. The Magnus expansion is a universal finite type invariant of pure braids with values in the space of horizontal chord diagrams. The Conway polynomial composed with the short-circuit map from braids to knots gives rise to a series of finite type invariants of pure braids and thus factors through the Magnus map. We describe explicitly the resulting mapping from horizontal chord diagrams on 3 strands to univariate polynomials and evaluate it on the Drinfeld associator obtaining, conjecturally, a beautiful generating function whose coefficients are alternating sums of multiple zeta values.

1. INTRODUCTION

We assume that the reader is familiar with fundamentals of knot theory, braid groups and finite type (Vassiliev) invariants. All these preliminaries can be found, for instance, in [15].

The short-circuit closure of pure braids [10] induces a map from pure braids onto the set of (topological types of) oriented knots. Any Vassiliev invariant of knots thus becomes a finite-type invariant of pure braids. There is a universal finite type invariant of pure braids given by the Magnus expansion. In this paper, we will explicitly describe the map from horizontal chord diagrams on 3 strands obtained by factoring the Conway polynomial pulled back to pure 3-braids through Magnus expansion. The result is described by a peculiar combinatorial map from ordered partitions of an integer into non-ordered partitions of the same integer.

In Section 2 we say introductory words about the group of pure braids and give the construction of the Magnus expansion. Section 3 is devoted to the construction of the short-circuit closure relating braids to knots. In Section 4 we speak about the Conway polynomial of braids transferred from knots via short-circuit closure and state the main theorem, whose proof is given in Section 5. In Section 6 we try to evaluate the mapping obtained in the main theorem, on the Drinfeld associator and state the results of our computer calculations and the corresponding conjecture. Finally, Section 7 lists some open problems related to the material of the paper.

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2. PURE BRAIDS AND MAGNUS EXPANSION

Let P_m be the group of pure braids on m enumerated vertical strands with multiplication defined as the concatenation from top to bottom. It is generated by the elements x_{ij} , $1 \leq i < j \leq m$, representing one full positive twist between the i -th and the j -th strands with all the remaining strands of the braid being strictly vertical and placed behind these two:

$$x_{ij} = \begin{array}{c} \begin{array}{c} i \qquad \qquad j \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ i \qquad \qquad j \end{array} \end{array}$$

The diagram shows a braid with m vertical strands. Two strands, labeled i and j , are twisted around each other in a full positive twist. The strands are labeled i and j at both the top and bottom. Ellipses \dots are placed between the strands i and j at both the top and bottom, indicating other strands. The braid is enclosed in a rectangular frame with vertical lines on the sides and horizontal dotted lines at the top and bottom.

Defining relation between these generators can be found, for instance, in [2, 4]; we will not need them here.

There is a semi-direct decomposition [2]

$$P_m \cong F_{m-1} \rtimes \dots \rtimes F_2 \rtimes F_1,$$

where F_k is a free group on k generators, implemented in our case as the subgroup of P_m generated by the set $x_{1,k+1}, x_{2,k+1}, \dots, x_{k,k+1}$. This decomposition assures that any pure braid can be uniquely written in the *combed form* $\prod_s x_{i_s j_s}^{a_s}$ with $j_1 \geq j_2 \geq \dots$ where a_s are arbitrary nonzero integers and in this product no two identical generators follow each other (that is, the word is reduced).

The *Magnus expansion* is a map from P_m into the \mathbb{Z} -algebra of formal power series in $\binom{m}{2}$ non-commuting variables t_{ij} , $1 \leq i < j \leq m$, defined by

$$\mu_m(\beta) = \prod_s (1 + t_{i_s j_s})^{a_s},$$

if $\prod_s x_{i_s j_s}^{a_s}$ is the combed form of the braid β . Here the negative powers are understood as usual, according to the rule $(1 + t)^{-1} = 1 - t + t^2 - t^3 + \dots$, — this is why we need power series, not just polynomials, in the construction of μ_m .

Example. To compute the value $\mu_3(x_{12}x_{23})$, we first find the combed form of this braid

$$x_{12}x_{23} = x_{13}x_{23}x_{13}^{-1}x_{12}$$

and then write:

$$\begin{aligned} \mu_3(x_{12}x_{23}) &= (1 + t_{13})(1 + t_{23})(1 - t_{13} + t_{13}^2 - \dots)(1 + t_{12}) \\ &= 1 + t_{12} + t_{23} + t_{13}t_{23} - t_{23}t_{13} + t_{23}t_{12} + \dots \end{aligned}$$

From a broader perspective, it makes sense to consider the codomain of the mapping μ_m as the completed quotient of the algebra $\mathbb{Z}[\{t_{ij}\}_{1 \leq i < j \leq m}]$ over the ideal generated by the elements $[t_{ij}, t_{kl}]$ and $[t_{ij}, t_{ik} + t_{jk}]$ where all indices are supposed to be distinct and t_{pq} is understood as t_{qp} if $p > q$. We denote this algebra by $\mathcal{A}^h(m)$ and view its generating monomials as horizontal chord diagrams on m vertical strands (each variable t_{ij} represents a chord connecting the i -th and j -th strands; the product of variables is understood as vertical concatenation from top to bottom). Example:

$$t_{13}t_{23}^2t_{12} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} .$$

We will denote the completion of $\mathcal{A}^h(m)$, that is the corresponding algebra of formal series, by $\hat{\mathcal{A}}^h(m)$.

Call a horizontal chord diagram *descending*, if it is represented by a monomial $\prod_s t_{i_s j_s}^{a_s}$ satisfying $j_1 \geq j_2 \geq \dots$. By definition, the set of descending diagrams is in one-to-one correspondence with the set of positive combed braids P_m^+ (braids whose combed form contains only positive powers of the generators x_{ij}). The set of descending chord diagrams forms a basis of the free abelian group $\mathcal{A}^h(m)$ (see [11], Sec. 3-2)¹, therefore we have a module isomorphism $\mathbb{Z}P_m^+ \cong \mathcal{A}^h(m)$.

By an *invariant of braids*, we understand any mapping from the braid group P_m into an arbitrary set — we are interested only in its invariance under the braid isotopy, tacitly assumed in the definition of P_m , and not in the invariance under the renumbering of strands etc. For pure braids, just like in the classical case of knots, one can define the notion of finite type (Vassiliev) invariants, see [1, 11, 3]. It turns out that the Magnus expansion truncated to any degree n is a Vassiliev invariant of order n . Moreover, the following theorem holds:

Theorem. ([11, 13, 3]) *The mapping $\mu_m : P_m \rightarrow \hat{\mathcal{A}}^h(m)$ is a universal finite type invariant of pure braids in the sense that for any degree n invariant $f : P_m \rightarrow \mathbb{Z}$ there exists a map $g : \hat{\mathcal{A}}^h(m) \rightarrow \mathbb{Z}$ vanishing on all monomials of degree greater than n and such that $f = g \circ \mu_m$.*

Remark. In fact, one may define a universal finite type invariant of pure braids by sending each x_{ij} in the combed form into $1 + c_{ij}t_{ij} + T_{ij}$ where c_{ij} are any nonzero constants and T_{ij} are arbitrary series starting with degree greater than one. A remarkable instance of this construction (with values in $\hat{\mathcal{A}}^h(m) \otimes \mathbb{C}$) is provided by the Kontsevich integral ([1, 3]). Its advantage over

¹In a more general setting, this fact easily follows from Theorem 3.1 of [13]. It can also be proved by using non-commutative Gröbner bases (I thank A. Khoroshkin for teaching me the idea of this proof). Closely related formulations are scattered in the works by T. Kohno, V. Drinfeld, An. Kirillov, S. Yuzvinsky etc.

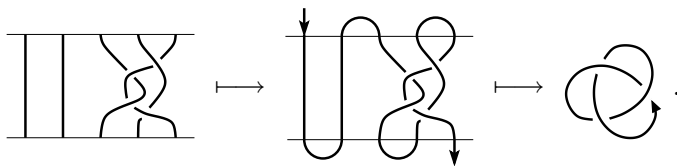
the usual Magnus expansion consists in multiplicativity; however, the definition of the Kontsevich integral is much more involved and its computation much more difficult; moreover, its value depends on the placement of the endpoints of the braid. For example, the Kontsevich integrals of the generating braids of the group P_3 , where the strand endpoints are collinear and equidistant, are infinite series with the following terms up to degree 2:

$$\begin{aligned} I(x_{12}) &= 1 - A + \frac{1}{2}A^2 - \frac{i \ln 2}{2\pi}[B, C] + \dots, \\ I(x_{13}) &= 1 - C + \frac{1}{2}C^2 + \frac{1}{2}[A, B] + \dots, \\ I(x_{23}) &= 1 - B + \frac{1}{2}B^2 + \frac{i \ln 2}{2\pi}[C, A] + \dots, \end{aligned}$$

where $A = t_{12}$, $B = t_{23}$, $C = t_{13}$ and we remind that $[A, B] = [B, C] = [C, A]$ according to the definition of $\hat{\mathcal{A}}^h(3)$. The reader may wish, by way of exercise, to check that these relations agree with the commutation relations in the group P_3 (expressed by the fact that the element $x_{12}x_{13}x_{23}$ is central, see [4]).

3. SHORT-CIRCUIT CLOSURE

Alongside with the ordinary (Artin) closure which turns braids into links, there is another operation of closing pure braids into oriented knots, called *short-circuit closure*, see [10, 3]. It is defined by connecting pairwise by short arcs the upper endpoints number $2i$ and $2i+1$ and the lower endpoints number $2i-1$ and $2i$ thus obtaining a long knot with two loose endpoints. Attaching an additional arc, one obtains a conventional compact knot. Orientation is chosen so that the leftmost strand of the braid is oriented downwards. For example:



(The non-oriented version of this operation, called *plat closure*, was studied earlier, see e.g. [2].)

It is easy to see that the short-circuit closures of braids with different number of strands are consistent with the inclusions $P_m \rightarrow P_{m+1}$ (adding a vertical strand on the right), so that we obtain a well-defined map \varkappa from the group $P_\infty := \cup_{m \geq 1} P_m$ to the set of oriented knots \mathcal{K} . A theorem of Mostovoy and Stanford asserts that this map is onto and that it identifies \mathcal{K} with double cosets of the group P_∞ over two special subgroups, see [10].

In the particular case $m = 3$, the image of $\varkappa_3 = \varkappa|_{P_3}$ coincides with the set of all 2-bridge (rational) knots (see [12] for an introduction to rational knots). Indeed, the short-circuit closure of the braid $x_{13}^{a_1}x_{23}^{b_1}\dots x_{13}^{a_k}x_{23}^{b_k}$, respectively $x_{13}^{a_1}x_{23}^{b_1}\dots x_{13}^{a_k}x_{23}^{b_k}x_{13}^{a_{k+1}}$, where all a_i, b_i are non-zero integers, is the rational knot corresponding to the continued fraction with denominators

$$(2a_1, -2b_1, \dots, 2a_k, -2b_k + 1),$$

respectively,

$$(2a_1, -2b_1, \dots, 2a_k, -2b_k, 2a_k + 1).$$

Now, a simple number-theoretic argument shows that any rational number with odd numerator and denominator has a continued fraction of this kind (the last number is odd, while all the previous ones are even). Rational knots, in distinction to links, correspond to rational fractions with odd denominators. If the numerator of the number p/q happens to be even, then the relevant knot is equivalent to knot $(p \pm q)/q$ (see [12]); our assertion is thus proved.

4. SYMBOL OF THE CONWAY POLYNOMIAL

By taking composition with the short-circuit closure, any invariant of knots can be converted into an invariant of braids. For example, the Conway polynomial of knots $\nabla : \mathcal{K} \rightarrow \mathbb{Z}[t]$ induces an invariant of pure braids $\nabla \circ \varkappa_m : P_m \rightarrow \mathbb{Z}[t]$ (the Conway polynomial of braids). For every n , the coefficient of t^{2n} in this polynomial is a Vassiliev invariant of degree $2n$. By the universality of the Magnus expansion, there is a mapping (the ‘‘symbol’’ of the Conway polynomial) $\chi_m : \hat{\mathcal{A}}^h(m) \rightarrow \mathbb{Z}[t]$ such that $\nabla \circ \varkappa_m = \chi_m \circ \mu_m$. We succeeded in finding an explicit description of the symbol only for pure braids on 3 strands.

The following theorem describes the values of the mapping $\chi = \chi_3$ on descending chord diagrams (which form a basis of the free abelian group $\mathcal{A}^h(3)$). Let $A = t_{12}$, $B = t_{23}$, $C = t_{13}$.

Theorem. *Any descending chord diagram on three strands is a (positive) word in the letters A, B, C where all A 's come at the end. We claim that for any words w, w_1, w_2*

- (1) $\chi(wA) = 0$.
- (2) $\chi(Bw) = 0$.
- (3) $\chi(w_1B^2w_2) = 0$.

Assertions (1), (2), (3) leave us with just two kinds of words:

$$C^{c_1}B \cdot \dots \cdot C^{c_{k-1}}BC^{c_k}$$

and

$$C^{c_1}B \cdot \dots \cdot C^{c_{k-1}}BC^{c_k}B$$

which we encode respectively by $[c_1, \dots, c_k]$ and $[c_1, \dots, c_k]'$.

(4) The values of χ on the elements of the second kind are reduced to its values on the elements of the first kind:

$$\chi([c_1, \dots, c_k]') = t^{-2} \chi([c_1, \dots, c_k, 1]).$$

It thus remains to determine χ on the elements $[c_1, \dots, c_k]$.

(5) We have

$$\chi([c_1, \dots, c_k]) = (-1)^{k-1} \left(\prod_{i=1}^{k-1} p_1 p_{c_i-1} \right) \cdot p_{c_k},$$

where $p_s = \chi([s])$ is a sequence of polynomials in t that can be defined recursively by $p_0 = 1$, $p_1 = t^2$ and $p_{s+2} = t^2(p_s + p_{s+1})$ for $s \geq 0$. In particular, the value of χ on the empty chord diagram (unit of the algebra $\hat{\mathcal{A}}^h(3)$) is 1.

Remark 1. Note that the polynomial $p_k = \chi([k])$ is equal to $t^k \nabla(T_{k+1,2})$ where the letter T denotes the torus link with given parameters (in the case when this is a 2-component link, correct orientations of the components must be chosen) and can be written explicitly as

$$p_k = \sum_{k/2 \leq j \leq k} \binom{j}{2j-k} t^{2j}.$$

Remark 2. The image of χ belongs to the commutative algebra generated by polynomials p_1, p_2 etc, whose additive basis can be identified with (unordered) partitions. In this setting, the map χ is defined by a transformation of ordered partitions into unordered partitions according to the rule

$$[c_1, \dots, c_k] \mapsto (1^{k-1}, c_1 - 1, \dots, c_{k-1} - 1, c_k).$$

Examples.

$$\begin{aligned} \chi(1) &= 1, \\ \chi(B) &= 0, \\ \chi(C) &= t^2, \\ \chi(CB) &= -t^2, \\ \chi(BC) &= 0, \\ \chi(C^3 BC^3) &= -p_1 p_2 p_3 = -t^2(t^4 + t^2)(t^6 + 2t^4). \end{aligned}$$

5. PROOF OF THE THEOREM

We must find a map χ rendering commutative the diagram

$$\begin{array}{ccc} P_3 & \xrightarrow{\mu_3} & \hat{\mathcal{A}}^h(3) \\ \downarrow \varkappa & & \downarrow \chi \\ \mathcal{K} & \xrightarrow{\nabla} & \mathbb{Z}[[t]] \end{array}$$

Extend the Magnus expansion linearly to the map $\widehat{\mathbb{Z}P_3} \rightarrow \hat{\mathcal{A}}^h(3)$ denoted by the same letter μ_3 . We will prove the theorem by finding a right inverse of μ_3 , that is a mapping $\nu_3 : \hat{\mathcal{A}}^h(3) \rightarrow \widehat{\mathbb{Z}P_3}$ such that $\mu_3 \circ \nu_3 = \text{id}$. Indeed, the set of decreasing chord diagrams on 3 strands is in one-to-one correspondence with the set of positive braids P_3^+ . The correspondence is defined simply by setting $x_{ij} \leftrightarrow t_{ij}$. Identifying a word w in x_{ij} with the corresponding word in t_{ij} , we see that

$$\mu_3(w) = \sum_{w' \subseteq w} w'$$

for any positive word w . It is easy to check that the inverse of this map $\widehat{\mathbb{Z}P_3^+} \rightarrow \hat{\mathcal{A}}^h(3)$ is given by the formula

$$\nu_3(w) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|} w',$$

where the absolute value of a word denotes its length (or total exponent). The diagram

$$\begin{array}{ccc} \widehat{\mathbb{Z}P_3} & \begin{array}{c} \xrightarrow{\mu_3} \\ \xleftarrow{\nu_3} \end{array} & \hat{\mathcal{A}}^h(3) \\ \downarrow \varkappa & & \downarrow \chi \\ \widehat{\mathbb{Z}\mathcal{K}} & \xrightarrow{\nabla} & \mathbb{Z}[[t]] \end{array}$$

shows that we have $\chi = \nabla \circ \varkappa \circ \nu_3$ and, consequently,

$$\chi(w) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|} \nabla(\varkappa(w')),$$

where w' , a word in the letters t_{ij} , is understood, via the mentioned identification, as a word in the letters x_{ij} , that is, as a positive pure braid.

We will consecutively prove the five assertions of the theorem by applying this expression for χ and splitting the sum over all 2^n subwords $w' \subseteq w$ into appropriate subsums of 2, 4, ..., 2^k terms.

(1). Split the sum into pairs $\pm(\nabla(\varkappa(w'A)) - \nabla(\varkappa(w')))$ and notice that the knots $\varkappa(w'A)$ and $\varkappa(w')$ are isotopic.

(2). The same argument for the pairs of knots $\varkappa(Bw')$ and $\varkappa(w')$.

(3). The sum giving $\chi(w_1 B^2 w_2)$ consists of quadruples defined by a choice of subwords $w'_1 \subseteq w_1$, $w'_2 \subseteq w_2$:

$$\pm(\nabla(\varkappa(w'_1 B^2 w'_2)) - 2\nabla(\varkappa(w'_1 B w'_2)) + \nabla(\varkappa(w'_1 w'_2))).$$

We shall prove that any such quadruple sums to zero.

Indeed, the defining skein relation for the Conway polynomial implies that

$$\nabla(\text{braid 1}) - \nabla(\text{braid 2}) = -t\nabla(\text{crossing}) = \nabla(\text{braid 3}) - \nabla(\text{braid 4}),$$

where the braids corresponding to the words w'_1 and w'_2 are depicted as rectangular boxes.

Therefore,

$$\nabla(\text{braid 1}) - 2\nabla(\text{braid 2}) + \nabla(\text{braid 3}) = 0,$$

as required.

(4). We will prove that for any word w we have $\chi(wBC) = t^2\chi(wB)$. Indeed,

$$\begin{aligned} \chi(wB) &= \sum_{w' \subseteq w} (-1)^{|w| - |w'|} (\nabla(w'B) - \nabla(w')), \\ \chi(wBC) &= \sum_{w' \subseteq w} (-1)^{|w| - |w'|} (\nabla(w'BC) - \nabla(w'B) - \nabla(w'C) + \nabla(w')). \end{aligned}$$

Now, the needed assertion follows from the identity

$$\nabla(w'BC) - \nabla(w'C) = (t^2 + 1)(\nabla(w'B) - \nabla(w')),$$

which is proved by applying the Conway skein relation:

$$\nabla(\text{braid 1}) - \nabla(\text{braid 2}) = -t\nabla(\text{crossing}),$$

$$\nabla(\text{trefoil}) - \nabla(\text{strand}) = -t \nabla(\text{strand with loop})$$

and noticing that taking the sum of a link with the trefoil knot (which happens in the pictures on the right) leads to the multiplication of the corresponding polynomial by $(t^2 + 1)$.

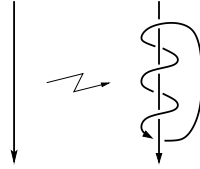
(5). Here we must show that $\chi(C^n Bw) = -p_1 p_{n-1} \chi(w)$ where w is an arbitrary word in C and B . Indeed, let us split the alternating sum for $\chi(C^n Bw)$ into the parts corresponding to a fixed subword $w' \subseteq w$:

$$\chi(C^n Bw) = \sum_{w' \subseteq w} (-1)^{|w| - |w'|} \sum_{l=1}^n (-1)^{n-l} \binom{n}{l} (\nabla \varkappa(C^l Bw') - \nabla \varkappa(C^l w')).$$

Using the Conway skein relations on a proper crossing, we get:

$$\nabla \varkappa(C^l Bw') - \nabla \varkappa(C^l w') = -t \nabla(K_{(l)}),$$

where $K = \varkappa(w')$ and $K_{(l)}$ denotes the oriented 2-component link obtained from the oriented knot K by adding a trivial l -linked component to K according to the picture (example for $l = 3$):



Lemma. For any knot K and any natural number l , we have

$$\nabla(K_{(l)}) = t(q_0 + q_1 + \dots + q_{l-1}) \nabla(K),$$

where q_s is the Conway polynomial of the torus knot of type $(2, 2s + 1)$ given explicitly by

$$q_s = \sum_{j=0}^s \binom{s+j}{s-j} t^{2j}.$$

This equality is proved, as usual, by recursively applying the Conway skein relation. Substituting it into the previous formula for $\chi(C^n Bw)$, we get

$$-t^2 \cdot \sum_{l=1}^n (-1)^{n-l} \binom{n}{l} \sum_{s=0}^{l-1} q_s \cdot \sum_{w' \subseteq w} (-1)^{|w| - |w'|} \nabla(\varkappa(w')),$$

and it remains to show that the middle term of this product is equal to p_{n-1} . Indeed, it is easily transformed to the form

$$\sum_{s=0}^{n-1} (-1)^{n-1-s} \binom{n-1}{s} q_s$$

or, recalling the expression for q_s , to

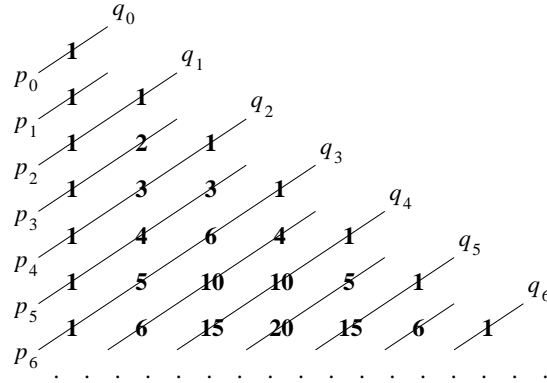
$$\sum_{s=0}^{n-1} \sum_{j=0}^s (-1)^{n-1-s} \binom{n-1}{s} \binom{s+j}{s-j} t^{2j}.$$

Changing the order of summation, this can be rewritten as

$$(-1)^{n-1} \sum_{j=0}^{n-1} \left[\sum_{s=j}^{n-1} (-1)^s \binom{n-1}{s} \binom{s+j}{2j} \right] t^{2j}.$$

An application of the product summation formula (5.24) from [7] to the sum over s inside the brackets gives $(-1)^{n-1} \binom{j}{2j-n+1}$ thus proving the required assertion.

Remark. The coefficients of the polynomials p_n and q_n can be read off the Pascal triangle in this way:



6. EVALUATION ON THE ASSOCIATOR

The Drinfeld associator [4, 9] is a remarkable element of the algebra $\hat{\mathcal{A}}^h(3)$, given by an infinite series in the (non-commuting) variables $a = A/(2\pi i)$, $b = B/(2\pi i)$ with coefficients in the algebra of multiple zeta values (MZV, see [8]):

$$\begin{aligned} \Phi = & 1 - \zeta_2[a, b] - \zeta_3([a, [a, b]] + [b, [a, b]]) \\ & - \zeta_4[a, [a, [a, b]]] - \zeta_{3,1}[b, [a, [a, b]]] - \zeta_{2,1,1}[b, [b, [a, b]]] + \frac{1}{2} \zeta_2^2[a, b]^2 \dots \end{aligned}$$

(see [5] for an explicit expansion of Φ up to degree 12).

Taking the value of the complexified mapping $\chi_{\mathbb{C}} : \hat{\mathcal{A}}^h(3) \otimes \mathbb{C} \rightarrow \mathbb{C}[t]$ on this element Φ , we obtain the following result:

Conjecture.

$$\begin{aligned} \chi_{\mathbb{C}}(\Phi) &= -\zeta_2 T^2 + (-\zeta_3 + \zeta_{2,2}) T^4 + (-\zeta_4 + \zeta_{2,3} + \zeta_{3,2} - \zeta_{2,2,2}) T^6 + \dots \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^k \zeta_{n+k}^{(k)} \right) T^{2n}, \end{aligned}$$

where $T = t/(2\pi i)$, the numbers $\zeta_{l_1, \dots, l_k} = \zeta(l_1, \dots, l_k)$ are multiple zeta values and $\zeta_m^{(k)}$ is the short-hand notation for the sum of all ζ_{l_1, \dots, l_k} where $l_i \geq 2$ for $i = 1, \dots, k$ and $l_1 + l_2 + \dots + l_k = m$.

We have checked this formula by computer up to T^{10} (see [5]) using the table of relations between MZV's provided in [14]. It is an interesting remark that, when evaluated numerically, the coefficients of this polynomial:

$$\begin{aligned} &-1.644934 T^2 - 0.390314 T^4 - 0.332698 T^6 - 0.312405 T^8 \\ &-0.303958 T^{10} - 0.300153 T^{12} - 0.298365 T^{14} - 0.297505 T^{16} + \dots \end{aligned}$$

seem to tend to a limit whose nature remains unclear.

7. OPEN PROBLEMS

- (1) For what triples of reduced rational fractions $p_1/q_1, p_2/q_2, p_3/q_3$ where both numerators and denominators form arithmetical progressions, the values of the Conway polynomial on the corresponding rational knots also form an arithmetical progression? Our proof of assertion 3 of the main theorem (see page 8) gives an abnormally big number of such triples, for instance, $(\frac{3}{11}, \frac{13}{31}, \frac{23}{51})$, $(\frac{5}{11}, \frac{9}{19}, \frac{13}{27})$, $(\frac{5}{17}, \frac{11}{41}, \frac{17}{65})$, $(\frac{13}{75}, \frac{19}{111}, \frac{25}{147})$, yet the claim about arithmetical progressions is not true in general.
- (2) Generalize the main theorem (Section 4) in two directions: (A) to pure braids with an arbitrary number of strands, (B) to the HOMFLY polynomial which is a generalization of the Conway polynomial.
- (3) This is related to the Remark on page 3. Describe all (or some) triples of formal series P, Q, R in $\hat{\mathcal{A}}^h(3) \otimes \mathbb{C}$ starting with terms of degree higher than one, such that the correspondence $x_{12} \mapsto 1 + t_{12} + P$, $x_{23} \mapsto 1 + t_{23} + Q$, $x_{13} \mapsto 1 + t_{13} + R$ defines a group homomorphism.
- (4) Prove the conjecture from Section 6. Find similar facts for other horizontal associators and relate them to the action of the Grothendieck–Teichmüller group (see [4, 6]).

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