Introduction to the mechanics of Cosserat media

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Plan of lectures

- Some facts from the tensorial algebra
- How to deduce equations of a continuum
- Basic laws of mechanics
- Elastic continua: classic, full Cosserat, reduced Cosserat
- Basic laws for elastic continua. Integral and local forms.
- Nonlinear full Cosserat continuum. Equations
- Nonlinear reduced Cosserat continuum. Equations
- Linear full Cosserat continuum. Equations
- Linear isotropic full Cosserat continuum. Equations
- Linear reduced Cosserat continuum. Equations
- Linear isotropic reduced Cosserat continuum. Equations

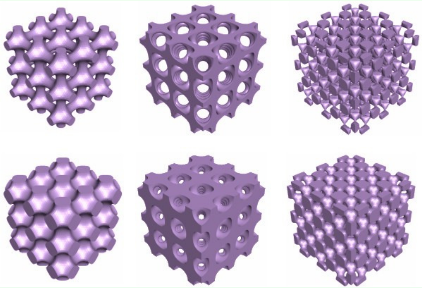
Lectures 1,2.

What is the mechanics of the Cosserat media?

Some facts from the tensorial algebra

Complex materials

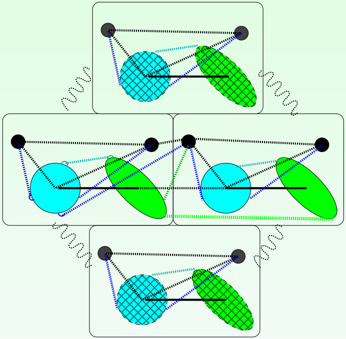
Classical continuum is a continuum of point masses that only can move. However, the reality is much richer! Advanced applied science deals with



3D acoustic metamaterials with effective negative elastic moduli / density for some frequencies. Control of wave beams, acoustic cloaking, noise reduction, ...

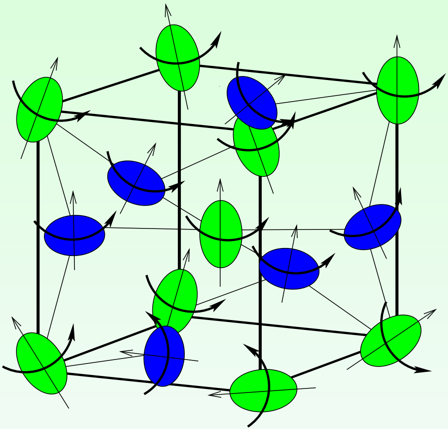
What is mechanics of the Cosserat media?

Cosserat medium is a continuum whose point bodies (particles) have rotational degrees of freedom.



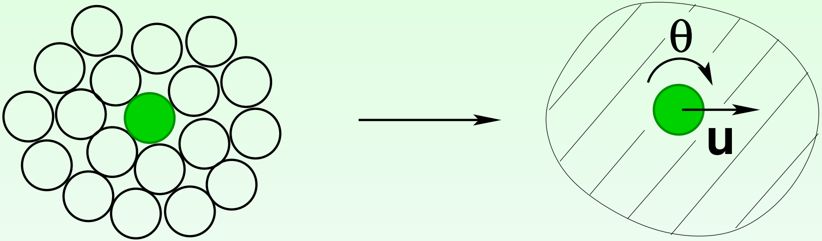
Hand-made smart materials

What is mechanics of the Cosserat media?



Magnetic materials (Kelvin's medium — special Cosserat medium with particle posessing large spin)

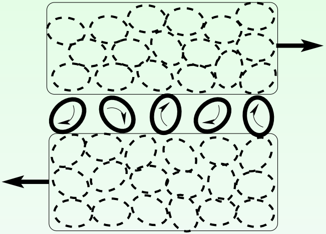
What is mechanics of the Cosserat media?



Granular and particulate materials

What is mechanics of the Cosserat media?

Cosserat media: heterogeneous materials with granular structure, composites under loading that causes rotation of (sufficiently rigid) grains (superplastic materials, acoustic metamaterials)



Shear in a granular/particulate medium

What is mechanics of the Cosserat media?

Limitations: Cosserat medium is a particular case of complex medium. Its point-body is rigid. There are other more complex media, e.g. where a point-body is deformable (protein chains, porous media, etc.) It is only a first step to the world of enriched continua. In this course we will make an introduction to the **elastic** Cosserat media. **No temperature or heat** effects are considered.

Methods: Theory is based on the fundamental laws of mechanics (balance of forces, couples, energy) and, for inelastic media, 2nd law of the thermodynamics. Symmetry considerations and material objectivity (frame indifference). Another branch is the microstructural approach. Experimental methods: under development. We need experiments to determine the moduli. Most of them are based on the experiments on waves (mechanics of magnetic and piezoelectric materials, mechanics of granular materials, rotational seismology...)

Your suggestions?

What do we need to know to describe a behaviour of a Cosserat-like material?

Scope of the course

Continuum deforms. Therefore we cannot find out its motion only from equilibrium or dynamic equations. Constitutive equations tell us how the medium reacts to the stresses in each point. They are needed to solve any problem where there are deformations.

- Mathematical technique: **tensorial algebra**. (Brief overview.)
- **Basic equations** of the elastic Cosserat media (constitutive and dynamic equations)
- **Waves** in the elastic Cosserat continua

Tensorial algebra. Plan

- Tensors. Definition
- Co-ordinates and direct tensorial notation
- Polar and axial vectors and tensors
- Tensor invariants
- Important identities
- Accompanying vector of an antisymmetric tensor
- Orthogonal tensors. Rotation tensor

Literature: books by P.A. Zhilin and A.I. Lurie on tensorial algebra

Tensorial product. Diades and tensors

Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$

Operation $\otimes$: tensorial product. Linear in its arguments.

Dyadic $\mathbf{a}\otimes\mathbf{b}$ is a linear object with respect to both vectors:

$(\alpha\mathbf{a}+\beta\mathbf{b})\otimes\mathbf{c} = \alpha\mathbf{a}\otimes\mathbf{c}+\beta\mathbf{b}\otimes\mathbf{c}$, $\mathbf{c}\otimes(\alpha\mathbf{a}+\beta\mathbf{b}) = \alpha\mathbf{c}\otimes\mathbf{a}+\beta\mathbf{c}\otimes\mathbf{b}$.

This is an ordered pair: $\mathbf{a}\otimes\mathbf{b} \neq \mathbf{b}\otimes\mathbf{a}$ (generally speaking).

A tensor of second rank is a sum of dyadics:

$$\mathbf{A} = \mathbf{a}\otimes\mathbf{b}+\mathbf{c}\otimes\mathbf{d}+\mathbf{e}\otimes\mathbf{f}+\cdots = \sum_{i=1}^{n}\mathbf{a}_{(i)}\otimes\mathbf{b}^{(i)}$$

Questions

1) How to introduce a tensor of the 4th rank?

2) We will work with 3D vectors. What is the minimal n such that any tensor of the 2nd rank can be represented as $\sum_{i=1}^{n}\mathbf{c}_{(i)}\otimes\mathbf{d}^{(i)}$?

3) How to introduce and represent an identity tensor?

Answer 1. Tensor of the 4th rank

${}^4\mathbf{A} = \mathbf{a}_1 \otimes \mathbf{b}_1 \otimes \mathbf{c}_1 \otimes \mathbf{d}_1 + \mathbf{a}_2 \otimes \mathbf{b}_2 \otimes \mathbf{c}_2 \otimes \mathbf{d}_2 + \ldots$ — a sum of polyadics of the 4th rank. Polyadycs are linear in their arguments and the permutation of vectores is not allowed (generally speaking)

Answer 2. Co-ordinates

We see that tensors are linear with respect to the vectors of their dyadics. We choose an orthonormal basis in 3D space: $\mathbf{i}_k$, $k = \overline{1,3}$. Any dyadics can be represented as

$$\mathbf{a}\otimes\mathbf{b} = \sum_{m=1}^{3} a^m \mathbf{i}_m \otimes \sum_{n=1}^{3} b^n \mathbf{i}_n = \sum_{m=1}^{3} \sum_{n=1}^{3} a^m b^n \mathbf{i}_m \otimes \mathbf{i}_n.$$

The numbers $L^{mn} = a^m b^n$ form a matrix of co-ordinates of the dyadic $\mathbf{a}\otimes\mathbf{b}$ in the basis $\mathbf{i}_k$.

Express a^n, b^n in terms of $\mathbf{a}, \mathbf{b}, \mathbf{i}_k$ and L^{mn} in terms of $\mathbf{a}\otimes\mathbf{b}, \mathbf{i}_k$.

Any tensor of the 2nd rank is a sum of dyadics: $\mathbf{A} = \sum_{j=1}^{J} \mathbf{a}_j \otimes \mathbf{b}^j$. Let us proceed with all the dyadics the same. We have

$$\mathbf{A} = \sum_{j=1}^{J} \sum_{m=1}^{3} \sum_{n=1}^{3} a_j^m \mathbf{i}_m \otimes b^{j\,n} \mathbf{i}_n = \sum_{m=1}^{3} \sum_{n=1}^{3} \Big(\sum_{j=1}^{J} a_j^m b^{j\,m}\Big) \mathbf{i}_m \otimes \mathbf{i}_n.$$

In 3D space any tensor of the 2^{nd} rank is a sum of $\leqslant 9$ dyadics.

Answer 3. Identity tensor

$\mathbf{E}$ is the identity tensor if for any tensor of the 2nd rank $\mathbf{A}$ it holds $\mathbf{A}\cdot\mathbf{E}=\mathbf{E}\cdot\mathbf{A}=\mathbf{A}$.

Let us prove that $\mathbf{E}$ exists. Choose an orthonormal basis $\mathbf{i}_k$.

$\mathbf{E}=\sum_{k=1}^{3}\mathbf{i}_k\otimes\mathbf{i}_k$. Indeed, $\mathbf{A}\cdot\mathbf{E}=\sum_{m,n=1}^{3}\mathbf{A}^{mn}\mathbf{i}_m\mathbf{i}_n\cdot\sum_{k=1}^{3}\mathbf{i}_k\otimes\mathbf{i}_k=$

$\sum_{m,n=1}^{3}\sum_{k=1}^{3}\mathbf{A}^{mn}\mathbf{i}_m\delta_{kn}\otimes\mathbf{i}_k=\sum_{m,n=1}^{3}\mathbf{A}^{mn}\mathbf{i}_m\otimes\mathbf{i}_n=\mathbf{A}$.

In the same way we prove that $\mathbf{E}\cdot\mathbf{A}=\mathbf{A}$.

Prove that $\mathbf{E}$ is unique. Suppose that there exist $\mathbf{E}_1,\mathbf{E}_2$, both identity tensors. In this case $\mathbf{E}_1=\mathbf{E}_1\cdot\mathbf{E}_2=\mathbf{E}_2$.

Verify that for tensor $\mathbf{a}$ of any rank (including 1, i.e. for vectors) $\mathbf{a}\cdot\mathbf{E}=\mathbf{E}\cdot\mathbf{a}=\mathbf{a}$.

Co-ordinates

$\mathbf{A} = A^{mn}\mathbf{i}_m\mathbf{i}_n$. The coefficients A^{mn} form a matrix of co-ordinates in the basis $\mathbf{i}_m$.
Let us omit the sign $\sum$ and sum in repeated indices of the Roman alphabet from 1 to 3 (in Greek indices from 1 to 2) if one is subscript and another superscript.
A tensor $\mathbf{A}$ does not change when we change the basis $\mathbf{i}_k$. Its co-ordinates change.
A tensor that represents a physical object, generally speaking, depends on the system of reference (it is a physical thing), but never depends on the system of co-ordinates (which is a mathematical thing that we choose arbitrarily).

Direct tensorial notation

This allows us to perform calculus in the simplest way. We see that we can omit the symbol $\otimes$. In this case $\mathbf{ab}$ is a dyadic $\mathbf{a}\otimes\mathbf{b}$. We can do it since $\lambda\otimes\mathbf{a} = \lambda\mathbf{a}$, if λ is a scalar.
We shall use the notation $\mathbf{A}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{A}$ for scalar products of tensor $\mathbf{A}$ and vector $\mathbf{b}$, and notation $\mathbf{A}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{A}$ for vectorial products (e.g. $\mathbf{A}\times\mathbf{b} = \mathbf{a}^k\otimes\mathbf{a}'_k\times\mathbf{b}$).
Example 1. $\boldsymbol{\Theta} = \lambda\mathbf{k}\otimes\mathbf{k} + \mu(\mathbf{E} - \mathbf{k}\otimes\mathbf{k})$ — tensor of inertia of a body with axial symmetry about the axis $\mathbf{k}$. Let us calculate its moment of inertia with respect to the axis $\mathbf{n} = (\mathbf{i}+\mathbf{j})\sqrt{2}/2$:
$\mathbf{n}\cdot\boldsymbol{\Theta}\cdot\mathbf{n} = \mathbf{n}\cdot(\lambda\mathbf{k}\otimes\mathbf{k} + \mu(\mathbf{E} - \mathbf{k}\otimes\mathbf{k}))\cdot\mathbf{n} =$
$\mathbf{n}\cdot\lambda\mathbf{k}\otimes\mathbf{k}\cdot\mathbf{n} + \mu\mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n} - \mu\mathbf{n}\cdot\mathbf{k}\otimes\mathbf{k}\cdot\mathbf{n} = 0 + \mu\mathbf{n}\cdot\mathbf{n} - 0 = \mu.$
Examples 2,3. $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\otimes\mathbf{b}\cdot\mathbf{c} = \mathbf{a}\mathbf{b}\cdot\mathbf{c}$;
Let us omit $\otimes$: $(\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}\mathbf{b}\cdot\mathbf{c} = \mathbf{a}\mathbf{b}\cdot\mathbf{c}$.
$\mathbf{c}\times\mathbf{a}\otimes\mathbf{b} = \mathbf{c}\times\mathbf{a}\otimes\mathbf{b}$. Omitting $\otimes$, we have $\mathbf{c}\times\mathbf{ab} = \mathbf{c}\times\mathbf{a}\mathbf{b}$.

Direct tensorial notation

Introduce $\mathbf{A} \cdot \mathbf{B}, \mathbf{A} \times \mathbf{B}, \mathbf{A}^\top, \mathbf{A} \cdot\cdot\, \mathbf{B}$ etc. for dyadics. (Exercise 1: write down the definition for tensors in general case, verify that we can omit the symbol $\otimes$ and that we simply may forget about the brackets.)

$(\mathbf{a}\otimes\mathbf{b}) \cdot (\mathbf{c}\otimes\mathbf{d}) = \mathbf{a}\otimes(\mathbf{b} \cdot \mathbf{c})\mathbf{d} = \mathbf{b} \cdot \mathbf{c}\, \mathbf{a}\otimes\mathbf{d}$

$(\mathbf{a}\otimes\mathbf{b}) \times (\mathbf{c}\otimes\mathbf{d}) = \mathbf{a}\otimes(\mathbf{b} \times \mathbf{c})\otimes\mathbf{d}$

$(\mathbf{a}\otimes\mathbf{b})^\top = \mathbf{b}\otimes\mathbf{a}$

$(\mathbf{a}\otimes\mathbf{b}) \cdot\cdot\, (\mathbf{c}\otimes\mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})\mathbf{d} = \mathbf{a} \cdot \mathbf{d}\, \mathbf{b} \cdot \mathbf{c}$

In USA they use the operation $\mathbf{A} : \mathbf{B} = \mathbf{A} \cdot\cdot\, \mathbf{B}^\top$ (write down for dyadics)

$\mathrm{tr}\, \mathbf{A} = \mathbf{A} \cdot\cdot\, \mathbf{E}$

Vectorial invariant $[\mathbf{a}\otimes\mathbf{b}]_\times = \mathbf{a} \times \mathbf{b}$

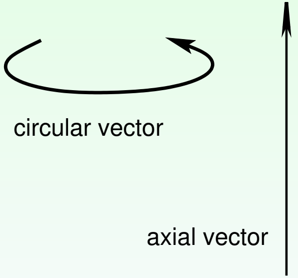
Dual bases and co-ordinates

Let $\mathbf{i}_k$ be a basis. The dual basis $\mathbf{i}^k$ is defined by: $\mathbf{i}_k \cdot \mathbf{i}^n = \delta_k^n$. Delta of Kronecker $\delta_k^n = 1$ if $k = n$, and $\delta_k^n = 0$ if $k \neq n$.
Verify that if $\mathbf{i}^k$ is dual for $\mathbf{i}_k$, then $\mathbf{i}_k$ is dual for $\mathbf{i}^k$, and that an orthonormal basis is dual for itself.
Let $\mathbf{a}$ be a vector. Let us find its co-ordinates in both bases. Look for $a^{(k)}$ such that $\mathbf{a} = a^{(k)}\mathbf{i}_k$. $\qquad | \cdot \mathbf{i}^s$
$\mathbf{a} \cdot \mathbf{i}^s = a^{(k)}\mathbf{i}_k \cdot \mathbf{i}^s = a^{(k)}\delta_k^s = a^{(s)}$.
Co-ordinates in the dual basis: $\mathbf{a} \cdot \mathbf{i}_s = a_{(k)}\mathbf{i}^k \cdot \mathbf{i}_s = a_{(k)}\delta_s^k = a_{(s)}$.
Verify that $\mathbf{i}_k\mathbf{i}^k$ is the identity tensor, if $\mathbf{i}_k$ is any basis.
When changing the basis, the co-ordinates of a vector or a tensor change. When they change in the same way as the basis, they are "covariant", we use a subscript for notation. When they change in the same way as dual basis, they are called "contravariant", and we use superscripts. (Give examples. Express a^i via a_i and $\mathbf{i}_k \cdot \mathbf{i}_s$.)

Polar and axial vectors

Vectors are mathematical objects that can reflect something physical, for instance, correspond to a translational displacement, or to a rotation. To describe a translation in space in a certain direction for a certain distance we use polar vectors (they have direction and absolute value).

To describe a rotation we need "spin-vectors", or "circular vectors" introduced by P.A. Zhilin (a circular arrow, its direction corresponds to the direction of rotation, and its longitude to the absolute value of rotation). We put in correspondence to a circular vector a straight vector of the same longitude (axial vector), for instance, using the "right hand screw rule"



Polar and axial vectors

We introduce an axial vector because it is easy to work with it, but we could also define it by a "left hand screw rule". Choosing the rule, we define the orientation of the system of reference and we must perform all the operations with the same orientation. This orientation does not correspond to any physical reality, this is only our arbitrary choice. Axial vectors are also called "pseudovectors".

A vector is a *polar* vector if it does not depend on the orientation of the system of reference.

A vector is an *axial* vector, if the change the orientation of the system of reference changes its direction to the opposite, and its longitude does not change.

One cannot add polar vectors to axial ones (the absolute value of the sum would depend on the orientation). This is so since polar vectors correspond to the translation, and axial vectors to the rotation.

Polar and axial vectors

Remark 1: Polar and axial vectors differ in the physical sense.
Remark 2: The orientation of the system of reference has nothing to do with the orientation of the system of co-ordinates which we use. Having chosen any orientation of the system of reference we may use no system of co-ordinates at all, or we may orient it in the same or opposite way.
Questions: 1) how to introduce a polar or axial tensor / scalar?
2) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be a) polar; b) axial; c) polar and axial vectors. What type have 1) vector $\mathbf{a} \times \mathbf{b}$ 2) scalar $\mathbf{a} \cdot \mathbf{b}$ 3) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$? Does the result change if $\mathbf{a}$, $\mathbf{b}, \mathbf{c}$ are tensors?
Exercises.
2. $\mathbf{x} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{x} + \alpha = 0$. Vector $\mathbf{a}$ and scalar α are given. Find:
1) vector $\mathbf{x}$ (general solution) 2) solution with minimal and maximal absolute values.
3. $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ (vectors), find $\mathbf{x}$ and $|\mathbf{x}|_{\min}$.

Determinant

$$\det \mathbf{A} = \frac{[(\mathbf{A}\cdot\mathbf{a})\times(\mathbf{A}\cdot\mathbf{b})]\cdot(\mathbf{A}\cdot\mathbf{c})}{[\mathbf{a}\times\mathbf{b}]\cdot\mathbf{c}}$$

Prove that if $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\neq 0$ (these vectores are lineally independent), $\det\mathbf{A}$ does not depend on the choice of $\mathbf{a},\mathbf{b},\mathbf{c}$. Prove:
$\det\mathbf{A}^\top = \det\mathbf{A}$, $\det(\mathbf{A}\cdot\mathbf{B}) = \det\mathbf{A}\det\mathbf{B}$, $\det\mathbf{A}^{-1} = \frac{1}{\det \mathrm{A}}$.
How to define $\mathbf{A}^{-1}$?
Find $\mathbf{E}^{-1}$ and $(\Theta^i\mathbf{e}_i\mathbf{e}_i)^{-1}$, where $\mathbf{e}_i$ is an orthonormal basis.

Always remember:
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is an oriented volume of a parallelepiped generated by vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. If they lie in the same plane (in particular, if some of them coincide), it is 0. Cyclic permutations of vectors do not change it, and others change its sign.

Invariants of the 2nd rank tensor

$\mathrm{I}_1(\mathbf{A}) = \operatorname{tr}\mathbf{A} = \mathrm{A}^{\mathrm{mn}}\mathbf{i}_{\mathrm{m}} \cdot \mathbf{i}_{\mathrm{n}} = \mathbf{A} \cdot\cdot\, \mathbf{E}$.
$\mathrm{I}_2(\mathbf{A}) = ((\operatorname{tr}\mathbf{A})^2 - \operatorname{tr}\mathbf{A}^2)/2$,
$\mathrm{I}_3 = \det(\mathbf{A}) = \frac{1}{6}(\operatorname{tr}\mathbf{A})^3 - \frac{1}{2}\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{A}^2 + \frac{1}{3}\operatorname{tr}\mathbf{A}^3$.
For a symmetric tensor the physical sense of invariants is related to perimeter, surface and volume of a rectangular parallepiped whose sides are equal to the tensor eigenvalues
Prove:
$\operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{A}^\top$, $\operatorname{tr}(\mathbf{A}\cdot\mathbf{B}) = \operatorname{tr}(\mathbf{B}\cdot\mathbf{A}) = \mathbf{A} \cdot\cdot\, \mathbf{B}$, $\operatorname{tr}(\mathbf{A}\cdot\mathbf{B}) = \operatorname{tr}(\mathbf{A}^\top\cdot\mathbf{B}^\top)$.

** Identities **

Prove what you can.
Identity of Cayley–Hamilton

$$-\mathbf{A}^3 + \mathrm{I}_1(\mathbf{A})\mathbf{A}^2 - \mathrm{I}_2(\mathbf{A})\mathbf{A} + \mathrm{I}_3(\mathbf{A})\mathbf{E} = 0$$

How to express other degrees (positive and negative) of $\mathbf{A}$ via $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3$, using this identity?
$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$.
If $\det \mathbf{A} \neq 0$,

$$(\mathbf{A} \cdot \mathbf{a}) \times (\mathbf{A} \cdot \mathbf{b}) = (\det \mathbf{A})\mathbf{A}^{-\top} \cdot (\mathbf{a} \times \mathbf{b}).$$

If $\mathbf{S} = \mathbf{S}^\top$, $\mathbf{T} = -\mathbf{T}^\top = \mathbf{t} \times \mathbf{E} \Longrightarrow \det(\mathbf{S} + \mathbf{T}) = \det \mathbf{S} + \mathbf{t} \cdot \mathbf{S} \cdot \mathbf{t}$.

Antisymmetric tensor. Accompanying vector

If $\mathbf{A} = -\mathbf{A}^\top$, there exist a vector $\mathbf{a}$ such that $\mathbf{A} = \mathbf{a} \times \mathbf{E}$.
$[\mathbf{A}]_\times = -2\mathbf{a}$. Prove this.
$\mathbf{a}$ is the accompanying vector of $\mathbf{A}$.

Orthogonal tensors

If for any vector $\mathbf{x}$

$$|\mathbf{Q}\cdot\mathbf{x}| = |\mathbf{x}|, \qquad (1)$$

$\mathbf{Q}$ is an orthogonal tensor. Prove that it is the same that to require $\mathbf{Q}\cdot\mathbf{Q}^\top = \mathbf{E}$, that $\det\mathbf{Q} = \pm 1$ and that $\mathbf{Q}_1\cdot\mathbf{Q}_2$ is orthogonal if $\mathbf{Q}_1, \mathbf{Q}_2$ are orthogonal.

An orthogonal tensor does not change the angles between vectors.

$(\mathbf{Q}\cdot\mathbf{a})\cdot(\mathbf{Q}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$.

Proof: $(\mathbf{Q}\cdot\mathbf{a})\cdot(\mathbf{Q}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{Q}^\top\cdot\mathbf{Q}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{E}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$.

Note:

$[(\mathbf{Q}\cdot\mathbf{e}_1)\times(\mathbf{Q}\cdot\mathbf{e}_2)]\cdot(\mathbf{Q}\cdot\mathbf{e}_3) = \pm[\mathbf{e}_1\times\mathbf{e}_2]\cdot\mathbf{e}_3$.

If $\det\mathbf{Q} = 1$, the tensor $\mathbf{Q}$ does not change the orientation of a vector basis (rotation tensor, or tensor of turn in Zhilin's terminology).

If $\det\mathbf{Q} = -1$, it changes the orientation of a triadic (tensor of reflection).

** Orthogonal tensors. Identities **

Prove what you can

$$\mathbf{Q}\cdot(\mathbf{a}\times\mathbf{Q}^\top)=\mathbf{Q}\cdot(\mathbf{a}\times\mathbf{E})\cdot\mathbf{Q}^\top=\det\mathbf{Q}[(\mathbf{Q}\cdot\mathbf{a})\times\mathbf{E}],$$

$$(\mathbf{Q}\times\mathbf{a})\cdot\mathbf{Q}^\top=\mathbf{Q}\cdot(\mathbf{E}\times\mathbf{a})\cdot\mathbf{Q}^\top=\det\mathbf{Q}[(\mathbf{Q}\cdot\mathbf{a})\times\mathbf{E}],$$

$$\mathrm{tr}\,(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^\top)=\mathrm{tr}\,\mathbf{A},$$

$$\det(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^\top)=\det\mathbf{A},$$

$$\mathrm{I}_2(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^\top)=\mathrm{I}_2(\mathbf{A}),$$

$$\mathrm{tr}\,((\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^\top)^{\mathrm{n}})=\mathrm{tr}\,(\mathbf{A}^{\mathrm{n}})$$

Rotation tensor

Rotation tensor is an orthogonal tensor whose determinant is 1:
$\mathbf{P} \cdot \mathbf{P}^\top = \mathbf{E}$, $\det \mathbf{P} = 1$.
Theorem by Euler: any rotation tensor (except $\mathbf{E}$) can be represented in a unique way as

$$\mathbf{P}(\theta \mathbf{m}) = (1 - \cos\theta)\mathbf{m}\,\mathbf{m} + \cos\theta \mathbf{E} + \sin\theta\, \mathbf{m} \times \mathbf{E}, \quad -\pi < \theta < \pi,$$

θ is an angle of rotation and $\mathbf{m}$ is an axis of rotation (fixed vector of $\mathbf{P}$).
Verify: $\mathbf{P}(\theta\mathbf{m}) \cdot \mathbf{m} = \mathbf{m} \cdot \mathbf{P}(\theta\mathbf{m}) = \mathbf{m}$.
Calculate $\mathbf{P}(\theta\mathbf{m}) \cdot \mathbf{n}$, where $\mathbf{n} \cdot \mathbf{m} = 0$.
If $\mathbf{d}_k$ is a basis and $\mathbf{D}_k = \mathbf{P} \cdot \mathbf{d}_k$, it holds $\mathbf{P} = \mathbf{D}_k \mathbf{d}^k$ (verify it).

Rotation tensor. Properties

Is $\mathbf{P}_1 \cdot \mathbf{P}_2$ equal to $\mathbf{P}_2 \cdot \mathbf{P}_1$ or not?
Answer: sometimes. Generally speaking, NO (draw examples). It is true only when axes of rotation coincide. (Prove if you have a wish.)
Calculate the rotation vector for tensor $\mathbf{P}_2 \cdot \mathbf{P}_1$ for the case when the axes of rotations 1,2 coincide.
A rotation tensor is represented as a composition of three rotations about three fixed axes (the second does not coincide neither with the first one nor with the third one):

$$\mathbf{P} = \mathbf{P}_3(\psi \mathbf{m}_0) \cdot \mathbf{P}_2(\theta \mathbf{n}_0) \cdot \mathbf{P}_1(\varphi \mathbf{l}_0) \qquad (2)$$

If $\mathbf{m}_0 = \mathbf{l}_0$, $\mathbf{m}_0 \cdot \mathbf{n}_0 = \mathbf{l}_0 \cdot \mathbf{n}_0 = 0$, ψ, θ, φ are angles of precession, nutation, proper rotation.

Angular velocity and angular strains

Equation of Poisson: $\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}$. Prove that ω exists and find it. (Help: Prove that $\dot{\mathbf{P}} \cdot \mathbf{P}^\top$ is an antisymmetric tensor. Calculate its accompanying vector.)

Spatial analogue for the equation of Poisson: if q^i are co-ordinates, $\partial_i = \frac{\partial}{\partial q^i}, \quad \partial_i \mathbf{P} = \mathbf{\Phi}_i \times \mathbf{P}$.

1) Prove that if $\mathbf{P} = \mathbf{P}_2 \cdot \mathbf{P}_1$, then $\boldsymbol{\omega} = \boldsymbol{\omega}_2 + \mathbf{P}_2 \cdot \boldsymbol{\omega}_1$.

2) Calculate the angular velocity $\boldsymbol{\omega}$ for the case when the rotation axes $\mathbf{P}_1$ and $\mathbf{P}_2$ coincide.

3) Obtain the formula for $\boldsymbol{\omega}$ in terms of the angles of precession, nutation, proper rotation.

4) **Hometask:** prove that

$$\partial_i \boldsymbol{\omega} = \dot{\mathbf{\Phi}}_i + \mathbf{\Phi}_i \times \boldsymbol{\omega}. \tag{3}$$

Infinitesimal rotation

If the angle of rotation θ is infinitesimal,
$\mathbf{P} \approx \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}$, $\boldsymbol{\theta} = \theta\mathbf{m}$, $\boldsymbol{\omega} \approx \dot{\boldsymbol{\theta}}$. Obtain it from Euler theorem.
Calculate $\mathbf{P}_2 \cdot \mathbf{P}_1$ and the angular velocity if $\mathbf{P}_1, \mathbf{P}_2$ are small rotations with rotation vectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$, respectively.

Derivatives

If scalar U is a function of vector $\boldsymbol{\theta}$,

$$dU = d\boldsymbol{\theta} \cdot \frac{\partial U}{\partial \boldsymbol{\theta}}.$$

If scalar U is a function of tensor $\mathbf{A}$,

$$dU = d\mathbf{A}^\top \cdot\cdot \left(\frac{\partial U}{\partial \mathbf{A}}\right)$$

If tensor $\boldsymbol{\tau}$ is a function of tensor $\mathbf{A}$,

$$d\boldsymbol{\tau} = d\mathbf{A}^\top \cdot\cdot \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{A}}$$

Lecture 3.

Test.

What is the mechanics of the Cosserat media?

Test

Prove that if $\mathbf{P}$ is a rotation tensor and $\mathbf{a}, \mathbf{b}$ are vectors, then

$$\mathbf{P} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{P} \cdot \mathbf{a}) \times (\mathbf{P} \cdot \mathbf{b}).$$

Test: help

Use the definition of the determinant of a tensor

$$\det \mathbf{A} = \frac{[(\mathbf{A} \cdot \mathbf{a}) \times (\mathbf{A} \cdot \mathbf{b})] \cdot (\mathbf{A} \cdot \mathbf{c})}{[\mathbf{a} \times \mathbf{b}] \cdot \mathbf{c}},$$

where $\mathbf{A}$ is a tensor of the 2nd rank, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any vectors, and the expression for the rotation tensor

$$\mathbf{P} = \mathbf{D}_k \mathbf{d}^k,$$

where $\mathbf{d}^k$ is the dual basis for a vectorial basis $\mathbf{d}_k$, and $\mathbf{D}_k = \mathbf{P} \cdot \mathbf{d}_k$.

How to deduce equations of an elastic continuum

The fundamental laws give us a general frame. Using symmetry considerations, we essentially reduce it.

- balance of forces (1st law of dynamics of Euler, balance of impulse)
- balance of couples (2nd law of dynamics of Euler, balance of kinetic moments)
- balance of energy
- principle of material objectivity (frame indifference)
- 2nd law of thermodynamics satisfied (elasticity, no heat)

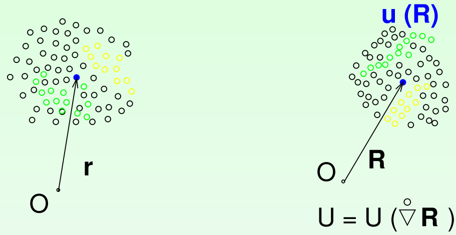
We write down the laws of balance in the integral form for a representative volume of the medium. We obtain its local form. Combining the balance of energy with dynamic laws, we obtain its form that depends only on **internal** stresses and strains. This lets us to express the stresses via the strain energy and strain tensors (constitutive equations).

How to deduce equations of the elastic continuum

The principle of material objectivity (frame indifference) (independence on the system of reference) is: if a piece of material in any system of reference performs a rigid motion, the stresses rotate in the same way and do not change with the rigid translation.
It does not matter of the system of reference is inertial or not.
Physically this means that if an observer moves or walks around the material, this material does not change its constitutive behaviour. This law yields in very important restrictions for the strain energy (it cannot change under rigid motion).
Linearity (if it is the case) and symmetry give more restrictions. For instance we obtain with this reasoning equations of an elastic classical linear isotropic medium (with Poisson coefficient and Young modulus depending on each material).
For inelastic media we have to use 2nd law of thermodynamics.

Models of continua

Classic medium: continuum that consists of mass points



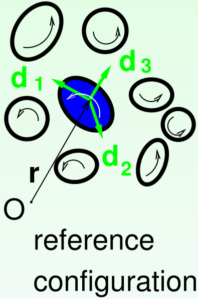
reference configuration
(before the deformation)

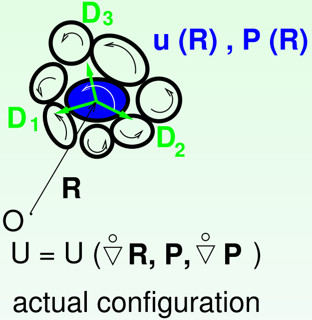
actual configuration
(after the deformation)

U — elastic energy, $\overset{\circ}{\nabla}$ — nabla operator with respect to $\mathbf{r}$,
$\mathbf{u}$ — translational displacement.
Inertial characteristics of each point: mass density ρ.
Each point is subjected to forces.

Models of continua

Cosserat medium: continuum consisting of infinitesimal rigid bodies. At each point there are two fields: displacement $\mathbf{u}$ and rotation tensor $\mathbf{P}$ such that $\mathbf{P} \cdot \mathbf{d}_k = \mathbf{D}_k$.

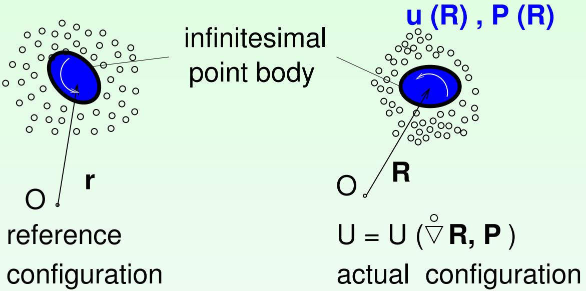




Inertial characteristics of each point: mass density ρ and density of tensor of inertia $\rho\mathbf{I}$. Each point is subjected to forces and couples.

Models of continua

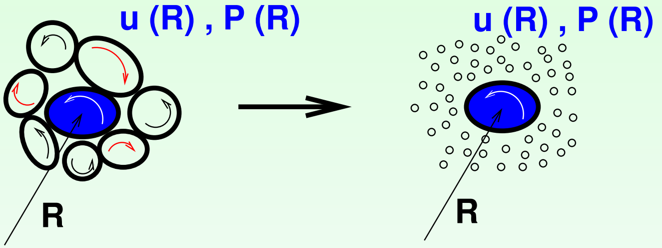
Reduced Cosserat medium: Cosserat medium that does not react to the gradient of rotation.



Rotations and translations are independent kinematically, but the strain energy does not depend on $\overset{\circ}{\nabla}\mathbf{P}$.
Inertial characteristics of each point: mass density ρ, density of tensor of inertia $\rho\mathbf{I}$. Each point is subjected to forces and couples.

Models of continua

Reduced Cosserat medium: why do we choose this model for a granular material?



$$U = U\left(\overset{\circ}{\nabla}\mathbf{R},\ \mathbf{P},\ \overset{\circ}{\nabla}\mathbf{P}\right) \qquad U = U\left(\overset{\circ}{\nabla}\mathbf{R},\ \mathbf{P}\right)$$

There is no "rotational spring" that tries to reduce the relative rotation of particles $\Longrightarrow$ there is no ordered structure of rotations.

References. (Take critically even the best works!)

Cauchy, Green: classical nonlinear elasticity
Brothers **Cosserat**: Cosserat medium
Eringen, Kafadar: full Cosserat continuum
Zhilin: method to obtain constitutive equations via the balance of energy
Many works on Cosserat continua: Green, Naghdi, Rivlin, Erbay, Suhubi, Nowacki, Palmov, Aero,...
Books to read by: Zhilin; Eremeyev, Lebedev, Altenbach; Eringen; Maugin; Nowacki; Erofeyev;...
Granular media in terms of full Cosserat continua: Vardoulakis, Besdo, Metrikine, Askes, Suiker, de Borst, Sulem
Granular media as a linear reduced isotropic Cosserat medium: **Schwartz, Johnson, Feng**
Waves in the linear elastic reduced Cosserat medium: Herman, Kulesh, Shardakov, Grekova

Balance of momentum

Existence of the stress tensor

Balance of forces: 1st law of dynamics by Euler

We write down the laws of dynamics in inertial systems of reference. Consider a material volume V with a surface S. Suppose that there is no volumetric income of impulse.

Balance of forces, global form

$$(\int\limits_V \rho \mathbf{v} dV)^{\cdot} = \int\limits_V \rho \mathbf{F} dV + \int\limits_S \mathbf{T}_{(\mathrm{n})} dS \tag{4}$$

$()^{\cdot}$ is a material derivative with respect to time (we follow the same point bodies), $\mathbf{F}$ is the density of external volumetric force, $\mathbf{v} = \dot{\mathbf{R}}$, $\mathbf{R}$ position vector of the centre of mass of a point body, $\mathbf{T}_{(\mathrm{n})}$ the force acting upon a unit surface S with vector of normal $\mathbf{n}$ (from the part of other point bodies outside of the volume V).

Cosserat medium. Stress tensors

If $\mathbf{T}_{(\mathrm{n})}$ is a force acting upon a unit surface with normal $\mathbf{n}$ from the outer part of the volume, there exist a stress tensor (of forces) $\boldsymbol{\tau}$ such that

Cauchy stress tensor, definition

$$\mathbf{T}_{(\mathrm{n})} = \mathbf{n} \cdot \boldsymbol{\tau}$$

If $\mathbf{M}_{(\mathrm{n})}$ is a couple (torque) acting upon a unit surface with normal $\mathbf{n}$, there exist a tensor $\boldsymbol{\mu}$ such that

Cauchy couple stress tensor, definition

$$\mathbf{M}_{(\mathrm{n})} = \mathbf{n} \cdot \boldsymbol{\mu}$$

Stress tensor $\boldsymbol{\tau}$ and couple tensor $\boldsymbol{\mu}$ do not depend on the normal $\mathbf{n}$.
$\boldsymbol{\tau}$ produces power on $\nabla\mathbf{v} + \boldsymbol{\omega} \times \mathbf{E}$ ($\nabla\mathbf{v}$ gradient of the translational velocity in the actual configuration, $\mathbf{v} = \dot{\mathbf{u}}$); $\boldsymbol{\mu}$ produces power on $\nabla\boldsymbol{\omega}$ (gradient of the angular velocity in the actual configuration).

Existence of the stress tensor

Cauchy's proof is based on the balance of forces for a small tetrahedron (see A.I. Lurie, "Theory of elasticity").

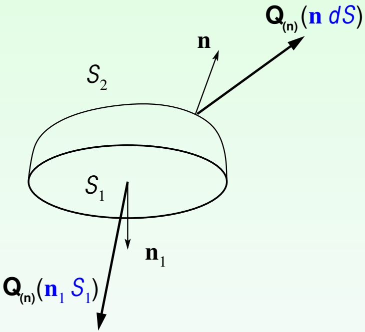
Proof. Let us consider an infinitesimal volume V limited by a closed surface S that has a flat part S_1 ($S = S_1 \cup S_2$). Then $V = o(S)$.

We shall use in the proof:

1. balance of force for V and S_1
2. that surface S is closed
3. $V = o(S)$ at $V \to 0$.

S is closed $\Longrightarrow \oint\limits_S \mathbf{n} dS = \mathbf{0} \Longrightarrow$

$$-\int\limits_{S_2} \mathbf{n} dS = \int\limits_{S_1} \mathbf{n} dS = \mathbf{n}_1 S_1$$



Arguments of the function $\mathbf{Q}_{(n)}(\mathbf{n} dS)$.

$$\mathbf{Q}_{(n)}(\mathbf{n} dS) = \mathbf{T}_{(n)} dS$$

Existence of the stress tensor

Balance of force for the surface $S_1 \Longrightarrow$

$$\mathbf{Q}_{(\mathrm{n})}(-\mathbf{n}_1 S_1) = -\mathbf{Q}_{(\mathrm{n})}(\mathbf{n}_1 S_1)$$

Balance of forces for V (if there are no singularities in $\mathbf{F}, \dot{\mathbf{v}}$):

$$\oint_S \mathbf{Q}_{(\mathrm{n})} = \int_V \rho(\dot{\mathbf{v}} - \mathbf{F})dV = O(V) = o(S)$$

Up to the next orders

$$\int_{S_2} \mathbf{Q}_{(\mathrm{n})}(\mathbf{n}dS) = -\int_{S_1} \mathbf{Q}_{(\mathrm{n})} = -\mathbf{Q}_{(\mathrm{n})}(\mathbf{n}_1 S_1)$$

$$= \mathbf{Q}_{(\mathrm{n})}(-\mathbf{n}_1 S_1) = \mathbf{Q}_{(\mathrm{n})}(\int_{S_2} \mathbf{n}dS)$$

Existence of the stress tensor

Thus we have for any S_2 (such that $S_2 >> V$ at $V \to 0$)

$$\mathbf{Q}_{(\mathrm{n})}\Big(\int\limits_{S_2} \mathbf{n} dS\Big) = \int\limits_{S_2} \mathbf{Q}_{(\mathrm{n})}(\mathbf{n} dS)$$

Therefore $\mathbf{Q}_{(\mathrm{n})} = \mathbf{T}_{(\mathrm{n})} dS$ is a linear function of $\mathbf{n} dS$, and, consequently, the traction $\mathbf{T}_{(\mathrm{n})}$ is the linear function of $\mathbf{n} \Longrightarrow$ there exist a 2nd rank tensor $\boldsymbol{\tau}$ such that

$$\mathbf{T}_{(\mathrm{n})} = \mathbf{n} \cdot \boldsymbol{\tau}.$$

$\boldsymbol{\tau}$ is called the *stress tensor*.

Divergence theorem (Ostrogradsky theorem, Gauss theorem) for 3D space

For a volume V which is compact and has a piecewise smooth boundary S with normal $\mathbf{n}$, if $\mathbf{G}$ is a continuously differentiable tensor field defined on a neighborhood of V, then we have:

$$\int_V \nabla \cdot \mathbf{G}\, dV = \oint_S \mathbf{n} \cdot \mathbf{G}\, dS$$

Lagrange (1762), Gauss (1813), Ostrogradsky (1826, the first proof of the general theorem), Green (1828)...

Cosserat and classical media.
Local form of the balance of forces

The 1st law of dynamics by Euler can be rewritten using the fact that $\mathbf{T}_{(\mathrm{n})} = \mathbf{n} \cdot \boldsymbol{\tau}$ and the divergence theorem as

$$\int\limits_V (\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} - \rho \dot{\mathbf{v}}) dV = \mathbf{0}.$$

Since V is arbitrary, we may obtain the local form of this law

Balance of forces, local form

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \dot{\mathbf{v}}. \tag{5}$$

NB: All this and what we discuss further is valid only if functions are sufficiently smooth (continuous or "piece-wise" continuous).

NB: Operator gradient in the reference configuration (e.g. inicial) $\overset{\circ}{\nabla} \neq \nabla$ (operador gradient in actual configuration).

Balance of forces. Summary

- We formulate balance of force in the global form for a selected volume inside the continuum (separating volume – external – forces and surface – from the other part of the continuum – forces)
- We prove the existence of the Cauchy stress tensor $\boldsymbol{\tau}$ if there are no singularities in external forces and accelerations
- Using the Gauss theorem, we pass from surface integrals to the volume integrals, from surfance forces to the stress tensor and obtain the local form of the balance of forces

Balance of kinetic moment. Existence of the couple stress tensor

Comments on the balance of moments

To write down the balance of moments we have to choose two points: reference point (a fixed point or a centre of mass of the body) and the centre of reduction.

Kinetic moment consists of moment of momentum (impulse) that depends on the reference point, and of proper kinetic moment that does not depend on the reference point.

The full moment consists of the moment of force and couple (proper moment, torque). Torque does not depend on the reference point, and the moment of force does. Torque makes the body rotate about the centre of reduction. Thus torque depends on it. The full moment does not depend on the centre of reduction, though depends on the reference point.

Comments on the balance of moments

Here we take the origin of an inertial system of reference as the reference point and the centre of mass of a point-body as the reduction centre. Mass density of the moment of momentum equals $\mathbf{R} \times \mathbf{v}$, $\mathbf{v} = \dot{\mathbf{R}}$. Proper kinetic moment equals $\mathbf{I} \cdot \boldsymbol{\omega}$, $\boldsymbol{\omega}$ is the angular velocity of the point body ($\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}$, $\mathbf{P}$ — rotation tensor), $\mathbf{I} = \mathbf{P} \cdot \mathbf{I}_0 \cdot \mathbf{P}^\top$ is the tensor of inertia of the point body,

The mass density of the full external moment acting upon a point body in the volume consists of the mass density of moment of force $\mathbf{R} \times \mathbf{F}$ and the mass density of torque (couple) $\mathbf{L}$.

The full moment acting upon the unit surface (part of the surface limiting volume V) from the part of the material outside of V, equals the moment of force $\mathbf{R} \times \mathbf{T}_{(\mathrm{n})}$ and the torque $\mathbf{M}_{(\mathrm{n})}$.

Balance of moments: 2nd law of dynamics by Euler

Suppose that there is no income of the volumetric kinetic moment. We write down balance of moments taking origin as the reference point. The moment is calculated relatively to the centre of mass of a point body.

Balance of kinetic moment, global form

$$\left(\int_V \rho \mathbf{K}_2 dV\right)^{\cdot} = \int_V \rho(\mathbf{R} \times \mathbf{F} + \mathbf{L}) dV + \int_S (\mathbf{R} \times \mathbf{T}_{(\mathrm{n})} + \mathbf{M}_{(\mathrm{n})}) dS \qquad (6)$$

The density of kinetic moment $\mathbf{K}_2 = \mathbf{R} \times \mathbf{v} + \mathbf{I} \cdot \boldsymbol{\omega}$, $\mathbf{L}$ is the density of the external volumetric torque, $\mathbf{M}_{(\mathrm{n})}$ the torque acting upon a unit surface S with vector of normal $\mathbf{n}$ (from the part of other point bodies outside of the volume V).

Couple stress tensor

Hometask: prove that there exist a 2nd rank tensor $\boldsymbol{\mu}$ (*couple stress tensor*) such that $\mathbf{M}_{(n)} = \mathbf{n} \cdot \boldsymbol{\mu}$.

Classical medium: balance of moments = symmetry of the stress tensor

In the classical medium $\mathbf{I} = \mathbf{0}$, $\mathbf{M}_{(n)} = \mathbf{0}$, $\mathbf{L} = \mathbf{0}$. Using the divergence theorem (theorem by Ostrogradsky–Gauss) and the 1st law of dynamics by Euler obtain that the 2nd law of dynamics by Euler is reduced to $\boldsymbol{\tau} = \boldsymbol{\tau}^{\top}$.

Classical medium

The 2nd law of dynamics by Euler reduces to

$$\Big(\int\limits_V \rho(\mathbf{R}\times\mathbf{v})dV\Big)^{\boldsymbol{\cdot}} = \int\limits_V \rho(\mathbf{R}\times\mathbf{F})dV + \int\limits_S \mathbf{R}\times\mathbf{T}_{(\mathrm{n})}dS$$

We rewrite it using that $\mathbf{T}_{(\mathrm{n})} = \mathbf{n}\cdot\boldsymbol{\tau}$ and the divergence theorem as

$$\int\limits_V (-\nabla\cdot(\boldsymbol{\tau}\times\mathbf{R}) + \rho\mathbf{R}\times\mathbf{F} - \rho\mathbf{R}\times\dot{\mathbf{v}})dV = \mathbf{0}.$$

It is easy to show that $\nabla\cdot(\boldsymbol{\tau}\times\mathbf{R}) = (\nabla\cdot\boldsymbol{\tau})\times\mathbf{R} - \boldsymbol{\tau}_\times$. We see that due to the 1st law of dynamics by Euler almost all the terms vanish, and we obtain

$$\int\limits_V \boldsymbol{\tau}_\times dV = \mathbf{0}.$$

Since V is arbitrary, it gives us $\boldsymbol{\tau}_\times = \mathbf{0}$, or $\boldsymbol{\tau} = \boldsymbol{\tau}^\top$.

Cosserat medium. 2nd law of dynamics by Euler, local form.

balance of forces + Gauss theorem + global form of balance of moments $\Longrightarrow$

Balance of kinetic moment, local form

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho \mathbf{L} = \rho (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot}. \tag{7}$$

Proof. $\mathbf{T}_{(\mathrm{n})} = \mathbf{n} \cdot \boldsymbol{\tau}$, $\mathbf{M}_{(\mathrm{n})} = \mathbf{n} \cdot \boldsymbol{\mu}$, therefore

$$\int_V (-\nabla \cdot (\boldsymbol{\tau} \times \mathbf{R}) + \nabla \cdot \boldsymbol{\mu} + \rho(\mathbf{R} \times \mathbf{F} + \mathbf{L} - (\mathbf{R} \times \mathbf{v} + \mathbf{I} \cdot \boldsymbol{\omega})^{\cdot}))dV = \mathbf{0}.$$

$$\int_V (\mathbf{R} \times \nabla \cdot \boldsymbol{\tau} + \boldsymbol{\tau}_{\times} + \nabla \cdot \boldsymbol{\mu} + \rho(\mathbf{R} \times \mathbf{F} + \mathbf{L} - (\mathbf{R} \times \mathbf{v})^{\cdot} - (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot}))dV = \mathbf{0}.$$

$$\mathbf{R} \times \text{balance of force (5)} \Longrightarrow \int_V (\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho \mathbf{L} - \rho (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot})dV = \mathbf{0}.$$

V is arbitrary $\Longrightarrow$ (7)

Balance of moments. Summary

- We formulate balance of moment in the global form for a selected volume inside the continuum (separating volume – external – moments and surface – from the other part of the continuum – moments)
- We prove the existence of the Cauchy couple stress tensor $\boldsymbol{\mu}$ if there are no singularities in external forces, moments and accelerations
- Using the Gauss theorem, we pass from surface integrals to the volume integrals, from surfance forces and moments to the stress tensor and couple stress tensors and obtain the local form of the balance of moments

Balance of energy.

Continue with the Cosserat medium. Balance of energy

The mass density of the kinetic energy is $K = \mathbf{v}^2/2 + \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}/2$. The density of the strain energy U depends on the deformation in the medium.

Balance of energy, global form

$$\Big(\int_V \rho(K+U)dV\Big)^{\cdot} = \int_V \rho(\mathbf{F}\cdot\mathbf{v} + \mathbf{L}\cdot\boldsymbol{\omega})dV + \int_S (\mathbf{T}_{(\mathrm{n})}\cdot\mathbf{v} + \mathbf{M}_{(\mathrm{n})}\cdot\boldsymbol{\omega})dS \quad (8)$$

This is true for elastic media in the absence of heat effects. Generally speaking, we have the contribution of heat apart from work of forces and couples.

We want to eliminate external forces and couples from this law using the laws of dynamics. We will have only stresses, strains and strain energy. (For inelastic media there will be a contribution of the flux of heat and we will need the 2nd law of thermodynamics to write down the constitutive equations.)

Balance of energy. Local form

Exercise 5. Prove that $\boldsymbol{\omega} \cdot (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot} = \dot{\boldsymbol{\omega}} \cdot \mathbf{I} \cdot \boldsymbol{\omega}$.
Lemma. It holds $\rho\dot{K} = (\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{F}) \cdot \mathbf{v} + (\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho\mathbf{L}) \cdot \boldsymbol{\omega}$.
Proof. We will use the previous exercises and the laws of dynamics.

$$\rho\dot{K} = \frac{1}{2}\rho(\mathbf{v} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega})^{\cdot} = \frac{1}{2}\rho(2\mathbf{v} \cdot \dot{\mathbf{v}} + \dot{\boldsymbol{\omega}} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot})$$
$$= \rho(\dot{\mathbf{v}} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot}) = (\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{F}) \cdot \mathbf{v} + (\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho\mathbf{L}) \cdot \boldsymbol{\omega}$$

Theorem. 1st law of thermodynamics (8) can be rewritten as

Balance of energy. Local form

$$\rho\dot{U} = \boldsymbol{\tau}^\top \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^\top \cdot\cdot \nabla\boldsymbol{\omega} \tag{9}$$

(Gauss theorem + global form of the balance of energy + balance of force + balance of moments)
NB: If nothing produces power on $\nabla\boldsymbol{\omega}$, then $\boldsymbol{\mu} = \mathbf{0}$, and $\boldsymbol{\tau}$ can be asymmetric. This kind of continuum is called “reduced Cosserat medium”.

Balance of energy. Local form

Proof. We shall use the previous exercise, representation of forces and torques via Cauchy tensors, and divergence theorem.
Rewrite the right part of (8) as

$$\int_V \rho(\mathbf{F}\cdot\mathbf{v}+\mathbf{L}\cdot\boldsymbol{\omega})dV + \int_S (\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{v}+\mathbf{n}\cdot\boldsymbol{\mu}\cdot\boldsymbol{\omega})dS$$

$$= \int_V \rho(\mathbf{F}\cdot\mathbf{v}+\mathbf{L}\cdot\boldsymbol{\omega})dV + \int_V (\nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{v})+\nabla\cdot(\boldsymbol{\mu}\cdot\boldsymbol{\omega}))dV$$

$$= \int_V (\rho(\mathbf{F}\cdot\mathbf{v}+\mathbf{L}\cdot\boldsymbol{\omega})+(\nabla\cdot\boldsymbol{\tau})\cdot\mathbf{v}+\mathbf{i}^k\cdot\boldsymbol{\tau}\cdot\partial_k\mathbf{v}+(\nabla\cdot\boldsymbol{\mu})\cdot\boldsymbol{\omega}+\mathbf{i}^k\cdot\boldsymbol{\mu}\cdot\partial_k\boldsymbol{\omega}))dV$$

$$= \int_V ((\nabla\cdot\boldsymbol{\tau}+\rho\mathbf{F})\cdot\mathbf{v}+\boldsymbol{\tau}^\top\cdot\cdot\nabla\mathbf{v}+(\nabla\cdot\boldsymbol{\mu}+\rho\mathbf{L})\cdot\boldsymbol{\omega}+\boldsymbol{\mu}^\top\cdot\cdot\nabla\boldsymbol{\omega}))dV$$

Balance of energy. Local form

Using the lemma

$$\rho\dot{K} = (\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{F}) \cdot \mathbf{v} + (\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho\mathbf{L}) \cdot \boldsymbol{\omega}$$

proved above, we see that (8) takes form

$$\int\limits_V \rho\dot{U} dV = \int\limits_V (\boldsymbol{\tau}^\top \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^\top \cdot\cdot \nabla\boldsymbol{\omega}) dV$$

Since it holds for an arbitrary volume V, we obtain

$$\rho\dot{U} = \boldsymbol{\tau}^\top \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^\top \cdot\cdot \nabla\boldsymbol{\omega}$$

Quod erat demonstrandum.

Lecture 6.

Material objectivity (frame indifference)

Strain tensors of the Cosserat medium

Balance of energy in terms of "energetic tensors"

Constitutive equations
(sketch)

In a few words

During last lectures we have written

- the balance of forces, balance of moments and balance of energy for a material volume V limited by surface S (integral form)
- the local form (at a point) of the balance of force, using the divergence theorem by Ostrogradsky–Gauss
- the local form of the balance of moments, using the divergence theorem and the balance of force (in the local form)
- the local form of the balance of energy, using the divergence theorem and the balances of force and moments in the local form. External forces and moments do not enter there.

Now we will formulate

- the principle of material objectivity (frame indifference).

Principle of material frame indifference (Noll, 1958)

If a piece of material performs a rigid motion, its stresses rotate together with it and do not depend on the translation.

It does not matter if the system of reference is inertial or not. In other words, if the observer moves or rotates, the material does not change its physical behaviour.

Mathematically: if in the material there exist stresses $\boldsymbol{\tau}(\mathbf{R}), \boldsymbol{\mu}(\mathbf{R})$ and it performs the motion $\mathbf{R}' = \mathbf{Q}(t) \cdot (\mathbf{R} - \mathbf{R}_c) + \mathbf{R}_c + \mathbf{R}_0(t)$, $\mathbf{P}' = \mathbf{Q} \cdot \mathbf{P}$, where $\mathbf{R}'$ is the position vector and $\mathbf{P}'$ is a rotation of point body of the material subjected to the rigid motion, the rotation tensor $\mathbf{Q}(t)$, $\mathbf{R}_0(t)$, $\mathbf{R}_c$ do not depend on $\mathbf{R}$, then stresses in this motion will be equal to $\boldsymbol{\tau}' = \mathbf{Q} \cdot \boldsymbol{\tau} \cdot \mathbf{Q}^\top$, $\boldsymbol{\mu}' = \mathbf{Q} \cdot \boldsymbol{\mu} \cdot \mathbf{Q}^\top$ (stresses are materially objective, frame indifferent).

NB: If the stresses depend only on the strain tensors which are frame indifferent, i.e. rotate together with the piece of material when it performs a rigid motion, and do not depend on any anisotropic tensorial constants, the principle holds.

Material frame indifference

Exercise 6. Prove that any 2nd rank tensor, a function of a frame indifferent 2nd tensor is also frame indifferent.
Help: expand it in series in its argument.

Note: $\overset{\circ}{\nabla}\mathbf{R}$ is not frame indifferent. Under the rigid motion with a rotation tensor $\mathbf{Q}$ we have $\overset{\circ}{\nabla}\mathbf{R}' = \overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top$.
Proof.

$$\overset{\circ}{\nabla}\mathbf{R}' = \overset{\circ}{\nabla}(\mathbf{Q}\cdot(\mathbf{R}-\mathbf{R}_c)) + \overset{\circ}{\nabla}(\mathbf{R}_c+\mathbf{R}_o) = \overset{\circ}{\nabla}(\mathbf{Q}\cdot\mathbf{R})$$
$$= \mathbf{i}^k\mathbf{Q}\cdot\overset{\circ}{\partial}_k\mathbf{R} = \mathbf{i}^k\overset{\circ}{\partial}_k\mathbf{R}\cdot\mathbf{Q}^\top = \overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top.$$

It yields that constitutive equation (in classical elasticity) $\boldsymbol{\tau} = {}^4\mathbf{C}\cdot\cdot\overset{\circ}{\nabla}\mathbf{R}$ cannot be valid even for the isotropic theory. However, the right Cauchy–Green strain tensor $\boldsymbol{\varepsilon} = (\overset{\circ}{\nabla}\mathbf{R})^\top\cdot\overset{\circ}{\nabla}\mathbf{R}$ is frame indifferent (check it!), and any isotropic $\boldsymbol{\tau}(\boldsymbol{\varepsilon})$ is also frame indifferent.

Material frame indifference and linearity

When we have nonlinear equations, under some conditions we may linearize them near a certain state.
Question: what happens with the requirement of the material objectivity (frame indifference) for the linear theory?
Answer: we have to require the same, but for infinitesimal rotations.

Strain tensors in the Cosserat medium

The Poisson equation is $\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}$. If we differentiate it not with respect to time, but in space (with respect to co-ordinate q^i), we have $\partial_i \mathbf{P} = \mathbf{\Phi}_i \times \mathbf{P}$.

<u>Exercise 7</u>: prove that $\partial_i \boldsymbol{\omega} = \dot{\mathbf{\Phi}}_i + \mathbf{\Phi}_i \times \boldsymbol{\omega}$.

Define the Cosserat deformation tensor $\mathbf{A} = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P}$ and transposed wryness tensor $\mathbf{K} = \mathbf{r}^i \mathbf{\Phi}_i \cdot \mathbf{P}$.

<u>Exercise 8</u>. Find out an invariant expression for $\mathbf{K}$.

<u>Exercise 9</u>(!). Prove that $\mathbf{A}, \mathbf{K}$ do not change under rigid motion.

Proof for $\mathbf{A}$.

$$\mathbf{A}' = \overset{\circ}{\nabla}\mathbf{R}' \cdot \mathbf{P}' = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{Q}^\top \cdot \mathbf{Q} \cdot \mathbf{P} = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P} = \mathbf{A}$$

Balance of energy. Local form with “energetic tensors”. Constitutive equations

Exercise 10: prove that the law of balance of energy can be rewritten as

$$\rho \dot{U} = \boldsymbol{\tau}_*^\top \cdot\cdot\, \dot{\mathbf{A}} + \boldsymbol{\mu}_*^\top \cdot\cdot\, \dot{\mathbf{K}}, \tag{10}$$

where we introduce “energetic tensors” $\boldsymbol{\tau}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P}$, $\boldsymbol{\mu}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\mu} \cdot \mathbf{P}$. (Pavel Zhilin for 2D)

Constitutive equations

Consider a hyperelastic medium: elastic energy is a function of strain tensors. If $U = U(\mathbf{A}, \mathbf{K})$, we obtain from (10)

$$\boldsymbol{\tau}_* = \rho \frac{\partial U}{\partial \mathbf{A}} \ , \qquad \boldsymbol{\mu}_* = \rho \frac{\partial U}{\partial \mathbf{K}} \ . \tag{11}$$

These equations are called "constitutive equations": it is the relation between internal forces/torques and deformations in the medium. They do not depend on nothing external.

Check that since $\mathbf{A}, \mathbf{K}$ do not change under rigid motion, $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are frame indifferent.

Questions

1) Is it true that equations (11) satisfy the principle of material objectivity (frame indifference)? Why?

2) Can we choose any $U(\mathbf{A}, \mathbf{K})$ and state that such a material may exist? (e.g., its existence does not violate basic principles). Why?

3) Have we now a closed system of equations? If we have initial and/or boundary conditions, can we resolve any problem?

4) Can other constitutive equations of the elastic Cosserat medium, that do not enter in this frame, exist?

5) What to do with stability?

6) Has it to be simpler everything for the classical medium in the sense of material objectivity (frame indifference)?

Lecture 7.

Material frame indifference

Strain tensors of the Cosserat medium

Balance of energy in terms of “energetic tensors”

Constitutive equations
(full and reduced Cosserat media)
(continuation)

Strain tensors in the Cosserat medium

Exercise 9 from lecture 6: $\mathbf{K}$ does not change under rigid motion.
Proof.

$$\partial_i \mathbf{P}' = (\partial_i(\mathbf{Q}\cdot\mathbf{P})) = \mathbf{Q}\cdot\partial_i\mathbf{P} = \mathbf{Q}\cdot(\mathbf{\Phi}_i\times\mathbf{P}) = (\mathbf{Q}\cdot\mathbf{\Phi}_i)\times(\mathbf{Q}\cdot\mathbf{P}) = (\mathbf{Q}\cdot\mathbf{\Phi}_i)\times\mathbf{P}'$$

$$\Longrightarrow \mathbf{\Phi}_i' = \mathbf{Q}\cdot\mathbf{\Phi}_i = \mathbf{\Phi}_i\cdot\mathbf{Q}^\top.$$

$$\mathbf{K}' = \mathbf{r}^i\mathbf{\Phi}_i'\cdot\mathbf{P}' = \mathbf{r}^i\mathbf{\Phi}_i\cdot\mathbf{Q}^\top\cdot\mathbf{Q}\cdot\mathbf{P} = \mathbf{r}^i\mathbf{\Phi}_i\cdot\mathbf{P} = \mathbf{K}$$

Note: If f is a function of $\mathbf{A}, \mathbf{K}$ and not of any other types of deformation, f does not change under rigid motion, and $\frac{\partial f}{\partial \mathrm{A}}$, $\frac{\partial f}{\partial \mathrm{K}}$ do not change under rigid motion.

Note: For any vector $\mathbf{w}$ it holds $\overset{\circ}{\nabla}\mathbf{w} = \overset{\circ}{\nabla}\mathbf{R}\cdot\nabla\mathbf{w}$
(another form: $\overset{\circ}{\nabla}\mathbf{R}^{-1}\cdot\overset{\circ}{\nabla}\mathbf{w} = \nabla\mathbf{w}$).
Proof. The increment of $\mathbf{w}$ in space

$$d\mathbf{r}\cdot\overset{\circ}{\nabla}\mathbf{w} = d\mathbf{R}\cdot\nabla\mathbf{w} = d\mathbf{r}\cdot(\overset{\circ}{\nabla}\mathbf{R})\cdot\nabla\mathbf{w} \quad\Longrightarrow\quad \overset{\circ}{\nabla}\mathbf{w} = \overset{\circ}{\nabla}\mathbf{R}\cdot\nabla\mathbf{w}$$

Balance of energy. Local form with “energetic tensors”.

Theorem (following the formulation and proof by Pavel Zhilin for 2D):
The law of balance of energy can be rewritten as

$$\rho \dot{U} = \boldsymbol{\tau}_*^\top \cdot\cdot \dot{\mathbf{A}} + \boldsymbol{\mu}_*^\top \cdot\cdot \dot{\mathbf{K}},$$

where “energetic tensors”

$$\boldsymbol{\tau}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P}, \qquad \boldsymbol{\mu}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\mu} \cdot \mathbf{P}. \tag{12}$$

Proof.
We use the local form of balance of energy

$$\rho \dot{U} = \boldsymbol{\tau}^\top \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^\top \cdot\cdot \nabla \boldsymbol{\omega} = \boldsymbol{\tau}^\top \cdot\cdot (\nabla \mathbf{v} + \boldsymbol{\omega} \times \mathbf{E}) + \boldsymbol{\mu}^\top \cdot\cdot \nabla \boldsymbol{\omega},$$

the Poisson equation $\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}$,
the relation $\partial_i \boldsymbol{\omega} = \dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i \times \boldsymbol{\omega}$ proved as an exercise,
and identities $(\mathbf{X} \cdot \mathbf{Y}) \cdot\cdot \mathbf{Z} = \mathbf{X} \cdot\cdot (\mathbf{Y} \cdot \mathbf{Z}) = \mathbf{Y} \cdot\cdot (\mathbf{Z} \cdot \mathbf{X})$,
$\mathbf{X} \cdot\cdot (\mathbf{w} \times \mathbf{E}) = \mathbf{X}_\times \cdot \mathbf{w}$, and $\overset{\circ}{\nabla}\mathbf{R}^{-1} \cdot \overset{\circ}{\nabla}\mathbf{w} = \nabla \mathbf{w}$.

Balance of energy. Local form with "energetic tensors".

$$\dot{\mathbf{A}} = (\overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P})^{\cdot} = \overset{\circ}{\nabla}\mathbf{v} \cdot \mathbf{P} + \overset{\circ}{\nabla}\mathbf{R} \cdot (\boldsymbol{\omega} \times \mathbf{P}) = (\overset{\circ}{\nabla}\mathbf{v} + (\overset{\circ}{\nabla}\mathbf{R}) \cdot (\boldsymbol{\omega} \times \mathbf{E})) \cdot \mathbf{P}$$

(By the way we obtained $\dot{\mathbf{A}} = (\overset{\circ}{\nabla}\mathbf{v} + \overset{\circ}{\nabla}\mathbf{R} \times \boldsymbol{\omega}) \cdot \mathbf{P}$.)

$$\begin{aligned}
\boldsymbol{\tau}_*^\top \cdot\cdot\, \dot{\mathbf{A}} &= (\overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P})^\top \cdot\cdot\, ((\overset{\circ}{\nabla}\mathbf{v} + (\overset{\circ}{\nabla}\mathbf{R}) \cdot (\boldsymbol{\omega} \times \mathbf{E})) \cdot \mathbf{P}) \\
&= (\mathbf{P}^\top \cdot \boldsymbol{\tau}^\top \cdot (\overset{\circ}{\nabla}\mathbf{R}^{-1})) \cdot\cdot\, ((\overset{\circ}{\nabla}\mathbf{v} + (\overset{\circ}{\nabla}\mathbf{R}) \cdot (\boldsymbol{\omega} \times \mathbf{E})) \cdot \mathbf{P}) \\
&= \boldsymbol{\tau}^\top \cdot\cdot\, (\overset{\circ}{\nabla}\mathbf{R}^{-1} \cdot \overset{\circ}{\nabla}\mathbf{v}) + \boldsymbol{\tau}^\top \cdot\cdot\, (\boldsymbol{\omega} \times \mathbf{E}) = \boldsymbol{\tau}^\top \cdot\cdot\, \nabla\mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega}.
\end{aligned}$$

Balance of energy. Local form with "energetic tensors".

$$\dot{\mathbf{K}} = (\mathbf{r}^i \boldsymbol{\Phi}_i \cdot \mathbf{P})^{\cdot} = \mathbf{r}^i \dot{\boldsymbol{\Phi}}_i \cdot \mathbf{P} + \mathbf{r}^i \boldsymbol{\Phi}_i \cdot (\boldsymbol{\omega} \times \mathbf{P})$$
$$= \mathbf{r}^i (\partial_i \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\Phi}_i) \cdot \mathbf{P} + \mathbf{r}^i \boldsymbol{\Phi}_i \cdot (\boldsymbol{\omega} \times \mathbf{P})$$
$$= (\overset{\circ}{\nabla} \boldsymbol{\omega} + \mathbf{r}^i \boldsymbol{\omega} \times \boldsymbol{\Phi}_i) \cdot \mathbf{P} + \mathbf{r}^i \boldsymbol{\Phi}_i \cdot (\boldsymbol{\omega} \times \mathbf{P})$$
$$= (\overset{\circ}{\nabla} \boldsymbol{\omega} - \mathbf{r}^i \boldsymbol{\Phi}_i \times \boldsymbol{\omega}) \cdot \mathbf{P} + \mathbf{r}^i (\boldsymbol{\Phi}_i \times \boldsymbol{\omega}) \cdot \mathbf{P} = \overset{\circ}{\nabla} \boldsymbol{\omega} \cdot \mathbf{P}.$$

$$\boldsymbol{\mu}_*^\top \cdot\cdot \dot{\mathbf{K}} = (\overset{\circ}{\nabla} \mathbf{R}^{-\top} \cdot \boldsymbol{\mu} \cdot \mathbf{P})^\top \cdot\cdot (\overset{\circ}{\nabla} \boldsymbol{\omega} \cdot \mathbf{P})$$
$$= (\mathbf{P}^\top \cdot \boldsymbol{\mu}^\top \cdot \overset{\circ}{\nabla} \mathbf{R}^{-1}) \cdot\cdot (\overset{\circ}{\nabla} \boldsymbol{\omega} \cdot \mathbf{P}) = \boldsymbol{\mu}^\top \cdot\cdot \nabla \boldsymbol{\omega}.$$

Thus we have

$$\boldsymbol{\tau}_*^\top \cdot\cdot \dot{\mathbf{A}} + \boldsymbol{\mu}_*^\top \cdot\cdot \dot{\mathbf{K}} = \boldsymbol{\tau}^\top \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^\top \cdot\cdot \nabla \boldsymbol{\omega} = \rho \dot{U}, \qquad (13)$$

quod erat demonstrandum.

Constitutive equations of the full Cosserat medium

Consider a hyperelastic medium: elastic energy is a function of strain tensors. If $U = U(\mathbf{A}, \mathbf{K})$, we obtain from (10)

$$\boldsymbol{\tau}_* = \rho \frac{\partial U}{\partial \mathbf{A}} \ , \qquad \boldsymbol{\mu}_* = \rho \frac{\partial U}{\partial \mathbf{K}} \ .$$

These equations are called "constitutive equations": it is the relation between internal forces/torques and deformations in the medium. They do not depend on nothing external.
Write down the constitutive equations in terms of Cauchy stress and couple stress tensors, using $\boldsymbol{\tau} = \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \boldsymbol{\tau}_* \cdot \mathbf{P}^\top, \quad \boldsymbol{\mu} = \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \boldsymbol{\mu}_* \cdot \mathbf{P}^\top$:

$$\boldsymbol{\tau} = \rho \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^\top, \qquad \boldsymbol{\mu} = \rho \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathbf{K}} \cdot \mathbf{P}^\top. \tag{14}$$

Reduced Cosserat medium. Constitutive equations.

$$\rho \dot{U} = \boldsymbol{\tau}^\top \cdot\cdot (\nabla \mathbf{v} + \boldsymbol{\omega} \times \mathbf{E}) = \boldsymbol{\tau}_*^\top \cdot\cdot \dot{\mathbf{A}},$$

Strain energy depends only on $\mathbf{A} = \overset{\circ}{\nabla} \mathbf{R} \cdot \mathbf{P}$ (the Cosserat deformation tensor): $U = U(\mathbf{A})$.

$$\boldsymbol{\tau}_* = \rho \frac{\partial U}{\partial \mathbf{A}},$$

$$\boldsymbol{\tau} = \rho \overset{\circ}{\nabla} \mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^\top,$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_* = \mathbf{0}. \qquad (15)$$

Questions

1) Is it true that equations (14) satisfy the principle of material objectivity (frame indifference)? Why?

2) Can we choose any $U(\mathbf{A}, \mathbf{K})$ and state that such a material may exist? (e.g., its existence does not violate basic principles). Why?

3) Have we now a closed system of equations? If we have initial and/or boundary conditions, can we resolve any problem?

4) Can other constitutive equations of the elastic Cosserat medium, that do not enter in this frame, exist?

5) What to do with stability?

6) Has it to be simpler everything for the classical medium in the sense of material objectivity (frame indifference)?

Answers

1) Yes; 2) Yes; 3) Yes, if we know $U(\mathbf{A}, \mathbf{K})$; who knows.
4) Only hypoelastic (or with singularities?)
5) The condition of the stability of the material is not a basic principle. There may exist unstable materials even in the sense of translation. Examples: explosion, phase transitions, possibly flow surfaces. Instability in the sense of rotation not necessarily yields in the destruction of the material. Perhaps there are exist regimes of unstable rotations. There are many works on stability in nonlinear classical elasticity and much less for the Cosserat media.
6) Not much. The energy cannot depend on rotations of the material. A frame indifferent strain tensor is $\overset{\circ}{\nabla}\mathbf{R}^\top \cdot \overset{\circ}{\nabla}\mathbf{R}$ (right Cauchy–Green strain tensor). A strain tensor not influenced by rigid motion is $\overset{\circ}{\nabla}\mathbf{R} \cdot \overset{\circ}{\nabla}\mathbf{R}^\top$ (left Cauchy–Green strain measure).

Material objectivity (frame indifference) of $\boldsymbol{\tau}, \boldsymbol{\mu}$

Frame indifference: Noll, 1958.

Consider a material subjected to the strains $\mathbf{A}, \mathbf{K}$ with stresses $\boldsymbol{\tau}$ and couple stresses $\boldsymbol{\mu}$ in our frame of references, and the same material under the same strains subjected to the rigid motion $\mathbf{R}' = \mathbf{Q}(t) \cdot (\mathbf{R} - \mathbf{R}_c) + \mathbf{R}_c + \mathbf{R}_0(t)$, $\mathbf{P}' = \mathbf{Q} \cdot \mathbf{P}$. Then

$$\boldsymbol{\tau}' = \rho \overset{\circ}{\nabla} \mathbf{R}'^{\top} \cdot \frac{\partial U}{\partial \mathbf{A}'} \cdot \mathbf{P}'^{\top} = \rho (\overset{\circ}{\nabla} \mathbf{R} \cdot \mathbf{Q}^{\top})^{\top} \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot (\mathbf{Q} \cdot \mathbf{P})^{\top}$$
$$= \rho \mathbf{Q} \cdot \overset{\circ}{\nabla} \mathbf{R}^{\top} \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^{\top} \cdot \mathbf{Q}^{\top} = \mathbf{Q} \cdot \boldsymbol{\tau} \cdot \mathbf{Q}^{\top}.$$

Analogously

$$\boldsymbol{\mu}' = \mathbf{Q} \cdot \boldsymbol{\mu} \cdot \mathbf{Q}^{\top}. \tag{16}$$

Therefore for any $U(\mathbf{A}, \mathbf{K})$ the principle of material frame indifference holds. All the fundamental laws are satisfied.

Second law of thermodynamics

In hyperelasticity we suppose that U is determined by the state of material, i.e. by strain tensors $\mathbf{A}, \mathbf{K}$. The work of mechanical forces in a cyclic process (passing from a certain state (*) of material to the same state (*)) is equal to the change of the strain energy $U(\mathbf{A}_*, \mathbf{K}_*) - U(\mathbf{A}_*, \mathbf{K}_*)$, which is zero. Since we consider no heat effects (e.g. adiabatic processes), no energy is lost, and the second law of thermodynamics holds identically (equality as a particular case of the non-strict inequality).

Classical elastic medium

The law of balance of energy for the classical elastic medium reduces to

$$\rho \dot{U} = \boldsymbol{\tau} \cdot\cdot \nabla \mathbf{v}, \tag{17}$$

$\boldsymbol{\tau} = \boldsymbol{\tau}^\top$ (due to the balance of moments for the classical medium).

The left Cauchy–Green strain tensor $\mathcal{E} = \overset{\circ}{\nabla}\mathbf{R} \cdot (\overset{\circ}{\nabla}\mathbf{R})^\top = \mathcal{E}^\top$ has 6 independent components (as $\boldsymbol{\tau}$) and does not change under rigid motion, so in a hyperelastic medium $U = U(\mathcal{E})$. Calculating $\dot{U}(\mathcal{E})$, after transformations (see next page) we have

$$\rho \dot{U} = 2\rho(\overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathcal{E}} \cdot \overset{\circ}{\nabla}\mathbf{R}) \cdot\cdot \nabla \mathbf{v}, \tag{18}$$

therefore

$$\boldsymbol{\tau} = 2\rho(\overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathcal{E}} \cdot \overset{\circ}{\nabla}\mathbf{R}). \tag{19}$$

We see that $\boldsymbol{\tau}$ is frame indifferent (since $\overset{\circ}{\nabla}\mathbf{R}' = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{Q}^\top$).

Classical elastic medium (proofs for the previous page)

1. Let us calculate $\dot{U}(\mathcal{E})$, remembering that the contraction of a symmetric and antisymmetric tensor is zero (if $\mathbf{U}^\top = \mathbf{U}$, $\mathbf{W}^\top = -\mathbf{W}$, then $\mathbf{U}\cdot\cdot\,\mathbf{W} = 0$), $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\top \Longrightarrow \frac{\partial U}{\partial \mathcal{E}} = \frac{\partial U}{\partial \mathcal{E}}^\top$.

$$\dot{U}(\mathcal{E}) = \left(\frac{\partial U}{\partial \mathcal{E}}\right)^\top \cdot\cdot\,\dot{\mathcal{E}} = \left(\frac{\partial U}{\partial \mathcal{E}}\right)^\top \cdot\cdot\,(\overset{\circ}{\nabla}\mathbf{v}\cdot\overset{\circ}{\nabla}\mathbf{R}^\top + \overset{\circ}{\nabla}\mathbf{R}\cdot\overset{\circ}{\nabla}\mathbf{v}^\top)$$
$$= 2\left(\frac{\partial U}{\partial \mathcal{E}}\right)\cdot\cdot\,(\overset{\circ}{\nabla}\mathbf{v}\cdot\overset{\circ}{\nabla}\mathbf{R}^\top) = 2(\overset{\circ}{\nabla}\mathbf{R}^\top\cdot\frac{\partial U}{\partial \mathcal{E}})\cdot\cdot\,\overset{\circ}{\nabla}\mathbf{v}$$
$$= 2\overset{\circ}{\nabla}\mathbf{R}^\top\cdot\frac{\partial U}{\partial \mathcal{E}}\cdot\cdot\,(\overset{\circ}{\nabla}\mathbf{R}\cdot\nabla\mathbf{v}) = 2(\overset{\circ}{\nabla}\mathbf{R}^\top\cdot\frac{\partial U}{\partial \mathcal{E}}\cdot\overset{\circ}{\nabla}\mathbf{R})\cdot\cdot\,(\nabla\mathbf{v})$$

2. $$\boldsymbol{\tau}' = \rho(\overset{\circ}{\nabla}\mathbf{R}'^\top)\cdot\frac{\partial U(\mathcal{E}')}{\partial \mathcal{E}'}\cdot\overset{\circ}{\nabla}\mathbf{R}' = \rho(\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top)^\top\cdot\frac{\partial U(\mathcal{E})}{\partial \mathcal{E}}\cdot\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top$$
$$= \rho\mathbf{Q}\cdot\overset{\circ}{\nabla}\mathbf{R}^\top\cdot\frac{\partial U}{\partial \boldsymbol{\varepsilon}}\cdot\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top = \mathbf{Q}\cdot\boldsymbol{\tau}\cdot\mathbf{Q}^\top.$$

Lecture 8.

Linear full and reduced Cosserat media

Linear Cosserat medium. Linearization in kinematics

Small displacements $\mathbf{u} = \mathbf{R} - \mathbf{r} = o(1)$ and rotations $\theta = o(1)$.

Lemma 1. A tensor of infinitesimal rotation looks as $\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}$, $\boldsymbol{\theta}$ is the infinitisemal rotation vector. Its angular velocity $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$.

Proof. Indeed, at $\theta \to 0$

$$\mathbf{P} = (1 - \cos\theta)\mathbf{mm} + \cos\theta\mathbf{E} + \sin\theta\mathbf{m} \times \mathbf{E} \overset{\theta \to 0}{\approx} \mathbf{E} + \theta\mathbf{m} \times \mathbf{E}$$
$$= \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}, \text{ where } \boldsymbol{\theta} = \theta\mathbf{m}.$$

$\dot{\mathbf{P}} \approx \dot{\boldsymbol{\theta}} \times \mathbf{E} \approx \dot{\boldsymbol{\theta}} \times (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \approx \dot{\boldsymbol{\theta}} \times \mathbf{P}$.

Lemma 2. $\nabla = \overset{\circ}{\nabla} + o(1)$

Proof. Let f be an arbitrary function depending on $\mathbf{R}$.

$$df = d\mathbf{r} \cdot \overset{\circ}{\nabla} f = d\mathbf{R} \cdot \nabla f = (d\mathbf{r} + d\mathbf{u}) \cdot \nabla f.$$

Therefore $d\mathbf{r} \cdot (\overset{\circ}{\nabla} f - \nabla f) = d\mathbf{u} \cdot \nabla f$. Since $du << dr$, we have $\overset{\circ}{\nabla} f - \nabla f = o(1)$. It is so for each f and $d\mathbf{r} \Longrightarrow \overset{\circ}{\nabla} = \nabla + o(1)$.

Linear Cosserat medium. Strain tensors

Theorem. In the linear approximation in the vicinity of the natural configuration (zero stresses)

$$\rho \approx \rho_0(1 - \overset{\circ}{\nabla} \cdot \mathbf{u}), \qquad \mathbf{A} \approx \mathbf{E} + \nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}, \qquad \mathbf{K} \approx \nabla \boldsymbol{\theta}, \qquad (20)$$

$\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are linear functions of $\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}$ and $\nabla \boldsymbol{\theta}$.

Proof. $\rho = \rho_0 / \det \overset{\circ}{\nabla} \mathbf{R} \approx \rho_0(1 - \overset{\circ}{\nabla} \cdot \mathbf{u})$.

$$\mathbf{A} = \overset{\circ}{\nabla} \mathbf{R} \cdot \mathbf{P} \approx \overset{\circ}{\nabla}(\mathbf{r} + \mathbf{u}) \cdot (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E})$$
$$= (\mathbf{E} + \overset{\circ}{\nabla} \mathbf{u}) \cdot (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \approx \mathbf{E} + \nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}.$$

$$\boldsymbol{\Phi}_i = -\frac{1}{2}[\partial_i \mathbf{P} \cdot \mathbf{P}^\top]_\times \approx -\frac{1}{2}[\partial_i (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \cdot (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E})^\top]_\times$$
$$\approx -\frac{1}{2}[(\partial_i \boldsymbol{\theta}) \times \mathbf{E}]_\times = \partial_i \boldsymbol{\theta}$$

$$\mathbf{K} = \mathbf{r}^i \boldsymbol{\Phi}_i \cdot \mathbf{P} \approx \mathbf{r}^i \partial_i \boldsymbol{\theta} \cdot (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \approx \overset{\circ}{\nabla} \boldsymbol{\theta} \approx \nabla \boldsymbol{\theta}$$

Linear Cosserat medium. Stress tensor

$$\boldsymbol{\tau} = \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \boldsymbol{\tau}_* \cdot \mathbf{P}^\top \approx (\mathbf{E} + \overset{\circ}{\nabla}\mathbf{u})^\top \cdot \boldsymbol{\tau}_*(\mathbf{A}, \mathbf{K}) \cdot (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E})^\top$$
$$⚠ \approx [\boldsymbol{\tau}_*]_0 + \nabla\mathbf{u}^\top \cdot [\boldsymbol{\tau}_*]_0 - [\boldsymbol{\tau}_*]_0 \times \boldsymbol{\theta}$$
$$+ (\nabla\mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^\top \cdot\cdot \left[\frac{\partial \boldsymbol{\tau}_*}{\partial \mathbf{A}}\right]_0 + \nabla\boldsymbol{\theta}^\top \cdot\cdot \left[\frac{\partial \boldsymbol{\tau}_*}{\partial \mathbf{K}}\right]_0$$

⚠ This step is not always possible: for instance, if U is can be expanded into the series in $\sqrt{\mathbf{A}\cdot\mathbf{A}^\top}, \sqrt{\mathbf{K}\cdot\mathbf{K}^\top}$ (with non-zero odd terms). There are media where the linear theory cannot be applied even for small strains. Here we suppose it to be valid. $U(\mathbf{A}, \mathbf{K})$ has to be "nice enough" for this. It is a strong hypothesis.

If are in the natural configuration ($[\boldsymbol{\tau}]_0 = \mathbf{0}$), then $[\boldsymbol{\tau}_*]_0$, and $\boldsymbol{\tau}$ is a linear function of linear strain tensors.

Linear Cosserat medium. Couple stress tensor

Analogously, if U can be expanded into the series at least up to the second term of the magnitude in the vicinity of natural configuration,

$$\boldsymbol{\mu} \approx [\boldsymbol{\mu}_*]_0 + \overset{\circ}{\nabla}\mathbf{u}^\top \cdot [\boldsymbol{\mu}_*]_0 - [\boldsymbol{\mu}_*]_0 \times \boldsymbol{\theta} + (\nabla\mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^\top \cdot\cdot \left[\frac{\partial \boldsymbol{\mu}_*}{\partial \mathbf{A}}\right]_0 + \nabla\boldsymbol{\theta}^\top \cdot\cdot \left[\frac{\partial \boldsymbol{\mu}_*}{\partial \mathbf{K}}\right]_0$$

If the reference configuration is natural, first three terms in the expression for $\boldsymbol{\mu}$ vanish.

Linear Cosserat medium. Constitutive equations

Another way to obtain constitutive equations is to expand the law of balance of energy (10) into the series with respect to the linear strain tensors. Take into account that near the natural configuration $\boldsymbol{\tau}_* = \boldsymbol{\tau} + o^2(1) = o(1), \quad \boldsymbol{\mu}_* = \boldsymbol{\mu} + o^2(1) = o(1)$, both of them functions of the linear strain tensors, and we have to keep only cuadratic terms in U whose approximation is also a function of the linear strain tensors.
In the vicinity of natural configuration for such U nonlinear constitutive equations (14) give in the linear approximation

$$\boldsymbol{\tau} = \rho_0 \frac{\partial \overset{\square}{U}}{\partial(\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})}, \qquad \boldsymbol{\mu} = \rho_0 \frac{\partial \overset{\square}{U}}{\partial \nabla \boldsymbol{\theta}}, \tag{21}$$

$\overset{\square}{U}$ is the quadratic approximation of U expanded in $\mathbf{u}, \boldsymbol{\theta}$.

Linear Cosserat medium

Note: $(\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^{\cdot} = \nabla \mathbf{v} + \boldsymbol{\omega} \times \mathbf{E}, \quad (\nabla \boldsymbol{\theta})^{\cdot} = \nabla \boldsymbol{\omega}.$

Note: Nonlinear $\boldsymbol{\tau}, \boldsymbol{\mu}$ work on material derivatives of linear strain tensors $\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}, \quad \nabla \boldsymbol{\theta}.$

Note: In the reduced Cosserat medium in the vicinity of the natural configuration

$$\boldsymbol{\tau} = \rho_0 \frac{\partial U}{\partial (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})} = {}^4\mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}), \qquad \boldsymbol{\mu} = \mathbf{0}.$$

Linear classical elastic medium

We linearize the nonlinear constitutive equation near the natural configuration, provided that U can be expanded into series in $\mathcal{E}$ at least up to $o^2(1)$.

$$\boldsymbol{\tau} = 2\rho(\overset{\circ}{\nabla}\mathbf{R}^{\top} \cdot \frac{\partial U}{\partial \mathcal{E}} \cdot \overset{\circ}{\nabla}\mathbf{R}).$$

$\overset{\circ}{\nabla}\mathbf{R} = \mathbf{E} + \overset{\circ}{\nabla}\mathbf{u} \approx \mathbf{E} + \nabla\mathbf{u}$,

$$\mathcal{E} = \overset{\circ}{\nabla}\mathbf{R} \cdot \overset{\circ}{\nabla}\mathbf{R}^{\top} \approx (\mathbf{E} + \nabla\mathbf{u}) \cdot (\mathbf{E} + \nabla\mathbf{u})^{\top} = \mathbf{E} + 2(\nabla\mathbf{u})^{S} + o^2(1).$$

In the natural configuration $\left[\frac{\partial U}{\partial \mathcal{E}}\right]_0 = \mathbf{0}$, therefore

$$\boldsymbol{\tau} \approx 4\rho_0 \left[\frac{\partial^2 U}{\partial \mathcal{E}^2}\right]_0 \cdot\cdot (\nabla\mathbf{u})^{S}.$$

Linear Cosserat medium

Questions.

1) Are linear stress and couple stress tensors frame indifferent?

2) Can $\boldsymbol{\tau}, \boldsymbol{\mu}$ exist that depend in a nonlinear way on linear strain tensors?

Linear Cosserat medium. Answers

1) NO. (Prove that stresses do not rotate when the material is subjected to a final rigid rotation.) However, they are frame indifferent in the infinitesimal sense (for any rigid translation and infinitesimal rotation). Also prove that.

Linear Cosserat medium. Answers

2) Often it is not correct (even if published in journals with high impact factor). Vice versa yes; we can have stresses that depend linearly on nonlinear $\mathbf{A}, \mathbf{K}$ ("physically linear and geometrically nonlinear material"). If we consider a physically nonlinear but geometrically linear material, often it yields that when linearizing equations (11) we have kept some terms and neglected other ones of the same order. (Find examples.) There are exceptions: "piece-wise linear equations" for heteromodular media, non-linearizable equations near 0 (for instance, with ρU function of $\sqrt{\overset{\circ}{\nabla}\boldsymbol{\theta} \cdot \overset{\circ}{\nabla}\boldsymbol{\theta}^{\top}}$, cases when there are other small parameters, cases when the equation in reality is one-dimensional etc.) Each time we must specify the orders of magnitude and explain why we can neglect one term and keep another one.

Exercises

Exercise 11.
Verify that $\mathbf{X}, \mathbf{Z}$ are polar, $X_{mnkl} = X_{klmn}$, $Z_{mnkl} = Z_{klmn}$, $\mathbf{Y}$ is axial, and if they are isotropic, then $\mathbf{Y} = \mathbf{0}$, and each of $\mathbf{X}$ and $\mathbf{Z}$ contains three independent constants. Obtain that they have a form

$$\mathbf{X} = \lambda \mathbf{EE} + 2\mu(\mathbf{i}_m \mathbf{i}_n)^S (\mathbf{i}^m \mathbf{i}^n)^S + 2\alpha(\mathbf{i}_m \mathbf{i}_n)^A (\mathbf{i}^m \mathbf{i}^n)^A, \tag{22}$$

$$\mathbf{Z} = \beta \mathbf{EE} + 2\gamma(\mathbf{i}_m \mathbf{i}_n)^S (\mathbf{i}^m \mathbf{i}^n)^S + 2\varepsilon(\mathbf{i}_m \mathbf{i}_n)^A (\mathbf{i}^m \mathbf{i}^n)^A. \tag{23}$$

Exercise 12.
Verify that in the linear case $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$ and $(\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot} = \mathbf{I}_0 \cdot \ddot{\boldsymbol{\theta}}$, $\rho = \rho_0$, where $\mathbf{I}_0$ is $\mathbf{I}$ in the reference configuration.

Lecture 9.

Isotropic full and reduced Cosserat media

Brief overview of the course.
Ideas and equations

Look ahead: wave propagation

Linear isotropic Cosserat medium. Elastic energy

For any linear Cosserat medium the elastic energy looks as

$$U = \frac{1}{\rho_0}\Big(\frac{1}{2}(\nabla\mathbf{u} + \boldsymbol{\theta}\times\mathbf{E})^{\top}\cdot\cdot\,\mathbf{X}\cdot\cdot\,(\nabla\mathbf{u} + \boldsymbol{\theta}\times\mathbf{E}) + \\ (\nabla\mathbf{u} + \boldsymbol{\theta}\times\mathbf{E})^{\top}\cdot\cdot\,\mathbf{Y}\cdot\cdot\,\nabla\boldsymbol{\theta} + \frac{1}{2}\nabla\boldsymbol{\theta}^{\top}\cdot\cdot\,\mathbf{Z}\cdot\cdot\,\nabla\boldsymbol{\theta}\Big)$$

Principle by Curie–Neumann yields: for an isotropic material $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are isotropic.

Isotropic linear Cosserat medium. Elastic energy

Lemma: An axial isotropic 4th rank tensor is zero.
Proof. An isotropic tensor of the fourth rank $Y^{mnkl}\mathbf{i}_m\mathbf{i}_n\mathbf{i}_k\mathbf{i}_l$ does not change under any rotation. Let us choose polar unit vectors $\mathbf{i}_1, \mathbf{i}_2$, and the axial vector $\mathbf{i}_3 = \mathbf{i}_1 \times \mathbf{i}_2$: an orthonormal basis, oriented by the right hand screw rule.
Rotate it at π about $\mathbf{i}_1$. Then $\mathbf{i}_2$ will change to $-\mathbf{i}_2$, $\mathbf{i}_3$ to $-\mathbf{i}_3$, and $\mathbf{i}_1$ will not change. Then rotate it at π about $\mathbf{i}_2$, etc. Since rotated tensors are equal to the initial one, non-zero components must have even number of repeating indices (Y^{1122}, Y^{2332}, Y^{1111} etc.)
Now if this tensor is axial, the change of orientation of space will change its sign. At the same time it will change sign of all vector products, and in particular of $\mathbf{i}_3 = \mathbf{i}_1 \times \mathbf{i}_2$. Since in all non-zero components each index repeats even number of times, change of sign of $\mathbf{i}_3$ will not influence $\mathbf{Y}$. On the other hand, it has to change the sign. Thus an axial fourth rank isotropic tensor equals zero.

Isotropic linear Cosserat medium. Constitutive equations

Full Cosserat continuum:

$$\boldsymbol{\tau} = \rho_0 \frac{\partial U}{\partial(\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})} = \mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})$$
$$= \lambda \nabla \cdot \mathbf{u}\, \mathbf{E} + 2\mu \nabla \mathbf{u}^S + 2\alpha (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A,$$

$$\boldsymbol{\mu} = \rho_0 \frac{\partial U}{\partial \nabla \boldsymbol{\theta}} = \mathbf{Z} \cdot\cdot \nabla \boldsymbol{\theta} = \beta \nabla \cdot \boldsymbol{\theta}\, \mathbf{E} + 2\gamma \nabla \boldsymbol{\theta}^S + 2\varepsilon (\nabla \boldsymbol{\theta})^A.$$

Reduced Cosserat continuum:

$$\boldsymbol{\tau} = \rho_0 \frac{\partial U}{\partial(\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})} = \mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})$$
$$= \lambda \nabla \cdot \mathbf{u}\, \mathbf{E} + 2\mu \nabla \mathbf{u}^S + 2\alpha (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A,$$

$\boldsymbol{\mu} = \mathbf{0}$. For anisotropic media expressions for $\boldsymbol{\tau}$ may differ in reduced and full Cosserat continuum.

Isotropic linear Cosserat medium. Dynamic equations

Exercise 12 (solution). Verify that in the linear case $(\mathbf{I}\cdot\boldsymbol{\omega})^{\cdot} = \mathbf{I}_0\cdot\ddot{\boldsymbol{\theta}}$.
Solution.
$(\mathbf{E}+\boldsymbol{\theta}\times\mathbf{E})^{\cdot} = \dot{\boldsymbol{\theta}}\times(\mathbf{E}+\boldsymbol{\theta}\times\mathbf{E}) + o^2(1)$. Therefore by Poisson equation $\dot{\boldsymbol{\theta}}$ is the angular velocity for the infinitesimal turn $\mathbf{E}+\boldsymbol{\theta}\times\mathbf{E}$.
$(\mathbf{I}\cdot\boldsymbol{\omega})^{\cdot} \approx ((\mathbf{E}+\boldsymbol{\theta}\times\mathbf{E})\cdot\mathbf{I}_0\cdot(\mathbf{E}+\boldsymbol{\theta}\times\mathbf{E})^{\top}\cdot\dot{\boldsymbol{\theta}})^{\cdot} = \mathbf{I}_0\cdot\ddot{\boldsymbol{\theta}} + o^2(1)$.
Substitute the expressions for $\boldsymbol{\tau}, \boldsymbol{\mu}$ in dynamic laws and obtain their form in displacements:

$$(\lambda+2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu+\alpha)\nabla\times(\nabla\times\mathbf{u}) + 2\alpha\nabla\times\boldsymbol{\theta} + \rho\mathbf{F} = \rho\ddot{\mathbf{u}},$$

$$(\beta+2\gamma)\nabla\nabla\cdot\boldsymbol{\theta} - (\gamma+\varepsilon)\nabla\times(\nabla\times\boldsymbol{\theta}) + 2\alpha(\nabla\times\mathbf{u} - 2\boldsymbol{\theta}) + \rho\mathbf{L} = \rho\mathbf{I}_0\cdot\ddot{\boldsymbol{\theta}}.$$

Verify that if elastic rotational constants $\alpha, \beta, \gamma, \varepsilon$ and tensor of inertia $\mathbf{I}$ are zero, we have equations of a classical isotropic linear elastic medium.

Nota bene: here (up to the high order terms that do not enter in the equations) $\rho = \rho_0$.

Equations of the linear isotropic reduced Cosserat medium

We can obtain it from the equations for the full linear isotropic Cosserat medium, considering $\beta, \gamma, \varepsilon$ to be zero.

$$
\begin{aligned}
(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu+\alpha)\nabla\times(\nabla\times\mathbf{u}) + 2\alpha\nabla\times\boldsymbol{\theta} &= \rho\ddot{\mathbf{u}} \\
2\alpha\nabla\times\mathbf{u} - 4\alpha\boldsymbol{\theta} &= \rho I\ddot{\boldsymbol{\theta}}
\end{aligned}
$$

If $\alpha = 0$, we have a classical linear isotropic elastic medium.

Brief overview of the course. Ideas

- We considered elastic Cosserat continua: media, where particles (point bodies) can move and rotate, possess density and tensor of inertia and are subjected to forces and couples
- We introduced stress tensor and couple stress tensor. Force acting upon a point body in the medium depends on the stress tensor, and full moment depend both on stress and couple stress
- We have written fundamental laws of mechanics for these continua: balance of force, balance of moments, balance of energy. From their integral form (for a material volume V) we passed to the local form by means of theorem by Ostrogradsky–Gauss.
- Second law of thermodynamics holds identically for elasticity

Brief overview of the course. Ideas

- Combining the law of balance of energy with other laws of balance, we obtained the frame for the constitutive equations (how stresses depend on the deformations in the medium). Strain tensors (Cosserat deformation tensor, transposed wryness tensor) appear naturally in the balance of energy.
- Nonlinear stress and couple stress work on derivatives of linear strain tensors
- Cosserat deformation tensor and transposed wryness tensor do not change under rigid motion
- We checked that stress and couple stress obeying these constitutive equations, are frame indifferent (no additional elastic energy appears in the material if we walk around it)

Brief overview of the course. Ideas

- We linearised these equations (under supposition that it is allowed)
- We have written the equations for the linear isotropic Cosserat medium
- We considered also reduced Cosserat medium, where rotations and translations are also independent, but the couple stress is zero (nothing works on the gradient of angular velocity)
- We obtained also constitutive equations, linear and nonlinear, for classical elasticity

Brief overview of the course. Equations

General theory:

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}} \tag{24}$$

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho \mathbf{L} = (\mathbf{I} \cdot \boldsymbol{\omega})^{\cdot} \tag{25}$$

$$\boldsymbol{\tau} = \overset{\circ}{\nabla} \mathbf{R}^\top \cdot \rho \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^\top, \qquad \boldsymbol{\mu} = \overset{\circ}{\nabla} \mathbf{R}^\top \cdot \rho \frac{\partial U}{\partial \mathbf{K}} \cdot \mathbf{P}^\top. \tag{26}$$

$$\mathbf{A} = \overset{\circ}{\nabla} \mathbf{R} \cdot \mathbf{P}, \qquad \mathbf{K} = \mathbf{r}^i \boldsymbol{\Phi}_i \cdot \mathbf{P}, \quad \partial_i \mathbf{P} = \boldsymbol{\Phi}_i \times \mathbf{P}. \tag{27}$$

Reduced Cosserat theory:

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}} \tag{28}$$

$$\boldsymbol{\tau}_\times + \rho \mathbf{L} = (\rho \mathbf{I} \cdot \boldsymbol{\omega})^{\cdot} \tag{29}$$

$$\boldsymbol{\tau} = \rho \overset{\circ}{\nabla} \mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^\top, \qquad \boldsymbol{\mu} = \mathbf{0} \tag{30}$$

Brief overview of the course. Equations

Linear theory (natural configuration, zero initial stresses):

$$\boldsymbol{\tau} = \mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}) + \mathbf{Y} \cdot\cdot (\nabla \boldsymbol{\theta}) \tag{31}$$

$$\boldsymbol{\mu} = ((\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^\top \cdot\cdot \mathbf{Y})^\top + \mathbf{Z} \cdot\cdot \nabla \boldsymbol{\theta} \tag{32}$$

Reduced Cosserat theory:

$$\boldsymbol{\tau} = \mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}), \quad \boldsymbol{\mu} = \mathbf{0}. \tag{33}$$

Linear isotropic theory: $\mathbf{Y} = \mathbf{0}$, $\mathbf{X}$ and $\mathbf{Z}$ are determined by 3 independent constants each one.
Full linear Cosserat isotropic theory: 6 constants.
Reduced linear Cosserat isotropic theory: 3 constants.
Linear classical isotropic elasticity: 2 constants.

Brief overview of the course.

Full Cosserat theory: $\boldsymbol{\tau}, \boldsymbol{\mu}$. Stress tensor is not symmetric: $\boldsymbol{\tau} \neq \boldsymbol{\tau}^\top$.
Reduced Cosserat theory: $\boldsymbol{\tau} \neq \boldsymbol{\tau}^\top, \quad \boldsymbol{\mu} = \mathbf{0}$.
Classical elasticity: $\boldsymbol{\tau} = \boldsymbol{\tau}^\top, \quad \boldsymbol{\mu} = \mathbf{0}$.

Годится ли определяющее уравнение для полной и редуцированной сред Коссера?

1. Надо проверять уравнение для малых или больших перемещений и поворотов? Или для тех и других?
   Если есть $\mathbf{A}, \mathbf{K}, \mathbf{P}$ — для больших. Если есть $\boldsymbol{\theta}$ — для малых.
2. Проверяем уравнение на материальную объективность (в большом или малом). Если нет ее — уравнение неправильное. Если есть — может годиться для полной среды Коссера (подобрать константу). Тогда идем дальше.
   1. Если требуется проверить для малых и больших деформаций, сначала проверяем для малых. Если для них не годится — уравнение всегда неверно.
   2. Если тензоры напряжений малы, их материальная объективность = неизменности при малых жестких движениях
3. Проверяем на полярность / аксиальность. $\boldsymbol{\tau}, \mathbf{A}$ — полярны. $\mathbf{K}, \boldsymbol{\mu}$ — аксиальны. Подбираем тип константы.
4. В редуцированной среде Коссера нет зависимости от градиента поворота. Также $\boldsymbol{\mu} = \mathbf{0}$. В классической среде нет $\boldsymbol{\mu}$, зависимости от поворота или его градиента.

Свойства тензоров деформаций (при жестких движениях материала)

Для **больших** деформаций

$$\overset{\circ}{\nabla}\mathbf{R}' = \overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^{\top}, \quad \mathbf{P} = \mathbf{Q}\cdot\mathbf{P}, \quad \mathbf{A}' = \mathbf{A}, \quad \mathbf{K}' = \mathbf{K}.$$

$\overset{\circ}{\nabla}\mathbf{u}, \nabla\mathbf{u},$ их функции $(\overset{\circ}{\nabla}\mathbf{u}^S, \nabla\mathbf{u}^S, \overset{\circ}{\nabla}\mathbf{u}^A, \nabla\mathbf{u}^A, \overset{\circ}{\nabla}\times\mathbf{u}, \nabla\times\mathbf{u}, \overset{\circ}{\nabla}\cdot\mathbf{u}, \nabla\cdot\mathbf{u})$

не материально объективны при **больших** жестких движениях.

Для **малых** деформаций

$$\overset{\circ}{\nabla}\mathbf{u}, \nabla\mathbf{u}, \overset{\circ}{\nabla}\mathbf{u}^A, \nabla\mathbf{u}^A, \overset{\circ}{\nabla}\times\mathbf{u}, \nabla\times\mathbf{u}, \boldsymbol{\theta}, |\boldsymbol{\theta}|$$

не материально объективны и меняются при **малых** жестких движениях.

$$\overset{\circ}{\nabla}\mathbf{u} + \boldsymbol{\theta}\times\mathbf{E}, \overset{\circ}{\nabla}\boldsymbol{\theta}, \nabla\mathbf{u} + \boldsymbol{\theta}\times\mathbf{E}, \nabla\boldsymbol{\theta}, \text{ их функции}$$

$$(\overset{\circ}{\nabla}\mathbf{u}^S, \nabla\mathbf{u}^S, \overset{\circ}{\nabla}\cdot\mathbf{u}, \nabla\cdot\mathbf{u}, \nabla\mathbf{u}^A + \boldsymbol{\theta}\times\mathbf{E}, \boldsymbol{\theta} - \nabla\times\mathbf{u}/2,$$

$$\overset{\circ}{\nabla}\mathbf{u}^A + \boldsymbol{\theta}\times\mathbf{E}, \boldsymbol{\theta} - \overset{\circ}{\nabla}\times\mathbf{u}/2, \dots)$$

материально объективны и неизменны при **малых** жестких

Lecture 10.

General ideas.

How will we see a reduced Cosserat medium if we are "rotationally daltonic"?

Viscoelastic linear Cosserat medium

Constrained Cosserat media

General ideas

Equations for all the materials must satisfy

- balance of force
- balance of moment
- balance of energy
- the second law of thermodynamics
- material objectivity

Elastic media undergoing adiabatic or isothermic processes satisfy the second law of thermodynamics identically.
To obtain the constitutive equations for elastic continua, we use the balance of energy (modified taking into account balance of force and balance of moment).
Together with dynamic laws we obtain a closed system of equations.

What happens if we cannot observe rotations in the reduced linear elastic Cosserat medium?

Isotropic case:

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - (\mu + \alpha)\nabla \times (\nabla \times \mathbf{u}) + 2\alpha\nabla \times \boldsymbol{\theta} = \rho\ddot{\mathbf{u}}$$
$$4\alpha\boldsymbol{\theta} + \rho I\ddot{\boldsymbol{\theta}} = 2\alpha\nabla \times \mathbf{u}$$

Consider $2\alpha\nabla \times \mathbf{u}$ as an external moment for the mathematical pendulum with the rotation vector $\boldsymbol{\theta}$ and integrate the second equation in time. After some math we obtain

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{i\omega_0 t} + \nabla \times \tilde{\mathbf{u}}/2,$$

where $\qquad \omega_0^2 = \dfrac{4\alpha}{\rho I}, \qquad \tilde{\mathbf{u}} = \omega_0 \displaystyle\int_{-\infty}^{t} \mathbf{u}(\tau) \sin \omega_0 (t - \tau) d\tau.$

Linear reduced Cosserat medium by eyes of a rotationally daltonic scientist

We substitute the expression for $\boldsymbol{\theta}$ into the balance of force:

$$(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu+\alpha)\nabla\times(\nabla\times\mathbf{u}) + \alpha\nabla\times(\nabla\times\tilde{\mathbf{u}}) + 2\alpha\nabla\times\boldsymbol{\theta}_0 e^{i\omega_0 t} = \rho\ddot{\mathbf{u}}.$$

We can rewrite it as

$$\nabla\cdot\hat{\boldsymbol{\tau}} + 2\alpha\nabla\times\boldsymbol{\theta}_0 e^{i\omega_0 t} = \rho\,\ddot{\mathbf{u}},$$

$$\hat{\boldsymbol{\tau}} \stackrel{\text{def}}{=} \boldsymbol{\tau}_c + 2\alpha(\nabla\cdot\tilde{\mathbf{u}}\,\mathbf{E} - \nabla\tilde{\mathbf{u}}^S) = \hat{\boldsymbol{\tau}}^\top,$$

$\boldsymbol{\tau}_c = \lambda'\nabla\cdot\mathbf{u}\,\mathbf{E} + 2\mu'\nabla\mathbf{u}^S$ is a stress tensor in the classical isotropic elastic medium with Lamé constants $\lambda' = \lambda - 2\alpha$, $\mu' = \mu + \alpha$.

Linear reduced isotropic Cosserat medium by eyes of a rotationally daltonic scientist

$$\hat{\boldsymbol{\tau}} \stackrel{\text{def}}{=} \boldsymbol{\tau}_c + 2\alpha(\nabla \cdot \tilde{\mathbf{u}}\,\mathbf{E} - \nabla \tilde{\mathbf{u}}^S) = \hat{\boldsymbol{\tau}}^\top.$$

We will see such a medium as a classical but history-depedent medium, possibly with some unpredictable harmonic external body forces with frequency ω_0.
If we consider external body forces and moments for the reduced Cosserat medium, we will see that the expression for the effective body force in the effective classical medium will have a time-dependent contribution from the rotor of body moment.

Linear reduced anisotropic Cosserat medium by eyes of a rotationally daltonic scientist

If only the translational part of the stress is anisotropic ($\boldsymbol{\tau} = \mathbf{C} \cdot\cdot \nabla \mathbf{u}^S + 2\alpha(\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A$), we obtain the similar interpretation with

$$\hat{\boldsymbol{\tau}} = \mathbf{C} \cdot\cdot \nabla \mathbf{u}^S - 2\alpha \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\mathbf{E} + 2\alpha(\nabla(\mathbf{u} - \tilde{\mathbf{u}}))^S = \hat{\boldsymbol{\tau}}^\top.$$

If the coupling $\mathbf{N}$ between $\nabla \mathbf{u}^S$ and $(\mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A$ is present

$$\rho U = \frac{1}{2} \nabla \mathbf{u}^S \cdot\cdot \mathbf{C} \cdot\cdot \nabla \mathbf{u}^S + \nabla \mathbf{u}^S \cdot\cdot \mathbf{N} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A + (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A \cdot\cdot \boldsymbol{\alpha} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^A,$$

even if only $\mathbf{N}$ is anisotropic, such an interpretation does not exist. We can find a symmetric effective $\hat{\boldsymbol{\tau}}$, but it will depend on $\nabla \mathbf{u}^A$ and $\nabla \tilde{\mathbf{u}}^A$, so it will not possess material objectivity in the linear approximation.
See for details Grekova, 2012, Mathematics and mechanics of solids

Constrained Cosserat medium

Sometimes a kinematical hypothesis $\boldsymbol{\theta} \equiv \nabla \times \mathbf{u}/2$ is accepted. Such a medium is called *constrained* (linear) Cosserat medium. For the nonlinear case this hypothesis is different.
Balance of energy for the Cosserat medium

$$\rho \dot{U} = \boldsymbol{\tau}^S \cdot\cdot \nabla \mathbf{v}^S - \boldsymbol{\tau}^A \cdot\cdot (\nabla \mathbf{v} + \boldsymbol{\omega} \times \mathbf{E})^A + \boldsymbol{\mu}^\top \cdot\cdot \nabla \boldsymbol{\omega}.$$

For the constrained theory $\nabla \mathbf{v}^A + \boldsymbol{\omega} \times \mathbf{E} = 0$, the red term does not exist, $\boldsymbol{\tau}^A$ cannot be determined from the constitutive equation!
In any theories, if a kinematical constraint is accepted, stresses working on the corresponding strain rate cannot be determined from the constitutive equation.
They must be determined from dynamic laws.

Constrained Cosserat medium

Is it important for the balance of force and moment that the kinematical constraint is accepted?

Balance of force

$$\nabla \cdot (\boldsymbol{\tau}^S + \boldsymbol{\tau}^A) + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}.$$

$$\nabla \cdot \boldsymbol{\tau}^S - \nabla \times \boldsymbol{\tau}_\times /2 + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}.$$

Balance of moment

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho \mathbf{L} = (\rho \mathbf{I} \cdot \boldsymbol{\omega})^{\cdot} \equiv (\rho \mathbf{I} \cdot (\nabla \times \mathbf{v}/2))^{\cdot}.$$

For the linear or isotropic case the dynamic term equals $\rho \mathbf{I} \cdot (\nabla \times \dot{\mathbf{v}})/2$,

Constrained Cosserat medium

Balance of moment

$$\boldsymbol{\tau}_{\times} = \rho \mathbf{I} \cdot (\nabla \times \dot{\mathbf{v}})/2 - \nabla \cdot \boldsymbol{\mu} - \rho \mathbf{L}.$$

Balance of force

$$\nabla \cdot \boldsymbol{\tau}^S - \nabla \times (\rho \mathbf{I} \cdot (\nabla \times \dot{\mathbf{v}})/2 - \nabla \cdot \boldsymbol{\mu} - \rho \mathbf{L})/2 + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}.$$

$$\nabla \cdot \boldsymbol{\tau}^S + \nabla \times (\nabla \cdot \boldsymbol{\mu}/2) + \rho \mathbf{F} + \nabla \times (\rho \mathbf{L})/2 = \rho \ddot{\mathbf{u}} + \nabla \times (\rho \mathbf{I} \cdot (\nabla \times \ddot{\mathbf{u}}))/4.$$

$\boldsymbol{\tau}^S$ is determined via constitutive equations.
$\boldsymbol{\mu}$ is determined via constitutive equations.
If rotational strain enters into equations, it must be substituted by translational strain, using the kinematical hypothesis. Everything is in terms of translations, but higher gradients are present.

Full isotropic constrained linear elastic Cosserat model

Constitutive equations for the isotropic case

$$\boldsymbol{\mu} = \mathbf{Z} \cdot\cdot \nabla\boldsymbol{\theta} \equiv \mathbf{Z} \cdot\cdot (\nabla\nabla \times \mathbf{u})/2,$$

$$\boldsymbol{\tau}^S = \mathbf{C} \cdot\cdot \nabla\mathbf{u}^S.$$

Balance of force takes the form

$$\nabla \cdot (\mathbf{C} \cdot\cdot \nabla\mathbf{u}^S) + \nabla \times (\nabla \cdot (\mathbf{Z} \cdot\cdot (\nabla\nabla \times \mathbf{u})/4)) \\ + \rho\mathbf{F} + \nabla \times (\rho\mathbf{L})/2 = \rho\ddot{\mathbf{u}} + \nabla \times (\rho\mathbf{I} \cdot (\nabla \times \ddot{\mathbf{u}}))/4.$$

This is a second-gradient theory.
Only two moduli in $\mathbf{Z}$ enter into the constitutive equations, since $\beta\mathbf{E}\mathbf{E} \cdot\cdot \nabla\nabla \times \mathbf{u} = 0$.
What about balance of moment? What happens with it?
Exercise: obtain the equations for the homogeneous case, $\mathbf{I} = I\mathbf{E}$.

Reduced constrained Cosserat model

$$\boldsymbol{\mu} = \mathbf{0}$$

$$\nabla \cdot \boldsymbol{\tau}^S + \rho \mathbf{F} + \nabla \times (\rho \mathbf{L})/2 = \rho \ddot{\mathbf{u}} + \nabla \times ((\rho \mathbf{I} \cdot (\nabla \times \dot{\mathbf{u}}))^{\cdot})/4.$$

No rotations enter into the constitutive equation for $\boldsymbol{\tau}^S$. We obtain almost a classical theory but with a very strange dynamic term derivated with respect to the space co-ordinates.
Note that here we did not use anything but the laws of balance of force and moment and kinematical constraint $\boldsymbol{\omega} = \nabla \times \dot{\mathbf{u}}/2$.

Is it true in the nonlinear case?
In the inelastic case?
In anisotropic case?
In inhomogeneous case?

Reduced constrained Cosserat model

The answer is: yes for the linear case! For the nonlinear case material objectivity fails.
For spherical density of the tensor of inertia $\rho I \mathbf{E}$, equal for all body points,

$$\nabla \cdot \boldsymbol{\tau}^S = \rho \ddot{\mathbf{u}} + \rho I (\nabla \nabla \cdot \ddot{\mathbf{u}} - \triangle \ddot{\mathbf{u}})/4$$

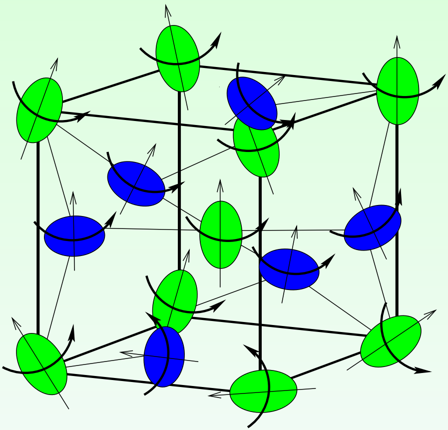
For the linear isotropic case in absence of external loads

$$\boldsymbol{\tau}^S = \lambda \nabla \cdot \mathbf{u} \mathbf{E} + 2\mu (\nabla \mathbf{u})^S,$$

$$\lambda \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times (\nabla \times \mathbf{u}) = \rho \ddot{\mathbf{u}} + \rho I (\nabla \nabla \cdot \ddot{\mathbf{u}} - \triangle \ddot{\mathbf{u}})/4$$

Note that the highest order of the space derivative is the same as in the classical theory. Here we do not obtain the second gradient theory.

Kelvin's medium



Kelvin's medium — special Cosserat medium with particle posessing large spin

Kelvin's medium

In Kelvin's medium body points possess rotational symmetry (angle φ, axis $\mathbf{m}$), and no stresses are caused by their proper rotation. The main term in the kinetic moment is due to dynamic spin.
This is a Cosserat medium with special type of kinetic moment and $U(\mathbf{A}, \mathbf{K})$ such that

$$\frac{\partial U}{\partial \varphi} = 0, \quad \frac{\partial U}{\partial \nabla \varphi} = \mathbf{0}.$$

One can be prove that it is necessary and sufficient for U to be a function of

$$\mathcal{G} = \mathbf{A} \cdot \mathbf{A}^\top, \quad \mathcal{F} = \mathbf{A} \cdot (\mathbf{E} - \mathbf{m}_0 \mathbf{m}_0) \cdot \mathbf{K}^\top, \quad \boldsymbol{\gamma} = \mathbf{A} \cdot \mathbf{m}_0$$

or any other 14 **independent** strain measures, functions of these ones. Here $\mathbf{m}_0$ be the unit vector of the body axis in the reference configurarion, $\mathbf{m} = \mathbf{P} \cdot \mathbf{m}_0$ in the actual configuration.

Exercise: check that $\mathcal{G}, \mathcal{F}, \boldsymbol{\gamma}$ do not depend on φ and $\nabla \varphi$.

Kelvin's medium

Lord Kelvin suggested an idea of such a medium to describe electromagnetic phenomena.
Linear equations: Gavrilov, Grekova and Zhilin (1996), nonlinear: Grekova and Zhilin (1996–1999).

Density of inertia tensor: $\rho(I\mathbf{mm} + I_1(\mathbf{E} - \mathbf{mm}))$.

Dynamic term: $I, I_1 = o(1), \dot{\varphi}$ is large, $\rho I\dot{\varphi} = O(1) \Longrightarrow$

$$\rho(\mathbf{I}\cdot\boldsymbol{\omega})^{\cdot} \approx \boldsymbol{\omega} \times \rho I\dot{\varphi}\mathbf{m}.$$

Balance of force:

$$\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{F} = \rho\ddot{\mathbf{u}}.$$

Balance of moment:

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho\mathbf{L} = \boldsymbol{\omega} \times \rho I\dot{\varphi}\mathbf{m}.$$

Kelvin's medium

Sixth balance equation:

$$\frac{\partial U}{\partial \varphi} = 0 \quad \Longrightarrow \quad (\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times) \cdot \mathbf{m} = \mathbf{0}.$$

$$\frac{\partial U}{\partial \nabla \varphi} = 0 \quad \Longrightarrow \quad \boldsymbol{\mu} \cdot \mathbf{m} = \mathbf{0}.$$

Proof. Let $\mathbf{P} = \mathbf{P}_\perp \cdot \mathbf{P}_\varphi(\varphi \mathbf{m}_0)$, then $\boldsymbol{\omega} = \boldsymbol{\omega}_\perp + \dot{\varphi}\mathbf{m}$,
$\nabla \boldsymbol{\omega} = \nabla \boldsymbol{\omega}_\perp + \nabla \dot{\varphi}\mathbf{m} + \dot{\varphi} \nabla \mathbf{m}$.

$$\begin{aligned}\rho \dot{U} = \boldsymbol{\tau}^\top \cdot\cdot (\nabla \mathbf{v} + \boldsymbol{\omega}_\perp \times \mathbf{E} + \dot{\varphi}\mathbf{m} \times \mathbf{E}) + \boldsymbol{\mu}^\top \cdot\cdot (\nabla \boldsymbol{\omega}_\perp + \nabla \dot{\varphi}\mathbf{m} + \dot{\varphi}\nabla \mathbf{m}).\\ = \boldsymbol{\tau}^\top \cdot\cdot (\nabla \mathbf{v} + \boldsymbol{\omega}_\perp \times \mathbf{E}) + \boldsymbol{\mu}^\top \cdot\cdot \nabla \boldsymbol{\omega}_\perp\\ \dot{\varphi}(\boldsymbol{\tau} \cdot\cdot (\mathbf{m} \times \mathbf{E}) + \boldsymbol{\mu}^\top \cdot\cdot \nabla \mathbf{m}) + \nabla \dot{\varphi} \cdot (\boldsymbol{\mu} \cdot \mathbf{m}).\end{aligned}$$

$\frac{\partial U}{\partial \nabla \varphi} = \mathbf{0} \quad \Longrightarrow \frac{\partial \dot{U}}{\partial \nabla \dot{\varphi}} = 0 \Longrightarrow \quad \boldsymbol{\mu} \cdot \mathbf{m} = 0,$
$\frac{\partial U}{\partial \varphi} = 0 \quad \Longrightarrow \frac{\partial \dot{U}}{\partial \dot{\varphi}} = 0 \quad \Longrightarrow$

$$\boldsymbol{\tau}_\times \cdot \mathbf{m} = \boldsymbol{\mu}^\top \cdot\cdot \nabla \mathbf{m} = -(\nabla \cdot \boldsymbol{\mu}) \cdot \mathbf{m}.$$

Kelvin's medium

Constitutive equations:

$$\boldsymbol{\tau} = \overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial U}{\partial \boldsymbol{\mathcal{G}}}\cdot\mathbf{A} + \frac{\partial U}{\partial \boldsymbol{\gamma}}\mathbf{m}_0 + \left(\frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}\right)^{\top}\cdot\mathbf{K}\cdot(\mathbf{E} - \mathbf{m}_0\mathbf{m}_0)^{\top}\right)\cdot\mathbf{P}^{\top},$$

$$\boldsymbol{\mu} = \overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\,\frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}\cdot\mathbf{A}\cdot(\mathbf{E} - \mathbf{m}_0\mathbf{m}_0)^{\top}\cdot\mathbf{P}^{\top},$$

Kelvin's medium

It is important that we would have a complete set of independent strain measures. For instance, a set

$$\boldsymbol{\mathcal{G}}, \quad \boldsymbol{\gamma}, \quad \mathbf{K}\cdot(\mathbf{E}-\mathbf{m}_0\mathbf{m}_0)\cdot\mathbf{K}^\top \tag{34}$$

seems to be perfect. But it is not complete; there is another independent strain measure $\mathbf{m}_0\cdot\mathbf{K}\cdot(\mathbf{E}-\mathbf{m}_0\mathbf{m}_0)\cdot\mathbf{A}^\top\cdot\mathbf{m}_0$, on which the strain energy may depend.

There is an analogy between Kelvin's medium and magnetic media. Neglecting dependence on this strain measure means to forbid helicoidal magnetic materials.

Kelvin's medium

It has been shown that the dynamical laws and constitutive equations for Kelvin's medium with large angular velocity of proper rotation and infinitesimal inertia moments per unit mass of point-bodies, and for elastic ferromagnetic saturated insulators coincide. All angular quantities correspond to the magnetic subsystem and translational ones to the elastic subsystem.

Kelvin's medium

for Kelvin's medium	for ferromagnet
$\mathbf{u}$ is the translational displacement	
$\boldsymbol{\tau}$ is the stress tensor	
$\mathbf{m}$ is a unit vector	
of an axis of a point-body	of the magnetic moment $\mathbf{S}$
$\rho I \dot{\varphi}\, \mathbf{m} \longleftrightarrow \rho\, \mathbf{S} \,/\, \gamma = M\mathbf{m}/\gamma$ is the kinetic moment	
$\dot{\varphi}$ is the angular velocity of proper rotation of a point-body, I is the density of axial moment of inertia	$\mathbf{S}$ is magnetic moment, $M = \rho\,\lvert\mathbf{S}\rvert$ is magnetization, $\lvert\mathbf{S}\rvert = \mathbf{const}$ for saturation state, γ is gyromagnetic ratio

Kelvin's medium

for Kelvin's medium	for ferromagnet
$\boldsymbol{\tau}^A$ is related to the moment working on the rate of the rotation of a body point relatively to the surrounding continuum	$\boldsymbol{\tau}^A$ is caused by spin-lattice interaction
$\boldsymbol{\mu} \longleftrightarrow -\boldsymbol{\mathcal{B}} \times \mathbf{S}$	
$\boldsymbol{\mu}$ is the couple tensor	$\boldsymbol{\mathcal{B}}$ is the tensor of exchange interactions
$\mathbf{L} \longleftrightarrow \mathbf{B}^e \times \mathbf{S}$	
$\mathbf{L}$ is the volume density of an external moment	$\mathbf{B}^e$ is the external magnetic induction

Linear Kelvin's medium

Let $\mathbf{P} = (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \cdot \mathbf{P}_\varphi(\varphi \mathbf{m}_0)$, $\quad \boldsymbol{\theta} = o(1)$, $\quad \mathbf{u} = \mathbf{R} - \mathbf{r} = o(1)$. Then

$$[\mathbf{A}]_0 = \mathbf{P}_\varphi(\varphi \mathbf{m}_0), \qquad [\mathbf{K}]_0 = \mathbf{0},$$

$$[\mathbf{A}]_1 = (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}) \cdot \mathbf{P}_\varphi(\varphi \mathbf{m}_0), \qquad [\mathbf{K}]_1 = \nabla \boldsymbol{\theta} \cdot \mathbf{P}_\varphi(\varphi \mathbf{m}_0).$$

Linearized equations near the natural state, suppose it is at $\overset{\circ}{\nabla} \mathbf{m}_0 = \mathbf{0}$:

$$\boldsymbol{\tau} = \mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}) + \mathbf{Y} \cdot\cdot (\nabla \boldsymbol{\theta}) \tag{35}$$

$$\boldsymbol{\mu} = ((\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^\top \cdot\cdot \mathbf{Y})^\top + \mathbf{Z} \cdot\cdot \nabla \boldsymbol{\theta} \tag{36}$$

$$\mathbf{X} \cdot\cdot (\mathbf{m}_0 \times \mathbf{E}) = \mathbf{0}, \qquad \mathbf{Y} \cdot\cdot (\mathbf{m}_0 \times \mathbf{E}) = \mathbf{0}.$$

$(\boldsymbol{\theta} - \nabla \times \mathbf{u}/2) \cdot \mathbf{m}_0$ does not enter into equations
Direct way to obtain linear equations from the balance of energy: Gavrilov, Zhilin, 1996

Linear Kelvin's medium

Dynamic term:
$\boldsymbol{\omega} \approx \dot{\boldsymbol{\theta}} + \dot{\varphi}(\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \cdot \mathbf{m}_0 = \dot{\boldsymbol{\theta}} + \dot{\varphi}\mathbf{m}$.
$\boldsymbol{\omega} \times \rho l \dot{\varphi}\mathbf{m} \approx \dot{\boldsymbol{\theta}} \times \rho l \dot{\varphi}\mathbf{m}_0$.
Equations in displacements:

$$\nabla \cdot (\mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}) + \mathbf{Y} \cdot\cdot (\nabla \boldsymbol{\theta})) = \rho \ddot{\mathbf{u}},$$

$$\nabla \cdot (((\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E})^\top \cdot\cdot \mathbf{Y})^\top + \mathbf{Z} \cdot\cdot \nabla \boldsymbol{\theta}) + [\mathbf{X} \cdot\cdot (\nabla \mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}) + \mathbf{Y} \cdot\cdot (\nabla \boldsymbol{\theta})]_\times = \dot{\boldsymbol{\theta}} \times \rho l \dot{\varphi}\mathbf{m}_0.$$

They are the same as for the linear Cosserat medium, but: dynamic term in the balance of moment is different, and elastic tensors forbid $(\boldsymbol{\theta} - \nabla \times \mathbf{u}) \cdot \mathbf{m}_0$ enter into equations. $\mathbf{m} \approx \mathbf{m}_0$.

Simplest reduced linear Kelvin's medium

In the reduced Kelvin's medium by definition stresses do not work on $\nabla\dot{\boldsymbol{\theta}} \quad \Longrightarrow \boldsymbol{\mu} = \mathbf{0}$.

$$\rho U = \frac{1}{2}(\nabla \mathbf{u}^S \cdot\cdot\, \mathbf{C} \cdot\cdot\, \nabla \mathbf{u}^S + 4\alpha((\boldsymbol{\theta} - \nabla \times \mathbf{u}/2) \cdot (\mathbf{E} - \mathbf{mm}))^2).$$

Constitutive equations:

$$\boldsymbol{\tau} = \lambda \nabla \cdot \mathbf{u}\,\mathbf{E} + 2\mu(\nabla \mathbf{u})^S + 2\alpha(\boldsymbol{\theta} - \frac{1}{2}\nabla \times \mathbf{u}) \cdot (\mathbf{E} - \mathbf{mm}) \times \mathbf{E}.$$

Equations in displacements ($\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} \cdot (\mathbf{E} - \mathbf{mm})$, $b\tilde{\boldsymbol{\theta}}$ — external body moment):

$$\begin{aligned}(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times (\nabla \times \mathbf{u})& \\ - \alpha\nabla \times (\mathbf{E} - \mathbf{mm}) \cdot (\nabla \times \mathbf{u} - 2\boldsymbol{\theta}) &= \rho\ddot{\mathbf{u}}, \\ 4\alpha(\nabla \times \mathbf{u}/2 - \boldsymbol{\theta}) \cdot (\mathbf{E} - \mathbf{mm}) - b\,\tilde{\boldsymbol{\theta}} &= M\dot{\boldsymbol{\theta}} \times \mathbf{m}.\end{aligned}$$

Useful formulae and facts

$$\overset{\circ}{\nabla}\mathbf{r} = \nabla\mathbf{R} = \mathbf{E}$$

$$\overset{\circ}{\nabla} = \overset{\circ}{\nabla}\mathbf{R}\cdot\nabla = \nabla + \overset{\circ}{\nabla}\mathbf{u}\cdot\nabla \quad (= \nabla + o(1) \text{ if } \mathbf{u} = o(1))$$

If tensor $\boldsymbol{\lambda} = o(1), \quad (\mathbf{E}+\boldsymbol{\lambda})^{-1} \approx \mathbf{E} - \boldsymbol{\lambda}$

$$\overset{\circ}{\nabla}\mathbf{R}^{-1} \approx \mathbf{E} - \overset{\circ}{\nabla}\mathbf{u} \quad \text{if } \mathbf{u} = o(1)$$

$$\mathbf{Q} \approx \mathbf{E} + \mathbf{q}\times\mathbf{E} \quad \text{if } \mathbf{Q} \text{ is an infinitesimal rotation tensor}$$

$$(\mathbf{E} + \mathbf{q}\times\mathbf{E})^{-1} \approx \mathbf{E} - \mathbf{q}\times\mathbf{E} \quad \text{if } \mathbf{q} = o(1)$$

$$\mathbf{Q}\cdot\boldsymbol{\Lambda}\cdot\mathbf{Q}^{\top} \approx \boldsymbol{\Lambda} \text{ if } \boldsymbol{\Lambda} = o(1), \mathbf{Q} \text{ infinitesimal rotation tensor}$$

Useful formulae and facts

Rigid motion superposed upon a deformation $\mathbf{R}, \mathbf{P}$:
rigid rotation $\mathbf{Q}$ about a center $\mathbf{R}_c$, rigid displacement $\mathbf{R}_0$

$$\mathbf{R}' = \mathbf{Q}(t) \cdot (\mathbf{R} - \mathbf{R}_c) + \mathbf{R}_c + \mathbf{R}_0(t), \quad \mathbf{P}' = \mathbf{Q} \cdot \mathbf{P}$$

$$\overset{\circ}{\nabla}\mathbf{R}' = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{Q}^\top$$

$$\nabla\mathbf{u}' \approx \nabla\mathbf{u} - \mathbf{q} \times \mathbf{E},$$

$$\boldsymbol{\theta}' = \boldsymbol{\theta} + \mathbf{q},$$

where $\mathbf{u} = o(1), \quad \mathbf{q} = o(1), \quad \boldsymbol{\theta} = o(1),$

$$\mathbf{Q} \approx \mathbf{E} + \mathbf{q} \times \mathbf{E},$$

$$\mathbf{P} \approx \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}.$$

Useful formulae and facts

$\overset{\circ}{\nabla}\mathbf{u}^S$ is materially objective in the linear approximation and NOT in the nonlinear approximation

$\overset{\circ}{\nabla}\mathbf{u}^A$ is NOT materially objective even in the linear approximation

$\mathbf{A}, \mathbf{K}$ do not change under rigid motion

An infinitesimal tensor does not change under rigid motion
$\equiv$ it is materially objective in the linear appproximation
$(\mathbf{Q} \cdot \mathbf{\Lambda} \cdot \mathbf{Q}^\top \approx \mathbf{\Lambda})$
but NOT in the nonlinear sense

Example 1. $\nabla \mathbf{u}^A$ is not materially objective in any sense.

Proof. We consider two motions, $\mathbf{R}$ corresponds to a certain deformation, and $\mathbf{R}'$ to the same deformation but with superposed rigid motion.

$$\overset{\circ}{\nabla}\mathbf{u}'^A = \overset{\circ}{\nabla}(\mathbf{R}' - \mathbf{r})^A = (\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top - \mathbf{E})^A = (\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{Q}^\top)^A$$
$$= ((\overset{\circ}{\nabla}(\mathbf{r}+\mathbf{u}))\cdot\mathbf{Q}^\top)^A = ((\mathbf{E}+\overset{\circ}{\nabla}\mathbf{u})\cdot\mathbf{Q}^\top)^A = (\overset{\circ}{\nabla}\mathbf{u}\cdot\mathbf{Q}^\top)^A - \mathbf{Q}^A$$
$$= (\overset{\circ}{\nabla}\mathbf{u}\cdot\mathbf{Q}^\top - \mathbf{Q}\cdot\overset{\circ}{\nabla}\mathbf{u}^\top)/2 - \mathbf{Q}^A \neq \mathbf{Q}\cdot\overset{\circ}{\nabla}\mathbf{u}^A\cdot\mathbf{Q}^\top.$$

So we see that it is not materially objective in the nonlinear sense. Let us make the linear approximation ($\mathbf{Q} \approx \mathbf{E} + \mathbf{q}\times\mathbf{E}$). Then

$$\overset{\circ}{\nabla}\mathbf{u}'^A = (\overset{\circ}{\nabla}\mathbf{u} - \overset{\circ}{\nabla}\mathbf{u}^\top)/2 - \mathbf{q}\times\mathbf{E}^A = \overset{\circ}{\nabla}\mathbf{u}^A - \mathbf{q}\times\mathbf{E} \approx \mathbf{Q}\cdot\overset{\circ}{\nabla}\mathbf{u}^A\cdot\mathbf{Q}^\top - \mathbf{q}\times\mathbf{E}$$

It is not rotated with the material even in the linear sense.

Example 2

Can $\boldsymbol{\tau} = \mathbf{C} \cdot\cdot \overset{\circ}{\nabla} \mathbf{u}^S$, where $C_{ijkl} = C_{klij} = C_{ijlk} \neq 0$, be a good constitutive equation for a nonlinear theory? For a linear theory?
Answer: no for nonlinear, yes for linear.
Proof. Let us check the material objectivity. Remember that $\mathbf{u} = \mathbf{R} - \mathbf{r}$.
Consider a piece of deformed material subjected to a rigid motion corresponding to deformations $\mathbf{u}'$, stress $\boldsymbol{\tau}'$.

$$
\begin{aligned}
\overset{\circ}{\nabla} \mathbf{u}'^S &= \overset{\circ}{\nabla} (\mathbf{R}' - \mathbf{r})^S = \overset{\circ}{\nabla} \mathbf{R}'^S - \mathbf{E} = (\overset{\circ}{\nabla} \mathbf{R} \cdot \mathbf{Q}^\top)^S - \mathbf{E} \\
&= \frac{1}{2}(\mathbf{Q} \cdot \overset{\circ}{\nabla} \mathbf{R}^\top + \overset{\circ}{\nabla} \mathbf{R} \cdot \mathbf{Q}^\top) - \mathbf{E} = \frac{1}{2}(\mathbf{Q} \cdot \overset{\circ}{\nabla} \mathbf{u}^\top + \overset{\circ}{\nabla} \mathbf{u} \cdot \mathbf{Q}^\top) + \mathbf{Q}^S - \mathbf{E}
\end{aligned}
$$

Since $\overset{\circ}{\nabla} \mathbf{u}'^S \neq \mathbf{Q} \cdot \overset{\circ}{\nabla} \mathbf{u}^S \cdot \mathbf{Q}^\top$, it is not materially objective strain tensor.
Also it changes under rigid motion.

We see that $\boldsymbol{\tau}' = \mathbf{C} \cdot\cdot \overset{\circ}{\nabla} \mathbf{u}'^S \neq \mathbf{Q} \cdot (\mathbf{C} \cdot\cdot \overset{\circ}{\nabla} \mathbf{u}^S) \cdot \mathbf{Q}^\top$.

Useful facts (proved previously)

- If energetic stress tensors $\boldsymbol{\tau}_*$, $\boldsymbol{\mu}_*$ do not change under rigid motion, then Cauchy stress tensor $\boldsymbol{\tau}$ and Cauchy couple stress tensor $\boldsymbol{\mu}$ are materially objective (rotate together with the piece of material).
- If $\boldsymbol{\tau}_*$, $\boldsymbol{\mu}_*$ depend only on strain measures that do not change under rigid motion, themselves they do not change under it, and $\boldsymbol{\tau}, \boldsymbol{\mu}$ are materially objective.
- If $\boldsymbol{\tau}, \boldsymbol{\mu}$ depend only on materially objective strain measures, they are materially objective.

Example

In the linear approximation, using $\mathbf{Q} \approx \mathbf{E} + \mathbf{q} \times \mathbf{E}$, we obtain $\overset{\circ}{\nabla} \mathbf{u}'^S \approx \overset{\circ}{\nabla} \mathbf{u}^S \approx \mathbf{Q} \cdot \overset{\circ}{\nabla} \mathbf{u}^S \cdot \mathbf{Q}^\top$ up to $o^2(1)$:

$$\begin{aligned}
\overset{\circ}{\nabla} \mathbf{u}'^S &= \frac{1}{2}(\mathbf{Q} \cdot \overset{\circ}{\nabla} \mathbf{u}^\top + \overset{\circ}{\nabla} \mathbf{u} \cdot \mathbf{Q}^\top) + \mathbf{Q}^S - \mathbf{E} \\
&\approx \frac{1}{2}((\mathbf{E} + \mathbf{q} \times \mathbf{E}) \cdot \overset{\circ}{\nabla} \mathbf{u}^\top + \overset{\circ}{\nabla} \mathbf{u} \cdot (\mathbf{E} - \mathbf{q} \times \mathbf{E})) + (\mathbf{E} + \mathbf{q} \times \mathbf{E})^S - \mathbf{E} \\
&\approx \frac{1}{2}(\overset{\circ}{\nabla} \mathbf{u}^\top + \overset{\circ}{\nabla} \mathbf{u}) + \mathbf{E} - \mathbf{E} = \overset{\circ}{\nabla} \mathbf{u}^S
\end{aligned}$$

Then

$$\begin{aligned}
\boldsymbol{\tau}' = \mathbf{C} \cdot\cdot (\overset{\circ}{\nabla} \mathbf{u})^S &\approx (\mathbf{E} + \mathbf{q} \times \mathbf{E}) \cdot (\mathbf{C} \cdot\cdot \overset{\circ}{\nabla} \mathbf{u}^S) \cdot (\mathbf{E} - \mathbf{q} \times \mathbf{E}) \\
&\approx \mathbf{Q} \cdot (\mathbf{C} \cdot\cdot \overset{\circ}{\nabla} \mathbf{u}^S) \cdot \mathbf{Q}^\top = \mathbf{Q} \cdot \boldsymbol{\tau} \cdot \mathbf{Q}^\top
\end{aligned}$$

posesses material objectivity in the linear approximation.

How to check if constitutive equation is a good one

1. Understand if it is a linear one or nonlinear one, or can be both. If we have $\mathbf{A}, \mathbf{K}, \overset{\circ}{\nabla}\mathbf{R}, \mathbf{P}$, this is a nonlinear equation. If we have $\boldsymbol{\theta}$, we have defined it only for the linear case.
2. Check the material objectivity (in the linear or nonlinear sense). If it is not satisfied, it is never valid. $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are rotated by rigid motion, U does not change.
3. If it is satisfied, it is a good equation for the full Cosserat medium.
4. For the classical medium we have no rotations, and $\boldsymbol{\tau} = \boldsymbol{\tau}^T$.
5. For the reduced Cosserat medium $\boldsymbol{\mu} = \mathbf{0}$, and $\boldsymbol{\tau}$ and U does not depend on gradient of rotation.
6. For the constrained (pseudo-) Cosserat medium we consider only the linear case, and $\boldsymbol{\tau}^A$ cannot be found from constitutive equations.
7. For Kelvin's medium either we check that $\boldsymbol{\tau}, \boldsymbol{\mu}, U$ do not depend on $\varphi, \nabla\varphi$, or $(\boldsymbol{\tau}_\times + \nabla \cdot \boldsymbol{\mu}) \cdot \mathbf{m} = 0, \boldsymbol{\mu} \cdot \mathbf{m} = \mathbf{0}$.

Properties of strain tensors

1. $\mathbf{A}, \mathbf{K}$ do not change under ridid motion
2. $\overset{\circ}{\nabla}\mathbf{R}' = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{Q}^\top$, $\mathbf{P}' = \mathbf{Q} \cdot \mathbf{P}$
3. $\nabla\mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}, \nabla\boldsymbol{\theta}$, $\overset{\circ}{\nabla}\mathbf{u} + \boldsymbol{\theta} \times \mathbf{E}, \overset{\circ}{\nabla}\boldsymbol{\theta}$, $\overset{\circ}{\nabla}\mathbf{u}^S, \nabla\mathbf{u}^S, \nabla \cdot \mathbf{u}, \overset{\circ}{\nabla} \cdot \mathbf{u}$, $\overset{\circ}{\nabla}\mathbf{u}^A + \boldsymbol{\theta} \times \mathbf{E}, \nabla\mathbf{u}^A + \boldsymbol{\theta} \times \mathbf{E}$, $\boldsymbol{\theta} - \nabla \times \mathbf{u}/2$, $\boldsymbol{\theta} - \overset{\circ}{\nabla} \times \mathbf{u}/2$ are materially objective and do not change under rigid motion in the linear sense. They have no any of these properties in the nonlinear sense ("bad" ones)
4. $\nabla \times \mathbf{u}$, $\mathbf{u}$, $\boldsymbol{\theta}$ and their absolute values are not valid in any sense (linear / nonlinear). They change under rigid motion and are not materially objective

Waves in linear Cosserat-type media

Simple mathematical facts

If $\mathbf{f} = \mathbf{f}_0 e^{i(\omega t + \mathrm{k}\cdot\mathrm{r})}$, then

$$\dot{\mathbf{f}} = i\omega\mathbf{f},$$

$$\nabla\mathbf{f} = i\mathbf{k}\mathbf{f}.$$

Indeed, if $\mathbf{r} = q^s\mathbf{e}_s$, and $\mathbf{k} = k^s\mathbf{e}_s$, then

$$\nabla\mathbf{f} = \frac{\partial}{\partial q_s}\mathbf{e}_s\mathbf{f}_0 e^{i(\omega t + k^s q_s)} = ik^s\mathbf{e}_s\mathbf{f}_0 e^{i(\omega t + k^s q_s)} = i\mathbf{k}\mathbf{f}_0 e^{i(\omega t + \mathrm{k}\cdot\mathrm{r})}$$

Reminiscences: linear elastic classical medium

In absence of external loads

$$\boldsymbol{\tau} = \mathbf{C} \cdot\cdot (\nabla \mathbf{u})^S, \qquad \nabla \cdot \boldsymbol{\tau} = \rho \ddot{\mathbf{u}}.$$

Equations in displacements:

$$\nabla \cdot (\mathbf{C} \cdot\cdot \nabla \mathbf{u}^S) = \rho \ddot{\mathbf{u}}.$$

Looking for the solution $\mathbf{u} = \mathbf{u}_0 e^{i(\omega t + \mathrm{k} \cdot \mathrm{r})}$, we obtain

$$i\mathbf{k} \cdot (\mathbf{C} \cdot\cdot i(\mathbf{k}\mathbf{u}_0)^S) = -\omega^2 \rho \mathbf{u}_0.$$

$$(\rho\omega^2 \mathbf{E} - \mathbf{k} \cdot \mathbf{C} \cdot \mathbf{k}) \cdot \mathbf{u}_0 = \mathbf{0}.$$

Denote $k = |\mathbf{k}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$.

$$(\rho\omega^2 \mathbf{E} - k^2 \hat{\mathbf{k}} \cdot \mathbf{C} \cdot \hat{\mathbf{k}}) \cdot \mathbf{u}_0 = \mathbf{0}.$$

Reminiscences: linear elastic classical medium

Since $\hat{\mathbf{k}} \cdot \mathbf{C} \cdot \hat{\mathbf{k}}$ is a symmetric tensor, it can be represented as $C_i \mathbf{e}_i \mathbf{e}_i$, eigenvectors and eigenvalues in the anisotropic case depend on the direction of wave propagation $\hat{\mathbf{k}}$.

Then we obtain three possible solutions: $\mathbf{u}_0 = u_0 \mathbf{e}_i$,

$$\rho\omega^2 = k^2 C_i \qquad \Longleftrightarrow \qquad \omega = \sqrt{C_i/\rho}\, k.$$

These are straight lines.

In the isotropic case, eigenvectors are $\hat{\mathbf{k}}$ (compression wave, velocity C_l) and any vector orthogonal to $\hat{\mathbf{k}}$ (shear wave, velocity C_s), and corresponding velocities are

$$C_l = \sqrt{(\lambda + 2\mu)/\rho}, \qquad C_s = \sqrt{\mu/\rho} < C_l.$$

Dispersion relations for infinite linear elastic classical medium are always straight lines

Dynamic equations for the linear elastic isotropic reduced Cosserat medium (free oscillations)

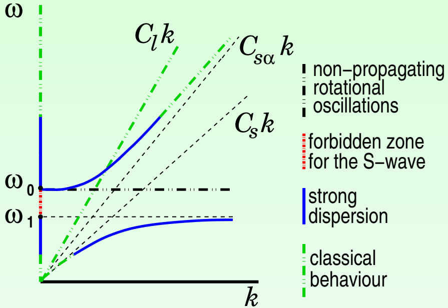
For Cosserat-type media, there will be not more than one straight line...

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - (\mu + \alpha)\nabla \times (\nabla \times \mathbf{u}) + 2\alpha\nabla \times \boldsymbol{\theta} = \rho\ddot{\mathbf{u}}$$

$$2\alpha\nabla \times \mathbf{u} - 4\alpha\boldsymbol{\theta} = \rho\mathbf{I} \cdot \ddot{\boldsymbol{\theta}} \quad \stackrel{\text{if } \mathbf{I} = I\mathbf{E}}{=} \quad \rho I\ddot{\boldsymbol{\theta}}$$

If $\alpha = 0$, we have the classical isotropic elastic medium.

Linear isotropic reduced Cosserat medium: dispersion



$\omega_0^2 = 4\alpha/(\rho_0 I)$,
$\omega_1^2 = \omega_0^2/(1+\alpha/\mu)$,
$C_s^2 = \mu/\rho$,
$C_{s\alpha}^2 = (\mu+\alpha)/\rho$,
$C_l^2 = (\lambda+2\mu)/\rho$.

There exist a band gap, where shear–rotational waves do not propagate (single negative acoustic metamaterial).

$$k^2 = \frac{\omega^2}{C_s^2}\frac{(1-\omega^2/\omega_0^2)}{(1-\omega^2/\omega_1^2)}.$$

This happens since some elastic connections are broken (the medium does not react to $\nabla\boldsymbol{\theta}$.) In this zone waves will be localised.

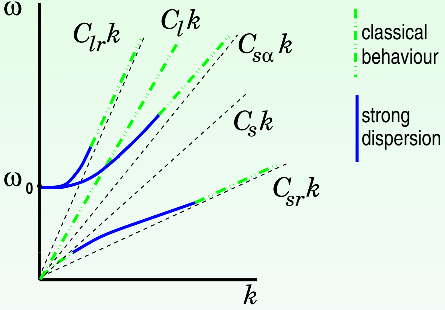
Isotropic linear Cosserat medium. Dynamic equations

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - (\mu + \alpha)\nabla \times (\nabla \times \mathbf{u}) + 2\alpha\nabla \times \boldsymbol{\theta} + \rho\mathbf{F} = \rho\ddot{\mathbf{u}},$$

$$(\beta + 2\gamma)\nabla\nabla \cdot \boldsymbol{\theta} - (\gamma + \varepsilon)\nabla \times (\nabla \times \boldsymbol{\theta}) + 2\alpha(\nabla \times \mathbf{u} - 2\boldsymbol{\theta}) + \rho\mathbf{L} = \rho\mathbf{I} \cdot \ddot{\boldsymbol{\theta}}.$$

Linear isotropic Cosserat medium with spherical tensor of inertia. Dispersion curves

If $\mathbf{I} = I\mathbf{E}$, the inertial term in the balance of moment equals $\rho I \ddot{\boldsymbol{\theta}}$. Consider free oscillations ($\mathbf{F} = \mathbf{0}, \mathbf{L} = \mathbf{0}$). For 3D plane waves we obtain the following dispersion curves:



$$C_l^2 = (\lambda + 2\mu)/\rho,$$
$$C_{lr}^2 = (\beta + 2\gamma)/\rho I,$$
$$C_s^2 = \mu/\rho,$$
$$C_{s\alpha}^2 = (\mu + \alpha)/\rho,$$
$$C_{sr}^2 = (\gamma + \varepsilon)/\rho I,$$
$$\omega_0^2 = 4\alpha/(\rho_0 I).$$

Linear isotropic reduced Cosserat medium: dispersion relations

We look for the solution of dynamic equations for the reduced Cosserat medium as $\mathbf{u} = \mathbf{u}_0 e^{i(\mathrm{k}\cdot\mathrm{r}+\omega t)}$, $\quad \boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{i(\mathrm{k}\cdot\mathrm{r}+\omega t)}$. Then we have to change the operator ∇ to $i\mathbf{k}$, and time derivatives to $i\omega$.

$$-(\lambda+2\mu)\mathbf{k}\mathbf{k}\cdot\mathbf{u} + (\mu+\alpha)\mathbf{k}\times(\mathbf{k}\times\mathbf{u}) + 2i\alpha\mathbf{k}\times\boldsymbol{\theta} = -\omega^2\rho\mathbf{u}$$
$$2i\alpha\mathbf{k}\times\mathbf{u} - 4\alpha\boldsymbol{\theta} = -\omega^2\rho I\boldsymbol{\theta}$$

We express $\boldsymbol{\theta}$ via $\mathbf{u}$ from the last equation, introducing $\omega_0^2 = 4\alpha/(\rho I)$. At $\omega \neq \omega_0$

$$\boldsymbol{\theta} = \frac{i\omega_0^2}{2(\omega_0^2-\omega^2)}\mathbf{k}\times\mathbf{u}. \tag{37}$$

Consider separately the case $\omega = \omega_0$. We have a possible solution $\mathbf{u} = \mathbf{0}$, $\boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{i\omega_0 t}$, or $\boldsymbol{\theta} = \boldsymbol{\theta}_0(\mathbf{r})e^{i\omega_0 t}$, if $\nabla\times\boldsymbol{\theta}_0(\mathbf{r}) = \mathbf{0}$.

Linear isotropic reduced Cosserat medium: dispersion relations

We substitute $\boldsymbol{\theta}$, expressed via $\mathbf{u}$, to the balance of force:

$$-(\lambda + 2\mu)\mathbf{k}\mathbf{k} \cdot \mathbf{u} + (\mu + \alpha \frac{\omega^2}{\omega^2 - \omega_0^2})\mathbf{k} \times (\mathbf{k} \times \mathbf{u}) = -\omega^2 \rho \mathbf{u}.$$

Note that $\mathbf{k} \times (\mathbf{k} \times \mathbf{u}) = k^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{E}) \cdot \mathbf{u}$, where $\hat{\mathbf{k}} = \mathbf{k}/k$, $k = |\mathbf{k}|$.

We can write down $\rho\omega^2\mathbf{E} = \rho\omega^2\hat{\mathbf{k}}\hat{\mathbf{k}} + \rho\omega^2(\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}})$.

Linear isotropic reduced Cosserat medium: dispersion relations

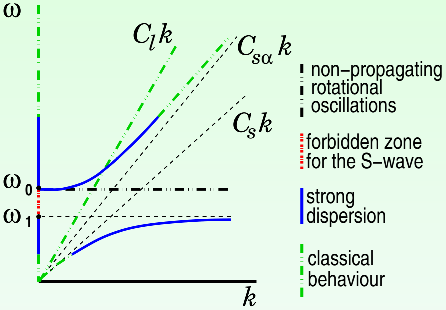
We obtain

$$(\rho\omega^2 - k^2(\lambda + 2\mu))\hat{\mathbf{k}}\hat{\mathbf{k}} \cdot \mathbf{u} + (\rho\omega^2 - k^2(\mu + \alpha\frac{\omega^2}{\omega^2 - \omega_0^2}))(\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \mathbf{u} = \mathbf{0}.$$

We see that the compression wave is the same as in the classical medium, and the dispersion relation for the shear–rotational wave is

$$\rho\omega^2 = k^2(\mu + \alpha\frac{\omega^2}{\omega^2 - \omega_0^2}) \quad \Longleftrightarrow \quad k^2 = \frac{\omega^2}{C_s^2}\frac{(1 - \omega^2/\omega_0^2)}{(1 - \omega^2/\omega_1^2)}.$$

Reduced isotropic Cosserat medium: dispersion graph



$$\omega_0^2 = 4\alpha/(\rho_0 I),$$
$$\omega_1^2 = \omega_0^2/(1+\alpha/\mu),$$
$$C_s^2 = \mu/\rho,$$
$$C_{s\alpha}^2 = (\mu+\alpha)/\rho,$$
$$C_l^2 = (\lambda+2\mu)/\rho.$$

This is a localisation

If we consider point sources $\mathbf{F} = \mathbf{F}_0 e^{i\omega t}, \mathbf{L} = \mathbf{L}_0 e^{i\omega t}$, and $\omega_1 \leqslant \omega < \omega_0$, then a part of wave is localised near the source (Grekova, Kulesh, Herman, BSSA, 2009)
At ω_0 there are resonant phenomena (if point torque is applied) or stronger localisation (for point force).

Linear isotropic full Cosserat medium: dispersion relations

For zero loads

$$(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu+\alpha)\nabla\times(\nabla\times\mathbf{u}) + 2\alpha\nabla\times\boldsymbol{\theta} = \rho\ddot{\mathbf{u}},$$

$$(\beta + 2\gamma)\nabla\nabla\cdot\boldsymbol{\theta} - (\gamma+\varepsilon)\nabla\times(\nabla\times\boldsymbol{\theta}) + 2\alpha(\nabla\times\mathbf{u} - 2\boldsymbol{\theta}) = \rho I\ddot{\boldsymbol{\theta}}.$$

We look for solutions of dynamic equations in the form $\mathbf{u} = \mathbf{u}_0 e^{i(\mathrm{k}\cdot\mathrm{r}+\omega t)}$, $\boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{i(\mathrm{k}\cdot\mathrm{r}+\omega t)}$. Again we change ∇ to $i\mathbf{k}$, and time derivatives to $i\omega$. We have

$$-(\lambda + 2\mu)\mathbf{k}\mathbf{k}\cdot\mathbf{u} + (\mu+\alpha)\mathbf{k}\times(\mathbf{k}\times\mathbf{u}) + 2i\alpha\mathbf{k}\times\boldsymbol{\theta} = -\rho\omega^2\mathbf{u}, \tag{38}$$

$$-(\beta + 2\gamma)\mathbf{k}\mathbf{k}\cdot\boldsymbol{\theta} + (\gamma+\varepsilon)\mathbf{k}\times(\mathbf{k}\times\boldsymbol{\theta}) + 2\alpha(i\mathbf{k}\times\mathbf{u} - 2\boldsymbol{\theta}) = -\rho I\omega^2\boldsymbol{\theta}. \tag{39}$$

If we try to eliminate $\boldsymbol{\theta}$ from the balance of force using the balance of moment, we will have more difficulties than for the reduced Cosserat medium. What to do?

Linear isotropic full Cosserat medium: dispersion relations

Note that we only have to express $\mathbf{k} \times \boldsymbol{\theta}$ via $\mathbf{u}$. Calculating the cross product of (39) and $\mathbf{k}$, we obtain

$$(\gamma + \varepsilon)\mathbf{k} \times (\mathbf{k} \times (\mathbf{k} \times \boldsymbol{\theta})) + 2\alpha(i\mathbf{k} \times (\mathbf{k} \times \mathbf{u}) - 2\mathbf{k} \times \boldsymbol{\theta}) = -\rho I \omega^2 \mathbf{k} \times \boldsymbol{\theta}.$$

Note that $\mathbf{k} \times (\mathbf{k} \times (\mathbf{k} \times \boldsymbol{\theta})) = -k^2(\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot (\mathbf{k} \times \boldsymbol{\theta}) = -k^2 \mathbf{k} \times \boldsymbol{\theta}$
We have

$$2i\alpha\mathbf{k} \times (\mathbf{k} \times \mathbf{u}) = (k^2(\gamma + \varepsilon) + 4\alpha - \rho I \omega^2)\mathbf{k} \times \boldsymbol{\theta}.$$

Dividing by ρI, we obtain

$$i\omega_0^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{u})/2 = (k^2 C_{sr}^2 + \omega_0^2 - \omega^2)\mathbf{k} \times \boldsymbol{\theta}.$$

Linear isotropic full Cosserat medium: dispersion relations

We will show that this cannot be true:

$$\omega^2 = \omega_0^2 + C_{sr}^2 k^2. \tag{40}$$

In the reduced Cosserat medium it was a straight line $\omega = \omega_0$. We see that this yields $\mathbf{u} \cdot (\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) = \mathbf{0}$. Substituting to (39) these equations, we obtain $\mathbf{k} \cdot \boldsymbol{\theta} = 0$.

Indeed, $\mathbf{k} \times \mathbf{u} = \mathbf{0}$, dividing (39) by ρI, we obtain

$$-C_{lr}^2 \mathbf{k}\mathbf{k} \cdot \boldsymbol{\theta} - C_{sr}^2 k^2 (\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \boldsymbol{\theta} - \omega_0^2 \boldsymbol{\theta} = -\omega^2 \boldsymbol{\theta}, \tag{41}$$

and taking into account (40) we have $(C_{sr}^2 - C_{lr}^2)\mathbf{k}\mathbf{k} \cdot \boldsymbol{\theta} = \mathbf{0}$. We will not consider the case $C_{sr} = C_{lr}$.

Linear isotropic full Cosserat medium: dispersion relations

Then we have $\mathbf{k} \cdot \boldsymbol{\theta} = 0$.
At the same time (38) in this case can be written as

$$(\rho\omega^2 - k^2(\lambda + 2\mu))\mathbf{k}\mathbf{k} \cdot \mathbf{u} + 2i\alpha\mathbf{k} \times \boldsymbol{\theta} = \mathbf{0},$$

which is due to (40)$\cdot\mathbf{k}$ gives us $\mathbf{k} \cdot \mathbf{u} = 0$. Then $\mathbf{k} \times \boldsymbol{\theta} = \mathbf{0}$, and therefore $\boldsymbol{\theta} = 0$.
Contrary to the reduced Cosserat medium, there are no free oscillations.

Linear isotropic full Cosserat medium: dispersion relations

Continue to eliminate $\boldsymbol{\theta}$ from the balance of force. We can divide both sides

$$i\omega_0^2 \mathbf{k}\times(\mathbf{k}\times\mathbf{u})/2 = (k^2 C_{sr}^2 + \omega_0^2 - \omega^2)\mathbf{k}\times\boldsymbol{\theta}.$$

by the multiplier at $\boldsymbol{\theta}$:

$$2i\alpha\mathbf{k}\times\boldsymbol{\theta} = -\frac{\alpha\omega_0^2}{(k^2 C_{sr}^2 + \omega_0^2 - \omega^2)}\mathbf{k}\times(\mathbf{k}\times\mathbf{u}).$$

Substitute it into the balance of force (38):

$$-(\lambda+2\mu)\mathbf{k}\mathbf{k}\cdot\mathbf{u} + (\mu + \alpha\frac{k^2 C_{sr}^2 - \omega^2}{k^2 C_{sr}^2 + \omega_0^2 - \omega^2})\mathbf{k}\times(\mathbf{k}\times\mathbf{u}) = -\rho\omega^2\mathbf{u}.$$

Linear isotropic full Cosserat medium: dispersion relations. Compression wave

We obtain the spectral problem for $\mathbf{u}$

$$(\rho\omega^2 - k^2(\lambda + 2\mu))\hat{\mathbf{k}}\hat{\mathbf{k}} \cdot \mathbf{u} + (\rho\omega^2 - k^2(\mu + \alpha \frac{k^2 C_{sr}^2 - \omega^2}{k^2 C_{sr}^2 + \omega_0^2 - \omega^2}))(\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \mathbf{u} = \mathbf{0}.$$

1. Longitudinal acoustic branch. We see that compression wave in the isotropic full linear elastic Cosserat medium is the same as in the classical medium (dispersion relation $\omega = C_l k$, $\quad C_l^2 = (\lambda + 2\mu)/\rho$, $\mathbf{u}_0 = u_0\hat{\mathbf{k}}$, from (39) it follows $\boldsymbol{\theta} = 0$).

Linear isotropic full Cosserat medium: dispersion relations

2,3. Transverse acoustic and optical branches Shear–rotational wave ($\mathbf{u} \cdot \mathbf{k} = 0$) corresponds to the branches of the equation

$$\rho\omega^2 = k^2(\mu + \alpha \frac{k^2 C_{sr}^2 - \omega^2}{k^2 C_{sr}^2 + \omega_0^2 - \omega^2}),$$

which gives us

$$\omega^4 - \omega^2(k^2(C_{sr}^2 + C_{s\alpha}^2) + \omega_0^2) + k^2 C_s^2 \omega_0^2 + k^4 C_{s\alpha}^2 C_{sr}^2 = 0.$$

This biquadratic equations can be resolved with respect to ω or k. Note that at $k \to \infty$ it gives $\omega \approx C_{sr} k$ or $\omega \approx C_{s\alpha} k$. It has no horizontal asymptotes, contrary to the reduced Cosserat medium, since the elastic energy is positively defined. At small k either $\omega \approx C_s k$, or $\omega^2 \approx \omega_0^2 + k^2(C_{sr}^2 + C_{s\alpha}^2 - C_s^2) = \omega_0^2 + k^2(C_{sr}^2 + \alpha/\rho)$, i.e. for the upper branch we have a cut-off frequency ω_0, and the lower branch starts as the shear wave in the classical medium.

Linear isotropic full Cosserat medium: dispersion relations. Shear-rotational wave

The expression for ω takes form

$$2\omega^2 = \omega_0^2 + k^2(C_{sr}^2 + C_{s\alpha}^2) \pm \sqrt{(\omega_0^2 + k^2(C_{sr}^2 - C_{s\alpha}^2))^2 + 4\omega_0^2 k^2 \alpha/\rho}$$

We can obtain from (39), that for the shear–rotational wave $\boldsymbol{\theta} \cdot \mathbf{k} = 0$.

Linear isotropic full Cosserat medium: dispersion relations. Longitudinal rotational wave

4. Longitudinal optical branch. Longitudinal rotational branch. As we saw, $\boldsymbol{\theta} \cdot \mathbf{k}$ does not influence the spectral problem for $\mathbf{u}$, but there exist free plane waves with $\boldsymbol{\theta}$ parallel to $\mathbf{k}$. Calculate (39)$\cdot\mathbf{k}$:

$$((\omega^2 - \omega_0^2) - k^2(\beta + 2\gamma)/(\rho I))\boldsymbol{\theta} \cdot \mathbf{k} = 0$$

We obtain the dispersion relation

$$\omega^2 = \omega_0^2 + C_{lr}^2 k^2$$

Here $\mathbf{u}_0 = \mathbf{0}$, $\boldsymbol{\theta}_0 = \theta_0 \hat{\mathbf{k}}$. Another way to obtain the dispersion relations: Eringen, 1999, Microcontinuum field theories, pp. 147–150.

Reduced linear constrained elastic Cosserat medium. Dispersion relation

$$\nabla \cdot \boldsymbol{\tau}^S = \nabla \times (I \nabla \times \ddot{\mathbf{u}})/4 + \rho \ddot{\mathbf{u}}$$

$$\boldsymbol{\tau}_\times = I \ddot{\boldsymbol{\theta}} = I \nabla \times \ddot{\mathbf{u}}/2.$$

First we choose an isotropic homogeneous elastic model

$$\boldsymbol{\tau}^S = \lambda \nabla \cdot \mathbf{u} \mathbf{E} + 2\mu (\nabla \mathbf{u})^S$$

Equations in displacements:

$$(\lambda + 2\mu)(\nabla \nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) = \nabla \times (I \nabla \times \ddot{\mathbf{u}})/4 + \rho \ddot{\mathbf{u}}$$

Reduced linear constrained isotropic elastic Cosserat medium. Plane shear wave

P-wave is classical for the isotropic case. It is separated

Plane wave solution: $\mathbf{u} = \mathbf{u}_0 e^{i(\mathrm{k}\cdot\mathrm{r}+\omega t)}$.

Shear wave in the Fourier domain ($\mathbf{u}\cdot\mathbf{k} = 0$):

$$\mu\mathbf{k}\times(\mathbf{k}\times\mathbf{u}) = -\omega^2\rho\mathbf{u} - I\omega^2 k^2\mathbf{u}/4.$$

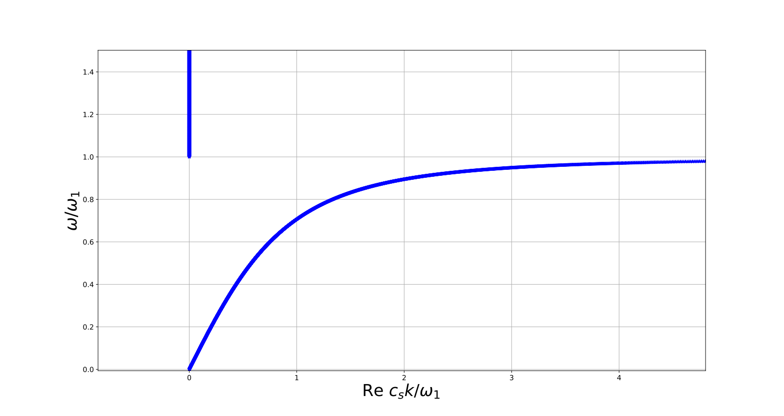
Dispersion relation:

$$C_s^2 k^2 = \left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)^{-1}, \quad C_s^2 = \frac{\mu}{\rho}, \quad \omega_1^2 = \frac{4\mu}{I}.$$

Reduced linear constrained isotropic elastic Cosserat medium. Plane shear wave

Dispersion relation: $$\left(\frac{C_s}{\omega_1}k\right)^2 = \frac{\omega^2}{\omega_1^2 - \omega^2}$$

Single negative acoustic metamaterial



Elastic case with the simplest anisotropic coupling term (axial symmetry, axis $\mathbf{n}$)

Constitutive equations:

$$U = U_{\text{isotropic}} + N(\nabla\mathbf{u})^S \cdot\cdot\, \mathbf{Enn} \cdot\cdot\, \nabla\mathbf{u}^S = U_{\text{isotropic}} + N(\nabla\cdot\mathbf{u})\,\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}$$

$$\boldsymbol{\tau}^S = \lambda\nabla\cdot\mathbf{u} + 2\mu(\nabla\mathbf{u})^S + N\mathbf{En}\cdot\nabla\mathbf{u}\cdot\mathbf{n} + N\nabla\cdot\mathbf{u}\,\mathbf{nn}$$

Dynamic equations in displacements (after Fourier transform):

$$-(\lambda+2\mu)\mathbf{kk}\cdot\mathbf{u} + \mu\mathbf{k}\times(\mathbf{k}\times\mathbf{u}) - N\mathbf{kku}\cdot\cdot\,\mathbf{nn} - N\mathbf{n}\cdot\mathbf{kk}\cdot\mathbf{un}$$
$$= -\omega^2\rho\mathbf{u} - I\omega^2k^2\mathbf{u}/4$$

Elastic case with the simplest anisotropic coupling term (axial symmetry, axis $\mathbf{n}$)

Denote $\hat{\mathbf{k}} = \mathbf{k}/k$, then $\mathbf{n} = \mathbf{n} \cdot \hat{\mathbf{k}}\hat{\mathbf{k}} + \tilde{\mathbf{n}}$, $\quad \tilde{\mathbf{n}} = (\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \mathbf{n}$.

Spectral problem:

$$
\begin{aligned}
&[(\rho\omega^2 - (\lambda + 2\mu)k^2 - 2Nk^2(\hat{\mathbf{k}} \cdot \mathbf{n})^2)\hat{\mathbf{k}}\hat{\mathbf{k}} \\
&\qquad + (\rho\omega^2 - \mu k^2 + \frac{I}{4}\omega^2 k^2)(\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \\
&\qquad\qquad - Nk^2(\hat{\mathbf{k}} \cdot \mathbf{n})(\tilde{\mathbf{n}}\hat{\mathbf{k}} + \hat{\mathbf{k}}\tilde{\mathbf{n}})] \cdot \mathbf{u} = \mathbf{0}.
\end{aligned}
$$

There exist a shear wave: $\mathbf{u} \,||\, (\mathbf{n} \times \hat{\mathbf{k}})$.

The same dispersion relation as for the isotropic case.

Elastic case with the simplest anisotropic coupling term (axial symmetry, axis $\mathbf{n}$). Special directions of wave propagation

If $\mathbf{n} \cdot \hat{\mathbf{k}} = \pm 1$ or $\mathbf{n} \cdot \hat{\mathbf{k}} = 0$,

1. anisotropic term for the shear wave disappears, and its dispersion relation is the same as for the isotropic medium
2. longitudinal wave is non-dispersive, with constant velocity C_l,

$$c_p^2 = \frac{\lambda + 2\mu + 2N}{\rho} \quad \text{for} \quad \hat{\mathbf{k}} || \hat{\mathbf{n}},$$

$$c_p^2 = \frac{\lambda + 2\mu}{\rho} = C_l^2 \quad \text{for} \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = 0.$$

Elastic case (axial symmetry, axis $\mathbf{n}$). Mixed wave

If neither $\mathbf{n} \cdot \hat{\mathbf{k}} \neq \pm 1$ nor $\mathbf{n} \cdot \hat{\mathbf{k}} \neq 0$, longitudinal wave and shear wave with $\mathbf{u}$ not parallel to $\hat{\mathbf{n}} \times \hat{\mathbf{k}}$ do not exist. Waves become mixed.

Dispersion relation for the mixed wave

$$\omega^4 \Big(1 + \frac{C_s^2 k^2}{\omega_1^2}\Big) - \omega^2 k^2 \Big(c_p^2 \Big(1 + \frac{C_s^2 k^2}{\omega_1^2}\Big) + C_s^2\Big) + k^4 \big(c_p^2 C_s^2 - c_n^4 (\mathbf{n} \cdot \hat{\mathbf{k}})^2 |\mathbf{n} \times \hat{\mathbf{k}}|^2\big) = 0.$$

$$\omega^2 = \frac{k^2}{2} \left(c_p^2 + \frac{C_s^2}{1 + \frac{C_s^2 k^2}{\omega_1^2}} \pm \sqrt{ \left(c_p^2 - \frac{C_s^2}{1 + \frac{C_s^2 k^2}{\omega_1^2}} \right)^2 + \frac{4 c_n^4 (\mathbf{n} \cdot \mathbf{k})^2 (\mathbf{n} \times \mathbf{k})^2}{1 + \frac{C_s^2 k^2}{\omega_1^2}} } \right)$$

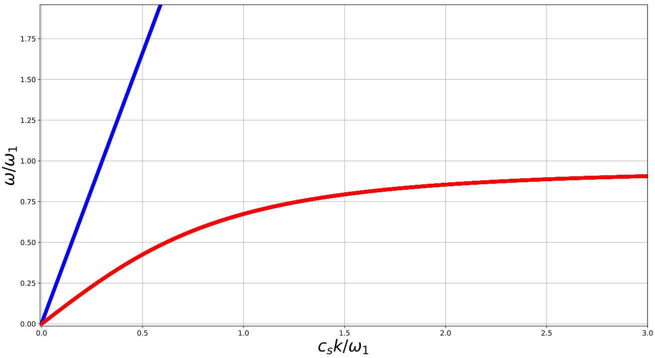
Here $c_n^2 = N/\rho$, $\quad c_p^2 = C_l^2 + 2N(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})^2/\rho$, $\quad C_l^2 = (\lambda + 2\mu)/\rho$.

Example of dispersion curves ($C_l^2/C_s^2 = 3, c_N^2/C_s^2 = 1.4$, $\mathbf{n}\cdot\mathbf{k} = \sqrt{2}/2$.)

Horizontal asymptote $\omega = \omega_* < \omega_1$, at $\omega \to \infty$ we have $\omega \approx c_p k$

Both curves are dispersive but for the upper one we hardly notice this.

They are different but look very similar to the isotropic case.



Linear isotropic viscoelastic constrained reduced Cosserat medium

We consider the simplest viscoelastic model (Eringen; Sorokin's body). Without proof: It satisfies all the balance laws, material objectivity in the linear approximation and the 2nd law of thermodynamics.

$$\boldsymbol{\tau}^S = \lambda \nabla \cdot \mathbf{u} + 2\mu (\nabla \mathbf{u})^S + \lambda\kappa \nabla \cdot \dot{\mathbf{u}} + 2\mu\nu (\nabla \dot{\mathbf{u}})^S$$

We substitute it into the balance of force and obtain equations in displacements:

$$(\lambda + 2\mu)(\nabla\nabla \cdot (\mathbf{u} + \kappa\dot{\mathbf{u}})) - \mu \nabla \times (\nabla \times (\mathbf{u} + \nu\dot{\mathbf{u}})) = \nabla \times (I \nabla \times \ddot{\mathbf{u}})/4 + \rho\ddot{\mathbf{u}}$$

Plane waves in the viscoelastic constrained medium

We look for the solution $\mathbf{u} = \mathbf{u}_0 e^{i(\mathrm{k}\cdot\mathrm{r}+\omega t)}$. All elastic constants are now complex and linear in frequency: $\mu(1 + i\omega\nu)$ instead of μ etc.
As above, the pressure wave is separated (isotropy!) and classical.
Shear wave ($\mathbf{u} \cdot \mathbf{k} = 0$):

$$\mu(1 + i\nu\omega)\mathbf{k} \times (\mathbf{k} \times \mathbf{u}) = -\omega^2\rho\mathbf{u} - I\omega^2 k^2\mathbf{u}/4.$$

Dispersion relation:

$$C_s^2 k^2 = \left(\frac{1 + i\nu\omega}{\omega^2} - \frac{1}{\omega_1^2}\right)^{-1},$$

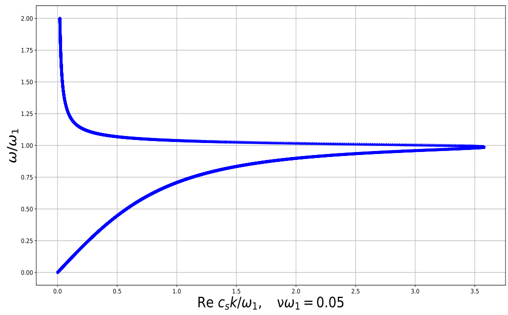
$$C_s^2 = \frac{\mu}{\rho}, \quad \omega_1^2 = \frac{4\mu}{I}.$$

Plane waves in the viscoelastic constrained medium

There is one positive root for $(\operatorname{Re} k)^2$

$$(\operatorname{Re} k)^2 = \frac{\omega^2}{C_s^2} \cdot \frac{\omega_1^2 - \omega^2 + \sqrt{z}}{2z}, \qquad z = (\omega_1^2 - \omega^2)^2 + \nu^2\omega^2 > 0.$$

No band gap. Decreasing part of the dispersion graph. $\operatorname{Re} k$ is limited



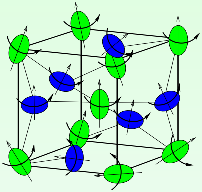
Dissipation changes the type of metamaterial and favours wave propagation.

Similar effect for reduced linear isotropic Cosserat medium (Grekova, Abreu, Piatysheva, 2019).

It destroys localisation

Waves in the simplest reduced Kelvin's medium / simplest gyrocontinuum

Equations in displacements ($\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} \cdot (\mathbf{E} - \mathbf{mm})$):



$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times (\nabla \times \mathbf{u}) - \alpha\nabla \times (\mathbf{E} - \mathbf{mm}) \cdot (\nabla \times \mathbf{u} - 2\boldsymbol{\theta}) = \rho\ddot{\mathbf{u}},$$

$$4\alpha(\nabla \times \mathbf{u}/2 - \boldsymbol{\theta}) \cdot (\mathbf{E} - \mathbf{mm}) - b\,\tilde{\boldsymbol{\theta}} = M\dot{\boldsymbol{\theta}} \times \mathbf{m}.$$

Here λ, μ are Lamé constants, α is the elastic rotational constant, $M = \rho I\dot{\varphi}$ is the dynamic spin, $b\tilde{\boldsymbol{\theta}}$ external body moment.

P-wave is classical:

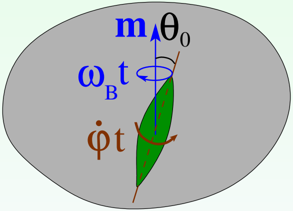
$\nabla \cdot \mathbf{u}$ enters only in balance of force.

Special regime at frequency $\omega_B = \frac{4\alpha+b}{M}$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{m}\}$ be a right-handed orthonormal basis.
A **special regime of motion**: $\mathbf{u} \equiv \mathbf{0}$, $\nabla \times (\theta_0(\mathbf{e}_1 + i\mathbf{e}_2)) = \mathbf{0}$,

$$\tilde{\boldsymbol{\theta}} = \theta_0(\mathbf{e}_1 + i\mathbf{e}_2)e^{i\omega_B t}$$

This is a regular precession with the nutation angle θ_0, velocity of precession $-\omega_B$ and velocity of proper rotation $\dot{\varphi}$.



Spectral problem for the shear wave

Let $\hat{\mathbf{k}} = \mathbf{k}/k$, $\mathbf{u}_s = (\mathbf{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \cdot \mathbf{u}$.
Shear wave:

$$\mu \mathbf{k} \times (\mathbf{k} \times \mathbf{u}_s) + 2i\alpha \mathbf{k} \times ((\mathbf{E} - \mathbf{mm}) \cdot (\tilde{\boldsymbol{\theta}} - i\mathbf{k} \times \mathbf{u}_s/2)) = -\rho\omega^2 \mathbf{u}_s,$$

$$(\mathbf{E} - \mathbf{mm}) \cdot (\mathbf{k}i \times \mathbf{u}_s)/2 = \frac{\omega_B}{\omega_0}\tilde{\boldsymbol{\theta}} + i\frac{\omega}{\omega_0}\tilde{\boldsymbol{\theta}} \times \mathbf{m},$$

where $\omega_0 = \frac{4\alpha}{M}$. In absence of external torque $\omega_0 = \omega_B$.

Multiplying the balance of moment by tensor $((\mathbf{E} - \mathbf{mm}) - i\frac{\omega}{\omega_B}\mathbf{m} \times \mathbf{E})^\top$,

we express $\tilde{\boldsymbol{\theta}}$ in terms of $\mathbf{u}$ at $\omega \neq \omega_B$:

$$\tilde{\boldsymbol{\theta}} = -\frac{\omega_0\omega_B}{\omega^2 - \omega_B^2}(i(\mathbf{E} - \mathbf{mm}) \cdot (\mathbf{k} \times \mathbf{u}_s) - \frac{\omega}{\omega_B}\mathbf{m} \times (\mathbf{k} \times \mathbf{u}_s)).$$

Reduced spectral problem for the shear–rotational wave

Substitute $\tilde{\boldsymbol{\theta}}(\mathbf{u})$ in the balance of force. If $\omega \neq \omega_B$, we obtain

Reduced spectral problem

$$(\rho\omega^2 - (\mu + \alpha(1 + \frac{\omega_0\omega_B}{\omega^2 - \omega_B^2}))k^2)\mathbf{u}_s \\ + \alpha k^2(1 + \frac{\omega_0\omega_B}{\omega^2 - \omega_B^2})(\hat{\mathbf{k}} \times \mathbf{m})(\hat{\mathbf{k}} \times \mathbf{m}) \cdot \mathbf{u}_s \\ - i\alpha k^2 \frac{\omega_0\omega}{\omega^2 - \omega_B^2}\hat{\mathbf{k}} \cdot \mathbf{m}(\hat{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{u}_s = \mathbf{0}.$$

Spectral problem. $\mathbf{k} \cdot \mathbf{m} = 0$

Denote $\mathbf{e} = \hat{\mathbf{k}} \times \mathbf{m}$.

Shear wave propagating perpendicular to rotor axes $\mathbf{m}$:

$$(\rho\omega^2 - \mu k^2)\mathbf{ee} \cdot \mathbf{u}_s + (\rho\omega^2 - (\mu + \alpha(1 + \frac{\omega_0\omega_B}{\omega^2 - \omega_B^2}))k^2)\mathbf{mm} \cdot \mathbf{u}_s = \mathbf{0}.$$

At $\mathbf{u}_s = u_0\mathbf{e}$

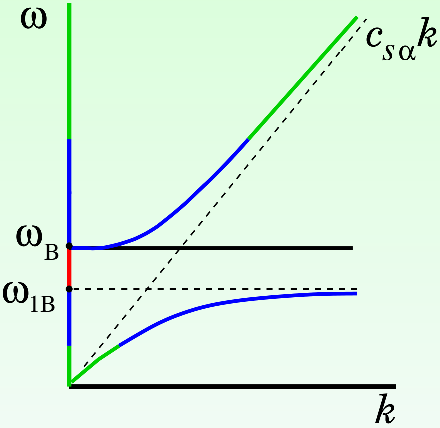
this is a classical shear wave: $\omega = C_s k, \quad C_s^2 = \mu/\rho.$

At $\mathbf{u}_s = u_0\mathbf{m}$

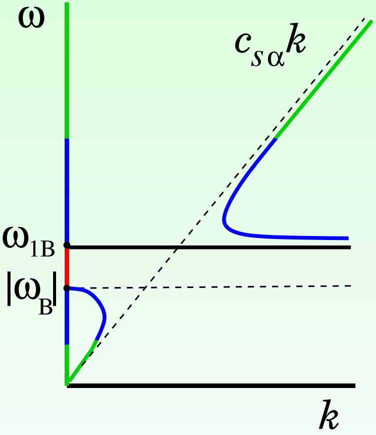
$$k^2 = \frac{\omega^2}{C_{s\alpha}^2}\frac{\omega^2 - \omega_B^2}{\omega^2 - \omega_{1B}^2}$$

Low frequency velocity $\in [C_s; C_{s\alpha})$ at $b \geqslant 0$,
less than C_s or disappears at $b < 0$ $C_{s\alpha}^2 = (\mu + \alpha)/\rho$
$\omega_{1B}^2 = \omega_B(\omega_B - \alpha\omega_0/(\mu + \alpha)) < \omega_B^2$

Dispersion curves: plane waves propagating orthogonal to rotors' axes



$4\alpha + b > 0$
Lower branch disappears at
$-1 < \frac{b}{4\alpha} < \frac{1}{1+\alpha/\mu}$



$4\alpha + b < 0$

Case $\hat{\mathbf{k}}||\mathbf{m}$

Spectral problem for shear waves propagating along rotors' axes

$$(\rho\omega^2 - (\mu + \alpha(1 + \frac{\omega_0\omega_B}{\omega^2 - \omega_B^2}))k^2)\mathbf{u}_s \mp i\alpha k^2 \frac{\omega_0\omega}{\omega^2 - \omega_B^2}(\hat{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{u}_s = \mathbf{0}.$$

Wave with circular polarization:

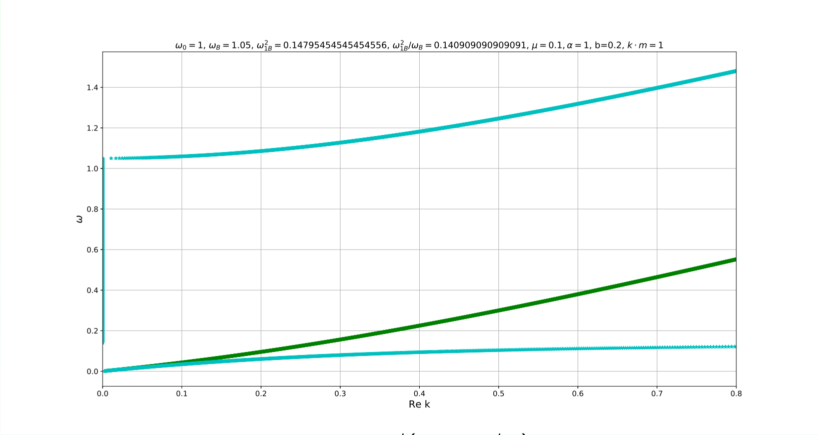
$\mathbf{u}_s = \mathbf{u}_0(\mathbf{e}_1 \pm i\mathbf{e}_2)$,

$$k^2 = \frac{\omega^2}{C_{s\alpha}^2}\frac{\omega + \omega_B}{\omega + \omega_{1B}^2/\omega_B}, \qquad k^2 = \frac{\omega^2}{C_{s\alpha}^2}\frac{\omega - \omega_B}{\omega - \omega_{1B}^2/\omega_B},$$

Case $\hat{\mathbf{k}} || \mathbf{m}$. Shear wave propagates along rotors' axes

At $b > 0$ (stabilizing external torque)

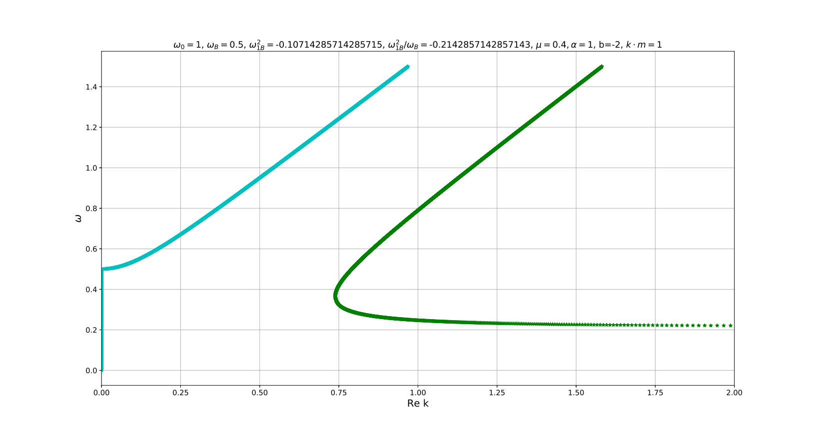
one of the branches has a band gap, and another one not.



$$b > -4\alpha/(1 + \alpha/\mu).$$

Case $\hat{\mathbf{k}}||\mathbf{m}, -4\alpha < b < 0, \omega_{1B}^2 < 0$

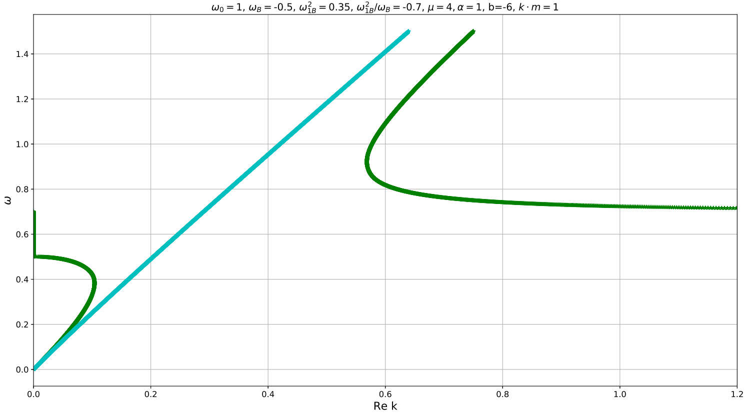
$-4\alpha < b \leqslant -4\alpha/(1+\alpha/\mu)$.



At $b < 0, \omega_{1B}^2 < 0$ (negative external torque)

there are bands where the medium is single/double negative acoustic metamaterial for different branches.

Shear–rotational wave with circular polarisation propagates parallel to rotors' axes, $b < -4\alpha$



General case: $\hat{\mathbf{k}} \cdot \mathbf{m} \neq 0$, $\hat{\mathbf{k}} \neq \pm\mathbf{m}$

The wave is polarized. We look for $\beta(\omega)$ such that $\mathbf{e}_1 + i\beta\mathbf{e}_2$ would be an eigen vector for the reduced spectral problem. We obtain the equation for β:

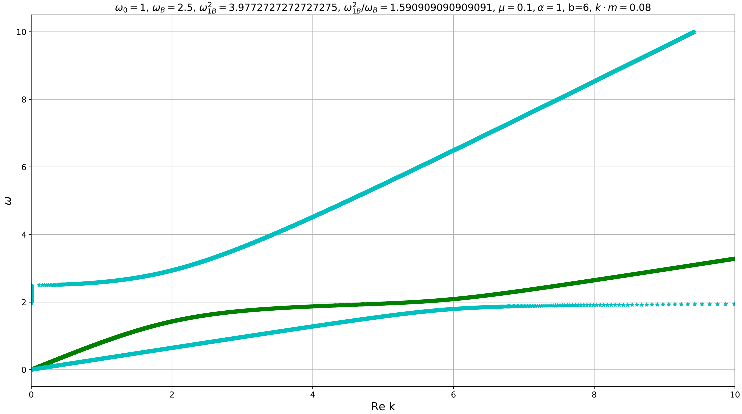
$$\beta^2 + 2\beta\xi - 1 = 0 \quad \Longleftrightarrow \quad \beta(\omega) = -\xi \pm \sqrt{\xi^2 + 1},$$

$$\xi = \frac{1}{2}\frac{(\hat{\mathbf{k}} \times \mathbf{m})^2}{\hat{\mathbf{k}} \cdot \mathbf{m}}\left(\frac{\omega}{\omega_0} - \frac{\omega_B}{\omega}\frac{b}{4\alpha}\right)$$

$$k^2 = \frac{\omega^2}{C_{s\alpha}^2}\frac{\omega - \omega_B^2}{\omega^2 - \omega_{1B}^2 + \beta_{\mp}\omega\omega_B\,\hat{\mathbf{k}} \cdot \mathbf{m}\,(1 - \frac{\omega_{1B}^2}{\omega_B^2})}$$

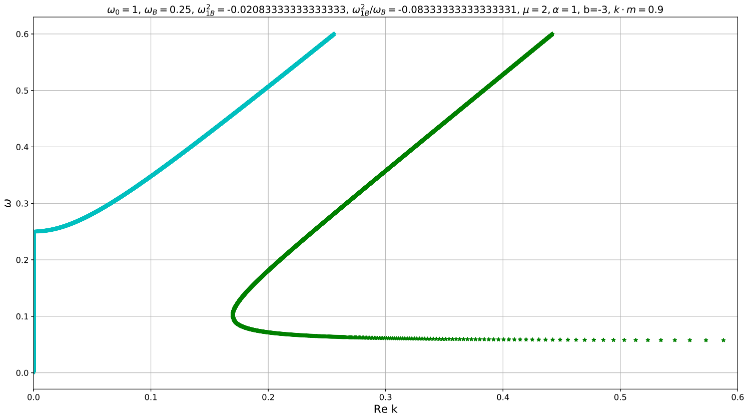
At $b \geqslant 0$ there is one branch with a band gap and another one without it. At $b < 0$ if the stability is not violated there may exist zones with decreasing part $\omega(k)$.

Positive follower torque



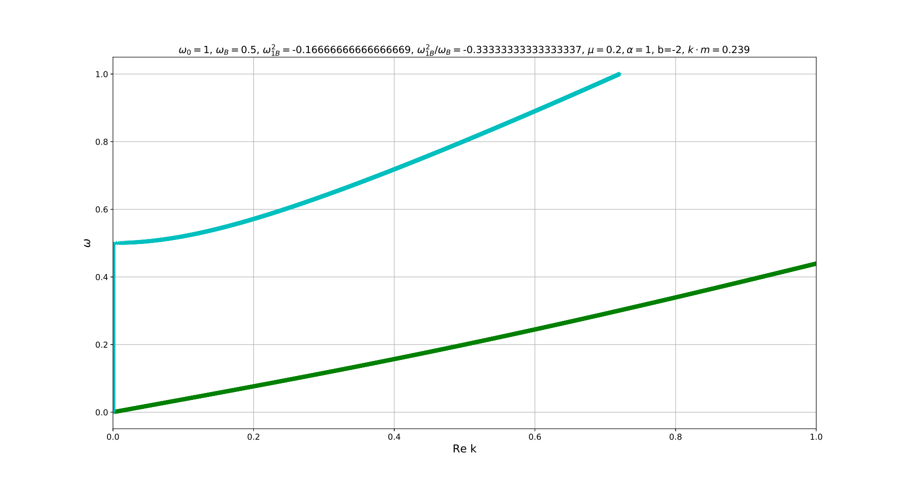
Shear–rotational wave with elliptic frequency-dependent polarisation,
$\mathbf{k} \cdot \mathbf{m} = 0.008$, $b > 0$.

Negative follower torque



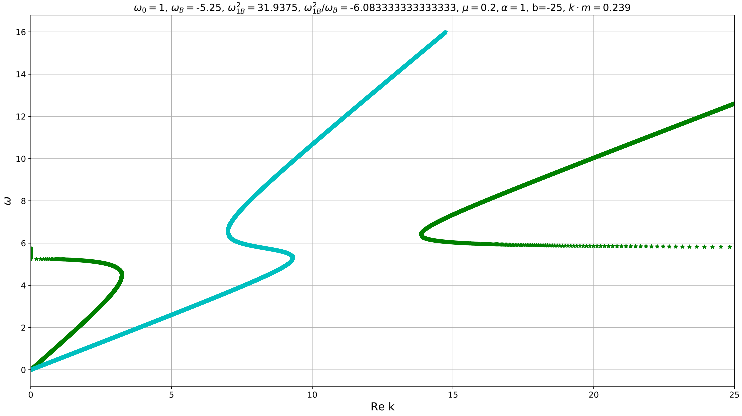
$$-4\alpha < b < -4\alpha/(1 + (\mathbf{k}\cdot\mathbf{m})^2\alpha/\mu),\ \mathbf{k}\cdot\mathbf{m} = 0.9.$$

Negative follower torque



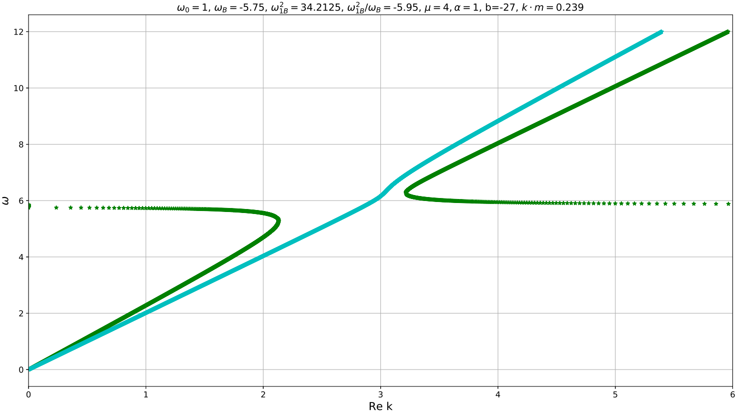
$\mathbf{k} \cdot \mathbf{m} = 0.239, \ b < -4\alpha$

Negative follower torque



$\mathbf{k} \cdot \mathbf{m} = 0.239,\ b < -4\alpha.$

Negative follower torque

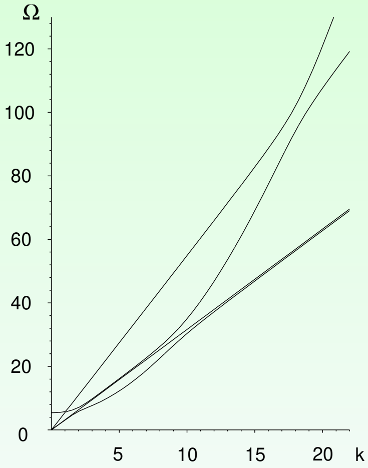


$\mathbf{k} \cdot \mathbf{m} = 0.239,\ b < -4\alpha$

Reduced continua with dynamic spin: conclusions

- P-wave is classical
- A regular precession of body points with fixed mass centers at frequency ω_B may take place
- Shear-rotational wave propagating perpendicular to the rotors' axes is plane, has a classical branch and a branch with a band gap limited by ω_B from above
- Other shear–rotational waves have dispersion, polarisation (circular if the wave vector is parallel to rotors and elliptic, frequency dependent, if not)
- Under positive external torque one of the branches has a band gap limited by ω_B from above, and another one not
- ω_B depends on the external follower torque, by its means we may somewhat control the band gap
- under negative external follower torque we may change qualitatively dispersion curves, obtaining the decreasing part

Some facts about waves in full Kelvin's medium



An example of dispersion curves

- No band gaps: the medium is not reduced
- Interesting: intersection curves of partial (rotational and translational) dispersion curves. Magnetoacoustic resonance. For simple roots of partial dispersion relations coupled curves look there like hyperbolas. Strong dispersion and mixed eigen vector. If we excite elastic subsystem, we obtain magnetic (rotational) wave and vice versa. If there is any nonlinear coupling, it is crucial at these points.

Waves. Conclusions

- In classical elastic medium dispersion relations are straight lines
- In full (non-reduced) Cosserat and Kelvin's media there are no band gaps
- In reduced constrained Cosserat, reduced Cosserat and reduced Kelvin's media almost always there exist band gaps (the media are single negative acoustic metamaterials for these frequencies) and for some cases — decreasing part of dispersion curves (double negative acoustic metamaterials)
- In reduced Cosserat medium there exist plane harmonic waves
- In continua with dynamic spin most harmonic waves are polarised
- Viscosity may favour the wave propagation destroying a band gap, and may change the type of acoustic metamaterial

Exam questions (1st part — mathematical facts)

1. Polar and axial tensors, vectors, scalars
2. Antisymmetric tensor. Accompanying vector
3. Rotation tensor. Definition. Poisson equation
4. Infinitesimal rotation (vector, tensor, their relation).
5. Angular strains $\mathbf{\Phi}_i$. Their expression via spatial derivatives of the rotation tensor.
6. Nabla operator ∇ in actual configuration (position vector $\mathbf{R}$), nabla operator $\overset{\circ}{\nabla}$ in the initial configuration (position vector $\mathbf{r}$). Prove: $\overset{\circ}{\nabla} = \overset{\circ}{\nabla}\mathbf{R} \cdot \nabla$.
7. Cosserat deformation tensor $\mathbf{A} = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P}$ and its linearization.
8. Transposed wryness tensor $\mathbf{K} = \mathbf{r}^i \mathbf{\Phi}_i \cdot \mathbf{P}$ and its linearization.

Exam questions (2nd part — basic equations)

1. Principles that we use to write down basic equations of the elastic medium.
2. Balance of forces for Cosserat and reduced Cosserat continuum.
3. Balance of moments for Cosserat and reduced Cosserat continuum.
4. Balance of energy for Cosserat and reduced Cosserat continuum.
5. Principle of material objectivity (material frame indifference).
6. Constitutive equations for elastic Cosserat and reduced Cosserat continua.
7. Kelvin's medium as a specific Cosserat medium.
8. Constrained (reduced and full) Cosserat medium, its difference from the full and reduced non-constrained Cosserat continua.

Exam questions (3rd part — harmonic waves)

1. Plane waves in an infinite 3D linear classical elasticity.
2. Plane waves in an infinite 3D linear reduced Cosserat medium
3. Difference between dispersion graphs of the three elastic isotropic continua: classical one, reduced and full Cosserat medium
4. Difference between dispersion graphs of the three elastic isotropic continua: constrained reduced Cosserat medium, reduced Cosserat medium and full Cosserat medium
5. Influence of anisotropic term coupling volumetric and shear deformations on the plane waves in the classical elasticity and constrained reduced elastic Cosserat medium
6. Plane waves in the isotropic reduced elastic and viscoelastic constrained reduced Cosserat media: difference in dispersion graphs, acoustic metamaterials of different type.
7. Difference between harmonic waves in the simplest elastic reduced Kelvin's medium and elastic isotropic reduced Cosserat medium
8. Difference between dispersion graphs of the full and reduced Kelvin's medium. Points of magnetoacoustic resonance