

ACTION OF HECKE OPERATORS ON PRODUCTS OF IGUSA THETA CONSTANTS WITH RATIONAL CHARACTERISTICS

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ABSTRACT. We derive explicit formulas for the images of products of Igusa theta constants with rational characteristics under the action of regular Hecke operators. In particular, we prove that when the class number of the sum of $2k$ squares is one, images of products of $2k$ Igusa theta constants with rational characteristics under the action of regular Hecke operators are, in general, linear combinations of similar products with explicitly given coefficients.

INTRODUCTION

In 1980 H.Yoshida ([Yo(80)], p.243) proposed a two-dimensional analog of the famous Shimura-Taniyama relation between Hasse zeta functions of elliptic curves over the field of rational numbers and Hecke zeta functions of elliptic modular forms. In confirmation of the Yoshida conjecture, R.Salvati-Manni and J.Top have considered in [SM-T(93)] a number of products of four Igusa theta constants with rational characteristics which (in part hypothetically) are eigenfunctions of all regular Hecke operators with Andrianov zeta function coinciding (up to a finite number of Euler factors) with Hasse zeta function of appropriate two-dimensional Abelian variety.

In this paper we begin to study transformation properties of products of even number of Igusa theta constants with rational characteristics considered as Siegel modular forms. More specifically, we investigate their transformations under modular substitutions and under the action of regular Hecke operators. For this we interpret products of theta constants as multiple theta functions of sums of squares

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and then apply explicit transformation formulas for the action of modular substitutions and regular Hecke operators on theta functions of integral quadratic forms (obtained in [An(95)] and [An(96)], respectively). In particular, if the class number of the sum of $2k$ squares is equal to one, we obtain explicit transformation formulas expressing images of the products of $2k$ theta constants with rational characteristics under the action of regular Hecke operators in the form of linear combinations of similar products (Theorem 4.1).

In forthcoming works we hope to extend our study in order to investigate the question of construction of common eigenfunctions for all regular Hecke operators on spaces spanned by products of theta constants with rational characteristics and, possibly, compute corresponding zeta functions.

Notation. We reserve the letters \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} for the set of positive rational integers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. \mathbb{A}_n^m is the set of all $m \times n$ -matrices with entries in a set \mathbb{A} , $\mathbb{A}^n = \mathbb{A}_1^n$, and $\mathbb{A}_n = \mathbb{A}_n^1$.

If M is a matrix, tM always denotes the transpose of M . If entries of M belong to \mathbb{C} , then \overline{M} is the matrix with complex conjugate entries. For a symmetric matrix Q we write $Q[M] = {}^tMQM$ if the product on the right is defined. 1_g is the unit matrix of order g , and

$$(0.1) \quad J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

We denote by

$$\mathbb{E}^m = \left\{ Q = (q_{\alpha\beta}) \in \mathbb{Z}_m^m \mid {}^tQ = Q, \quad q_{11}, q_{22}, \dots, q_{mm} \in 2\mathbb{Z} \right\}$$

the set of all *even matrices* of order m , i.e. the set of matrices of integral quadratic forms in m variables $\mathbf{q}(X) = \frac{1}{2} {}^tXQX$, where ${}^tX = (x_1, \dots, x_m)$.

We recall that the *level* q of an invertible matrix $Q \in \mathbb{E}^m$ (and that of the corresponding form) is the least positive integer satisfying $qQ^{-1} \in \mathbb{E}^m$.

§1. PRODUCTS OF THETA CONSTANTS AS THETA FUNCTIONS FOR SUMS OF SQUARES

Let $\mathbf{m} = (\mathbf{m}', \mathbf{m}'') \in \mathbb{C}_{2g}$ with $\mathbf{m}', \mathbf{m}'' \in \mathbb{C}_g$ be a complex $2g$ -row vector, and Z belongs to the upper half-plane of genus g :

$$\mathbb{H}^g = \left\{ Z = X + iY \in \mathbb{C}_g^g \mid {}^tZ = Z, \quad Y > 0 \right\}.$$

The *Igusa theta constant with characteristic* \mathbf{m} is defined by

$$\theta_{\mathbf{m}}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}_g} \exp(\pi i \{(\mathbf{n} + \mathbf{m}')Z^t(\mathbf{n} + \mathbf{m}') + 2(\mathbf{n} + \mathbf{m}')^t \mathbf{m}''\}).$$

If product AB of two matrices A and B is defined and is a square matrix, then clearly the product BA is also defined and is a square matrix with the same trace as that of AB . It follows that the theta constant can be rewritten in the form

$$(1.1) \quad \theta_{\mathbf{m}}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}_g} \mathbf{e}\{Z^t(\mathbf{n} + \mathbf{m}')(\mathbf{n} + \mathbf{m}') + 2^t \mathbf{m}''(\mathbf{n} + \mathbf{m}')\},$$

where for a square matrix A we set

$$(1.2) \quad \mathbf{e}\{A\} = \exp(\pi i \cdot \sigma(A)),$$

and $\sigma(A)$ denotes the trace of A . Hence, it is easy to see that the product of r theta constants with characteristics $\mathbf{m}_1, \dots, \mathbf{m}_r$ can be written in the form

$$(1.3) \quad \prod_{j=1}^r \theta_{\mathbf{m}_j}(Z)$$

$$= \sum_{\mathbf{n}_1, \dots, \mathbf{n}_r \in \mathbb{Z}_g} \mathbf{e}\left\{ \sum_{j=1}^r (Z^t(\mathbf{n}_j + \mathbf{m}'_j)(\mathbf{n}_j + \mathbf{m}'_j) + 2^t \mathbf{m}''_j(\mathbf{n}_j + \mathbf{m}'_j)) \right\}$$

$$= \sum_{N \in \mathbb{Z}_g^r} \mathbf{e}\{Z^t(N + M')(N + M') + 2^t M''(N + M')\},$$

where we set

$$N = \begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_r \end{pmatrix}, \quad M' = \begin{pmatrix} \mathbf{m}'_1 \\ \vdots \\ \mathbf{m}'_r \end{pmatrix}, \quad \text{and} \quad M'' = \begin{pmatrix} \mathbf{m}''_1 \\ \vdots \\ \mathbf{m}''_r \end{pmatrix}.$$

A product of the form (1.3) will be called the *theta product of genus* g *with the characteristic matrix* $M = (M', M'')$. The following criterion for the vanishing of theta products is a direct consequence of the Igusa result on vanishing of theta constants ([Ig(72)], Theorem 1, p. 174).

Lemma 1.1. *The theta product with the characteristic matrix $M = (M', M'') \in \mathbb{C}_{2g}^r$ is identically equal to 0, if and only if there is a row $\mathbf{m}_j = (\mathbf{m}'_j, \mathbf{m}''_j)$ of M satisfying*

$$\mathbf{m}_j \in \mathbb{Z}_{2g}, \quad 2\mathbf{m}'_j \mathbf{m}''_j \notin \mathbb{Z}.$$

The main purpose of this paper is to study the action of Hecke operators on the products of theta constants with rational characteristics. The action of Hecke operators on *theta functions of integral positive definite quadratic forms* of the form

$$(1.4) \quad \Theta(V, Z; Q) = \sum_{N \in \mathbb{Z}_g^r} e\{Z \cdot Q[N - V_2] + 2 \cdot {}^t V_1 Q N - {}^t V_1 Q V_2\},$$

where $V = (V_1, V_2)$ with $V_1, V_2 \in \mathbb{C}_g^r$ is the *characteristic matrix* of the theta function, $Z \in \mathbb{H}^g$, and where Q is an even positive definite matrix of order r , i.e. the matrix of an integral positive definite quadratic form, was considered in [An(96)]. The series (1.5) converges absolutely and uniformly on compact subsets of the complex space $\mathbb{C}_{2g}^r \times \mathbb{H}^g$ and therefore determines a complex-analytic function on the space. The formula (1.3) implies that the product of theta constants can be expressed through the theta function (1.4) of the quadratic form $\mathbf{q}_r(X) = x_1^2 + \dots + x_r^2$ with the matrix $Q_r = 2 \cdot 1_r$ by the formula

$$\begin{aligned} & e\{-{}^t M'' M'\} \prod_{j=1}^r \theta_{\mathbf{m}_j}(Z) \\ &= \sum_{N \in \mathbb{Z}_g^r} e\left\{\frac{1}{2}Z(2 \cdot 1_r)[N + M'] + 2 {}^t\left(\frac{1}{2}M''\right)(2 \cdot 1_r)N + {}^t\left(\frac{1}{2}M''\right)(2 \cdot 1_r)M'\right\} \\ &= \Theta\left(\left(\frac{1}{2}M'', -M'\right), \frac{1}{2}Z; Q_r\right), \end{aligned}$$

which we shall use in the form

$$(1.5) \quad \theta(Z, M) = \prod_{j=1}^r \theta_{\mathbf{m}_j}(Z) = \delta(M) \cdot \Theta\left(V(M), \frac{1}{2}Z; Q_r\right),$$

where, for $M = (M', M'')$, we set

$$(1.6) \quad \delta(M) = e\{{}^t M'' M'\}$$

and

$$(1.7) \quad V(M) = (V_1(M), V_2(M)) = \left(\frac{1}{2}M'', -M'\right) = M \begin{pmatrix} 1_g & 0 \\ 0 & 2 \cdot 1_g \end{pmatrix}^{-1} J_g$$

The following property of function $\delta(M)$ on rational M is an easy consequence of definition:

$$(1.8) \quad \delta(M + M_1) = \delta(M) \quad \text{if } dM \in \mathbb{Z}_{2g}^r \text{ and } M_1 \in 2d\mathbb{Z}_{2g}^r \text{ with } d \in \mathbb{N}.$$

§2. MODULAR TRANSFORMATIONS OF THETA FUNCTIONS AND THETA PRODUCTS

Here we shall recall basic transformation formulas of the theta functions (1.4) of integral positive definite quadratic forms and then specialize the formulas to the case of sums of squares.

First of all, note that for any matrix $U \in GL_r(\mathbb{Z})$ we have $U\mathbb{Z}_g^r = \mathbb{Z}_g^r$ and so, by replacing N by UN in (1.5), we get the identity

$$\begin{aligned}
 (2.1) \quad \Theta(V, Z; Q) &= \sum_{N \in \mathbb{Z}_g^r} \mathbf{e}\{Z \cdot Q[U][N - U^{-1}V_2] + 2 \cdot {}^t(U^{-1}V_1)Q[U]N - {}^t(U^{-1}V_1)Q[U]U^{-1}V_2\} \\
 &= \Theta(U^{-1}V, Z, Q[U]).
 \end{aligned}$$

In the case $Q = Q_r$ it implies the formula

$$(2.2) \quad \boldsymbol{\theta}(Z, UM) = \boldsymbol{\theta}(Z, M) \quad \text{for every } U \in E_r,$$

where

$$(2.3) \quad E_r = E(Q_r) = \left\{ U \in GL_r(\mathbb{Z}) \mid {}^tUU = 1_r \right\}$$

is the *group of units of Q_r* . In fact, by (1.5) and (2.1), we have

$$\begin{aligned}
 \boldsymbol{\theta}(Z, UM) &= \mathbf{e}\{{}^t(UM'')UM'\} \cdot \Theta\left(V(UM), \frac{1}{2}Z; Q_r\right) \\
 &= \mathbf{e}\{{}^tM''M'\} \cdot \Theta\left(V(M), \frac{1}{2}Z; Q_r[U^{-1}]\right) = \boldsymbol{\theta}(Z, M).
 \end{aligned}$$

Next we mention that the function $\Theta(V, Z; Q)$ is quasiperiodic with respect to the lattice $(Q^{-1}\mathbb{Z}_g^r, \mathbb{Z}_g^r) \subset \mathbb{C}_{2g}^r$. More exactly, for every $T_1 \in Q^{-1}\mathbb{Z}_g^r$ and $T_2 \in \mathbb{Z}_g^r$, we have

$$\begin{aligned}
 &\Theta((V_1 + T_1, V_2 + T_2), Z; Q) \\
 &= \sum_{N \in \mathbb{Z}_g^r} \mathbf{e}\{Z \cdot Q[N - V_2 - T_2] + 2 \cdot {}^t(V_1 + T_1)Q(N - T_2 + T_2) - {}^t(V_1 + T_1)Q(V_2 + T_2)\} \\
 &= \mathbf{e}\{{}^tV_1QT_2 - {}^tT_1QV_2 + {}^tT_1QT_2\} \cdot \Theta((V_1, V_2), Z; Q),
 \end{aligned}$$

since the matrix $N' = N - T_2$ ranges over the set \mathbb{Z}_g^r along with N , and the matrices ${}^tT_1QN'$ are integral for all $N' \in \mathbb{Z}_g^r$. In particular, using the above identity with

$Q = Q_r$, and the formula (1.7), we conclude that for each integral $r \times 2g$ -matrix $S = (S', S'')$ we have

$$\begin{aligned}
(2.4) \quad \theta(Z, M + S) &= \mathbf{e}\{ {}^t(M'' + S'')(M' + S') \} \cdot \Theta \left(V(M) + V(S), \frac{1}{2}Z, Q \right) \\
&= \mathbf{e}\{ {}^t(M'' + S'')(M' + S') - {}^tM''S' + {}^tS''M' - {}^tS''S' \} \cdot \Theta \left(V(M), \frac{1}{2}Z, Q_r \right) \\
&= \mathbf{e}\{ 2 {}^tS''M' \} \cdot \theta(Z, M).
\end{aligned}$$

The following formulas for integral symplectic transformations of theta functions with respect to the variable Z are particular cases of formulas proved in [An(95)], Theorems 3.1 and 4.3.

Let Q be an even positive definite matrix of even order $r = 2k$ and q the level of Q . Then the theta function $\Theta(V, Z; Q)$ of genus g of the matrix Q satisfies functional equation

$$\begin{aligned}
(2.5) \quad \det(CZ + D)^{-k} \cdot \Theta(V, (AZ + B)(CZ + D)^{-1}; Q) \\
= \chi_Q(\mathcal{M}) \cdot \Theta(V \cdot {}^t\mathcal{M}^{-1}, Z; Q)
\end{aligned}$$

for every matrix $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of group

$$(2.6) \quad \Gamma_0^g(q) = \left\{ \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}_{2g}^{2g} \mid {}^t\mathcal{M}J_g\mathcal{M} = J_g, \quad C \equiv 0 \pmod{q} \right\},$$

where J_g is the matrix (0.1), with character χ of the group defined by conditions

$$\chi_Q \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \begin{cases} 1 & \text{if } q = 1 \\ \chi_Q(\det D) & \text{if } q > 1 \end{cases},$$

where χ_Q is the real Dirichlet character modulo q satisfying $\chi_Q(-1) = (-1)^k$, and

$$\chi_Q(p) = \left(\frac{(-1)^k \det Q}{p} \right) \quad (\text{the Legendre symbol})$$

for each odd prime number p not dividing q .

Specialization of these formulas to the case of sums of squares allows us to prove the following lemma.

Lemma 2.1. *The product (1.5) of even number $r = 2k$ of theta constants satisfies the functional equation*

$$(2.7) \quad \det(CZ + D)^{-k} \cdot \boldsymbol{\theta}((AZ + B)(CZ + D)^{-1}, M) \\ = \chi_r(\mathcal{M})\delta(M)\overline{\delta(M\mathcal{M})} \cdot \boldsymbol{\theta}(Z, M\mathcal{M}),$$

for every matrix $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of group

$$(2.8) \quad \Gamma_{00}^g(2) = \left\{ \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g = \Gamma_0^g(1) \mid B \equiv C \equiv 0 \pmod{2} \right\},$$

where

$$(2.9) \quad \chi_r(\mathcal{M}) = \chi_{Q_r}(\mathcal{M}) = \chi_4^k(\det D)$$

with nontrivial Dirichlet character χ_4 modulo 4, and where δ is the function (1.6).

Proof. By (1.5) we can write

$$\det(CZ + D)^{-k} \cdot \boldsymbol{\theta}((AZ + B)(CZ + D)^{-1}, M) \\ = \delta(M) \det \left(2C \left(\frac{1}{2}Z \right) + D \right)^{-k} \cdot \Theta \left(V(M), \left(A \left(\frac{1}{2}Z \right) + \frac{1}{2}B \right) (2C \left(\frac{1}{2}Z \right) + D)^{-1}; Q_r \right).$$

Since, clearly,

$$\mathcal{M}' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A & \frac{1}{2}B \\ 2C & D \end{pmatrix} = \omega \mathcal{M} \omega^{-1} \in \Gamma_0^g(4) \quad \text{with } \omega = \begin{pmatrix} 1_g & 0 \\ 0 & 2 \cdot 1_g \end{pmatrix},$$

and the level of Q_r is equal to 4, then by the formula (2.5) for the matrix \mathcal{M}' , the last expression is equal to

$$\delta(M) \chi_{Q_r}(\mathcal{M}') \cdot \Theta \left(V(M) {}^t(\mathcal{M}')^{-1}, \frac{1}{2}Z; Q_r \right).$$

By the relations (1.7) and by ${}^t\mathcal{M}'J\mathcal{M}' = J$ in the form $J {}^t(\mathcal{M}')^{-1}J^{-1} = \mathcal{M}'$, where $J = J_g$ is the matrix (0.1), we have

$$V(M) {}^t(\mathcal{M}')^{-1} = M\omega^{-1}J {}^t(\mathcal{M}')^{-1}J^{-1}\omega\omega^{-1}J = M\omega^{-1}\mathcal{M}'\omega\omega^{-1}J = V(M\mathcal{M}).$$

Hence,

$$\det(CZ + D)^{-k} \cdot \boldsymbol{\theta}((AZ + B)(CZ + D)^{-1}, M)$$

$$= \delta(M)\chi_{Q_r}(\det D)\overline{\delta(M\mathcal{M})}\delta(M\mathcal{M}) \cdot \Theta\left(V(M\mathcal{M}), \frac{1}{2}Z; Q_r\right),$$

which together with the formula (1.5) for $M\mathcal{M}$ in place of M and with above formulas for χ_{Q_r} prove the lemma. \triangle

Now we shall recall some definitions. The *general real positive symplectic group of genus g* consists of all real symplectic matrices of order $2g$ with positive *multipliers*:

$$(2.10) \quad \mathbb{G}^g = GSp_g^+(\mathbb{R}) = \left\{ \mathcal{M} \in \mathbb{R}_{2g}^{2g} \mid {}^t\mathcal{M}J_g\mathcal{M} = \mu(\mathcal{M})J_g, \quad \mu(\mathcal{M}) > 0 \right\},$$

where J_g is the matrix (0.1). It is a real Lie group which acts as a group of analytic automorphisms on the $g(g+1)/2$ -dimensional open complex variety \mathbb{H}^g by the rule

$$\mathbb{G}^g \ni \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto \mathcal{M}\langle Z \rangle = (AZ + B)(CZ + D)^{-1} \quad (Z \in \mathbb{H}^g).$$

By acting on the upper half-plane \mathbb{H}^g , the general symplectic group also operates on complex-valued functions F on \mathbb{H}^g by *Petersson operators of integral weights k* ,

$$(2.11) \quad \mathbb{G}^g \ni \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|_k\mathcal{M} = \det(CZ + D)^{-k}F(\mathcal{M}\langle Z \rangle).$$

The Petersson operators satisfy the rules

$$(2.12) \quad F|_k\mathcal{M}\mathcal{M}' = (F|_k\mathcal{M})|_k\mathcal{M}' \quad (\mathcal{M}, \mathcal{M}' \in \mathbb{G}^g).$$

Let Ω be a subgroup of \mathbb{G}^g commensurable with the *modular group of genus g* , $\Gamma^g = \Gamma_0^g(1)$, χ a *character* of Ω , that is a multiplicative homomorphism of Ω into nonzero complex numbers with kernel of finite index in Ω , and let k be an integral number. A complex-valued function F on \mathbb{H} is called (*Siegel*) *modular form of weight k and character χ for the group Ω* , if the following conditions are fulfilled:

- (i) F is a holomorphic function in $g(g+1)/2$ complex variables on \mathbb{H}^g ;
- (ii) For every matrix $\mathcal{M} \in \Omega$, the function F satisfies the functional equation

$$(2.13) \quad F|_k\mathcal{M} = \chi(\mathcal{M})F,$$

where $|_k$ is the Petersson operator of weight k ;

(iii) If $g = 1$, then every function $F|_kM$ with $M \in \Gamma^1$ is bounded on each subset of \mathbb{H}^1 of the form $\mathbb{H}_\varepsilon^1 = \{x + iy \in \mathbb{H}^1 \mid y \geq \varepsilon\}$ with $\varepsilon > 0$.

The set $\mathfrak{M}_k(\Omega, \chi)$ of all modular forms of weight k and character χ for the group Ω is clearly a linear space over the field \mathbb{C} . Each space $\mathfrak{M}_k(\Omega, \chi)$ has finite dimension over \mathbb{C} .

Theorem 2.2. *The product of even number $r = 2k$ of theta constants with rational characteristic matrix $M \in \frac{1}{d}\mathbb{Z}_{2g}^r$, where $d \in \mathbb{N}$, satisfies the functional equation*

$$(2.14) \quad \boldsymbol{\theta}(Z, M)|_k \mathcal{M} = \boldsymbol{\chi}_M(\mathcal{M})\boldsymbol{\theta}(Z, M) \quad \text{for every } \mathcal{M} \in \Gamma^g(d) \cap \Gamma_{00}^g(2),$$

where

$$\Gamma^g(d) = \left\{ \mathcal{M} \in \mathbb{Z}_{2g}^{2g} \mid {}^t \mathcal{M} J_g \mathcal{M} = J_g, \quad \mathcal{M} \equiv 1_{2g} \pmod{d} \right\},$$

is the principal congruence subgroup of level d of the modular group $\Gamma^g = \Gamma^g(1)$, $\Gamma_{00}^g(2)$ is the group (2.8), and where

$$(2.15) \quad \boldsymbol{\chi}_M(\mathcal{M}) = \boldsymbol{\chi}_r(\mathcal{M}) \mathbf{e}\{S(\mathcal{M}) {}^t M M\}$$

with

$$(2.16) \quad S(\mathcal{M}) = \begin{pmatrix} B + {}^t B - A {}^t B & {}^t D - A {}^t D \\ D - 1_g - C {}^t B & -C {}^t D \end{pmatrix} \quad \left(\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right);$$

the matrix $S(\mathcal{M})$ is symmetric.

If the product $\boldsymbol{\theta}(Z, M)$ is not identically zero, then the function $\boldsymbol{\chi}_M : \mathcal{M} \mapsto \boldsymbol{\chi}_M(\mathcal{M})$ is a character of the group $\Gamma^g(d) \cap \Gamma_{00}^g(2)$ coinciding on the subgroup $\Gamma^g(2d^2)$ with the character $\boldsymbol{\chi}_r$; the theta product is a modular form of weight k and character $\boldsymbol{\chi}_M$ for the group $\Gamma^g(d) \cap \Gamma_{00}^g(2)$.

Proof. Since $\mathcal{M} \in \Gamma_{00}^g(2)$, by the definition of the Petersson operators of weight k , we can rewrite the functional equation (2.7) in the form

$$\boldsymbol{\theta}(Z, M)|_k \mathcal{M} = \boldsymbol{\chi}_r(\mathcal{M}) \delta(M) \overline{\delta(M \mathcal{M})} \cdot \boldsymbol{\theta}(Z, M \mathcal{M}).$$

Since $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g(d)$ and the matrix $d\mathcal{M}$ is integral, we conclude that the matrix

$$\begin{aligned} T = (T', T'') &= M \mathcal{M} - M = M(\mathcal{M} - 1_{2g}) = (M', M'') \begin{pmatrix} A - 1_g & B \\ C & D - 1_g \end{pmatrix} \\ &= (M'(A - 1_g) + M''C, M'B + M''(D - 1_g)), \end{aligned}$$

where $M = (M', M'')$, is also integral. Therefore, by (2.4), we get

$$\boldsymbol{\theta}(Z, M \mathcal{M}) = \boldsymbol{\theta}(Z, M + T) = \mathbf{e}\{2 {}^t T'' M'\} \cdot \boldsymbol{\theta}(Z, M).$$

The formula (2.14) follows with

$$\boldsymbol{\chi}_M(\mathcal{M}) = \boldsymbol{\chi}_r(\mathcal{M}) \delta(M) \overline{\delta(M \mathcal{M})} \mathbf{e}\{2 {}^t (M'B + M''(D - 1_g)) M'\}.$$

In order to prove the formula (2.15), it is sufficient to show that

$$(2.17) \quad \delta(M)\overline{\delta(M\mathcal{M})}\mathbf{e}\{2{}^t(M'B + M''(D - 1_g))M'\} = \mathbf{e}\{S(\mathcal{M}){}^tMM\}$$

with the matrix $S(\mathcal{M})$ defined by (2.16). Let us set $\mathcal{M} = 1_{2g} + \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$, then, by the (1.6), we have

$$\begin{aligned} \delta(M\mathcal{M}) &= \delta\left((M', M'') \begin{pmatrix} 1_g + A' & B' \\ C' & 1_g + D' \end{pmatrix}\right) \\ &= \mathbf{e}\{{}^t(M'' + (M'B' + M''D'))(M' + (M'A' + M''C'))\}. \end{aligned}$$

whence

$$\begin{aligned} &\delta(M)\overline{\delta(M\mathcal{M})}\mathbf{e}\{2{}^t(M'B + M''(D - 1_g))M'\} \\ &= \mathbf{e}\{{}^tM''M' - {}^t(M'' + (M'B' + M''D'))(M' + (M'A' + M''C'))\} \\ &\quad \times \mathbf{e}\{2{}^t(M'B' + M''D')M'\} \\ &= \mathbf{e}\{{}^t(M'B' + M''D')M' - {}^tM''(M'A' + M''C')\} \\ &\quad \times \mathbf{e}\{-{}^t(M'B' + M''D')(M'A' + M''C')\}. \end{aligned}$$

By standard properties of traces of square matrices, the last expression can be rewritten in the form

$$\begin{aligned} &= \mathbf{e}\{{}^tM'(M'B' + M''D') - {}^tM''(M'A' + M''C')\}\mathbf{e}\left\{-{}^t\left(M \begin{pmatrix} B' \\ D' \end{pmatrix}\right)M \begin{pmatrix} A' \\ C' \end{pmatrix}\right\} \\ &= \mathbf{e}\left\{{}^tMM \begin{pmatrix} B' & -A' \\ D' & -C' \end{pmatrix} - ({}^tB', {}^tD'){}^tMM \begin{pmatrix} A' \\ C' \end{pmatrix}\right\} \\ &= \mathbf{e}\left\{\begin{pmatrix} B & 1_g - A \\ D - 1_g & -C \end{pmatrix}{}^tMM - \begin{pmatrix} A - 1_g \\ C \end{pmatrix}({}^tB, {}^tD - 1_g){}^tMM\right\} \\ &= \mathbf{e}\left\{\begin{pmatrix} B + {}^tB - A{}^tB & {}^tD - A{}^tD \\ D - 1_g - C{}^tB & -C{}^tD \end{pmatrix}{}^tMM\right\}, \end{aligned}$$

which proves the formulas (2.15)–(2.16) and (2.17). In order to prove that the matrix $S(\mathcal{M})$ is symmetric, we note that the $g \times g$ blocks A, B, C , and D of every matrix $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ complying with ${}^t\mathcal{M}J_g\mathcal{M} = J_g$ satisfy the relations

$$(2.18) \quad A{}^tB = B{}^tA, \quad C{}^tD = D{}^tC, \quad \text{and} \quad A{}^tD - B{}^tC = 1_g$$

(see, e.g., [An(87)], p. 6). It follows that the matrix $S(\mathcal{M})$ can be written in the form

$$(2.19) \quad S(\mathcal{M}) = \begin{pmatrix} B + {}^tB - A {}^tB & {}^tD - A {}^tD \\ D - D {}^tA & -C {}^tD \end{pmatrix}$$

and is symmetric.

If $\theta(Z, M)$ is not identically zero, it follows from (2.12) and (2.14) that the function $\chi_M : \mathcal{M} \mapsto \chi_M(\mathcal{M})$ is multiplicative. If $\mathcal{M} \in \Gamma^g(2d^2)$, then the matrix $S(\mathcal{M})$ is, clearly, divisible by $2d^2$. Therefore, the matrix $S(\mathcal{M}) {}^tMM$ is integral and divisible by 2, and so $\mathbf{e}\{S(\mathcal{M}) {}^tMM\} = 1$. The rest for $g > 1$ follows from the definition of modular forms; condition (iii) for $g = 1$ follows from [Ig(72)], Corollary, p.176. \triangle

§3. ACTION OF HECKE OPERATORS ON THETA FUNCTIONS

Here we shall recall basic notions and facts related to Hecke operators and specialize the transformation formulas of general harmonic theta functions under the action of regular Hecke operators obtained in [An(96)] to the case of theta functions (1.4). For details and proofs see the book [An(87)], Chapters 3 and 4, and [An(96)], §5.

Let Δ be a multiplicative semigroup and Ω a subgroup of Δ such that every double coset $\Omega\mathcal{M}\Omega$ of Δ modulo Ω is a finite union of left cosets $\Omega\mathcal{M}'$. Let us consider the vector space over a field, say, the field \mathbb{C} of complex numbers, consisting of all formal finite linear combinations with coefficients in \mathbb{C} of symbols $(\Omega\mathcal{M})$ with $\mathcal{M} \in \Delta$ which are in one-to-one correspondence with left cosets $\Omega\mathcal{M}$ of the set Δ modulo Ω . The group Ω naturally acts on this space by right multiplication defined on the symbols $(\Omega\mathcal{M})$ by

$$(\Omega\mathcal{M})\omega = (\Omega\mathcal{M}\omega), \quad (\mathcal{M} \in \Delta, \omega \in \Omega).$$

We denote by

$$\mathcal{H}(\Omega, \Delta) = HS_{\mathbb{C}}(\Omega, \Delta)$$

the subspace of all Ω -invariant elements. The multiplication of elements of $\mathcal{H}(\Omega, \Delta)$ given by the formula

$$\left(\sum_{\alpha} a_{\alpha}(\Omega\mathcal{M}_{\alpha})\right)\left(\sum_{\beta} b_{\beta}(\Omega\mathcal{N}_{\beta})\right) = \sum_{\alpha, \beta} a_{\alpha}b_{\beta}(\Omega\mathcal{M}_{\alpha}\mathcal{N}_{\beta})$$

does not depend on the choice of representatives $\mathcal{M}_{\alpha} \in \Omega\mathcal{M}_{\alpha}$ and $\mathcal{N}_{\beta} \in \Omega\mathcal{N}_{\beta}$, and turns $\mathcal{H}(\Omega, \Delta)$ into an associative algebra over \mathbb{C} with the unity element $(\Omega 1_{\Omega})$, called *the Hecke–Shimura ring* or *HS–ring of Δ relative to Ω (over \mathbb{C})*. Elements

$$(3.1) \quad (\mathcal{M}) = (\mathcal{M})_{\Omega} = \sum_{M_i \in \Omega \backslash \Omega\mathcal{M}\Omega} (\Omega M_i), \quad (\mathcal{M} \in \Delta)$$

are in one-to-one correspondence with double cosets of Δ modulo Ω , belong to $\mathcal{H}(\Omega, \Delta)$, and form a basis of the ring over \mathbb{C} . For brevity, the symbols $(\Omega\mathcal{M})$ and (\mathcal{M}) will be referred as *left* and *double classes (of Δ modulo Ω)*, respectively.

We consider the semigroup

$$\Sigma^g = \mathbb{G}^g \cap \mathbb{Z}_{2g}^{2g} = \left\{ \mathcal{M} \in \mathbb{Z}_{2g}^{2g} \mid {}^t\mathcal{M}J_g\mathcal{M} = \mu(\mathcal{M})J_g, \quad \mu(\mathcal{M}) > 0 \right\}$$

and its subsemigroups

$$(3.2) \quad \begin{aligned} \Sigma_q^g &= \left\{ \mathcal{M} \in \Sigma^g \mid \gcd(\mu(\mathcal{M}), q) = 1 \right\} \\ \Sigma_0^g(q) &= \left\{ \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma_q^g \mid C \equiv 0 \pmod{q} \right\}, \\ \Sigma_{00}^g(q) &= \left\{ \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma_q^g \mid B \equiv C \equiv 0 \pmod{q} \right\}, \\ \Sigma^g(q) &= \left\{ \mathcal{M} \in \Sigma_q^g \mid \mathcal{M} \equiv \begin{pmatrix} \mu(\mathcal{M})1_g & 0 \\ 0 & 1_g \end{pmatrix} \pmod{q} \right\} \end{aligned}$$

with $q \in \mathbb{N}$. According to [An(87)], Lemma 3.3.5 and similarly treated the case of groups $\Gamma_{00}^g(q)$, for every $q, q' \in \mathbb{N}$ with $q \mid q'$ the semigroups satisfy the relations

$$(3.3) \quad \Sigma_0^g(q) \cap \Sigma_{q'}^g = \Gamma_0^g(q) \Sigma^g(q') = \Sigma^g(q') \Gamma_0^g(q),$$

and

$$(3.4) \quad \Sigma_{00}^g(q) \cap \Sigma_{q'}^g = \Gamma_{00}^g(q) \Sigma^g(q') = \Sigma^g(q') \Gamma_{00}^g(q),$$

where

$$(3.5) \quad \Gamma_{00}^g(q) = \left\{ \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g = \Gamma_0^g(1) \mid B \equiv C \equiv 0 \pmod{q} \right\},$$

If Ω is a subgroup of finite index of the modular group Γ^g , then the pair (Ω, Σ^g) satisfies the conditions of the definition of *HS*-rings, as well as each pair (Ω, Δ) with $\Omega \subset \Delta \subset \Sigma^g$, and we can define corresponding Hecke–Shimura rings $\mathcal{H}(\Omega, \Delta)$. We shall say that a group Ω satisfying $\Gamma^g(q) \subset \Omega \subset \Gamma^g$ is *q-symmetric* if $\Omega \Sigma^g(q) = \Sigma^g(q) \Omega$. For such group Ω the *HS*-ring

$$\mathcal{H}_{reg}(\Omega) = \mathcal{H}(\Omega, R(\Omega)) \quad \text{with} \quad R(\Omega) = \Omega \Sigma^g(q) = \Sigma^g(q) \Omega$$

is called the *regular HS-ring of the group Ω of level q* . According to Theorem 3.3.3 of [An(87)], all regular *HS*-rings for given genus g and level q are isomorphic

to each other. In particular, by (3.3) and (3.4) with $q' = q$, the groups $\Gamma_0^g(q)$ and $\Gamma_{00}^g(q)$ are q -symmetric, and the regular HS -rings of the groups $\Gamma_0^g(q)$, $\Gamma_{00}^g(q)$, and $\Gamma^g(q)$,

$$(3.6) \quad \begin{aligned} \mathcal{H}_0^g(q) &= \mathcal{H}_{reg}(\Gamma_0^g(q)) = \mathcal{H}(\Gamma_0^g(q), \Sigma_0^g(q)), \\ \mathcal{H}_{00}^g(q) &= \mathcal{H}_{reg}(\Gamma_{00}^g(q)) = \mathcal{H}(\Gamma_{00}^g(q), \Sigma_{00}^g(q)), \\ \mathcal{H}^g(q) &= \mathcal{H}_{reg}(\Gamma^g(q)) = \mathcal{H}(\Gamma^g(q), \Sigma^g(q)) \end{aligned}$$

are naturally isomorphic. For example, the isomorphism of the first and the third rings can be defined as follows. Let

$$T' = \sum_{\alpha} a_{\alpha}(\Gamma_0^g(q)\mathcal{M}_{\alpha}) \in \mathcal{H}_0^g(q),$$

where the left classes $(\Gamma_0^g(q)\mathcal{M}_{\alpha})$ with $a_{\alpha} \neq 0$ are pairwise distinct. Using (3.3), without loss of generality one may assume that all representatives \mathcal{M}_{α} of the left cosets $\Gamma_0^g(q)\mathcal{M}_{\alpha}$ belong to $\Sigma^g(q)$. Then, obviously, we have

$$(3.7) \quad T = \sum_{\alpha} a_{\alpha}(\Gamma^g(q)\mathcal{M}_{\alpha}) \in \mathcal{H}^g(q),$$

and the map $T' \mapsto T$ is a homomorphic embedding of the ring $\mathcal{H}_0^g(q)$ into $\mathcal{H}^g(q)$. In fact, it is a ring isomorphism, and the inverse isomorphism is determined by the condition

$$\mathcal{H}^g(q) \ni \sum_{\beta} b_{\beta}(\Gamma^g(q)\mathcal{N}_{\beta}) \mapsto \sum_{\beta} b_{\beta}(\Gamma_0^g(q)\mathcal{N}_{\beta}).$$

Note also that the map

$$(3.8) \quad \mathcal{M} \mapsto \mathcal{M}' = \omega^{-1}\mathcal{M}\omega \quad \text{with } \omega = \omega^g(q) = \begin{pmatrix} 1_g & 0 \\ 0 & q \cdot 1_g \end{pmatrix}$$

defines an isomorphism of the pair $\Gamma_0^g(q^2) \subset \Sigma_0^g(q^2)$ with the pair $\Gamma_{00}^g(q) \subset \Sigma_{00}^g(q)$, which induces the isomorphism

$$(3.9) \quad \omega = \omega^g(q) : \mathcal{H}_0^g(q^2) \mapsto \mathcal{H}_{00}^g(q^2).$$

The isomorphisms of the rings allows one to transfer various constructions from one of the rings to another. For example, *Zharkovskaya homomorphisms* $\Psi^{g,n} = \Psi_{k,\chi}^{g,n}$ of the ring $\mathcal{H}_0^g(q)$ to $\mathcal{H}_0^n(q)$, where $g > n \geq 1$, k is an integer, and χ is a Dirichlet character modulo q satisfying $\chi(-1) = (-1)^k$, can be naturally carried

out to the corresponding *HS*-rings of principal congruence subgroups, so that we get the homomorphisms

$$\Psi^{g,n} = \Psi_{k,\chi}^{g,n} : \mathcal{H}^g(q) \mapsto \mathcal{H}^n(q), \quad (g > n \geq 1)$$

satisfying the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_0^g(q) & \xrightarrow{\eta} & \mathcal{H}^g(q) \\ \Psi^{g,r} \downarrow & & \Psi^{g,r} \downarrow \\ \mathcal{H}_0^n(q) & \xrightarrow{\eta} & \mathcal{H}^n(q) \end{array}$$

We remind that the Zharkovskaya map from genus g to genus n ,

$$(3.10) \quad \Psi^{g,n} = \Psi_{k,\chi}^{g,n} : \mathcal{H}_0^g(q) \mapsto \mathcal{H}_0^n(q),$$

can be defined in the following way. Let $T' = \sum_{\alpha} a_{\alpha}(\Gamma_0^g(q)\mathcal{M}_{\alpha}) \in \mathcal{H}_0^g(q)$. One can assume that each representative $\mathcal{M}_{\alpha} \in \Gamma_0^g(q) \setminus \Sigma_0^g(q)$ is chosen in the form

$$\mathcal{M}_{\alpha} = \begin{pmatrix} A_{\alpha} & B_{\alpha} \\ 0 & D_{\alpha} \end{pmatrix} \quad \text{with } D_{\alpha} = \begin{pmatrix} D'_{\alpha} & * \\ 0 & D''_{\alpha} \end{pmatrix} \quad \text{and } D'_{\alpha} \in \mathbb{Z}_n^n.$$

If $A_{\alpha} = \begin{pmatrix} A'_{\alpha} & * \\ * & * \end{pmatrix}$ and $B_{\alpha} = \begin{pmatrix} B'_{\alpha} & * \\ * & * \end{pmatrix}$ with $r \times r$ -blocks A'_{α} and B'_{α} , then

$$\mathcal{M}'_{\alpha} = \begin{pmatrix} A'_{\alpha} & B'_{\alpha} \\ 0 & D'_{\alpha} \end{pmatrix} \in \Sigma_0^r(q),$$

and we put

$$(3.11) \quad \Psi_{k,\chi}^{g,n}(T') = \sum_{\alpha} a_{\alpha} |\det D''_{\alpha}|^{-k} \chi^{-1}(|\det D'_{\alpha}|)(\Gamma_0^n(q)\mathcal{M}'_{\alpha}).$$

Hecke–Shimura rings act on modular forms and on theta functions by means of linear representation given by *Hecke operators*.

First, let us consider the space $\mathcal{F} = \mathcal{F}(r, g)$ of all real analytical functions

$$F = F(V, Z) : \mathbb{C}_{2g}^r \times \mathbb{H}^g \mapsto \mathbb{C}$$

with even $r = 2k$ and define action of the semigroup $\Sigma_0^g(q)$, where q is the level of an even positive definite matrix Q of order r , on \mathcal{F} by

$$(3.12) \quad \Sigma_0^g(q) \ni \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|_{\mathcal{M}} = j(\mathcal{M}, Z)^{-1} F(V \cdot {}^t\mathcal{M}, \mathcal{M}\langle Z \rangle),$$

here

$$j(\mathcal{M}, Z) = j_Q(\mathcal{M}, Z) = \chi_Q(\mathcal{M}) \det(CZ + D)^k,$$

χ_Q is defined in previous section character associated with the matrix Q , and $\mathcal{M}\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$. It is easy to see that the function $j(\mathcal{M}, Z)$ satisfies the relations

$$j(\mathcal{M}, \mathcal{M}'\langle Z \rangle) \cdot j(\mathcal{M}', Z) = j(\mathcal{M}\mathcal{M}', Z)$$

for every $\mathcal{M}, \mathcal{M}' \in \Sigma_0^g(q)$ and $Z \in \mathbb{H}^g$. Hence,

$$(3.13) \quad F|\mathcal{M}|\mathcal{M}' = F|\mathcal{M}\mathcal{M}' \quad (F \in \mathcal{F}, \quad \mathcal{M}, \mathcal{M}' \in \Sigma_0^g(q)).$$

This property of the operators $|\mathcal{M}$ allows one to define the standard representation of the Hecke–Shimura ring $\mathcal{H}_0^g(q) = \mathcal{H}(\Gamma_0^g(q), \Sigma_0^g(q))$ on the subspace

$$\mathcal{F}(\Gamma_0^g(q)) = \left\{ F \in \mathcal{F} \mid F|\gamma = F, \quad \forall \gamma \in \Gamma_0^g(q) \right\}$$

of all $\Gamma_0^g(q)$ -invariant functions of \mathcal{F} . Namely, if

$$(3.14) \quad T = \sum_{\alpha} a_{\alpha}(\Gamma_0^g(q)\mathcal{M}_{\alpha}) \in \mathcal{H}_0^g(q)$$

is an element of the ring $\mathcal{H}_0^g(q)$ and $F \in \mathcal{F}(\Gamma_0^g(q))$, then the function

$$(3.15) \quad F|T = \sum_{\alpha} a_{\alpha}F|\mathcal{M}_{\alpha}$$

does not depend on the choice of representatives $\mathcal{M}_{\alpha} \in \Gamma_0^g(q)\mathcal{M}_{\alpha}$ and again belongs to $\mathcal{F}(\Gamma_0^g(q))$. The operators $|T$ are clearly linear. The map $T \mapsto |T$ is linear and, as follows from (2.12) and the definition of multiplication in HS -rings, it satisfies $|T|T' = |TT'$. Thus, we get a linear representation of the ring $\mathcal{H}_0^g(q)$ on the space $\mathcal{F}(\Gamma_0^g(q))$. The operators $|T$ are called *Hecke operators*.

By (2.5), the theta function $\Theta(V, Z; Q)$ with an even positive definite matrix Q of even order $r = 2k$ and level q , considered as a function of V and Z , belongs to the space $\mathcal{F}(\Gamma_0^g(q))$. By the above, its image

$$(3.16) \quad \Theta(V, Z; Q)|T = \sum_{\alpha} a_{\alpha}j(\mathcal{M}_{\alpha}, Z)^{-1}\Theta(V \cdot {}^t\mathcal{M}_{\alpha}, \mathcal{M}_{\alpha}\langle Z \rangle; Q)$$

under the action of Hecke operator corresponding to the element (3.13) does not depend on the choice of representatives $\mathcal{M}_{\alpha} \in \Gamma_0^g(q)\mathcal{M}_{\alpha}$ and again belongs to $\mathcal{F}(\Gamma_0^g(q))$. Formulas proved in [An(96)], Theorem 4.1 express the image of a theta

function under the action of Hecke operators as a linear combination with constant coefficients of similar theta functions. In order to formulate the theorem we have to remind two related reductions.

The first reduction relates to each of HS -rings $\mathcal{H}(\Omega, \Sigma)$ with $\Omega \subset \Gamma^g$ and $\Sigma \subset \Sigma^g$. By the definition of the semigroup Σ^g , each matrix $M \in \Sigma^g$ satisfies the relation ${}^t\mathcal{M}J_g\mathcal{M} = \mu(M)J_g$, where $\mu(\mathcal{M})$ is a positive integer called the *multiplier* of \mathcal{M} . The multipliers satisfy the rules

$$\mu(\mathcal{M}\mathcal{M}') = \mu(\mathcal{M})\mu(\mathcal{M}') \quad (\mathcal{M}, \mathcal{M}' \in \Sigma^g), \quad \text{and} \quad \mu(\mathcal{M}) = 1 \quad \text{if} \quad \mathcal{M} \in \Gamma^g.$$

It follows that the function μ takes the same value on each left and each double coset of Σ modulo Ω , so one can speak of the *multipliers of the cosets*. We say that a nonzero formal finite linear combination T of left or double cosets of Σ modulo Ω is *homogeneous of multiplier* $\mu(T) = \mu$, if all of the entering cosets have the same multiplier μ . It is clear that every finite linear combination of the cosets is a sum of *homogeneous components* having different multipliers, and the components are uniquely determined. In particular, this allows one to reduce the consideration of arbitrary Hecke operators $|T$ to the case of homogeneous T .

Another reduction is related to special choice of representatives in left cosets $\Omega\mathcal{M}$ when $\Gamma_0^g(q) \subset \Omega$ and in $\Sigma \subset \Sigma_0^g(q)$. By Lemma 3.3.4 in [An(87)], each of the left cosets contains a representative of the form

$$(3.17) \quad \mathcal{M} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad \text{with} \quad A, B, D \in \mathbb{Z}_g^g, \quad {}^tAD = \mu(\mathcal{M})1_g, \quad {}^tBD = {}^tDB.$$

Such representatives are often convenient for computations with Hecke operators and will be referred as *triangular representatives*.

The following particular case of Theorem 4.1 [An(96)] expresses the images of theta-series (1.4) under the action of regular Hecke operators in the form of linear combinations of similar theta-series with coefficients explicitly given in terms of certain trigonometric sums.

Theorem 3.1. *Let Q be an even positive definite matrix of even order $r = 2k$ and level q . Let*

$$T = \sum_{\alpha} a_{\alpha}(\Gamma_0^g(q)\mathcal{M}_{\alpha})$$

be a homogeneous element of the Hecke–Shimura ring $\mathcal{H}_0^g(q)$ with $\mu(T) = \mu$. Let us assume that in the case $g < r$ the element T belongs to the image of the ring $\mathcal{H}_0^r(q)$ under the Zharkovskaya map (3.11) from genus r to genus g with $k = r/2$ and the character $\chi = \chi_Q$ defined in §2,

$$(3.18) \quad T = \Psi_{r/2, \chi_Q}^{r, g}(T') \quad \text{with} \quad T' \in \mathcal{H}_0^r(q) \quad (g < r).$$

Then the image of a theta-series (1.5) under the Hecke operator $|T$ can be written in the form

$$(3.19) \quad \Theta(V, Z; Q)|T = \sum_{D \in \Delta(Q, \mu)/\Lambda} I(D, Q, \Psi^{g,r}(T)) \Theta(\mu D^{-1}V, Z; \mu^{-1}Q[D]),$$

where

$$\Delta(Q, \mu) = \left\{ D \in \mathbb{Z}_r^r \mid \det D = \pm \mu^{r/2}, \mu^{-1}Q[D] \in \mathbb{E}^r \right\}, \quad \Lambda = \Lambda^r = GL_r(\mathbb{Z}),$$

$$(3.20) \quad \Psi^{g,r}(T) = \Psi_{r/2, \chi_Q}^{g,r}(T) = \begin{cases} \Psi_{r/2, \chi_Q}^{g,r}(T) & \text{if } g > r \\ T & \text{if } r = g \\ T' \in (\Psi_{r/2, \chi_Q}^{r,g})^{-1}(T) & \text{if } g < r, \end{cases}$$

$I(D, Q, T')$, for written in the triangular form elements

$$T' = \sum_{\beta} b_{\beta} \left(\Gamma_0^r(q) \begin{pmatrix} A_{\beta} & B_{\beta} \\ 0 & D_{\beta} \end{pmatrix} \right) \in \mathcal{H}_0^r(q) \quad ({}^t A_{\beta} D_{\beta} = \mu 1_r),$$

are trigonometric sums defined by

$$(3.21) \quad I(D, Q, T') = \sum_{\beta; D {}^t D_{\beta} \equiv 0 \pmod{\mu}} b_{\beta} |\det D_{\beta}|^{-r/2} \chi_Q^{-1}(|\det D_{\beta}|) \mathbf{e}\{\mu^{-2}Q[D] {}^t D_{\beta} B_{\beta}\},$$

with $\mathbf{e}\{\dots\}$ being the exponent (1.2).

As to the condition (3.18), it follows from [An(87), Proposition 5.1.14] that it can be replaced by a more explicit condition of the following lemma.

Lemma 3.2. *In the notation of the theorem, a nonzero homogeneous element $T \in \mathcal{H}_0^g(q)$ with multiplier $\mu(T) = \mu$ belongs to the image of the ring $\mathcal{H}_0^r(q)$ for $r > g$ under the Zharkovskaya map (3.10) from genus r to genus g with $k = r/2$ and $\chi = \chi_Q$ if and only if $g \geq k$, or $g < k$ and $\chi_Q(p) = -1$ for each prime divisor p of μ entering the prime numbers factorization of μ in an odd degree.*

It is sometimes convenient to rewrite the above transformation formulas in somewhat different form. In order to do this we have to make several preliminary remarks. We shall say that an even matrix Q' is *similar* to a nonsingular matrix $Q \in \mathbb{E}^r$, $Q' \sim Q$, if it can be written in the form $Q' = \mu^{-1}Q[D]$ with $D \in \Delta(Q, \mu)$,

where μ is coprime with the level q of Q . It easily follows from the definition that the relation of proper similarity is symmetric. Besides, it is clearly reflexive and transitive. Thus, the set of all nonsingular matrices of \mathbb{E}^r is disjoint union of *s-classes*

$$s\{Q\} = \left\{ Q' \in \mathbb{E}^r \mid Q' \sim Q \right\} \quad (Q \in \mathbb{E}^r, \det Q \neq 0).$$

It is easy to see that all matrices of the *s-class* of a matrix Q have the same signature, determinant, level, and divisor as those of Q , respectively. Further, we recall that two matrices Q, Q' of \mathbb{E}^r are said to be *equivalent*, $Q \simeq Q'$, if $Q' = Q[U]$ with $U \in \Lambda^r = GL_r(\mathbb{Z})$. All matrices properly equivalent to a matrix Q form the *e-class* of Q ,

$$e\{Q\} = \left\{ Q' = Q[U] \mid U \in \Lambda \right\}.$$

According to the reduction theory of integral quadratic forms, the *s-class* of every nonsingular matrix $Q \in \mathbb{E}^r$ is a finite union of the *e-classes*:

$$s\{Q\} = \bigcup_{j=1}^{h(Q)} e\{Q_j\}.$$

The quantity $h(Q)$ is called the *class number* of Q .

Now we can reformulate Theorem 3.1 in the following form:

Theorem 3.3. *With the notation and under assumptions of Theorem 3.1, the formula (3.19) can be rewritten in the form*

$$(3.22) \quad \Theta(V, Z; Q) | T \\ = \sum_{j=1}^{h^+(Q)} \sum_{D \in R(Q, \mu Q_j) / E(Q_j)} I(D, Q, \Psi^{g,r}(T)) \Theta(\mu D^{-1} V, Z, Q_j),$$

where $Q_1, \dots, Q_{h(Q)}$ is a system of representatives of all different *e-classes* contained in the *s-class* of Q ,

$$R(Q, Q') = \left\{ M \in \mathbb{Z}_r^r \mid Q[M] = Q' \right\} \quad (Q, Q' \in \mathbb{E}^r)$$

is the set of all integral representations of Q' by Q , and

$$E(Q') = R(Q', Q') \quad (Q' \in \mathbb{E}^r, \det Q' \neq 0)$$

denotes the group of all units of Q' .

In particular, if the class number $h(Q)$ of Q is equal to 1, then the formula (3.22) takes the form

$$(3.23) \quad \Theta(V, Z; Q)|T = \sum_{D \in R(Q, \mu Q)/E(Q)} I(D, Q, \Psi^{g,r}(T))\Theta(\mu D^{-1}V, Z, Q).$$

Proof. It is an easy consequence of the above definitions that the sums (3.21) are independent of the choice of representatives in the decomposition of T' and satisfy

$$(3.24) \quad I(UDU', Q, T') = I(D, Q[U], T') \quad \text{for all } U, U' \in GL_m(\mathbb{Z})$$

(see [An(96), Lemma 3.1]). From (2.1) and (3.24) we conclude that the term of the sum on the right of (3.19) corresponding to a matrix $D \in \Delta(Q, \mu)$ depends only on the coset $D\Lambda$. On the other hand, each of the matrices $\mu^{-1}Q[D]$ with $D \in \Delta(Q, \mu)$ is similar to Q and so is equivalent to one of the matrices $Q_1, \dots, Q_{h(Q)}$, say, $\mu^{-1}Q[D] \simeq Q_j$. This means that $\mu^{-1}Q[D][U] = \mu^{-1}Q[DU] = Q_j$ with $U \in \Lambda$. By replacing D by DU , we can assume that $D \in R(Q, Q_j)$. If $D' = DU'$ is another such matrix, then, clearly, $Q_j[U'] = Q_j$, whence $U' \in E(Q_j)$. This proves the formulas. \triangle

Note that the sums (3.21) were explicitly computed in [An(91)] and [An(93)] for certain generators of the rings $\mathcal{H}_0^r(q)$ including, in particular, all generators of the rings $\mathcal{H}_0^1(q)$ and $\mathcal{H}_0^2(q)$.

§4. ACTION OF HECKE OPERATORS ON THETA PRODUCTS

Here we shall consider the action of regular Hecke operators on products of even number $r = 2k$ of theta constants with rational characteristics considered as modular forms, assuming that the class number $h(Q_r)$ of the sum of r squares is equal to 1.

Let F belong to the space $\mathfrak{M}_k(\Omega) = \mathfrak{M}_k(\Omega, \mathbf{1})$ of modular forms of integral weight k and the trivial character $\chi = \mathbf{1}$ for a subgroup Ω of finite index in the modular group Γ^g , and let

$$T = \sum_i a_\alpha(\Omega \mathcal{M}_\alpha) \in \mathcal{H}(\Omega, \Sigma^g).$$

Then, as it easily follows from definitions of modular forms and HS -rings, and the properties (2.12), (2.13) of the Petersson operators (2.11), the function

$$(4.1) \quad F||T = F||_k T = \sum_\alpha a_\alpha F|_k \mathcal{M}_\alpha$$

does not depend on the choice of representatives $\mathcal{M}_\alpha \in \Omega\mathcal{M}_\alpha$ and again belongs to the space $\mathfrak{M}_k(\Omega)$. These operators are called *Hecke operators (of weight k for the group Ω)*. The Hecke operators corresponding to elements of regular *HS*-rings are called *regular*. It follows from the definition of multiplication in *HS*-rings and from (2.12) that the map $T \mapsto \|T$ is a linear representation of the ring $\mathcal{H}(\Omega)$ on the space $\mathfrak{M}_k(\Omega)$.

According to Theorem 2.2, if $r = 2k$, $M \in \frac{1}{d}\mathbb{Z}_{2g}^r$, then the theta product $\theta(Z, M)$ is a modular form of weight k and character χ_r for the group $\Gamma^g(2d^2)$. Assuming in addition that d is even, we have the inclusion

$$(4.2) \quad \theta(Z, M) \in \mathfrak{M}_k(\Gamma^g(2d^2)).$$

In this case the action (4.1) of the ring $\mathcal{H}^g(2d^2)$ on the modular form $\theta(Z, M)$ is correctly defined. In order to compute the corresponding images we assume also that $h(Q_r) = 1$ and use the formulas (3.23) for the image $\Theta(V', Z')|T' = \Theta(V', Z', Q_r)|T'$, where $V' = V(M)$, $Z' = \frac{1}{2}Z$, and

$$(4.3) \quad T' = \sum_{\alpha} a_{\alpha}(\Gamma_0^g(4)\mathcal{M}'_{\alpha}) \in \mathcal{H}_0^g(4),$$

is a homogeneous element of multiplier $\mu(T') = \mu$ coprime with d .

Under the assumptions of Theorem 3.3, by (3.23), we get the formula

$$(4.4) \quad \Theta(V', Z')|T' = \sum_{D \in R(Q_r, \mu Q_r)/E(Q_r)} I(D, Q_r, \Psi^{g,r}(T'))\Theta(\mu D^{-1}V', Z').$$

If $\mathcal{M}'_{\alpha} = \begin{pmatrix} A'_{\alpha} & B'_{\alpha} \\ C'_{\alpha} & D'_{\alpha} \end{pmatrix} \in \Sigma_0^g(4)$, then by (3.12) and (2.15) the left-hand side of (4.4) can be rewritten in the form

$$\Theta(V', Z')|T' = \sum_{\alpha} a_{\alpha} j_{Q_r}(\mathcal{M}'_{\alpha}, Z')^{-1} \Theta(V' \cdot {}^t\mathcal{M}'_{\alpha}, \mathcal{M}'_{\alpha}\langle Z' \rangle; Q).$$

Let us set

$$\mathcal{M}_{\alpha} = \begin{pmatrix} A_{\alpha} & B_{\alpha} \\ C_{\alpha} & D_{\alpha} \end{pmatrix} = \omega^{-1}\mathcal{M}'_{\alpha}\omega = \begin{pmatrix} A'_{\alpha} & 2B'_{\alpha} \\ \frac{1}{2}C'_{\alpha} & D'_{\alpha} \end{pmatrix} \in \omega^{-1}\Sigma_0^g(4)\omega \in \Sigma_{00}^g(2),$$

where $\omega = \omega^g(2) = \begin{pmatrix} 1_g & 0 \\ 0 & 2 \cdot 1_g \end{pmatrix}$, see (3.8). Then it follows directly from definitions that

$$j_{Q_r}(\mathcal{M}'_{\alpha}, Z') = \chi_r(\mathcal{M}_{\alpha}) \det(C_{\alpha}Z + D_{\alpha})^k,$$

where χ_r is the character (2.15), also

$$V' \cdot {}^t\mathcal{M}'_\alpha = V(M) \cdot {}^t\mathcal{M}'_\alpha = M\omega^{-1}J_g \cdot {}^t\mathcal{M}'_\alpha = M\mu\mathcal{M}_\alpha^{-1}\omega^{-1}J_g = V(M \cdot \mathcal{M}_\alpha^*),$$

where $\mathcal{M}_\alpha^* = \mu\mathcal{M}_\alpha^{-1}$, and we have used the relations (1.7) along with the fact that ${}^t\mathcal{M}'J_g\mathcal{M}' = \mu J_g$, and finally,

$$\mathcal{M}'_\alpha \langle Z' \rangle = \frac{1}{2} \mathcal{M}_\alpha \langle Z \rangle.$$

Hence, the left-hand side of (4.4) can be written in the form

$$\begin{aligned} & \sum_{\alpha} a_{\alpha} \chi_r(\mathcal{M}_{\alpha})^{-1} \overline{\delta(M\mathcal{M}_{\alpha}^*)} \det(C_{\alpha}Z + D_{\alpha})^{-k} \delta(M\mathcal{M}_{\alpha}^*) \Theta(V(M\mathcal{M}_{\alpha}^*), \frac{1}{2} \mathcal{M}_{\alpha} \langle Z \rangle) \\ &= \sum_{\alpha} a_{\alpha} \chi_r(\mathcal{M}_{\alpha})^{-1} \overline{\delta(M\mathcal{M}_{\alpha}^*)} \boldsymbol{\theta}(Z, M\mathcal{M}_{\alpha}^*)|_k \mathcal{M}_{\alpha}, \end{aligned}$$

where we have used the relation (1.5) and definition (2.11) of Petersson operators. On the other hand, since, clearly, $\mu D^{-1}V(M) = V(\mu D^{-1}M)$, the right-hand side of (4.4) can be rewritten in the form

$$\begin{aligned} & \sum_{D \in R(Q_r, \mu Q_r) / E(Q_r)} I(D, Q_r, \Psi^{g,r}(T')) \overline{\delta(\mu D^{-1}M)} \delta(\mu D^{-1}M) \Theta(V(\mu D^{-1}M), \frac{1}{2}Z) \\ &= \sum_{D \in R(Q_r, \mu Q_r) / E(Q_r)} I(D, Q_r, \Psi^{g,r}(T')) \overline{\delta(\mu D^{-1}M)} \boldsymbol{\theta}(Z, \mu D^{-1}M) \\ &= \sum_{D \in E(Q_r) \setminus R(Q_r, \mu Q_r)} I(\mu D^{-1}, Q_r, \Psi^{g,r}(T')) \overline{\delta(DM)} \boldsymbol{\theta}(Z, DM), \end{aligned}$$

since the mapping $D \mapsto \mu D^{-1}$ translates the set $R(Q_r, \mu Q_r)$ into itself and sends right cosets modulo $E(Q_r)$ to left cosets. Hence, the relation (4.4) takes the form

$$\begin{aligned} (4.5) \quad & \sum_{\alpha} a_{\alpha} \chi_r(\mathcal{M}_{\alpha})^{-1} \overline{\delta(M\mathcal{M}_{\alpha}^*)} \boldsymbol{\theta}(Z, M\mathcal{M}_{\alpha}^*)|_k \mathcal{M}_{\alpha} \\ &= \sum_{D \in E(Q_r) \setminus R(Q_r, \mu Q_r)} I(\mu D^{-1}, Q_r, \Psi^{g,r}(T')) \overline{\delta(DM)} \boldsymbol{\theta}(Z, DM). \end{aligned}$$

The linear combination

$$\omega(T') = \sum_{\alpha} a_{\alpha} (\omega^{-1} \Gamma_0^g(4) \omega \cdot \omega^{-1} \mathcal{M}'_{\alpha} \omega) = \sum_{\alpha} a_{\alpha} (\Gamma_{00}^g(2) \mathcal{M}_{\alpha})$$

belongs to the ring $\mathcal{H}_{00}^g(2)$ as the image of T' under the map (3.9) for $q = 2$. According to (3.4) for $q = 2$ and $q' = 2d^2$, we can assume without loss of generality that all representatives $\mathcal{M}_\alpha \in \Gamma_{00}^g(2)\mathcal{M}_\alpha$ belong to the semigroup $\Sigma^g(2d^2)$. Then the linear combination

$$T = \sum_{\alpha} a_{\alpha}(\Gamma^g(2d^2)\mathcal{M}_{\alpha})$$

belongs to the ring $\mathcal{H}^g(2d^2)$, it is a homogeneous element of multiplier $\mu(T) = \mu$, and each such element has the this form for a homogeneous element $T' \in \mathcal{H}_0^g(4)$ of multiplier $\mu(T') = \mu$. For $\mu \in \mathbb{N}$ we denote by $[\mu]^g$ and $[\mu]_g$ matrices of order $2g$ of the form

$$(4.6) \quad [\mu]^g = \begin{pmatrix} \mu \cdot 1_g & 0 \\ 0 & 1_g \end{pmatrix} \quad \text{and} \quad [\mu]_g = \begin{pmatrix} 1_g & 0 \\ 0 & \mu \cdot 1_g \end{pmatrix}.$$

Since $\mathcal{M}_\alpha \in \Sigma^g(2d^2)$, we have $\mathcal{M}_\alpha \equiv [\mu]^g \pmod{2d^2}$, hence $\mathcal{M}_\alpha^* \equiv [\mu]_g \pmod{2d^2}$, that is

$$\mathcal{M}_\alpha^* = [\mu]_g + 2d^2 \mathcal{N} \quad \text{with} \quad \mathcal{N} = \mathcal{N}_\alpha \in \mathbb{Z}_{2g}^{2g}$$

(it is easy to see that the map $\mathbb{G}^g \ni \mathcal{M} \mapsto \mathcal{M}^{-1}\mu(\mathcal{M})\mathcal{M}$ is an antiautomorphism of the semigroup Σ_q^g). It follows that $M\mathcal{M}_\alpha^* = M[\mu]_g + 2dR$ with the integral matrix $R = (dM\mathcal{N}) = (R', R'')$. Since the matrix dM is integral, by (1.8) we obtain $\overline{\delta(M\mathcal{M}_\alpha^*)} = \overline{\delta(M[\mu]_g)}$ and by (2.4) we have

$$\boldsymbol{\theta}(Z, M\mathcal{M}_\alpha^*) = \mathbf{e}(4d^t(R''M'))\boldsymbol{\theta}(Z, M[\mu]_g) = \boldsymbol{\theta}(Z, M[\mu]_g).$$

Therefore, since clearly $\chi_r(\mathcal{M}_\alpha) = 1$ for $\mathcal{M}_\alpha \in \Sigma^g(2d^2)$ with even d , we can rewrite the relation (4.5) in the form

$$(4.7) \quad \overline{\delta(M[\mu]_g)}\boldsymbol{\theta}(Z, M[\mu]_g) \parallel_k T \\ = \sum_{D \in E(Q_r) \setminus R(Q_r, \mu Q_r)} I(\mu D^{-1}, Q_r, \Psi^{g,r}(T')) \overline{\delta(DM)}\boldsymbol{\theta}(Z, DM).$$

After these considerations we come to the following theorem.

Theorem 4.1. *Let $r = 2k \in 2\mathbb{N}$ be such that the class number $h(Q_r)$ of the sum of r squares is equal to 1, $g \in \mathbb{N}$, and $M \in \frac{1}{d}\mathbb{Z}_{2g}^r$ with $d \in 2\mathbb{N}$. Let*

$$T = \sum_{\alpha} a_{\alpha}(\Gamma^g(2d^2)\mathcal{M}_{\alpha}) \in \mathcal{H}^g(2d^2)$$

be a homogeneous element with multiplier $\mu(T) = \mu$, such that for the corresponding element

$$T' = \sum_{\alpha} a_{\alpha}(\Gamma_0^g(4)(\omega\mathcal{M}_{\alpha}\omega^{-1})) \in \mathcal{H}_0^g(4) \quad \text{with} \quad \omega = \begin{pmatrix} 1_g & 0 \\ 0 & 2 \cdot 1_g \end{pmatrix}$$

there exists an element of the form $\Psi^{g,r}(T') \in \mathcal{H}_0^r(4)$ defined by (3.20). Then the theta product $\theta(Z, M)$ is a modular form of the space $\mathfrak{M}_k(\Gamma^g(2d^2))$, and its image under the Hecke operator $\|_k T$ is again a linear combination of theta products:

$$(4.8) \quad \theta(Z, M)\|_k T = \sum_{D \in E(Q_r) \setminus R(Q_r, \mu Q_r)} I(\mu D^{-1}, Q_r, \Psi^{g,r}(T')) \theta(Z, DM[\tilde{\mu}]_g),$$

where I are trigonometric sums defined by (3.21), $\tilde{\mu} \in \mathbb{N}$ is an inverse of μ modulo $2d^2$, and $[\tilde{\mu}]_g$ is defined by (4.6).

Proof. The first assertion follows from Theorem 2.2. The formula (4.8) follows from the formula (4.7), if we note that for $D \in R(Q_r, \mu Q_r)$,

$$\delta(DM) = \mathbf{e}\{ {}^t M'' {}^t D D M' \} = \mathbf{e}\{ \mu {}^t M'' M \} = \delta(M[\mu]_g)$$

by (1.6), and replace M by $M[\tilde{\mu}]_g$. \triangle

As to existence of elements $\Psi^{g,r}(T')$, specialization of Lemma 3.2 to the case $Q = Q_r$ and $\chi_Q = \chi_4$ gives us the following lemma.

Lemma 4.2. *Under the notation of Theorem 4.1, element $\Psi^{g,r}(T') \in \mathcal{H}_0^r(4)$ exists if and only if $g \geq k$, or $g < k$ and each prime divisor p of μ entering the prime numbers factorization of $\mu = \mu(T')$ in an odd degree satisfies the congruence $p \equiv 3 \pmod{4}$.*

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