ON INVARIANT SUBSPACES AND EIGENFUNCTIONS FOR REGULAR HECKE OPERATORS ON SPACES OF MULTIPLE THETA CONSTANTS

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Abstract. Invariant subspaces and eigenfunctions for regular Hecke operators actinng on spaces spanned by products of even number of Igusa theta constants with rational characteristics are constructed. For some of the eigenfunctions of genuses $g = 1$ and 2, the corresponding zeta functions of Hecke and Andrianov are explicitely calculated.

Introduction

The *Igusa theta constant of genus* $g \in N$ *with characteristic* $\mathbf{m} \in \mathbb{C}_{2g}$ is the function on the upper half-plane of genus g,

$$\mathbb{H}^g = \left\{ Z = X + iY \in \mathbb{C}_g^g \;\middle|\; {}^tZ = Z, \quad Y > 0 \right\},$$

defined by the series

$$\theta_{\mathbf{m}}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}_g} \exp \pi i \left\{ (\mathbf{n} + \mathbf{m}')Z\,{}^t(\mathbf{n} + \mathbf{m}') + 2(\mathbf{n} + \mathbf{m}')\,{}^t\mathbf{m}'' \right\},$$

where $\mathbf{m} = (\mathbf{m}', \mathbf{m}'')$ with $\mathbf{m}'$, $\mathbf{m}'' \in \mathbb{C}_g$.

In [AA-FA(04)], assuming that the class number of the quadratic form

$$\mathbf{q}_r(X) = {}^tXX = x_1^2 + \cdots + x_r^2 \qquad (X = {}^t(x_1, \ldots, x_r))$$

is equal to one, we have obtained explicit formulae for images of the products of even number r of Igusa theta constants with rational characteristics under regular Hecke operators in the form of linear combinations of similar products. In this paper we apply the formulae in order to construct invariant subspaces and eigenfunctions

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of regular Hecke operators on spaces spanned by the products. For some of the eigenfunctions of small genuses we calculate the corresponding zeta functions.

Notation. We reserve the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the set of positive rational integers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. $\mathbb{A}_n^m$ is the set of all $m \times n$-matrices with entries in a set $\mathbb{A}$, $\mathbb{A}^n = \mathbb{A}_1^n$, and $\mathbb{A}_n = \mathbb{A}_n^1$.

We denote by

$$\mathbb{E}^n = \left\{ Q = (q_{\alpha\beta}) \in \mathbb{Z}_n^n \,\middle|\, q_{\alpha\beta} = q_{\beta\alpha}, \quad q_{\alpha\alpha} \in 2\mathbb{Z} \qquad (\alpha, \beta = 1, \dots, n) \right\}$$

the set of all *even matrices* of order n, i.e. the set of mtrices of integral quadratic forms in n variables $\mathbf{q}(x_1, \dots, x_n) = \frac{1}{2} \sum_{\alpha\beta} q_{\alpha\beta} x_\alpha x_\beta$.

If M is a matrix, ${}^tM$ always denotes the transpose of M. We denote through this work by 1_g the unit matrix of order g, and use the matrix notation

$$J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}, \quad \omega_g(a) = \begin{pmatrix} 1_g & 0 \\ 0 & a1_g \end{pmatrix}, \quad \omega^g(a) = \begin{pmatrix} a1_g & 0 \\ 0 & 1_g \end{pmatrix},$$

where a is a scalar and $0 = 0_g$ the zero matrix of order g.

§1. Transformations of multiple theta constants

In this section we remind the basic definitions and transformation formulae for products of Igusa theta constants, which will be refered as *multiple theta constants* or *theta products*.

Note that if the products AB and BA two matrices A and B are both defined, then, clearly, the products are square matrices with equal traces. It follows that the theta constant $\theta_{\mathbf{m}}(Z)$ with $Z \in \mathbb{H}^g$ and $\mathbf{m} = (\mathbf{m}', \mathbf{m}'') \in (\mathbb{C}_g, \mathbb{C}_g)$ can be rewritten in the form

$$\theta_{\mathbf{m}}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}_g} \mathbf{e}\{Z\, {}^t(\mathbf{n} + \mathbf{m}')(\mathbf{n} + \mathbf{m}') + 2\, {}^t\mathbf{m}''(\mathbf{n} + \mathbf{m}')\}, \tag{1.1}$$

where for a square matrix A we set

$$\mathbf{e}\{A\} = \exp(\pi i \cdot \sigma(A)), \tag{1.2}$$

and $\sigma(A)$ denotes the trace of A. It follows that the product of r theta constants with characteristics $\mathbf{m}_1, \dots, \mathbf{m}_r$ can be written in the form

$$\boldsymbol{\theta}(Z, M) = \prod_{j=1}^{r} \theta_{\mathbf{m}_j}(Z) \tag{1.3}$$

$$= \sum_{\mathbf{n}_1,\dots,\mathbf{n}_r\in\mathbb{Z}_g} \mathbf{e}\Big\{\sum_{j=1}^{r}\big(Z\,{}^t(\mathbf{n}_j+\mathbf{m}'_j)(\mathbf{n}_j+\mathbf{m}'_j)+2\,{}^t\mathbf{m}_j{}''(\mathbf{n}_j+\mathbf{m}'_j)\big)\Big\}$$

$$= \sum_{N\in\mathbb{Z}_g^r} \mathbf{e}\big\{Z\,{}^t(N+M')(N+M')+2\,{}^tM''(N+M')\big\},$$

where we set

$$N=\begin{pmatrix}\mathbf{n}_1\\ \vdots\\ \mathbf{n}_r\end{pmatrix},\quad M'=\begin{pmatrix}\mathbf{m}'_1\\ \vdots\\ \mathbf{m}'_r\end{pmatrix},\quad \text{and}\quad M''=\begin{pmatrix}\mathbf{m}''_1\\ \vdots\\ \mathbf{m}''_r\end{pmatrix}.$$

A product of the form (1.3) will be called a *multiple theta constant of genus g with the characteristic matrix $M=(M',\,M'')$* or just *theta product*. It can be also considered as a theta function of genus g of the quadratic form $\mathbf{q}_r$.

The following criterion of the identical vanishing of multiple theta constants is a direct consequence of the Igusa result on vanishing of theta constants ([Ig(72)], Theorem 1, p. 174): the theta product with the characteristic matrix $M=(M',\,M'')\in\mathbb{C}^r_{2g}$ is equal to the constant 0, if and only if there is a row $\mathbf{m}_j=(\mathbf{m}'_j,\mathbf{m}''_j)$ of M satisfying

$$(1.4)\qquad 2\mathbf{m}_j\in\mathbb{Z}_{2g},\quad 2\mathbf{m}'_j\,{}^t\mathbf{m}''_j\notin\mathbb{Z}.$$

The transformation formulae of multiple theta constants $\boldsymbol{\theta}(Z,\,M)$ with respect to variable M easily follow from definition (see, for example, [AA-FA(04)], formulae (2.2), (2,4)). We have

$$(1.5)\quad \boldsymbol{\theta}(Z,\,UM)=\boldsymbol{\theta}(Z,\,M)\quad\text{for every}\quad U\in E_r=\Big\{U\in GL_r(\mathbb{Z})\;\Big|\;{}^tUU=1_r\Big\},$$

and

$$(1.6)\quad \boldsymbol{\theta}(Z,\,M+S)=\mathbf{e}\big\{2\,{}^tS''M'\big\}\boldsymbol{\theta}(Z,\,M)\quad\text{for every}\quad S=(S',\,S'')\in(\mathbb{Z}^r_g,\,\mathbb{Z}^r_g).$$

As to the variable Z, according to [AA-FA(04)], Lemma 2.1, the product of even number $r=2k$ of theta constants satisfies the functional equation

$$(1.7)\quad \det(CZ+D)^{-k}\boldsymbol{\theta}((AZ+B)(CZ+D)^{-1},\,M) = \boldsymbol{\chi}_r(\mathcal{M})\delta(M)\overline{\delta(M\mathcal{M})}\boldsymbol{\theta}(Z,\,M\mathcal{M}),$$

for every matrix $\mathcal{M}=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ of the group

$$(1.8)\quad \Gamma^g_{00}(2)=\Big\{\mathcal{M}=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathbb{Z}^{2g}_{2g}\;\Big|\;{}^t\mathcal{M}J_g\mathcal{M}=J_g,\,B\equiv C\equiv 0\pmod 2\Big\},$$

where

$$\boldsymbol{\chi}_r\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \chi_2^k(\det D) \tag{1.9}$$

with the nontrivial Dirichlet character χ_2 modulo 4, δ is the function of matrices $V = (V',\, V'') \in (\mathbb{C}_g^r,\, \mathbb{C}_g^r)$ defined by

$$\delta(V) = \delta(V',\, V'') = \mathbf{e}\{\,{}^tM'M''\}, \tag{1.10}$$

and bar denotes complex conjugation.

In order to formulate transformation formulae related to theta products with rational characteristic matrices, we have to recall some definitions. The connected component of general real symplectic group of genus g consisting of all real symplectic matrices of order $2g$ with positive *multipliers*,

$$\mathbb{G}^g = GSp_g^+(\mathbb{R}) = \left\{\mathcal{M} \in \mathbb{R}_{2g}^{2g} \;\middle|\; {}^t\mathcal{M}J_gM = \mu(\mathcal{M})J_g, \quad \mu(\mathcal{M}) > 0\right\}, \tag{1.11}$$

where the matrix J_g was defined in **Notation**, is a real Lie group acting as a group of analytic automorphisms on the $g(g+1)/2$-dimensional open complex variety $\mathbb{H}^g$ by the rule

$$\mathbb{G}^g \ni \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto \mathcal{M}\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \quad (Z \in \mathbb{H}^g).$$

Acting on the upper half-plane $\mathbb{H}^g$, the group operates also on complex-valued functions F on $\mathbb{H}^g$ by *Petersson operators of integral weights* k,

$$\mathbb{G}^g \ni \mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|_k\mathcal{M} = \det(CZ+D)^{-k}F(\mathcal{M}\langle Z\rangle). \tag{1.12}$$

The Petersson operators satisfy the rules

$$F|_k\mathcal{M}\mathcal{M}' = (F|_k\mathcal{M})|_k\mathcal{M}' \qquad (\mathcal{M}, \mathcal{M}' \in \mathbb{G}^g). \tag{1.13}$$

For a subgroup Ω of $\mathbb{G}^g$ commensurable with the *modular group of genus* g,

$$\Gamma^g = \left\{\mathcal{M} \in \mathbb{G}^g \;\middle|\; \mathcal{M},\, \mathcal{M}^{-1} \in \mathbb{Z}_{2g}^{2g}\right\}$$

a *character* $\boldsymbol{\chi}$ of Ω, i.e. a multiplicative homomorphism of Ω into nonzero complex numbers with the kernel of finite index in Ω, and an integral number k, we shall denote by $\mathfrak{M}_k(\Omega,\, \boldsymbol{\chi})$ the space of all (*Siegel*) *modular forms of weight* k *and character* $\boldsymbol{\chi}$ *for the group* Ω, i.e. the space of all complex-valued functions on $\mathbb{H}^g$ holomorphic in $g(g+1)/2$ complex variables, satisfying the functional equation

$$F|_k\mathcal{M} = \boldsymbol{\chi}(\mathcal{M})F \tag{1.14}$$

for every matrix $\mathcal{M} \in \Omega$, where $|_k$ is the Petersson operator of weight k, and regular at all cusps of Ω if $g = 1$.

Theorem 1. *($[AA-FA(04), Theorem\,2.2]$), the product of even number $r=2k$ of theta constants of genus $g\geq 1$ with rational characteristic matrix $M\in\frac{1}{d}\mathbb{Z}^r_{2g}$, where $d\in\mathbb{N}$, satisfies the functional equation*

$$\boldsymbol{\theta}(Z,\,M)|_k\mathcal{M}=\boldsymbol{\chi}_M(\mathcal{M})\boldsymbol{\theta}(Z,\,M)\quad\forall\mathcal{M}\in\Gamma^g(d)\bigcap\Gamma^g_{00}(2),\tag{1.15}$$

where

$$\Gamma^g(d)=\left\{\mathcal{M}\in\Gamma^g\;\middle|\;\mathcal{M}\equiv 1_{2g}\pmod d\right\},$$

is the principal congruence subgroup of level d of the modular group Γ^g, $\Gamma^g_{00}(2)$ is the group (1.8), and where

$$\boldsymbol{\chi}_M(\mathcal{M})=\boldsymbol{\chi}_r(\mathcal{M})\mathbf{e}\{S(\mathcal{M})\,{}^tMM\}\tag{1.16}$$

with the character $\boldsymbol{\chi}_r$ defined by (1.9) and a symmetric matrix $S(\mathcal{M})$ which can be written in the form

$$S(\mathcal{M})=\begin{pmatrix}B+{}^tB-A\,{}^tB & {}^tD-A\,{}^tD\\ D-1_g-C\,{}^tB & -C\,{}^tD\end{pmatrix}\qquad\left(\mathcal{M}=\begin{pmatrix}A&B\\C&D\end{pmatrix}\right).\tag{1.17}$$

If the theta product $\boldsymbol{\theta}(Z,\,M)$ is not identically zero, then the function $\boldsymbol{\chi}_M:\mathcal{M}\mapsto\boldsymbol{\chi}_M(\mathcal{M})$ is a character of the group $\Gamma^g(d)\bigcap\Gamma^g_{00}(2)$ coinciding on the subgroup $\Gamma^g(2d^2)$ with the character $\boldsymbol{\chi}_r$, and the theta product is a modular form of weight k and character $\boldsymbol{\chi}_M$ for the group $\Gamma^g(d)\bigcap\Gamma^g_{00}(2)$,

$$\boldsymbol{\theta}(Z,\,M)\in\mathfrak{M}_k(\Gamma^g(d)\bigcap\Gamma^g_{00}(2),\,\boldsymbol{\chi}_M),\tag{1.18}$$

if moreover d is even, then

$$\boldsymbol{\theta}(Z,\,M)\in\mathfrak{M}_k(\Gamma^g(2d^2),\,\mathbf{1})=\mathfrak{M}_k(\Gamma^g(2d^2)).\tag{1.19}$$

The last inclusion allows one to define the action of Hecke operators on the multiple theta constants.

We recall first the corresponding definitions. Let Δ be a multiplicative semigroup and Ω a subgroup of Δ. Let us consider the $\mathbb{C}$–linear space of all formal finite linear combinations with coefficients in $\mathbb{C}$ of symbols $(\Omega\mathcal{M})$, where $\mathcal{M}\in\Delta$, being in one-to-one correspondence with left cosets $\Omega\mathcal{M}$ of the set Δ modulo Ω, which are invariant with respect to all right multiplications by elements of Ω,

$$\begin{aligned}\mathcal{H}(\Omega,\,\Delta)&=\mathcal{H}_{\mathbb{C}}(\Omega,\,\Delta)\\&=\{T=\sum_\alpha a_\alpha(\Omega\mathcal{M}_\alpha)\;\Big|\;T\omega=\sum_\alpha a_\alpha(\Omega\mathcal{M}_\alpha\omega)=T\quad\forall\omega\in\Omega\}.\end{aligned}$$

The multiplication of elements

$$T=\sum_{\alpha}a_{\alpha}(\Omega\mathcal{M}_{\alpha}),\quad T'=\sum_{\beta}b_{\beta}(\Omega\mathcal{N}_{\beta}))\in\mathcal{H}(\Omega,\,\Delta)$$

defined by

$$TT'=\sum_{\alpha,\beta}a_{\alpha}b_{\beta}(\Omega\mathcal{M}_{\alpha}\mathcal{N}_{\beta})$$

does not depend on the choice of representatives $\mathcal{M}_{\alpha}\in\Omega\mathcal{M}_{\alpha}$ and $\mathcal{N}_{\beta}\in\Omega\mathcal{N}_{\beta}$ and turns the linear space $\mathcal{H}(\Omega,\,\Delta)$ into an associative $\mathbb{C}$-algebra with the unity element $(\Omega 1_{\Omega})$. The algebra is called *the Hecke–Shimura ring* or *HS–ring of a semigroup Δ relative to a subgroup $\Omega\subset\Delta$ (over the field $\mathbb{C}$)* If every double coset $\Omega\mathcal{M}\Omega$ is a finite union of left cosets $\Omega\mathcal{M}'$, then the linear combinations of the form

$$(\mathcal{M})=(\mathcal{M})_{\Omega}=\sum_{\mathcal{M}_i\in\Omega\mathcal{M}\Omega}(\Omega\mathcal{M}_i)\quad(\mathcal{M}\in\Delta),\tag{1.20}$$

being in one-to-one correspondence with double cosets of Δ modulo Ω, belong to $\mathcal{H}(\Omega,\,\Delta)$ and form a basis of the ring over $\mathbb{C}$. The symbols $(\Omega\mathcal{M})$ and $(\mathcal{M})$ will be refered as *left* and *double classes of Δ modulo Ω*, respectively.

We will be interested in HS–rings $\mathcal{H}(\Omega,\,\Delta)$ of semigroups Δ contained in the semigroup

$$\Sigma^g=\mathbb{G}^g\bigcap\mathbb{Z}^{2g}_{2g}=\left\{\mathcal{M}\in\mathbb{Z}^{2g}_{2g}\;\middle|\;{}^t\mathcal{M}J_g\mathcal{M}=\mu(\mathcal{M})J_g,\quad\mu(\mathcal{M})>0\right\},\tag{1.21}$$

of intergal symplectic matrices with positive multipliers $\mu(\mathcal{M})$ relative to congruence subgroups Ω of the modular group Γ^g. An element of such a ring is called *homogeneous of multiplier μ* if it is a linear combination of left or double classes consisting of matrices with the same multiplier μ.

If

$$T=\sum_{i}a_{\alpha}(\Omega\mathcal{M}_{\alpha})\in\mathcal{H}(\Omega,\,\Sigma^g)$$

and F is a function contained in the space $\mathfrak{M}_k(\Omega)=\mathfrak{M}_k(\Omega,\,\mathbf{1})$, then image of the function under the the action of *Hecke operator $\|_kT$ of weight k corresponding to* T is defined by

$$F\|_kT=\sum_{\alpha}a_{\alpha}F|_k\mathcal{M}_{\alpha},\tag{1.22}$$

where $|_k$ are the Petersson operators (1.12). It does not depend on the choice of representatives $\mathcal{M}_{\alpha}\in\Omega\mathcal{M}_{\alpha}$ and again belong to the space $\mathfrak{M}_k(\Omega)$. Product of elements of the HS–ring acts as product of the corresponding operators.

Specifically, since in the assumptions of Theorem 1 the theta product $\boldsymbol{\theta}(Z,\, M)$ belongs to the space $\mathfrak{M}_k(\Gamma^g(2d^2))$, we will consider the Hecke operators on this space corresponding to homogeneous elements T of the ring

$$\mathcal{H}^g(2d^2) = \mathcal{H}(\Gamma^g(2d^2),\, \Sigma^g(2d^2)), \tag{1.23}$$

where

$$\Sigma^g(q) = \left\{ \mathcal{M} \in \Sigma^g \;\middle|\; \gcd(\mu(M), q) = 1, \quad \mathcal{M} \equiv \omega^g(\mu(\mathcal{M})) \pmod q \right\} \tag{1.24}$$

(see **Notation**).

Theorem 2. *Suppose that an even number $r = 2k$ is such that the class number $h(\mathbf{q}_r)$ of the sum of r squares is equal to 1. Let, for $g \in \mathbb{N}$, $d \in 2\mathbb{N}$, and $M \in \frac{1}{d}\mathbb{Z}^r_{2g}$, $\boldsymbol{\theta}(Z,\, M)$ be the multiple theta product (1.3) of genus g with characteristic matrix M. Let $\mathbf{T}$ be an homogeneous element of the ring $\mathcal{H}^g(2d^2)$ with the multiplier $\mu = \mu(T)$ such that in the case $g < k$ each prime number p entering into the prime numbers factorization of μ in an odd degree satisfies the congruence $p \equiv 1 \pmod 4$.*

Then the image of the function $F = \boldsymbol{\theta}(Z,\, M)$ under the action of Hecke operator $\|_k\mathbf{T}$ is again a linear combination of theta products, and the linear combination can be written in the form

$$\boldsymbol{\theta}(Z,\, M)\|_k\mathbf{T} = \sum_{D \in \mathbf{E}^+_r \backslash \mathbf{S}^+_r(\mu)} c(D, \mathbf{T})\boldsymbol{\theta}(Z,\, DM\omega_g(\widetilde{\mu})) \tag{1.25}$$

with constant coefficients $c(D, \mathbf{T})$, where

$$\mathbf{S}^+_r(\mu) = \left\{ M \in \mathbb{Z}^r_r \;\middle|\; {}^tDD = \mu \cdot 1_r, \quad \det D > 0 \right\} \tag{1.26}$$

and

$$\mathbf{E}^+_r = \mathbf{S}^+_r(1) \tag{1.27}$$

denote the set of all proper integral similarities of the quadratic form $\mathbf{q}_r$ with the multiplier μ and the group of proper units of $\mathbf{q}_r$, respectively, matrices ω_g were defined in **Notation**, *and where $\widetilde{\mu} \in \mathbb{N}$ is an integral inverse of μ modulo $2d^2$.*

The coefficients $c(D, \mathbf{T})$ satisfy the relations

$$c(UDV, \mathbf{T}) = c(D, \mathbf{T}) \tag{1.28}$$
$$\text{for all } U \in GL_r(\mathbb{Z}) \text{ and } V \in \mathbf{E}_r = \left\{ M \in \mathbb{Z}^r_r \;\middle|\; {}^tDD = \mu \cdot 1_r \right\}.$$

Proof. It follows from [AA-FA(04)], Theorem 4.1 and Lemma 4.2 that in the assumptions of the theorem formulas (1.25) hold with summation over a set of representatives $\mathbf{E}_r\backslash\mathbf{S}_r(\mu)$, where $\mathbf{S}_r(\mu)$ is the set of all integral similarities of the quadratic form $\mathbf{q}_r$ with the multiplier μ. Since, clearly, each coset $\mathbf{E}_rD$ with $D\in\mathbf{S}_r(\mu)$ contains a representative D' with positive determinant and the coset $\mathbf{E}_r^+D'$ is uniquely determined by the coset $\mathbf{E}_rD$, the summation over $\mathbf{E}_r\backslash\mathbf{S}_r(\mu)$ can be replaced by the summation over $\mathbf{E}_r^+\backslash\mathbf{S}_r^+(\mu)$. △

§2. Computation of coefficients

Here we summarize the known results of computations of the coefficients $c(D,T)$ in formulae (1.25). In order to present an intelligible account, we have first to go into some details concerning Hecke–Shimura rings.

A subgroup Ω of the modular group Γ^g is said to be *q–symmetric* if it contains the principal congruence subgroup of level q,

$$\Omega\ni\Gamma^g(q)=\left\{\mathcal{M}\in\Gamma^g\;\middle|\;\mathcal{M}\equiv 1_{2g}\pmod{q}\right\},$$

and satisfies the condition $\Omega\Sigma^g(q)=\Sigma^g(q)\Omega$. For a q–symmetric group $\Omega\subset\Gamma^g$, the Hecke–Shimura ring $\mathcal{H}(\Omega,\,R_q(\Omega))$, where $R_q(\Omega)=\Omega\Sigma^g(q)=\Sigma^g(q)\Omega$, is called *$q$–regular HS–ring* of Ω. According to [An(87), theorems 3.3.3 and 3.3.7], all q–regular HS–rings of given genus g are commutative algebras isomorphic to each other.

One can show that groups $\Gamma^g(q')$ as well as groups

$$\Gamma_0^g(q')=\left\{\mathcal{M}=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma^g\;\middle|\;C\equiv 0\pmod{q}'\right\}\tag{2.1}$$

are all q–symmetric, provided that q' and q have the same prime divisors and q' divides q, in addition, $R_q(\Gamma^g(q'))=\Sigma^g(q')$, and

$$\begin{aligned}R_q(\Gamma_0^g(q'))&=\Sigma_0^g(q')\\&=\left\{M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Sigma^g\;\middle|\;\gcd(\mu(M),q')=1,\,C\equiv 0\pmod{q'}\right\}.\end{aligned}\tag{2.2}$$

In particular, the rings

$$\mathcal{H}_0^g(q)=\mathcal{H}(\Gamma_0^g(q),\,\Sigma_0^g(q)),\quad\text{and}\quad\mathcal{H}^g(q)=\mathcal{H}(\Gamma^g(q),\,\Sigma^g(q))\tag{2.3}$$

are isomorphic. Moreover, one can show directly from definitions that every map of the form

$$T=\sum_\alpha a_\alpha(\Gamma^g(q)\mathcal{M}_\alpha)\mapsto T'=\sum_\alpha a_\alpha\left(\Gamma_0^g(l)\omega_g(q')\mathcal{M}_\alpha\omega_g(q')^{-1}\right),\tag{2.4}$$

where q' divides q and l divides $q'q$, is a homomorphism of the rings

$$\mathcal{H}^g(q) \mapsto \mathcal{H}_0^g(l) \quad (q'|q,\, l|q'q). \tag{2.5}$$

This homomorphism determines isomorphisms of local p-subrings of the corresponding rings generated by double classes whose multipliers are degrees of a prime numbers p not dividing q,

$$\mathcal{H}_p^g(q) \mapsto \mathcal{H}_{0,p}^g(l) \quad (p \nmid q,\, q'|q,\, l|q'q) \tag{2.6}$$

as well as isomorphism of the subrings generated by such p-subrings.

Further, we remind that the *Zharkovskaya map from genus g to genus n*,

$$\Psi^{g,n} = \Psi_{k,\chi}^{g,n} : \mathcal{H}_0^g(q) \mapsto \mathcal{H}_0^n(q),$$

where k is an integer, and χ a Dirichlet character modulo q satisfying $\chi(-1) = (-1)^k$, can be defined in the following way. Let first $g > n \geq 1$, and $T = \sum_\alpha a_\alpha(\Gamma_0^g(q)\mathcal{M}_\alpha) \in \mathcal{H}_0^g(q)$. One can assume that each representative $\mathcal{M}_\alpha \in \Gamma_0^g(q)\backslash\Sigma_0^g(q)$ is choosen in the form

$$\mathcal{M}_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0 & D_\alpha \end{pmatrix} \quad \text{with } D_\alpha = \begin{pmatrix} D'_\alpha & * \\ 0 & D''_\alpha \end{pmatrix} \quad \text{and } D'_\alpha \in \mathbb{Z}_n^n.$$

If $A_\alpha = \begin{pmatrix} A'_\alpha & * \\ * & * \end{pmatrix}$ and $B_\alpha = \begin{pmatrix} B'_\alpha & * \\ * & * \end{pmatrix}$ with $r \times r$-blocks A'_α and B'_α, then

$$\mathcal{M}'_\alpha = \begin{pmatrix} A'_\alpha & B'_\alpha \\ 0 & D'_\alpha \end{pmatrix} \in \Sigma_0^r(q),$$

and we put

$$\Psi_{k,\chi}^{g,n}(T) = \sum_\alpha a_\alpha |\det D''_\alpha|^{-k} \chi^{-1}(|\det D''_\alpha|)(\Gamma_0^n(q)\mathcal{M}'_\alpha).$$

For $n = g$ we set $\Psi_{k,\chi}^{g,n}(T) = T$, and for $g < n$ define $\Psi_{k,\chi}^{g,n}(T)$ as an element of the inverse image $(\Psi_{k,\chi}^{n,g})^{-1}(T)$, if $T \in \Psi_{k,\chi}^{n,g}(\mathcal{H}_0^n(q))$.

Finally, for a nonsingular matrix $Q \in \mathbb{E}^r$ (see **Notation**), an integral $r\times r$-matrix D satisfying the condition $\det D = \pm\mu^{r/2}$ and such that $\mu^{-1}\,{}^tDQD \in \mathbb{E}^r$, and for written in the "triangular form" element

$$T' = \sum_\beta b_\beta \left(\Gamma_0^r(q) \begin{pmatrix} A_\beta & B_\beta \\ 0 & D_\beta \end{pmatrix}\right) \in \mathcal{H}_0^r(q) \qquad ({}^tA_\beta D_\beta = \mu 1_r),$$

where q is the level of Q, we define the trigonometric sums $I(D,Q,T')$ by

$$I(D,Q,T') = \sum_{\beta;\, D\,{}^tD_\beta\equiv 0 \pmod{\mu}} b_\beta |\det D_\beta|^{-r/2}\chi_Q^{-1}(|\det D_\beta|)\mathbf{e}\{\mu^{-2}Q[D]\,{}^tD_\beta B_\beta\},$$

with $\mathbf{e}\{\cdots\}$ being the exponent (1.2).

In that notation, Theorem 4.1 of [AA-FA(04)] gives the following formula for the coefficients $c(D,\mathbf{T})$:

$$c(D,\mathbf{T}) = I(\mu D^{-1}, Q_r, \Psi^{g,r}(T)), \tag{2.7}$$

where $Q_r = 2\cdot 1_r$ is the matrix of quadratic form $\mathbf{q}_r$ (of level 4), T is the image of $\mathbf{T}$ under the map (2.5) with $q = 2d^2$, $q' = 2$, and $l = 4$, $\Psi^{g,r}(T)$ is an image of T under the Zharkovskaya map $\Psi^{g,r} = \Psi^{g,r}_{k,\chi}$ with $k = r/2$ and $\chi = \chi_2^k$.

We remind that the HS–ring $\mathcal{H}_0^n(q)$ is generated over $\mathbb{C}$ by the following double classes of the form (1.20):

(2.8)
$$\begin{cases} T^n(p) = \big(diag(\underbrace{1,\dots,1}_{n},\underbrace{p,\dots,p}_{n})\big)_{\Gamma_0^n(q)}, \\ T_j^n(p^2) = \big(diag(\underbrace{1,\dots,1}_{n-j},\underbrace{p,\dots,p}_{j},\underbrace{p^2,\dots,p^2}_{n-j},\underbrace{p,\dots,p}_{j})\big)_{\Gamma_0^n(q)} \quad (j=1,\dots,n), \end{cases}$$

where p runs over all prime numbers not dividing q (see [An(87)], Theorem 3.3.23). The corresponding generators of the ring $\mathcal{H}^n(q)$ we will denote by

$$\mathbf{T}^n(p),\ \mathbf{T}_1^n(p^2),\dots,\mathbf{T}_n^n(p^2), \tag{2.9}$$

respectively. The following proposition summarize all known at present results of computations of sums (2.6) for the generators (2.8).

Proposition 3. *Let Q be an even positive definite matrix of even order $r = 2k$, q the level of Q, and χ_Q the Dirichlet character of the quadratic form with matrix Q. Then the sums* (2.6) *for elements $T^r(p)$, $T^r_{r-1}(p^2)$, and $T^r_r(p^2)$ with each prime number p not dividing q, can be computed by the following formulas:*

$$I(D,\,Q,\,T^r(p)) = \begin{cases} p^k\prod_{j=1}^k(1+\chi_Q(p)p^{-j}), & \text{if } D\in\Delta(Q,\,p) = \Lambda\omega_g(p)\Lambda, \\ 0, & \text{otherwise}; \end{cases} \tag{2.10}$$

$$(2.11)\qquad I(D,\,Q,\,T^r_{r-1}(p)) = \begin{cases} \chi_Q(p)p^{(2+r-r^2)/2}, \quad \text{if } D \in \Lambda D^r_{r-2,1}(p)\Lambda, \\ \alpha_r(p), \quad \text{if } D \in \Lambda(p1_r), \\ 0, \quad \text{otherwise}, \end{cases}$$

where $\Lambda = \Lambda^r = GL_r(\mathbb{Z})$, $D^r_{r-2,1}(p) = diag(1, \underbrace{p,\ldots,p}_{r-2}, p^2)$, *and*

$$\alpha_r(p) = \chi_Q(p)p^{(2+r-r^2)/2}\frac{(p^r-1)}{p-1} + p^{-r^2/2}(\chi_Q(p)p^{r/2}-1);$$

$$(2.12)\qquad I(D,\,Q,\,T^r_r(p^2)) = \begin{cases} p^{-r^2/2} & \text{if } D \in \Lambda(p1_r), \\ 0 & \text{otherwise}\ . \end{cases}$$

Proof. In [An(91), formula (2.19) and Lemma 5.1] the sums $\gamma(Q,\,D,\,T)$ similar to the sums (3.21) were defined and computed for $T = T^r(p)$. In [An(93), §2] the sums $\gamma(Q,\,D,\,T)$ were, in fact, computed for $T = T^r_r(p^2) = (p1_{2r})_{\Gamma^r_0(q)}$ and $T = T^r_{r-1}(p^2)$ (see also [An(87), Lemma 3.3.32] for the presentation of $T^r_{r-1}(p^2)$ used in [An(93)]). It directly follows from definitions of these sums that $I(D,\,Q,\,T) = \chi_Q(\mu)^r\mu^{r/2}\gamma(Q,\,\mu D^{-1}, T) = \mu^{r/2}\gamma(Q,\,\mu D^{-1}, T)$. The rest is clear. △

Note that the formulae of Proposition 3.4 determine sums $I(D,\,Q,\,T)$ for all generators of the rings $\mathcal{H}^1_0(q)$ and $\mathcal{H}^2_0(q)$.

We turn now to concrete formulae for the action on theta products of Hecke operators corresponding to certain coefficients of the spinor p-polynomials

$$(2.13)\qquad \mathbf{Q}^g_p(v) = \sum_{j=1}^{m}(-1)^j\mathbf{q}^g_j(p)v^j, \quad \text{where } m = 2^g,$$

over p-subrings of the rings $\mathcal{H}^g(q)$ for prime p not dividing q (for the case of isomorphic rings $\mathcal{H}^g_0(q)$ see, for example, [An(87), (3.3.78)]). The polynomials present considerable interest because after substituting $v = p^{-s}$ and replacing coefficients by eigenvalues of corresponding Hecke operators acting on an eigenfunction $F \in \mathfrak{M}_k(\Gamma^g(q))$ one gets denominators $Q_p(F,\,p^{-s})$ of p-factor of the spinor Euler product

$$(2.14)\qquad Z(F,\,s) = \prod_{p\nmid q}\mathbf{Q}_p(F,\,p^{-s})^{-1}.$$

relevant to the eigenfunction. For $g = 1$ it is the Hecke zeta function of the elliptic modular form F; for $g = 2$ the product determines the Andrianov zeta function of the eigenfunction F of genus 2.

We shall restrict ourselves to the action on theta products of Hecke operators corresponding to the elements $\mathbf{q}_1^g(p)$, $\mathbf{q}_{m-1}^g(p)$, $\mathbf{q}_m^g(p)$, and $\mathbf{q}_2^2(p)$. By [An(87), (3.3.81), (3.3.79), (3.3.80), and Exercise 3.3.38], in the notation (2.9) this coefficients can be written in the form

$$\mathbf{q}_1^g(p) = \mathbf{T}^g(p), \quad \mathbf{q}_{m-1}(p) = (p^{g(g+1)/2})^{(m/2)-1}\mathbf{T}^g(p), \qquad \mathbf{q}_m^g(p) = (p^{g(g+1)/2}\mathbf{T}_g^g(p^2))^{m/2}, \tag{2.15}$$

and

$$\mathbf{q}_2^2(p) = p\mathbf{T}_1^2(p^2) + p(p^2+1)\mathbf{T}_2^2(p^2). \tag{2.16}$$

Hence, it will be sufficient to consider the action of operators corresponding to elements $\mathbf{T}^g(p)$, $\mathbf{T}_g^g(p^2)$, and $\mathbf{T}_1^2(p^2)$.

First we shall consider the action of the Zharkovskaya map on the corresponding elements. For brevity we set

$$[p]_n = T_n^n(p^2) = (p1_{2g})_{\Gamma_0^n(q)}. \tag{2.17}$$

the

Lemma 4. *The following formulae hold for the action of the Zharkovskaya map $\Psi = \Psi_{k,\chi}^{n,n-1} : \mathcal{H}_0^n(q) \mapsto \mathcal{H}_0^{n-1}(q)$ on some of generators (2.8) for $n > 1$ and each prime number p not dividing q :*

$$\Psi^{n,n-1}(T^n(p)) = (1 + \bar{\chi}(p)p^{n-k})T^{n-1}(p); \tag{2.18}$$

$$\Psi^{n,n-1}([p]_n) = \bar{\chi}(p)p^{-k}[p]_{n-1}; \tag{2.19}$$

$$\Psi^{n,n-1}(T_{n-1}^n(p^2)) = \bar{\chi}(p)p^{1-k}T_{n-2}^{n-1}(p^2) + b_n(p)[p]_{n-1}, \tag{2.20}$$

where $b_n(p) = b_{n,k,\chi}(p) = \bar{\chi}(p^2)p^{2n-2k} + \bar{\chi}(p)(p-1)p^{-k} + 1$.

Remark 5. *The action of the Zharkovskaya map related to the action of Hecke operators on the spaces $\mathfrak{M}_k^n(q,\chi)$ was calculated in [An(87), §4.2.4]. However, applying the results of calculations, one have to take into account that the Hecke operators defined in [An(87)] by (2.4.11) and (2.4.12) have another normalization than one we use here and differ from the operators defined in [An(96)] by the equalities (1.10) with $l = 0$, (2.13), (2.14), and (2.20) with $Q = H$ and $P = 0$ on homogeneous elements of multiplicator μ by the factor $\chi(\mu^n)\mu^{nk-n(n+1)/2}$.*

Proof of the lemma. The formula (2.18) follows from Propositions 4.2.17 and 4.2.18 and formula (4.2.80) of [An(87)] applied to Hecke operators $\|_k\chi(p^n)p^{nk-n(n+1)/2}T^n(p)$ (see Remark 5). Formula (2.19) follows by similar arguments from Lemma 3.3.34 of [An(87)]. As to formula (2.20), the situation is slightly more complicated. First, by using factorization (3.5.69) of Theorem 3.5.23, formulae (3.5.34), (3.4.15), (3.5.33), and (3.3.61) of [An(87)], we get the relation

$$T^n_{n-1}(p^2) = -p^n[p]_n r^n_1(p) + (p^n - 1)[p]_n, \tag{2.21}$$

where $r^n_1(p)$ is the first coefficient of the Rankin p-polynomial $R^n_p(v)$ defined by formulae (3.5.15) and (3.5.16) of [An(87)]. Since $\mu(r^n_1(p)) = 1$, it follows from [An(87)], Theorem 4.2.18 and relation (4.2.82) that

$$\Psi^{n,n-1}(r^n_1(p)) = r^{n-1}_1(p) - \bar{\chi}(p)p^{n-k} - \chi(p)p^{k-n}. \tag{2.22}$$

Since the Zharkovskaya map is a ring homomorphism, formula (2.20) follows from (2.21), (2.19), (2.22) by an easy computation. △

Proposition 6. *In the notation and assumptions of Theorem* 2 *the image of the theta product* $\boldsymbol{\theta}(Z, M)$ *under the action of Hecke operators* $\|_k\mathbf{T}^g(p)$ *for every prime number* p *not dividing* d *and satisfying the congruence* $p \equiv 1 \pmod 4$ *if* $g < k$ *can be written in the form*

$$\boldsymbol{\theta}(Z, M)\|_k\mathbf{T}^g(p) = \gamma(g,r)\sum_{D\in\mathbf{E}^+_r\backslash\mathbf{S}^+_r(p)}\boldsymbol{\theta}(Z, DM\omega_g(\widetilde{p})), \tag{2.23}$$

where

$$\gamma(g,r) = \chi(p)^g p^{g(g+1)/2-kg} \times \begin{cases} \prod_{j=1}^{k-g}(1+\chi(p)p^{j-1})^{-1}, & \text{if } g<k;\\ 1, & \text{if } g=k;\\ \prod_{j=1}^{g-k}(1+\chi(p)p^{-i}), & \text{if } g>k,\end{cases}$$

and $\chi = \chi_r = \chi_2^k$ *is the character of the quadratic form* $\mathbf{q}_r$*.*

It follows that formulas (1.25) for $\mathbf{T} = \mathbf{T}^g(p)$ hold with the coefficients

$$c(D, \mathbf{T}^g(p)) = \gamma(g,r) \quad \forall D \in \mathbf{S}^+_r(p). \tag{2.24}$$

Proof. By (1.25), (2.7), and (2.15), we have

$$\boldsymbol{\theta}(Z, M)\|_k\mathbf{T}^g(p) = \sum_{D\in\mathbf{E}^+_r\backslash\mathbf{S}^+_r(p)} I(pD^{-1}, Q_r, \Psi^{g,r}(T^g(p)))\boldsymbol{\theta}(Z, DM\omega_g(\widetilde{p})),$$

where $\Psi^{g,r} = \Psi^{g,r}_{k,\chi}$ is the Zharkovskaya map with character χ.

Suppose first that $g < r$, then iteration of formula (2.18) give us the relation

$$\Psi^{r,g}(T^r(p)) = \left\{ \prod_{j=g+1}^{r} (1 + \overline{\chi}(p)p^{j-k}) \right\} T^g(p),$$

whence, since in our assumptions the product on the right is not zero, we get

$$\Psi^{g,r}(T^g(p)) = \left\{ \prod_{j=g+1}^{r} (1 + \overline{\chi}(p)p^{j-k}) \right\}^{-1} T^r(p)$$
$$= \chi(p)^{r-g} p^{(g-r)(g+1)/2} \left\{ \prod_{j=g+1}^{r} (1 + \chi(p)p^{k-j}) \right\}^{-1} T^r(p).$$

Note that if $D \in \mathbf{S}^+_r(p)$, i.e. D is an integral matrix of order $r = 2k$ satisfying ${}^tDD = p \cdot 1_r$, then $\det D = p^k$ and the matrix $pD^{-1} = {}^tD$ is integral. By the theory of elementary divisors for matrices over $\mathbb{Z}$, we conclude that the matrices D and pD^{-1} both belong to the double coset $\Lambda^r \begin{pmatrix} 1_r & 0 \\ 0 & p1_r \end{pmatrix} \Lambda^r$ of the group $\Lambda^r = GL_r(\mathbb{Z})$. Hence, using the above formulae together with formula (2.11), we obtain

$$\boldsymbol{\theta}(Z, M)\|_k \mathbf{T}^g(p) =$$
$$= \chi(p)^{r-g} p^{k+(g-r)(g+1)/2} \left\{ \prod_{j=1}^{k} (1 + \chi(p)p^{-j}) \right\}$$
$$\times \left\{ \prod_{j=g+1}^{r} (1 + \chi(p)p^{k-j}) \right\}^{-1} \sum_{D \in \mathbf{E}^+_r \backslash \mathbf{S}^+_r(p)} \boldsymbol{\theta}(Z, DM\omega_g(\widetilde{p})).$$

Since, clearly,

$$\chi(p)^{r-g} p^{k+(g-r)(g+1)/2} \left\{ \prod_{j=1}^{k} (1 + \chi(p)p^{-j}) \right\} \times \left\{ \prod_{j=g+1}^{r} (1 + \chi(p)p^{k-j}) \right\}^{-1}$$
$$= \chi(p)^{r-g} p^{g(g+1)/2-kg} \times \begin{cases} \prod_{j=1}^{k-g} (1 + \chi(p)p^{j-1})^{-1}, & \text{if } g < k; \\ 1, & \text{if } g = k; \\ \prod_{j=1}^{g-k} (1 + \chi(p)p^{-i}), & \text{if } g > k, \end{cases}$$

and $\chi(p) = \pm 1$, this proves the formula (2.23) in the case $g < r$. If $g > r$, then similarly, by using repeatedly (2.18), we get

$$\Psi^{g,r}(T^g(p)) = \left\{ \prod_{j=r+1}^{g} (1 + \overline{\chi}(p)p^{j-k}) \right\} T^r(p)$$

$$= \overline{\chi}(p)^{g-r} p^{(g-r)(g+1)/2} \left\{ \prod_{j=k+1}^{g-k} (1 + \chi(p)p^{-j}) \right\} T^r(p).$$

This together with (2.10) again prove the formulae (2.23). The case $g = r$ directly follows from (2.10). △

Corollary 7. *In the notation and assumptions of Theorem 2 the image of the theta product $\boldsymbol{\theta}(Z, M)$ under the action of Hecke operators $\|_k \mathbf{T}^g(p)$ for a prime number p not dividing d satisfies the rule*

$$\boldsymbol{\theta}(Z, M)\|_k \mathbf{T}^g(p) = 0 \quad \textit{if } g \geq k \quad \textit{and } p \equiv 3 \pmod 4. \tag{2.25}$$

Proof. The assertion follows from formula (2.23), because in this case the congruence ${}^tXX \equiv 0 \pmod p$ has no solutions in integral $r \times r$-matrices of rank k modulo p, and so the set $\mathbf{S}_r(p)$ is empty (see, e.g., [An(87)], Corollary A.2.15). △

Proposition 8. *In the notation and assumptions of Theorem 2 the image of the theta product $\boldsymbol{\theta}(Z, M)$ under the action of Hecke operators $\|_k \mathbf{T}^g_g(p^2)$ for every prime number p not dividing d can be written in the form*

$$\boldsymbol{\theta}(Z, M)\|_k \mathbf{T}^g_g(p^2) = \chi_r(p)^g p^{-kg} \boldsymbol{\theta}(Z, Mp\omega_g(\widetilde{p^2})). \tag{2.26}$$

It follows that formulas (1.25) for $\mathbf{T} = \mathbf{T}^g_g(p^2)$ hold with the coefficients

$$c(D, \mathbf{T}^g_g(p^2)) = \begin{cases} \chi_r(p)^g p^{-kg}, & \text{for } D \in \mathbf{E}^+_r(p1_r) \\ 0, & \text{for } D \notin \mathbf{E}^+_r(p1_r) \end{cases} \tag{2.27}$$

Proof. It follows from (2.19) that

$$\Psi^{g,r}([p]_g) = (\overline{\chi}(p)p^{-k})^{g-r}[p]_r = \chi(p)^g p^{-kg+2k^2}[p]_r,$$

where $[p]_g$ is the element (2.17) corresponding to $\mathbf{T}^g_g(p^2)$ and $\chi = \chi_r$. Hence, by Theorem 2, formulae (2.7) and (2.12), we can write

$$\boldsymbol{\theta}(Z, M)\|_k \mathbf{T}^g_g(p^2) = \chi(p)^g p^{-kg+2k^2-2k^2} \sum_{D \in \mathbf{E}^+_r \backslash \mathbf{S}^+_r(p^2) \bigcap \Lambda^r(p1_r)} \boldsymbol{\theta}(Z, DM\omega_g(\widetilde{p^2}))$$

$$= \chi(p)^g p^{-kg}\boldsymbol{\theta}(Z,\, Mp\omega_g(\widetilde{p^2})).\triangle$$

Note that the action of the operator $\|_k \mathbf{T}^g_g(p^2)$ can also be directly calculated on the basis of definitions, which, fortunately, gives the same result. Really, if we denote by $\rho(p)$ a matrix of Γ^g congruent modulo $2d^2$ to the matrix $\begin{pmatrix} p1_g & 0 \\ 0 & \widetilde{p}1_g \end{pmatrix}$, then one can take

$$\mathbf{T}^g_g(p^2) = (\rho(p)(p1_{2g}))_{\Gamma^g(2d^2)} = \left(\Gamma^g(2d^2)\,(\rho(p)(p1_{2g}))\right),$$

and so, by (1.22), (1.12), and (1.7), we have

$$\boldsymbol{\theta}(Z,\, M))\|_k \mathbf{T}^g_g(p^2) = \chi(\widetilde{p}^g)p^{-kg}\boldsymbol{\theta}(Z,\, M\rho(p)),$$

because, clearly, $\delta(M)\overline{\delta(M\rho(p))} = 1$.

Proposition 9. *In the notation and assumptions of Theorem* 2 *and Proposition* 3 *the image of the theta product* $\boldsymbol{\theta}(Z,\, M)$ *with* $g = 2$ *and even* $r \geq 2$ *under the action of Hecke operators* $\|_k \mathbf{T}^2_1(p^2)$ *can be written for every prime number* p *not dividing* d *in the form*

$$\text{(2.28)}\quad \boldsymbol{\theta}(Z,\, M)\|_k \mathbf{T}^2_1(p^2)$$
$$= p^{3-3k}\chi_r(p) \sum_{D\in \mathbf{E}^+_r \setminus (\mathbf{S}^+_r(p^2)\bigcap \Lambda^r D^r_{r-2,1}(p)\Lambda^r)} \boldsymbol{\theta}(Z,\, DM\omega_r(\widetilde{p^2})) + \beta_r(p)\boldsymbol{\theta}(Z,\, Mp\omega_r(\widetilde{p^2})),$$

where

$$\beta_r(p) = p^{2(1-k)^2}\alpha_r(p) - \begin{cases} 0, & \textit{if } r = 2 \\ \chi_r(p)p^{2-3k}\left(\sum_{i=0}^{r-1} p^i b_{r-i}(p)\right), & \textit{if } r > 2 \end{cases}$$

and constants $b_j(p)$ *were defined in Lemma* 4.

It follows that formulas (1.25) for $\mathbf{T} = \mathbf{T}^2_1(p^2)$ hold with the coefficients

$$\text{(2.29)}\qquad c(D,\, \mathbf{T}^2_1(p^2)) = \begin{cases} p^{3-3k}\chi_r(p), & \text{for } D \in \mathbf{S}^+_r(p^2)\bigcap \Lambda^r D^r_{r-2,1}(p)\Lambda^r \\ \beta_r(p), & \text{for } D \in \mathbf{E}^+_r(p1_r) \\ 0, & \text{in other cases .} \end{cases}$$

Proof. By induction from formulae (2.19) and (2.20) easily follow for $1 \leq j \leq r-2$ the relations $\Psi^{r,r-j}([p]_r) = a^j[p]_{r-j}$ and

$$\Psi^{r,r-j}(T^r_{r-1}(p^2)) = (ap)^j T^{r-j}_{r-j-1}(p^2) + a^{j-1}\left(\sum_{i=0}^{j-1} p^i b_{r-i}\right)[p]_{r-j},$$

where $a = \overline{\chi}(p)p^{-k}$. These relations with $j = r-2 \geq 1$ imply the relation

$$\Psi^{r,2}\left(T^r_{r-1}(p^2) - a^{-1}\left(\sum_{i=0}^{r-1} p^i b_{r-i}(p)\right)[p]_r\right) = (ap)^{r-2}T^2_1(p^2).$$

It follows that we can take

$$\Psi^{2,r}(T^2_1(p^2)) = (ap)^{2-r}\left(T^r_{r-1}(p^2) - a^{-1}\left(\sum_{i=0}^{r-1} p^i b_{r-i}(p)\right)[p]_r\right).$$

Now, by Theorem 2, formula (2.7), and Proposition 3,for $r > 2$ we obtain

$$\begin{aligned}
&\boldsymbol{\theta}(Z,\, M)\|_k \mathbf{T}^2_1(p^2) \\
&\quad= \sum_{D \in \mathbf{E}^+_r \backslash \mathbf{S}^+_r(p^2)} I(p^2D^{-1}, Q_r, \Psi^{2,r}(T^2_1(p^2)))\boldsymbol{\theta}(Z,\, DM\omega_r(\widetilde{p^2})) \\
&\qquad= (ap)^{2-r} \sum_{D \in \mathbf{E}^+_r \backslash \mathbf{S}^+_r(p^2)} \Bigg[I(p^2D^{-1}, Q_r, T^r_{r-1}(p^2))) \\
&\quad - a^{-1}\left(\sum_{i=0}^{r-1} p^i b_{r-i}(p)\right) I(p^2D^{-1}, Q_r, [\mathbf{p}]_r)\Bigg]\boldsymbol{\theta}(Z,\, DM\omega_r(\widetilde{p^2})) \\
&= (ap)^{2-r}p^{(2+r-r^2)/2}\chi_r(p) \sum_{D \in \mathbf{E}^+_r \backslash (\mathbf{S}^+_r(p^2) \bigcap \Lambda^r D^r_{r-2,1}(p)\Lambda^r)} \boldsymbol{\theta}(Z,\, DM\omega_r(\widetilde{p^2})) \\
&\qquad + (ap)^{2-r}\left(\alpha_r(p) - a^{-1}p^{-kr}\left(\sum_{i=0}^{r-1} p^i b_{r-i}(p)\right)\right)\boldsymbol{\theta}(Z,\, Mp\omega_r(\widetilde{p^2})).
\end{aligned}$$

Substituting here $a = \overline{\chi}(p)p^{-k} = \chi_r(p)p^{-k}$ and taking into account that $r = 2k$ and $\chi_r(p) = \pm 1$, we obtain the formula (2.28) for $r > 2$. But if $r = 2$, then, by the same reason, we have

$$\begin{aligned}
&\boldsymbol{\theta}(Z,\, M)\|_k \mathbf{T}^2_1(p^2) = \sum_{D \in \mathbf{E}^+_r \backslash \mathbf{S}^+_r(p^2)} I(p^2D^{-1}, Q_r, T^2_1(p^2))\boldsymbol{\theta}(Z,\, DM\omega_r(\widetilde{p^2})) \\
&\quad= \chi_2(p) \sum_{D \in \mathbf{E}^+_2 \backslash (\mathbf{S}^+_2(p^2) \bigcap \Lambda^2 D^2_{0,1}(p)\Lambda^2)} \boldsymbol{\theta}(Z,\, DM\omega_2(\widetilde{p^2})) + \alpha_2(p)\boldsymbol{\theta}(Z,\, Mp\omega_2(\widetilde{p^2})).
\end{aligned}$$

$\triangle$

§3. Congruence characters and Hecke invariant subspaces

In order to constrict search of invariant subspaces and eigenfunctions of Hecke operators, it turns out to be convenient to pass from the spaces $\mathfrak{M}_k^g(q) = \mathfrak{M}_k(\Gamma^g(q),\, 1)$ to subspaces of the form $\mathfrak{M}_k(\Omega,\, \chi)$ where Ω is a subgroup of the modular group containing $\Gamma^g(q)$ and χ a character of Ω trivial on $\Gamma^g(q)$. Generally speaking, nontrivial characters of this kind not necessarily exist. But if, for example, the quotient group $\Omega'\backslash\Omega$, where $\Gamma^g(q) \subset \Omega' \subset \Omega$, is Abelian, then a number of such characters exist and, moreover, since each finite-dimensional complex representation of a finite Abelian group is fully reducible, we have the direct sum decomposition

$$\mathfrak{M}_k(\Omega') = \bigoplus_{\chi \in Char(\Omega'\backslash\Omega)} \mathfrak{M}_k(\Omega,\, \chi). \tag{3.1}$$

where χ run through the group $Char(\Omega'\backslash\Omega)$ of characters of the factor group $\Omega'\backslash\Omega$ considered as characters of Ω trivial on Ω'. In this section we consider some intermediate subgroups Ω, $\Gamma^g(q) \subset \Omega \subset \Gamma^g$, with Abelian factor groups $\Gamma_0^g(q)\backslash\Omega$, describe corresponding characters, and examine their relation with regular Hecke operators. Then we shall consider characters $\boldsymbol{\chi}_M$ of the form (1.16) and their correlation with Abelian characters and Hecke operators.

Proposition 10. *Let q, q' be positive integers such that $q'|q|(q')^2$ and let $d = q/q'$. Then for every $g = 1, 2, \dots$ the map*

$$\varrho: \quad \Gamma^g(q') \ni \mathcal{M} = 1_{2g} + q'W \mapsto \varrho(\mathcal{M}) = W/d\mathbb{Z}_{2g}^{2g} \in \left(\mathbb{Z}/d\mathbb{Z}\right)_{2g}^{2g} \tag{3.2}$$

is an epimorphism of the group $\Gamma^g(q')$ on the additive Abelian group

$$W^g(d) = \left\{ W \in \left(\mathbb{Z}/d\mathbb{Z}\right)_{2g}^{2g} \;\middle|\; {}^tWJ \equiv -JW \pmod{d} \right\} \tag{3.3}$$

with the kernel $\Gamma^g(q)$, where $J = J_g$ is the matrix defined in **Notation**.

Proof. Each matrix $\mathcal{M} \in \Gamma^g$ is an integral matrix satisfying the relation

$${}^t\mathcal{M}J\mathcal{M} = J. \tag{3.4}$$

If $\mathcal{M} \in \Gamma^g(q')$, then, by setting in (3.4) $\mathcal{M} = 1_{2g} + q'W$ and passing to the congruence modulo q, we obtain

$$J = {}^t(1_{2g} + q'W)J(1_{2g} + q'W) \equiv J + q'\,{}^tWJ + q'JW \pmod{q},$$

whence ${}^tWJ + JW \equiv 0 \pmod{d}$. This proves that $\varrho(\Gamma^g(q')) \in W^g(d)$. It is clear that $\varrho(\mathcal{M}) = 0$ if and only if $\mathcal{M} \in \Gamma^g(q)$. Further, if matrices $\mathcal{M} = 1_g + q'W$ and

$\mathcal{M}' = 1_g + q'W'$ belong to $\Gamma^g(q')$, then $\mathcal{M}\mathcal{M}' \equiv 1_g + q'(W + W') \pmod q$, whence $\varrho(\mathcal{M}\mathcal{M}') = \varrho(\mathcal{M}) + \varrho(\mathcal{M}')$.

Finally, let us prove that the map ϱ is onto. If $W \in W^g(d)$ then the matrix $1_{2g} + q'W$ belongs to the group

$$(3.5) \quad Sp_g(\mathbb{Z}/q\mathbb{Z},\, q') = \left\{ \mathcal{M} \in (\mathbb{Z}/q\mathbb{Z})^{2g}_{2g} \;\middle|\; {}^t\mathcal{M}J\mathcal{M} \equiv J \pmod q, \quad \mathcal{M} \equiv 1_{2g} \pmod{q'} \right\}.$$

It is well known that the natural map modulo q

$$\Gamma^g = Sp_g(\mathbb{Z}) \mapsto Sp_g(\mathbb{Z}/q\mathbb{Z}) = Sp_g(\mathbb{Z}/q\mathbb{Z},\, 1)$$

is an epimorphism (see, for example, [An(87), Lemma 3.3.2(1)]). It follows that the map determines epimorphism

$$(3.6) \qquad \Gamma^g(q') \mapsto Sp_g(\mathbb{Z}/q\mathbb{Z},\, q') \qquad (q'|q).$$

Hence there is $\mathcal{M} \in \Gamma^g(q')$ with $\mathcal{M} \equiv 1_{2g} + q'W \pmod q$ and so $\varrho(\mathcal{M}) = W$. △

Now, in the assumptions of the proposition we are going to describe the group of characters of $\Gamma^g(q')$ trivial on $\Gamma^g(q)$. We denote by

$$(3.7) \qquad V^g(d) = \mathbb{E}^{2g}/d\mathbb{E}^{2g}$$

the set of *even matrices modulo d of order* $2g$ (see **Notation**). For $V \in V^g(d)$, we define the function $\widehat{V}$ on matrices $\mathcal{M} \in \Gamma^g(q')$ by

$$(3.8) \qquad \widehat{V}(\mathcal{M}) = \widehat{V}(1_{2g} + q'W) = \exp\left(\frac{\pi i}{d}\sigma(JWV)\right) = \mathbf{e}\left\{d^{-1}JWV\right\}.$$

Proposition 11. *In the notation and assumptions of Proposition* 10 *and notation* (3.7), (3.8)*, the map*

$$V^g(d) \ni V \mapsto \widehat{V}$$

is an isomorphism of the additive group $V^g(d)$ onto the group $\mathrm{Char}(\Gamma^g(q)\backslash\Gamma^g(q'))$ *of all characters of the group $\Gamma^g(q')$ trivial on $\Gamma^g(q)$.*

Proof. By definition, we can write

$$(3.9) \qquad \widehat{V}(\mathcal{M}) = e\left\{d^{-1}J\varrho(\mathcal{M})V\right\} \qquad (\mathcal{M} \in \Gamma^g(q'))$$

where ϱ is the map (3.2). If $\mathcal{M} \in \Gamma^g_0(q)$, we have $\varrho(\mathcal{M}) \equiv 0 \pmod d$, whence $\widehat{V}(\mathcal{M}) = 1$ for all $V \in V^g(d)$. Then, by Proposition 10, it follows that the function

$\widehat{V}$ is a character of $\Gamma^g(q')$ trivial on $\Gamma^g(q)$ for every $V \in V^g(d)$. It is also clear that $\widehat{V+V'} = \widehat{V}\widehat{V'}$.

It remains to show that each character $\chi \in$Char$(\Gamma^g(q')/\Gamma^g(q))$ has the form $\chi = \widehat{V}$ with $V \in V^g(d)$. In view of Proposition 10 it is sufficient to prove that each character of $W^g(d)$ has the form

$$W \mapsto \mathbf{e}\left\{d^{-1}JWV\right\} = exp\left(\frac{2\pi i}{d}\sum_{1\le i\le j\le 2g} w'_{ij}v_{ij}\right), \tag{3.10}$$

where $(w'_{ij}) = J\mathcal{W}$ and $\mathcal{V} = (v'_{ij}) \in V^g(d)$ with $v'_{ii} = 2v_{ii}$ and $v'_{ij} = v'_{ji} = v_{ij}$ for $1 \le i < j \le 2g$. By the definition of the sets $W^g(d)$, we conclude that the map

$$W \mapsto JW = (w'_{ij}) \tag{3.11}$$

is an isomorphism onto the group of all symmetric matrises of order $2g$ over $\mathbb{Z}/d\mathbb{Z}$ which is the direct sum of $2g(2g+1)/2 = g(2g+1)$ copies of the additive group $\mathbb{Z}/d\mathbb{Z}$. Since each character of the last group has the form $a \mapsto \exp(2\pi iab/d)$, it follows that each character of $W^g(d)$ has the form (3.10). $\triangle$

Now we shall turn to regular Hecke operators and examine their relations with the described congruence characters.

Proposition 12. *Let Ω be a q-symmetric subgroup of Γ^g, i.e. such that $\Gamma^g(q) \subset \Omega$ and $\Omega\Sigma^g(q) = \Sigma^g(q)\Omega$, and let μ be an integer coprime to q. Then the following statements are true:*

(1) *The map*

$$\overline{\mathcal{M}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & \mu B \\ \widetilde{\mu}C & D \end{pmatrix} = \omega^g(\mu)\overline{\mathcal{M}}\omega^g(\widetilde{\mu}) = \overline{\mathcal{M}}|\mu, \tag{3.12}$$

where $\overline{\mathcal{M}}$ denotes the class of $\mathcal{M} \in \Omega$ modulo $\Gamma^g(q)$ and $\widetilde{\mu}$ is an inverse of μ modulo q, is an automorphism of the factor group $\Gamma^g(q)\backslash\Omega$;

(2) *For every homogeneous element*

$$T = \sum_\alpha a_\alpha(\Gamma^g(q)\mathcal{M}_\alpha) \in \mathcal{H}^g(q)$$

with $\mu(T) = \mu$ and every character χ of Ω trivial on $\Gamma^g(q)$, the Hecke operator

$$\|_kT : F \mapsto F\|_kT = \sum_\alpha a_\alpha F|_k\mathcal{M}_\alpha$$

on the space $\mathfrak{M}_k(\Gamma^g(q))$, *where* $|_k\mathcal{M}$ *are the Petersson operators* (1.12), *maps the subspace* $\mathfrak{M}_k(\Omega,\,\chi)$ *into the subspace* $\mathfrak{M}_k(\Omega,\,\chi|\mu)$, *where*

$$(\chi|\mu)(\mathcal{M}) = \chi(\overline{\mathcal{M}}|\mu) \qquad (\mathcal{M}\in\Omega). \tag{3.13}$$

(3) *If in the notation and assumptions of Proposition* 11 *the character* χ *has the form* $\chi = \widehat{V}$, *then*

$$\chi|\mu = \widehat{V}|\mu = \widehat{V|\mu} \quad \text{with } V|\mu = \omega^g(\widetilde{\mu})V\omega_g(\mu), \tag{3.14}$$

An assertion similar to part (2) for the operators $\|_kT(\mu)$ was cited without proof in [SM-T(93), Lemma 3.1].

Proof. For brevity we set $\Gamma^g(q) = K$. If we take a matrix $\mathcal{N}\in R(K) = \Sigma^g(q)$ with $\mu(\mathcal{N}) = \mu$ and $\mathcal{M}\in\Omega$, then $\mathcal{N}\mathcal{M}\in R(K)\Omega = \Omega R(K)$, whence $\mathcal{N}\mathcal{M} = \mathcal{M}'\mathcal{N}'$ with $\mathcal{M}'\in\Omega$ and $\mathcal{N}'\in R(K)$. Passing to congruence modulo q, we get

$$\omega^g(\mu)\mathcal{M}\equiv\mathcal{M}'\omega^g(\mu) \pmod q \qquad\Longrightarrow\qquad \bar{\mathcal{M}}' = \overline{\mathcal{M}}|\mu.$$

It follows that the image of map (3.12) is contained in $\overline{\Omega} = \Gamma^n(q)\backslash S$. The rest of assertion (1) is clear.

In order to prove (2), it is sufficient to consider an element T of the form (1.20), i.e. $T = (\mathcal{N})_K$ with $\mathcal{N}\in\Sigma^g(q)$ and $\mu(\mathcal{N}) = \mu$. If $(\mathcal{N})_K = \sum_j(K\mathcal{N}_j)$ with $\mathcal{N}_j\in R(K)$, then, by [An(87), Theorem 3.3.3(4)], we get decomposition $(\mathcal{N})_\Omega = \sum_j(\Omega\mathcal{N}_j)$. It follows that for each $\mathcal{M}\in\Omega$ we have $\mathcal{N}_j\mathcal{M} = \mathcal{M}'_j\mathcal{N}_{j'}$, where $j\to j'$ is a permutation. Passing as above to congruences modulo q, we conclude that $\overline{\mathcal{N}}'_j = \overline{\mathcal{M}}|\mu$, where $\mu = \mu(N)$. Hence, for $F\in\mathfrak{M}_k(\Omega,\,\chi)$ we we obtain

$$F\|_k(\mathcal{N})_K\mathcal{M} = \sum_j F|_k\mathcal{N}_j\mathcal{M} = \sum_j F|_k\mathcal{M}'_j|_k\mathcal{N}_{j'} = \chi(\overline{\mathcal{M}}|\mu)F\|_k(\mathcal{N})_K.$$

Finally, the map $\chi\mapsto\chi|\mu$ defined by (3.12) and (3.13), clearly, replaces a character $\widehat{V}$ of the form (3.8) by the character $\widehat{V}|T$, given by

$$\begin{aligned}(\widehat{V}|\mu)(\mathcal{M}) &= \widehat{V}(\omega^g(\mu)\mathcal{M}\omega^g(\widetilde{\mu}) \\ &= \mathbf{e}\left\{d^{-1}J\omega^g(\mu)\varrho(\mathcal{M})\omega^g(\widetilde{\mu})V\right\}) = \mathbf{e}\left\{d^{-1}J\varrho(\mathcal{M})\omega^g(\widetilde{\mu})V\omega_g(\mu)\right\}.\triangle\end{aligned}$$

Now, let us turn to the characters $\boldsymbol{\chi}_M$ of the form (1.16). According to Theorem 1, if d is an even common denominator of the entries of M, then $\boldsymbol{\chi}_M$ is a character of the group $\Gamma^g(d)$ trivial on the subgroup $\Gamma^g(2d^2)$. If $g>1$ the factor group $\Gamma^g(2d^2)\backslash\Gamma^g(d)$ is non-Abelian ([SM-T(93)], Proposition 11), and so we can not directly apply Proposition . However, the two consecutive factor groups of the sequence

$$\Gamma^g(2d^2)\subset\Gamma^g(2d)\subset\Gamma^g(d)$$

are Abelian, which will allow us to apply Proposition 11 in two steps.

Proposition 13. *Let $\boldsymbol{\chi}_M$ and $\boldsymbol{\chi}_N$ be two characters of the group $\Gamma^g(d)$ of the form* (1.16), *where $M, N \in \frac{1}{d}\mathbb{Z}^r_{2g}$ with even r and d.*

If the matrices M and N satisfy the condition

$$ {}^tMM - {}^tNN \in \frac{1}{d}\mathbb{E}^{2g}, \tag{3.15}$$

then the corresponding characters are equal,

$$\boldsymbol{\chi}_M(\mathcal{M}) = \boldsymbol{\chi}_N(\mathcal{M}) \quad \forall \mathcal{M} \in \Gamma^g(d).$$

Conversely, if the characters are equal and $d \equiv 0 \pmod 4$, then the matrices M and N satisty the condition (3.15).

Proof. We start with preliminary remarks on the matrices $\mathcal{M}$ and $S(\mathcal{M})$. First of all, we note that for $\mathcal{M} = 1_{2g} + \mathcal{M}' = 1_{2g} + \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma^g(d)$ we can write

$$S(\mathcal{M}) = \begin{pmatrix} B & 1_g - A \\ D - 1_g & -C \end{pmatrix} - \begin{pmatrix} (A-1_g)\,{}^tB & (A-1_g)\,{}^t(D-1_g) \\ C\,{}^tB & C\,{}^t(D-1_g) \end{pmatrix}$$

$$= \begin{pmatrix} B' & -A' \\ D' & -C' \end{pmatrix} - \begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} = \mathcal{M}'J - \begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} \tag{3.16}$$

where the first matrix is divisible by d and, by (3.3), is symmetric modulo d^2, and the second matrix is divisible by d^2.

It follows from (1.16) that

$$\boldsymbol{\chi}_M(\mathcal{M})/\boldsymbol{\chi}_N(\mathcal{M}) = \mathbf{e}\left\{S(\mathcal{M})L\right\}, \tag{3.17}$$

where $L = {}^tNN - {}^tMM$.

Now, if the condition (3.15) is fulfilled, that is the matrix dL is even, then the matrix

$$d^2 \begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} L = \frac{1}{d} \begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} dL$$

is integral and divisible by 2. Hence, the quotient (3.17) with $S(\mathcal{M})$ taken in the form (3.16) is equal to $\mathbf{e}\{\mathcal{M}'JL\}$. Since the matrix $\mathcal{M}'J$ is symmetric modulo d^2, the integral matrix $\frac{1}{d}\mathcal{M}'J$ is symmetric modulo 2. Hence, the trace of the matrix $\frac{1}{d}\mathcal{M}'JdL$ is an even integer and the quotient (3.17) is equal to 1.

Conversely, if the quotient is equal to 1 for all $\mathcal{M} \in \Gamma^g(d)$, it is true for all $\mathcal{M} \in \Gamma^g(2d)$. But if $\mathcal{M} \in \Gamma^g(2d)$, the matrix $\mathcal{M}'$ is divisible by $2d$. Hence the matrix $\begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} L$ is integral and divisible by 4, and so the quotient (3.17) for $\mathcal{M} \in \Gamma^g(2d)$ is equal to $\mathbf{e}\{\mathcal{M}'JL\} = \mathbf{e}\{\frac{1}{d}(\mathcal{M}'/2)J/(2d^2L)\}$ with an even matrix $2d^2L$. By applying Proposition 11 to the quotient group $\Gamma^g(q)\backslash\Gamma^g(q')$ with $q = 2d^2$ and $q' = 2d$, we conclude that the character of the group given by

$$\mathcal{M} \mapsto \mathbf{e}\{\frac{1}{d}(\mathcal{M}'/2)J(2d^2L)\} = \mathbf{e}\{\frac{1}{d}\{J\rho(\mathcal{M})J(2d^2L)J^{-1}\}$$

can be trivial only if the matrix $J(2d^2L)J^{-1}$ is contained in $d\mathbb{E}^{2g}$, that is $L \in \frac{1}{2d}\mathbb{E}^{2g}$, since clearly $J^{-1}\mathbb{E}^{2g}J = {}^tJ\mathbb{E}^{2g}J = \mathbb{E}^{2g}$. Now, for $\mathcal{M} \in \Gamma^g(d)$, the matrix

$$\begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} L = \frac{1}{2d}\begin{pmatrix} A'\,{}^tB' & A'\,{}^tD' \\ C'\,{}^tB' & C'\,{}^tD' \end{pmatrix} 2dL$$

is integral and divisible by 2, since we have supposed that d is divisible by 4. It follows that the quotient (3.19), for $\mathcal{M} \in \Gamma^g(d)$, is equal to $\mathbf{e}\{\mathcal{M}'JL\} = \mathbf{e}\{\frac{1}{2}(\mathcal{M}'/d)J2dL\}$ with an even matrix $2dL$. By applying again Proposition 11 to the quotient group $\Gamma^g(q)\backslash\Gamma^g(q')$ this time with $q = 2d$ and $q' = d$, we similarly conclude that the quotient (3.19) can be trivial only if $2dL \in 2\mathbb{E}^g$. △

It looks likely that the condition $d \equiv 0 \pmod 4$ in the second part of the proposition can be omitted, but we could not prove it.

Finally, we shall consider relations of the characters $\boldsymbol{\chi}_M$ with Hecke operators.

Proposition 14. *Let $M = (M', M'') \in \frac{1}{d}\mathbb{Z}^r_{2g}$ with even d and $r = 2k$, and let $T \in \mathcal{H}^g(2d^2)$ be an homogeneous element with $\mu(T) = \mu$. Then the Hecke operator $\|_k T$ on the space $\mathfrak{M}_k(\Gamma^g(2d^2))$ maps the subspace $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M)$ into the subspace $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M|\mu)$ with the character $\boldsymbol{\chi}_M|\mu$ of the form*

$$(\boldsymbol{\chi}_M|\mu)(\mathcal{M}) = \boldsymbol{\chi}_r(\mathcal{M})\mathbf{e}\{S(\mathcal{M})\omega_g(\widetilde{\mu})\,{}^tMM\omega^g(\mu)\} \qquad (\mathcal{M} \in \Gamma^g(d))\,, \tag{3.18}$$

where $\widetilde{\mu}$ is an inverse of μ modulo $2d^2$.

The conditions

$$(\mu - 1)\,{}^tM'M',\ (\widetilde{\mu} - 1)\,{}^tM''M'' \in \frac{1}{d}\mathbb{E}^g \tag{3.19}$$

are sufficient for the equality $\boldsymbol{\chi}_M|\mu = \boldsymbol{\chi}_M$ of the characters; if d is divisible by 4, then the conditions (3.19) *are also necessary for the equality.*

Proof. By Proposition 12, we have

$$(\boldsymbol{\chi}_M|\mu)(\mathcal{M}) = \boldsymbol{\chi}_r(\mathcal{M})\mathbf{e}\{S(\omega^g(\mu)\mathcal{M}\omega^g(\widetilde{\mu}))\,{}^tMM\}.$$

If $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then, by (3.12), $\omega^g(\mu)\mathcal{M}\omega^g(\widetilde{\mu}) = \begin{pmatrix} A & \mu B \\ \widetilde{\mu}C & D \end{pmatrix}$, whence, by (1.17), we obtain

$$(3.20)\quad S(\omega^g(\mu)\mathcal{M}\omega^g(\widetilde{\mu})) = \begin{pmatrix} \mu(B + {}^tB - A\,{}^tB) & D - 1_g - C\,{}^tB \\ D - D\,{}^tA & -\widetilde{\mu}C\,{}^tD \end{pmatrix} = \omega^g(\mu)S(\mathcal{M})\omega_g(\widetilde{\mu}).$$

The formula (3.18) follows.

By (3.18) and Proposition 13, the condition

$$\omega^g(\mu)\,{}^tMM\omega_g(\widetilde{\mu}) - {}^tMM = \begin{pmatrix} (\mu-1)\,{}^tM'M' & (\mu\widetilde{\mu}-1)\,{}^tM'M'' \\ 0 & (\widetilde{\mu}-1)\,{}^tM''M'' \end{pmatrix} \in \frac{1}{d}\mathbb{E}^{2g}.$$

is sufficient for equality of the characters, and it is also necessary for the equality if d is divisible by 4. The proposition follows. △

Proposition 14 shows that in search of nonzero eigenfunctions for all regular Hecke operators on spaces $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M)$ one can restrict oneself, at least when d is divisible by 4, to the case of characters $\boldsymbol{\chi}_M$ with matrices M satisfying the conditions (3.19) for all μ coprime to d. Further reduction is related to an action of the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ of the ring $\mathbb{Z}/q\mathbb{Z}$ on spaces of modular forms for $\Gamma^g(q)$. Let us set

$$(3.21)\quad D^g(q) = \left\{ d(a) \in \Gamma^g \;\middle|\; d(a) \equiv \begin{pmatrix} a1_g & 0 \\ 0 & a^{-1}1_g \end{pmatrix} \pmod{q}, \quad \gcd(a,q) = 1 \right\}.$$

$D^g(q)$ is, clearly, a subgroup of Γ^g containing $\Gamma^g(q)$ as a normal subgroup, and the factor group $D^g(q)/\Gamma^g(q)$ is isomorphic to the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$. We shall denote by $\bar{d}(a) = d(\bar{a})$ the class of $d(a) \in D^g(q)$ in the factor group. It depends only on the class $\bar{a}$ of a modulo q. Operators $\|_k\bar{d}(a) = \|_k d(a)$ define an multiplicative action of the factor group on modular forms of weight k for the group $\Gamma^g(q)$.

Proposition 15. *Let $M = (M', M'') \in \frac{1}{d}\mathbb{Z}^r_{2g}$, where d and $r = 2k$ are even, be a matrix satisfying the conditions* (3.19) *for all μ coprime to d. Then each operator $\|_k\bar{d}(a) = \|_k d(a)$ with $d(a) \in D^g(2d^2)$ on the space $\mathfrak{M}_k(\Gamma^g(2d^2))$ maps the subspace $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M)$ into itself.*

Proof. If $F \in \mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi})$, then, by (1.13) and (1.14),

$$F\|_k d(a)\|_k d(a^{-1})\mathcal{M}d(a) = \boldsymbol{\chi}(\mathcal{M})F\|_k d(a) \qquad (\mathcal{M} \in \Gamma^g(d)),$$

hence $F\|_k d(a) \in \mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}')$ with $\boldsymbol{\chi}'(\mathcal{M}) = \boldsymbol{\chi}(d(a)\mathcal{M}d(a^{-1}))$. Applying it to the character $\boldsymbol{\chi} = \boldsymbol{\chi}_M$, since, for $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$\mathcal{M}' = d(a)\begin{pmatrix} A & B \\ C & D \end{pmatrix} d(a^{-1}) \equiv \begin{pmatrix} A & a^2 B \\ a^{-2}C & D \end{pmatrix} \pmod{2d^2},$$

we get, by Proposition 12, the relation

$$\begin{aligned}\boldsymbol{\chi}'_M(\mathcal{M}) &= \boldsymbol{\chi}_M(\mathcal{M}') = \boldsymbol{\chi}_r(\mathcal{M}')\mathbf{e}\{S(\mathcal{M}')\,{}^tMM\} \\ &= \boldsymbol{\chi}_r(\mathcal{M})\mathbf{e}\{S(\omega^g(a^2)\mathcal{M}\omega^g(\widetilde{a^2}))\,{}^tMM\},\end{aligned}$$

where $\widetilde{a}$ is an inverse of a modulo $2d^2$. By (3.20) and (3.19), the last expression is equal to $\boldsymbol{\chi}_r(\mathcal{M})\mathbf{e}\{S(\mathcal{M})\omega_g(\widetilde{a^2})\,{}^tMM\omega^g(a^2)\} = \boldsymbol{\chi}_M(\mathcal{M})$. △

The proposition implies that the group $(\mathbb{Z}/2d^2\mathbb{Z})^*$ acts by operators $\|_k\bar{d}(a)$ with $d(a) \in D^g(2d^2)$ on every space $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M)$, where M satisfies assumptions of the lemma. It follows that each such space is the direct sum

$$\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M) = \bigoplus_{\psi \in Char(\mathbb{Z}/2d^2\mathbb{Z})^*} \mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M; \psi), \tag{3.23}$$

where ψ ranges through the group of characters of the group $(\mathbb{Z}/2d^2\mathbb{Z})^*$, of the subspaces

$$\begin{aligned}&\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M; \psi) \\ &\quad= \{F \in \mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M) \mid F\|_k d(a) = \psi(a)F \quad (d(a) \in D^g(2d^2))\},\end{aligned} \tag{3.24}$$

and the map

$$\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M) \ni F \mapsto \sum_{a \in (\mathbb{Z}/2d^2\mathbb{Z})^*} \overline{\psi(a)} F\|_k d(a) \tag{3.25}$$

is projection of the space on the subspace $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M; \psi)$.

Proposition 16. *Each operator $\|_k d(a)$ with $d(a) \in D^g(2d^2)$ on the space $\mathfrak{M}_k(\Gamma^g(2d^2))$ commutes with each of the regular Hecke operators for $\Gamma^g(2d^2)$.*

Proof. It is sufficient to consider Hecke operators corresponding to elements $T = (\mathcal{M})_\Omega$ of the form (1.20) with $\mathcal{M} \in \Sigma^g(q)$, where $q = 2d^2$ and $\Omega = \Gamma^g(q)$. We note that the matrix $D(a) = ad(a)$ belongs to $\Sigma^g(q)$ and satisfies $\Omega D(a)\Omega = \Omega D(a)$. Since the ring $\mathcal{H}^g(q)$ is commutative, we have $(\mathcal{M})_\Omega(D(a))_\Omega = (D(a))_\Omega(\mathcal{M})_\Omega$. It follows that $\|_k T\|_k D(a)_\Omega = \|_k D(a)_\Omega\|_k T$. Since, clearly, $\|_k D(a)_\Omega = a^{-kg}\|_d(a)$, the proposition follows. △

The above propositions imply the following theorem.

Theorem 17. *For every matrix* $M = (M', M'') \in \frac{1}{d}\mathbb{Z}^r_{2g}$*, where* d *and* $r = 2k$ *are even, satisfying the conditions* (3.19) *for all* μ *coprime to* d*, and every character* ψ *of the group* $(\mathbb{Z}/2d^2\mathbb{Z})^*$*, the subspace* $\mathfrak{M}_k(\Gamma^g(d), \boldsymbol{\chi}_M; \psi) \subset \mathfrak{M}_k(\Gamma^g(2d^2))$ *is invariant with respect to all regular Hecke operators for the group* $\Gamma^g(2d^2)$*.*

The following lemma allows us to relate theta products to decompositions (3.23).

Lemma 18. *Let* $M = (M', M'') \in \frac{1}{d}\mathbb{Z}^r_{2g}$*, where* d *and* $r = 2k$ *are even, be a matrix satisfying the conditions* (3.20)*. Then*

$$\boldsymbol{\theta}(Z,\, M)\|_k d(b) = \chi_2^{kg}(b)\boldsymbol{\theta}(Z,\, (bM', \widetilde{b}M'')) \quad (\gcd(b, d) = 1), \tag{3.26}$$

where $\widetilde{b}$ *is an inverse of* b *modulo* $2d^2$*.*

Proof. By (1.7), we have

$$\boldsymbol{\theta}(Z,\, M)\|_k d(b) = \boldsymbol{\chi}_r(d(b))\delta(M)\overline{\delta(Md(b))}\boldsymbol{\theta}(Z,\, Md(b)).$$

By (1.9), we can write $\boldsymbol{\chi}_r(d(b)) = \chi_2(b)^{-kg} = \chi_2^{kg}(b)$. Since

$$Md(b) = (M', M'')\left(\begin{pmatrix} b1_g & 0 \\ 0 & \widetilde{b}1_g \end{pmatrix} + 2d^2N\right)$$

with an integral matrix N, from (1.10) we easily obtain equalities $\delta(Md(b)) = \delta(bM', \widetilde{b}M'') = \delta(M)$, and from (1.6) the relation $\boldsymbol{\theta}(Z,\, Md(b)) = \boldsymbol{\theta}(Z,\, (bM', \widetilde{b}M''))$. The relations (3.26) follows. $\triangle$

It follows from the lemma that, for each character $\psi \in Char(\mathbb{Z}/2d^2\mathbb{Z})^*$, the projection (3.25) of the theta product $\boldsymbol{\theta}(Z,\, M)$ can be written in the form

$$\boldsymbol{\theta}(Z,\, M;\, \psi) = \sum_{a \in (\mathbb{Z}/2d^2\mathbb{Z})^*} \overline{\psi(a)}\chi_2^{kg}(a)\boldsymbol{\theta}(Z,\, (aM', \widetilde{a}M'')). \tag{3.27}$$

§4. Representations of congruence similarity groups and Hecke eigenfunctions

According to Theorem 2, the image of a theta product $\boldsymbol{\theta}(Z,\, M)$ under the action of Hecke operator $\|_k T$ corresponding to regular homogeneous $T \in \mathcal{H}^g(2d^2)$ with multiplier $\mu(T) = \mu$ is a linear combination of theta products $\boldsymbol{\theta}(Z,\, DM\omega_g(\widetilde{\mu}))$ with D contained in the set $\mathbf{S}^+_r(\mu)$ of proper integral similarities of $\mathbf{q}_r$ with multiplier μ. Unions of the similarity sets with multipliers prime to an integer d,

$$\mathcal{S}^+_r[d] = \bigcup_{\mu \in \mathbb{N}, \gcd(\mu, d)=1} \mathbf{S}^+_r(\mu), \tag{4.1}$$

is a multiplicative semigroup, and it makes sense to consider action of the semigroups in appropriate abstract setting. For fixed d, g, $r \in \mathbb{N}$, let us denote by $\mathcal{V}(d) = \mathcal{V}^r_{2g}(d)$ the $\mathbb{C}$-linear space of all complex-valued function F on the set $\frac{1}{d}\mathbb{Z}^r_{2g}$ satisfying for every $M = (M', M'') \in (\frac{1}{d}\mathbb{Z}^r_g, \frac{1}{d}\mathbb{Z}^r_g)$ the conditions

$$F(M + S) = \mathbf{e}\{2\,{}^tS''M'\}F(M) \quad \text{for each} \quad S = (S', S'') \in (\mathbb{Z}^r_g, \mathbb{Z}^r_g), \tag{4.2}$$

and by $\mathcal{F}(d) = \mathcal{F}^r_{2g}(d)$ the subspace of all functions of $F \in \mathcal{V}(d)$ satisfying the conditions

$$F(UM) = F(M) \quad \text{for each} \quad U \in \mathbf{E}^+_r = \left\{U \in SL_r(\mathbb{Z}) \;\middle|\; {}^tUU = 1_r\right\}. \tag{4.3}$$

The conditions (4.2) imply that each function F of $\mathcal{V}(d)$ is uniquelly determined by its values on the finite set of all $r \times 2g$-matrices $M = (m_{ij})$ with entries of the form $m_{ij} = a_{ij}/d$, where a_{ij} are integral numbers satisfying $0 \le a_{ij} < d$. Therefore, the space $\mathcal{F}(d)$ is finite-dimentional. According to relations (1.5) and (1.6), the theta products $\boldsymbol{\theta}(Z, M)$ with fixed $Z \in \mathbb{H}^g$ and characteristic matrices $M \in \frac{1}{d}\mathbb{Z}^r_{2g}$ can be considered as elements of the subspace $\mathcal{F}^r_{2g}(d)$. For a matrix $D \in \mathcal{S}^+_r[d]$, we define the operator $\circ D$ on functions $F : \frac{1}{d}\mathbb{Z}^r_{2g} \mapsto \mathbb{C}$ by

$$\circ D : F \mapsto (F \circ D)(M) = F(DM\omega_g(\widetilde{\mu})) \tag{4.4}$$

where $\widetilde{\mu}$ is an integral inverse of μ modulo d^2.

Lemma 19. *If $F \in \mathcal{V}(d) = \mathcal{V}^r_{2g}(d)$ and $D \in \mathcal{S}^+_r[d]$, then the function $F \circ D$ depends only on D modulo d^2 and again belongs to $\mathcal{V}(d)$.*

Proof. If $D_1 \in \mathcal{S}^+_r[d]$, and $D_1 \equiv D \pmod{d^2}$, then clearly $\mu_1 = \mu(D_1) \equiv \mu = \mu(D)$ $\pmod{d^2}$ and $\widetilde{\mu}_1 \equiv \widetilde{\mu} \pmod{d^2}$. It follows that matrix $D_1M\omega_g(\widetilde{\mu}_1) - DM\omega_g(\widetilde{\mu})$ integral and divisible by d, hence, by (4.2), $(F \circ D_1)(M) = (F \circ D)(M)$. If $S = (S', S'') \in (\mathbb{Z}^r_g, \mathbb{Z}^r_g)$, then we have

$$\begin{aligned}(F \circ D)(M + S) &= F((DM', \widetilde{\mu}DM'') + (DS', \widetilde{\mu}DS''))\\ &= \mathbf{e}\{2\widetilde{\mu}\,{}^tS''\,{}^tDDM'\}F((DM', \widetilde{\mu}DM'')) = \mathbf{e}\{2\,{}^tS''M'\}(F \circ D)(M). \quad \triangle\end{aligned}$$

From the lemma we conclude that each operator $\circ D$ on the space $\mathcal{V}(d)$ does not depend on the choice of the inverse $\widetilde{\mu}$, maps the space into itself, and that the mapping $D \mapsto \circ D$ defines a linear representation of the semigroup $\mathcal{S}^+_r[d]$, on the space $\mathcal{V}(d)$. Moreover, the subspace $\mathcal{F}(d) \subset \mathcal{V}(d)$ can be characterized as the subspace of all $\mathbf{E}^+_r$-invariant functions of $\mathcal{V}(d)$. Therefore, according to the general scheme of definition of Hecke operators, we can define the standard linear representation of the Hecke–Shimura ring

$$\mathcal{L}_d^r = \mathcal{H}(\mathbf{E}_r^+, \mathcal{S}_r^+[d])$$

of the semigroup $\mathcal{S}_r^+[d]$ relative to the subgroup $\mathbf{E}_r^+$ (over $\mathbb{C}$) on the space $\mathcal{F}(d)$ by means of Hecke operators:

$$t = \sum_\alpha a_\alpha(\mathbf{E}_r D_\alpha) \in \mathcal{L}_d^r : F(M) \mapsto (F \circ t)(M) = \sum_\alpha a_\alpha (F \circ D_\alpha)(M). \tag{4.5}$$

Coming back to formulas of Theorem 2, we can now rewrite the right hand side of formula (1.25) in the terms of Hecke operators (4.5). Note, first of all, that, by Lemma 19, the inverse $\widetilde{\mu}$ of μ modulo $2d^2$ in Theorem 2 can be relaced by arbitrary inverse of μ modulo d^2. Since, by (1.28), the coefficients $c(D,T)$ on the right-hand side of formula (1.25) depend only on the double coset $\mathbf{E}_r^+ D \mathbf{E}_r^+$, one can write

$$\sum_{D \in \mathbf{E}_r^+ \backslash \mathbf{S}_r^+(\mu)} c(D,T)\boldsymbol{\theta}(Z,\, DM\omega_g(\widetilde{\mu})) = \sum_{D \in \mathbf{E}_r^+ \backslash \mathbf{S}_r^+(\mu)/\mathbf{E}_r^+} c(D,T) \sum_{D_\alpha \in \mathbf{E}_r^+ \backslash \mathbf{E}_r^+ D \mathbf{E}_r^+} \boldsymbol{\theta}(Z,\, M) \circ D_\alpha,$$

where operators $\circ D_\alpha$ on the right affect only argument M of the theta product. In the terms of Hecke operators (4.5) one can write

$$\sum_{D_\alpha \in \mathbf{E}_r^+ \backslash \mathbf{E}_r^+ D \mathbf{E}_r^+} \boldsymbol{\theta}(Z,\, M) \circ D_\alpha = \boldsymbol{\theta}(Z,\, M) \circ (D), \tag{4.6}$$

where

$$(D) = (D)_{\mathbf{E}^+} = \sum_{D_\alpha \in \mathbf{E}^+ \backslash \mathbf{E}^+ D \mathbf{E}^+} (\mathbf{E}^+ D_\alpha) \in \mathcal{L}_d^r$$

with $\mathbf{E}^+ = \mathbf{E}_r^+$ are elements of the form (1.20) corresponding to double cosets $\mathbf{E}^+ D \mathbf{E}^+$. Therefore, the formula (1.25) takes the shape

$$\boldsymbol{\theta}(Z,\, M)\|_k \mathbf{T} = \sum_{D \in \mathbf{E}_r^+ \backslash \mathbf{S}_r^+(\mu)/\mathbf{E}_r^+} c(D,\, T)\boldsymbol{\theta}(Z,\, M) \circ (D). \tag{4.7}$$

Note that Hecke operators in two parts of this relation affects different arguments of the theta product.

All matrices of the semigroup $\mathcal{S}_r^+[d]$ considered modulo the equivalence relation $D' \equiv D \pmod q$ with a fixed $q \in \mathbb{N}$ dividing a power of d form, obviously, finite *congruence similarity group* $\mathcal{S}^+_r[d]/(q)$ which can be considered as the homomorphic image of $\mathcal{S}^+_r[d]$ under the map $D \mapsto D$ modulo q. In this case the group $\mathbf{E}^+_r$ goes to the factor group

$$\mathbf{E}^+_r/(q) = \mathbf{E}_r/\left\{D \in \mathbf{E}^+_r \;\middle|\; D \equiv 1_r \pmod{q}\right\}. \tag{4.8}$$

Let $\mathcal{L}^r_d/(q) = \mathcal{H}(\mathbf{E}^+_r/(q), \mathcal{S}^+_r[d]/(q))$ be the HS–ring of pair $\mathbf{E}^+_r/(q)$, $\mathcal{S}^+_r[d]/(q)$. Since the semigroup $\mathcal{S}^+_r[d]/(q)$ is finite, the ring $\mathcal{L}^r_d/(q)$ is a finite-dimensional $\mathbb{C}$–algebra. The map $D \mapsto D$ modulo q defines, clearly, an epimorphism of the rings

$$/(q) : \mathcal{L}^r_d \mapsto \mathcal{L}^r_d/(q). \tag{4.9}$$

Since, by Lemma 19, the operators (4.4) on spaces $\mathcal{V}(d)$ depend only on D modulo d^2, each of the Hecke operators (4.6) depends in fact only on the image $(D)/(d^2)$ of the element (D) under the map (4.9) with $q = d^2$:

$$F \circ (D) = F \circ (D)/(d^2) \quad \forall F \in \mathcal{V}(d);$$

in particular,

$$\boldsymbol{\theta}(Z, M) \circ (D) = \boldsymbol{\theta}(Z, M) \circ (D)/(d^2). \tag{4.10}$$

The formulas (4.7) and (4.10) show that the theta product $\boldsymbol{\theta}(Z, M)$ is an eigenfunction of the Hecke operator $\|_k T$ if the function $M \mapsto F(M) = \boldsymbol{\theta}(Z, M)$ is an eigenfunction for every Hecke operator $\circ(D)/(d^2)$ with $(D)/(d^2) \in \mathcal{S}^+_r[d]/(d^2)$. Returning for a moment to the situation of a general HS–ring $\mathcal{H} = \mathcal{H}(\Omega, \Delta)$ of a semigroup Δ relative a subgroup Ω, we note that the left multiplication of elements of the space $\mathcal{C}$ consisting of all finite formal linear combinations (over $\mathbb{C}$) of left cosets of Δ modulo Ω by elements of $\mathcal{H}$ given by

$$\sum_\alpha a_\alpha(\Omega M_\alpha) \sum_\beta b_\beta(\Omega N_\beta) = \sum_{\alpha,\beta} a_\alpha b_\beta(\Omega M_\alpha N_\beta)$$
$$\Big(\sum_\alpha a_\alpha(\Omega M_\alpha) \in \mathcal{H}, \sum_\beta b_\beta(\Omega N_\beta) \in \mathcal{C}\Big)$$

is independent of the choice of representatives $M_\alpha \in \Omega M_\alpha$, $N_\beta \in \Omega N_\beta$, satisfies $(TT')L = T(T'L)$ for all $T, T' \in \mathcal{H}$ and $L \in \mathcal{C}$, and so defines a representation of the ring $\mathcal{H}$ on the space $\mathcal{C}$. In the case of the ring $\mathcal{H} = \mathcal{L}^r_d/(d^2)$ the corresponding space of left cosets

$$\left\{\sum_\beta b_\beta(\mathbf{E}^+_r/(d^2) \cdot D_\beta/(d^2)) \;\middle|\; b_\beta \in \mathbb{C}\right\}$$

is finite-dimensional, and so we get a finite-dimensional representation of the ring $\mathcal{L}^r_d/(d^2)$.

Lemma 20. *Let $\eta(D)$ be a function on the semigroup $\mathcal{S}_r^+[d]$ depending only on the left coset $\mathbf{E}_r^+/(d^2)\cdot D/(d^2)$ of D in $\mathcal{S}_r^+[d]/(d^2)$. Suppose that the linear combination of left cosets*

$$\sigma(\eta) = \sum_{D/(d^2)\in\mathbf{E}_r^+/(d^2)\backslash\mathcal{S}_r^+[d]/(d^2)} \overline{\eta(D)}(\mathbf{E}_r^+/(d^2)\cdot D_\beta/(d^2)), \tag{4.11}$$

where $\overline{\eta(D)}$ is the complex conjugate of $\eta(D)$, is an eigenfunction with respect to all left multiplications by elements of $\mathcal{L}_d^r/(d^2)$, in particular,

$$(D/(d^2))\sigma(\eta) = \lambda(D;\,\eta)\sigma(\eta).$$

Then in the assumptions and notation of Theorem 2 the linear combination of theta products

$$\boldsymbol{\theta}(Z,\,M;\,\eta) = \boldsymbol{\theta}(Z,\,M)\circ\sigma(\eta) = \sum_{D/(d^2)\in\mathbf{E}_r^+/(d^2)\backslash\mathcal{S}_r^+[d]/(d^2)} \eta(D)\boldsymbol{\theta}(Z,\,M)\circ D \tag{4.12}$$

is an eigenfunction for the Hecke operator $\|_kT$ with the eigenvalue

$$\lambda(T;\,\eta) = \sum_{D\in\mathbf{E}_r^+\backslash\mathbf{S}_r^+(\mu)/\mathbf{E}_r^+} c(D,\,T)\lambda(D;\,\eta). \tag{4.13}$$

Proof. By (4.12) and (4.7), we have

$$\begin{aligned}\boldsymbol{\theta}(Z,\,M;\,\eta)\|_kT & \sum_{D\in\mathbf{E}_r^+\backslash\mathbf{S}_r^+(\mu)/\mathbf{E}_r^+} c(D,\,T)\boldsymbol{\theta}(Z,\,M)\circ(D)\circ\sigma(\eta) \\ &= \sum_{D\in\mathbf{E}_r^+\backslash\mathbf{S}_r^+(\mu)/\mathbf{E}_r^+} c(D,\,T)\lambda(D;\,\eta)\boldsymbol{\theta}(Z,\,M)\circ\sigma(\eta)\end{aligned}$$

which proves the lemma. △

In order to illustrate applications of the above techniques, we consider in more detais the simplest case when $r=2$.

The following statements on proper similarities of the quadratic form $\mathbf{q}_2 = x_1^2+x_2^2$ are easy and direct consequences of definitions:

(i). $\mathcal{E} = \mathbf{E}_2^+ = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\}$;

(ii). $D = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathbf{S}_2^+(\mu) \leftrightarrow a, b\in\mathbb{Z},\ a^2+b^2=\mu,\ (c,d)=(-b,a)$;

(iii). The mapping

$$\mathcal{S} = \mathcal{S}_2^+[d] \ni D = \begin{pmatrix} a & b \\ * & * \end{pmatrix} \mapsto \xi(D) = a + \sqrt{-1}b \in \mathcal{O} = \mathbb{Z}[\sqrt{-1}]$$

is an isomorphism of the group $\mathcal{S}$ onto the subsemigroup $\mathcal{O}_{[d]}$ of Gauss integers coprime to d;

(iv). Each double coset $\mathcal{E}D\mathcal{E} \subset \mathcal{S}$ consists of a single left coset; if p is a prime number satisfying $p \equiv 1 \pmod 4$, then the set $\mathbf{S}_2^+(p)$ is the union of two different cosets $\mathcal{E}D(p)\mathcal{E} = \mathcal{E}D(p)$ and $\mathcal{E}D'(p)\mathcal{E} = \mathcal{E}D'(p)$, where

$$D(p) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad D'(p) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ with } a, b \in \mathbb{Z},\ a^2 + b^2 = p, \tag{4.14}$$

but if p is a prime number satisfying $p \equiv 3 \pmod 4$, then the set $\mathbf{S}_2^+(p)$ is empty;

(v). The mapping

$$(\mathcal{E}D) \mapsto \xi(\mathcal{E}D) = \xi(D)\mathcal{O}_{[d]},$$

extended by linearity on the HS–ring $\mathcal{L} = \mathcal{L}_d^2 = \mathcal{H}(\mathcal{E}, \mathcal{S})$, is an isomorphism of the ring onto the semigroup ring $\mathbb{C}[I(\mathcal{O}_{[d]})]$ of the semigroup $I(\mathcal{O}_{[d]}) = \mathcal{O}_{[d]}^*/\{\pm 1, \pm\sqrt{-1}\}$ of all nonzero ideals of $\mathcal{O}_{[d]}$.

(vi). The mapping $(\mathcal{E}D)/(q) \mapsto \xi(D)\mathcal{O}_d/q\mathcal{O}_{[d]}$, where $q \in \mathbb{N}$ divide a power of d, defines an isomorphism of the ring $\mathcal{L}/(q)$ with the ring of formal linear combinations of ideal classes of $\mathcal{O}_{[d]}$ modulo the equivalence relation $\alpha\mathcal{O}_{[d]} \sim \beta\mathcal{O}_{[d]} \pmod q$ if $\alpha \equiv \epsilon\beta \pmod q$ with a unit $\epsilon = \pm 1, \pm\sqrt{-1}$. The ideal classes form a finite Abelian group, the class group $H(I(\mathcal{O}_{[d]}/q))$, and the ring of formal linear combinations of the classes is just the group algebra of the class group (over $\mathbb{C}$). From these observations and Lemma 20 we obtain the following proposition.

Proposition 21. *Let η be a character of the ideal class group $H(I(\mathcal{O}_{[d]}/d^2))$ extended on $\mathcal{S}$. Then the linear combination* (4.11) *satisfies the condition of Lemma* 20 *with*

$$\lambda(D; \eta) = \eta(D) \quad (D \in \mathcal{S}),$$

and, in the assumptions and notation of Theorem 2*, the linear combination $\boldsymbol{\theta}(Z, M; \eta)$ of theta products of the form* (4.12) *is an eigenfunction of the Hecke operator $\|_1 T$ with the eigenvalue*

$$\lambda(T; \eta) = \sum_{(D) \in H(\mathcal{O}_{[d]}/d^2)} c(D, T)\eta(D). \tag{4.15}$$

Now, we are able to compute the Euler products (2.14) corresponding to the eigenfunction $\boldsymbol{\theta}(Z, M; \eta)$ for $g = 1$ and $g = 2$.

Proposition 22. *Let $F = \boldsymbol{\theta}(Z, M; \eta) \in \mathfrak{M}_1(\Gamma^1(2d^2))$ be a linear combination of theta products of the form* (4.12) *with $r = 2$ and $g = 1$, where η is choosen as indicated in Proposition* 21*. Then, in the assumptions of Theorem* 2*, F is an eigenfunction for all Hecke operators corresponding to elements of the HS–ring $\mathcal{H}^1(2d^2)$; Hecke zeta function of F has the form*

$$Z(F, s) = \prod_{\text{prime } p\nmid d} \mathbf{Q}_p(F, p^{-s})^{-1},$$

where

$$Q_p(F, v) = \begin{cases} (1-\eta(D(p))v)(1-\eta(D'(p))v), & \text{if } p \equiv 1 \pmod 4 \\ 1-\eta(p1_2)v^2, & \text{if } p \equiv 3 \pmod 4), \end{cases} \tag{4.16}$$

and $D(p)$ and $D'(p)$ are the matrices (4.14)*.*

Proof. It follows from Proposition 21 and Lemma 20 that F is an eigenfunction of all Hecke operators $\|_1 T$ for homogeneous $T \in \mathcal{H}^1(2d^2)$ with the eigenvalues

$$\lambda(T; \eta) = \sum_{D \in \mathbf{E}_2^+ \backslash \mathbf{S}_2^+(\mu(T))/\mathbf{E}_2^+} c(D, T)\eta(D). \tag{4.17}$$

For $T = \mathbf{T}^1(p) = \mathbf{q}_1^1(p)$, by (2.15),Proposition 6, and (iv), the formula (4.17) implies the relatons

$$\lambda(\mathbf{q}_1^1(p), \eta) = \begin{cases} \eta(D(p)) + \eta(D'(p)), & \text{if } p \equiv 1 \pmod 4 \\ 0, & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

Similarly, by (4.16), (2.15), and Proposition 8, we obtain

$$\lambda(\mathbf{q}_2^1(p), \eta) = p\lambda(\mathbf{T}_1^1(p^2)) = \chi_2(p)\eta(p1_2).$$

Since $\chi_2(p) = (-1)^{\frac{p-1}{2}}$, and

$$\eta(D(p))\eta(D'(p)) = \eta(D(p)D'(p)) = \eta(p1_2),$$

the formulas (4.16) follow. △

Proposition 23. *Let $G = \boldsymbol{\theta}(Z, M; \eta) \in \mathfrak{M}_1(\Gamma^2(2d^2))$ be a linear combination of theta products of the form* (4.12) *with $r = 2$ and $g = 2$, where η is choosen as indicated in Proposition* 22*. Then, in assumptions and notation of Theorem* 2*, G is* *an eigenfunction for all Hecke operators corresponding to elements of the HS–ring $\mathcal{H}^2(2d^2)$; Andrianov zeta function of G has the form*

$$Z(G,\,s)=\prod_{\text{prime } p\nmid d}\mathbf{Q}_p(G,\,p^{-s})^{-1},$$

where

$$(4.18)\quad Q_p(G,\,v)=\\ \begin{cases}(1-\eta(D(p))v)(1-\eta(D'(p))v)(1-\eta(D(p))pv)(1-\eta(D'(p))pv), \text{ if } p\equiv 1 \pmod 4,\\ (1-\eta(p1_2)v^2)(1-\eta(p1_2)p^2v^2), \text{ if } p\equiv -1 \pmod 4,\end{cases}$$

and $D(p)$ and $D'(p)$ are the matrices (4.14).

Proof. It again follows from Proposition 21 and Lemma 20 that F is an eigenfunction of all Hecke operators $\|_1T$ for homogeneous $T\in\mathcal{H}^2(2d^2)$ with the eigenvalues

$$\lambda(T;\,\eta)=\sum_{D\in\mathbf{E}_2^+\backslash\mathbf{S}_2^+(\mu(T))/\mathbf{E}_2^+}c(D,\,T)\eta(D). \tag{4.19}$$

For $T=\mathbf{T}^2(p)=\mathbf{q}_1^2(p)$, by (2.15),Proposition 6, and (iv), the formula (4.18) implies the relatons

$$\lambda(\mathbf{q}_1^2(p),\,\eta)=\begin{cases}(1+p)(\eta(D(p))+\eta(D'(p))), & \text{if } p\equiv 1 \pmod 4\\ 0, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

By (2.16), (2.29), and (2.27), we have

$$c(D,\,\mathbf{q}_2^2(p))=\begin{cases}\chi_2(p)p, & \text{if } D\in\mathbf{S}_2^+(p^2)\bigcap\Lambda^2D_{0,1}^2(p)\Lambda^2,\\ \chi_2(p)p^2+(\chi_2(p)+1)p+\chi_2(p), & \text{if } D\in\mathbf{E}_2^+(p1_2),\\ 0, & \text{otherwise,}\end{cases}$$

where $\Lambda^2=GL_2(\mathbb{Z})$. It is easy to see that the intersection $\mathbf{S}_2^+(p^2)\bigcap\Lambda^2D_{0,1}^2(p)\Lambda^2$ is the union of two different double cosets $\mathbf{E}_2^+D(p)^2\mathbf{E}_2^+$ and $\mathbf{E}_2^+D'(p)^2\mathbf{E}_2^+$, if $\chi_2(p)=1$, i.e. $p\equiv 1 \pmod 4$, and is empty, otherwise. Therefore, we obtain

$$\lambda(\mathbf{q}_2^2(p),\,\eta)=p\eta(D(p)^2)+p\eta(D'(p)^2)+(p+1)^2\eta(p1_2),$$

if $p\equiv 1 \pmod 4$, and

$$\lambda(\mathbf{q}_2^2(p))=-(p^2+1)\eta(p1_2),$$

if $p \equiv 3 \pmod 4$. Further, by (2.15), (2.24), and (2.27), we get

$$\lambda(\mathbf{q}_3^2(p), \eta) = \lambda(p^3\mathbf{T}_2^2(p)\mathbf{q}_1^2(p), \eta) \\ = p^3\lambda(\mathbf{T}_2^2(p^2), \eta)\lambda(\mathbf{q}_1^2(p), \eta) = p\eta(p1_2)\lambda(\mathbf{q}_1^2(p), \eta).$$

Finally, by (2.15) and (2.27), we conclude

$$\lambda(\mathbf{q}_4^2(p), \eta) = \lambda(p^3\mathbf{T}_2^2(p^2), \eta)^2 = p^2\eta(p^21_2).$$

The formulas (4.20) follow. △

Note that the zeta functions of eigenfunctions F and G considered in last two propositions for the same character η satisfy the relation

$$Z(G, s) = Z(F, s)Z(F, s-1).$$

The next and even more interesting case is, of course, the case $r = 4$ and $g = 2$, but to consider it one has first to sink into the depth of arithmetic of quaternions.

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