A DIFFERENTIAL MODEL OF GROWTH OF YOUNG DIAGRAMS

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1. Introduction

The starting point of the present paper is the result on the asymptotics of random growth of Young diagrams obtained in [1], [9]. The subject of these articles was a discrete time Markov process with the set of Young diagrams as a state space. At each stage of the process a new random box is attached to the current diagram. The definition of transition probabilities was motivated by asymptotic problems of representation theory of symmetric groups, see [4]. Taking into account the origins of the process in representation theory, we call it the *Plancherel Growth Process.*

In the process of growth, the area of a diagram increases to infinity. If a diagram is contracted uniformly, so that its area is normalized to unity, the edge of the diagram looks more and more like a continuous curve. The essential part of the result discovered and proved in [1], [9] is as follows: in the course of Plancherel Growth Process almost all of Young diagrams become uniformly close (assuming the area is normalized) to a common universal curve. In a natural coordinate system, the equation of the curve is

$$\tilde{\Omega}(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2 \end{cases} \tag{1}$$

(tilde distinguishes the curve (1) from a similar curve introduced in [1]). See Fig. 1 for its graph, along with a sample Young diagram of one hundred boxes obtained in a computer simulation of the Plancherel Growth Process.

During past fifteen years the appearance of the curve (1) seemed to be a specific isolated result. Nowadays the situation has changed: it is known that this very curve arises naturally in various problems of one-dimensional mathematical physics and function theory (see [5], [6], [7]). It was shown in [6] that the curve (1) describes the common asymptotics of interlacing roots for a wide class of orthogonal polynomials (including the classical polynomials of Tchebychef, Hermite and Laguerre). The same curve also determines the typical character of mutual separation of frequencies, when a linear constraint is imposed upon a linear mechanical system (see [6]). We establish in the present paper some new characterizations of the curve (1), relating the two contexts where it had been identified: in the Plancherel Growth Process of random Young diagrams, and in one-dimensional problems of mathematical physics.

The main idea is to replace the initial discrete time random growth process by a continuous time deterministic process with the same asymptotical behavior in the limit $t \to \infty$.

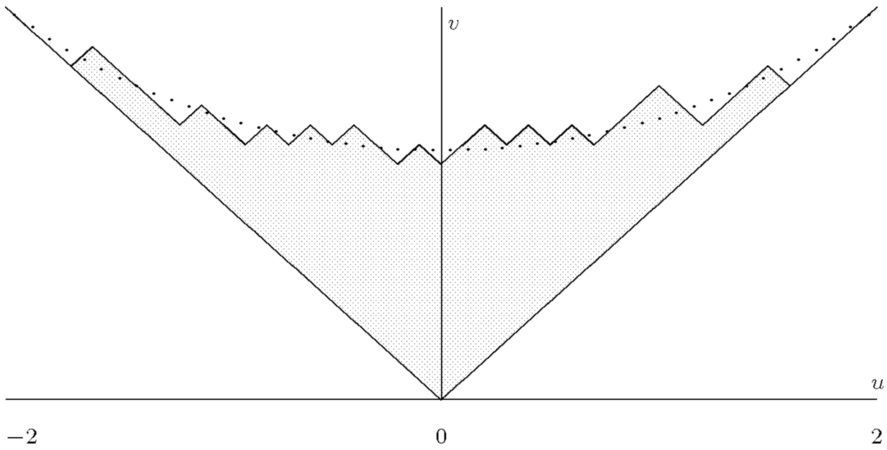

Fig. 1

The elimination of randomness is achieved, roughly speaking, as follows. Assume that in a particular state of Plancherel Growth Process a new box can be attached to a Young diagram at the locations $x_1, \dots, x_n$ with probabilities $\mu_1, \dots, \mu_n$. Then, instead of attaching randomly an entire box to one of these locations, we would like to attach simultaneously a fraction of a box (proportional to μ_k) at each of the locations x_k.

In order to implement the idea, we extend considerably the state space, using Young diagrams and the limiting curves of type (1) on equal grounds. See Section 3.1 for the precise definition of 'continuous diagrams' which we use as the unified concept. The extended space of continuous diagrams $\mathcal{D}$ is an infinite dimensional compact set, canonically homeomorphic to the space of probability distributions on an interval.

The correspondence, associating the probability distribution μ_ω to a diagram $\omega \in \mathcal{D}$, is by no means trivial. It was used, in a less explicit form, in connection with the Markov Moment Problem [8] (cf. [7]). In the present setup the map $\omega \mapsto \mu_\omega$ arises very naturally: if ω is a Young diagram, then μ_ω is its transition distribution in the Plancherel Growth Process. By continuity, the correspondence $\omega \mapsto \mu_\omega$ can be extended (see [5]) to general diagrams $\omega \in \mathcal{D}$. There is yet another interpretation of the correspondence $\omega \mapsto \mu_\omega$ in terms of partial fraction decompositions.

The history of growth of a general type diagram (analogous to Young tableau) can be described by a curve $\omega(\cdot, t)$; $1 < t < \infty$, in the space $\mathcal{D}$. It is understood that the diagram $\omega(\cdot, t)$ increases monotonically (in the ordering by the inclusion of subgraphs) as far as t increases. The infinitely thin layer between the graphs of diagrams $\omega(u, t_0)$ and $\omega(u, t_0 + dt)$ determines a one-dimensional distribution with the density $C\,\omega_t'(u, t_0)$ (where C is the normalization constant). One can think of it as of a tangent vector to the tableau $\omega(\cdot, t)$ at the point $t = t_0$. By equating the tangent and the transition distributions we derive the basic dynamic equation

$$d\mu_{\omega(\cdot,t)}(u) = C\,\omega_t'(u, t)\,dt. \tag{2}$$

A self-explanatory version of the equation (2) is the formula (4.2.3) below.

The equation (2) forces the diagram $\omega(\cdot, t)$ to grow in the direction of its transition distribution. Hence, one can expect that the asymptotics of solutions of the equation (2) is related to that of random Plancherel growth. Our main result (Theorem 4.5) confirms these expectations completely: the diagram $\tilde{\Omega}$ in (1) is, up to the change of scale, the only fixed point of the equation (2), and all other solutions tend to this point asymptotically.

In the course of our analysis we rewrite the equation (2) in several equivalent forms. Surprisingly, it appears that one of the equations, equivalent to (2) is the celebrated quasi-linear equation of the first order

$$R'_t + RR'_x = 0, \tag{3}$$

which describes the free motion of one-dimensional medium of non-interacting particles. This is one of the simplest equations implying non-linear phenomena, shock waves in particular. In terms of the equation (3), the solution corresponding to the curve (1) is the automodel solution $R(x,t) = r(x/\sqrt{t})/\sqrt{t}$, where

$$r(x) = \frac{1}{2}(x - \sqrt{x^2 - 4}). \tag{4}$$

It follows from the main Theorem that if the initial wave front $R(x, 1)$ is analytic at infinity then, in the course of evolution according to the equation (3), its shape (in some neighborhood of the point $x = \infty$) tends, as $t \to \infty$, to that of automodel solution (4).

The plan of the paper is as follows.

In Section 2 we describe the stochastic growth model of Young diagrams, the Plancherel Process, and recall the theorem on the limiting shape of large diagrams. Also, we briefly recall there the connection with the character theory of symmetric groups.

In Section 3 we replace the combinatorial objects, Young diagrams, by function - theoretical ones: continuous diagrams. For most general diagrams we describe transition distributions, generalizing the transition probabilities of the Plancherel Growth Markov process.

In Section 4, the main results of the paper are collected. The continuous analogue of Young tableau is defined, and the equation controlling the growth of continuous diagrams is introduced. We prove three theorems characterizing the diagram $\tilde{\Omega}$. It is shown in Theorem 4.3 that this diagram provides the unique automodel solution of the basic equation. According to Theorem 4.5, this solution is stable, and asymptotically attracts all other solutions. The Theorem 4.4 claims that $\tilde{\Omega}$ is the only diagram, up to the change of scale, for which the transition distribution coincides with the radial one (defined in Section 4.4 using simple geometric considerations).

The author is happy to thank A. M. Vershik for numerous helpful discussions on the subject of these notes, which we had since our first paper [1]. I also thank G. V. Rosenblum, who had explained to me the meaning of the equation (3).

2. The Plancherel growth of Young diagrams

We describe in this Section a probabilistic model of growth of Young diagrams (the Plancherel Growth Process) and recall the theorem on the asymptotic shape of large Young diagrams. Also, we remark on the relationship between the Plancherel Growth Process, and representation theory of finite symmetric groups and of the infinite group of finite permutations.

2.1 Young diagrams. By definition, a *partition* of a natural number N is any presentation of that number as a sum of weakly decreasing natural terms:

$$N = N_1 + N_2 + \dots, \quad N_1 \geq N_2 \geq \dots.$$

For a given number N the number of partitions $p(N)$ is finite, but it grows rapidly with N: by Hardy – Ramanujan Theorem, $p(N) = CN^{-1} \exp 2\pi\sqrt{N/6}$.

Young diagrams are in a one-to-one correspondence with partitions, and can be considered as visual geometric representation thereof.

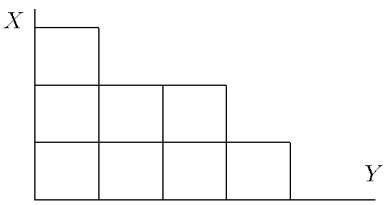

Fig. 2

Consider a quadrant $\{(X, Y) \in \mathbb{R}^+ \times \mathbb{R}^+\}$ in the plane, split by the net of straight lines $\{X = i\}_{i \in \mathbb{Z}^+}$, $\{Y = j\}_{j \in \mathbb{Z}^+}$ into rectangles, called *boxes.* We specify the position of a box by the coordinates (X, Y) of its central point. By definition, a Young diagram is a finite collection of boxes which contains with its every box (X, Y) all the boxes (X', Y') for which $X' < X$ and $Y' < Y$. The boxes of a diagram with the same first coordinate compose a *line.* The number of boxes in a line is called its *length.* The line lengths in a Young diagram are the terms of the corresponding partition of the total number of boxes N in this diagram. For instance, the Young diagram in Fig. 2 represents the partition $8 = 4 + 3 + 1$. Manifestly, the correspondence between the partitions and Young diagrams is bijective. Let us denote the set of Young diagrams with N boxes by $\mathcal{Y}_N$.

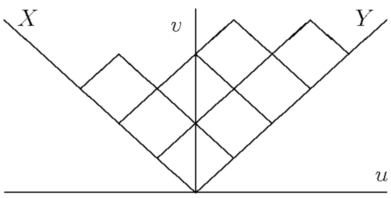

Fig. 3

In what follows, it will be convenient to draw Young diagrams rotated by 45° (see Fig. 3). Hence, the border of a diagram can be considered as a graph of a piecewise linear function $v = \lambda(u)$. More precisely, let us change the variables as follows:

$$u = Y - X, \quad v = Y + X. \tag{2.1.1}$$

In the new coordinates the border lines of boxes are determined by the lines $v+u = 2i$, $i \in \mathbb{Z}^+$ and $v - u = 2j$, $j \in \mathbb{Z}^+$, intersecting at the points (u, v) with integer coordinates of the same parity. The boxes of a diagram form the region

$$D_\lambda = \{(u, v) \colon |u| \le v < \lambda(u)\} \tag{2.1.2}$$

(we shall call it the *subgraph* of a diagram). The number of boxes can be written as an integral

$$N = \frac{1}{2} \int (\lambda(u) - |u|)\, du \tag{2.1.3}$$

(here and below the integrals with no explicitly indicated limits are taken over the whole real axis).

Summarizing the preceding discussion, let us emphasize the non-standard viewpoint on Young diagrams we admit: we identify Young diagrams with continuous piecewise linear functions $v = \lambda(u)$, for which

(i) $\lambda'(u) = \pm 1$,
(ii) $\lambda(u) = |u|$ for sufficiently large $|u|$,
(iii) the local extrema occur in integer points only.

Denote by $x_1, x_2, \ldots, x_n$ the consecutive minima points, and by $y_1, y_2, \ldots, y_{n-1}$ the interlacing maxima points. Then the pair of sequences

$$x_1 < y_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < x_n \tag{2.1.4}$$

determines a Young diagram uniquely. In particular, the total number of boxes is equal to

$$N = \sum_{1 \le i < j \le n} (y_i - x_i)(x_j - y_{j-1}), \tag{2.1.5}$$

and the length of the first row is x_n. The multiplicity of this row is $(x_n - y_{n-1})$. More generally, the Young diagram with the parameters (2.1.4) contains $(x_{k+1} - y_k)$ rows of length $\sum_{1 \le i \le k} (y_i - x_i)$, where $k = 1, 2, \ldots, n-1$.

The integer parameters $x = \{x_k\}_{k=1}^n$, $y = \{y_k\}_{k=1}^{n-1}$ are restricted by the inequalities (2.1.4), and by the relation

$$\sum_{k=1}^{n} x_k - \sum_{k=1}^{n-1} y_k = 0.$$

We call pairs of sequences x, y with these conditions *admissible* pairs. Any admissible pair determines uniquely a Young diagram with the corresponding extrema.

2.2 The Plancherel Growth Process. Let $\mathcal{Y}_N$ denote the set of Young diagrams with N boxes, and $\mathcal{Y}$ – the set of all Young diagrams (including the diagram $\lambda(u) \equiv |u|$ with empty subgraph). We endow $\mathcal{Y}$ with a partial order, comparing diagrams $\lambda, \Lambda \in \mathcal{Y}$ by inclusion of subgraphs (2.1.2): $\lambda \prec \Lambda$ iff $D_\lambda \subset D_\Lambda$. Let us write $\lambda \nearrow \Lambda$, if a diagram Λ is an immediate successor of a diagram λ; that means that the region D_Λ is obtained from D_λ by attaching exactly one new box. The abscissa $x = x(\lambda, \Lambda)$ of the center of this new box is necessarily one of the minima $x_1, \dots, x_n$ of the Young diagram λ. In the opposite direction, a Young diagram λ can grow to the shape of its immediate successor Λ by attaching a box with abscissa x_k, for any minimum x_k of the diagram λ.

The oriented graph with the vertex set $\mathcal{Y}$ and with the edges (λ, Λ), where $\lambda \nearrow \Lambda$, is called the *Young graph.* A *Young tableau* is an oriented path

$$t = (\lambda_0, \lambda_1, \dots, \lambda_N) \tag{2.2.1}$$

in the Young graph, starting with the empty diagram $\lambda_0 = \emptyset \in \mathcal{Y}_0$. Thus, a Young tableau records the history of growth of a Young diagram. Denote by $\mathcal{T}(\lambda)$ the set of Young tableaux with the last diagram $\lambda_N = \lambda$ (they say that the tableau has *shape* λ). We will also need the set of all infinite Young tableaux

$$\mathcal{T} = \{t = (\lambda_0, \lambda_1, \dots, \lambda_N, \dots)\}, \tag{2.2.2}$$

endowed with the topology of weak (coordinatewise) convergence. The space of tableau $\mathcal{T}$ is compact and totally disconnected.

Let us define a Markov chain on the Young graph, which we call here the *Plancherel Growth Process.*

The possible states of the process are Young diagrams $\lambda \in \mathcal{Y}$, and the admissible transitions are determined by the edges (λ, Λ) of Young graph. Thus, at any stage of the process, a new random box is attached to the current Young diagram.

Let the current state be a Young diagram λ with the extrema at the points $x_1 < y_1 < x_2 < \dots < x_{n-1} < y_{n-1} < x_n$. By definition, the probability to attach a new box over the minimum point x_k is

$$\mu_k = \prod_{i=1}^{k-1} \frac{x_k - y_i}{x_k - x_i} \prod_{j=k+1}^{n} \frac{y_{j-1} - x_k}{x_j - x_k}, \tag{2.2.3}$$

where $k = 1, \dots, n$.

Remark that all the numbers (2.2.3) are positive, and sum up to unity. In order to prove this, consider polynomials $P(x) = \prod (x - x_i)$ and $Q(x) = \prod (x - y_i)$, and note that the decomposition of the quotient $Q(x)/P(x)$ into partial fractions has the form

$$\prod_{i=1}^{n-1} (x - y_i) \prod_{i=1}^{n} (x - x_i)^{-1} = \sum_{k=1}^{n} \frac{\mu_k}{x - x_k}. \tag{2.2.4}$$

Multiplying both parts by x, and taking the limit as $x \to \infty$, one can see that $1 = \sum \mu_k$.

In Section 2.4 below we will explain the origins of the Plancherel Growth Process and its role in the character theory of the group $\mathfrak{S}_\infty$ of finite permutations of the set of natural numbers. Let us now recall the basic facts on its asymptotics.

2.3 The typical shape of large Young diagrams. The Markov chain constructed in Section 2.2 determines the probability measure M in the space $\mathcal{T}$ of infinite Young tableaux. Following [1], [3], let us call M the *Plancherel measure* (for the infinite symmetric group $\mathfrak{S}_\infty$). We shall recall below the Law of Large Numbers obtained in [1], [9] (see also [3]) describing the limiting shape of large Young diagrams for typical, with respect to the measure M, Young tableaux.

In the course of growth process, the area of the subgraph of a Young diagram (equal to the number of boxes) increases. In order to discuss the limiting shape of a growing Young diagram, let us normalize the area of its subgraph to unity by contracting the region (2.1.2) $\sqrt{N}$ times along both axes. In other words, instead of the N-th diagram λ_N in a tableau $t = (\lambda_0, \lambda_1, \dots, \lambda_N, \dots) \in \mathcal{T}$ we consider the contracted diagram

$$\hat{\lambda}_N(u) = \lambda_N(u\sqrt{N})/\sqrt{N}. \tag{2.3.1}$$

In fact, after renormalizing the area, the diagrams of almost all tableaux converge uniformly to a unique universal function, namely the function

$$\tilde{\Omega}(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2 \end{cases} \tag{2.3.2}$$

(this function is the same as the one used in [1], except for the scaling parameters).

Theorem [1]. *For almost all, with respect to the Plancherel measure M, Young tableaux*

$$t = (\lambda_0, \lambda_1, \dots, \lambda_N, \dots) \in \mathcal{T}$$

the limit

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \lambda_N(u\sqrt{N}) = \tilde{\Omega}(u) \tag{2.3.3}$$

does exist uniformly in $u \in \mathbb{R}$.

Instead of presenting the detailed proof (see [3]) we shall only recall that the function $\tilde{\Omega}$ arises here as a unique minimum (in fact, equal to zero) of the *hook integral*

$$\theta(\lambda) = 1 + \frac{1}{2} \iint_{u<v} \ln(v-u)\, d(u+\lambda(u))\, d(v-\lambda(v)) \tag{2.3.4}$$

under the constraint

$$A(\lambda) = \frac{1}{2} \iint_{u<v} d(u+\lambda(u))\, d(v-\lambda(v)) = 1. \tag{2.3.5}$$

2.4 Central growth and the characters of the group $\mathfrak{S}_\infty$. Let us explain the relationship of the Plancherel Growth Process to the character theory of symmetric groups.

First, the measure M is central in the following sense. We attach to a finite Young tableau $s = (\lambda_0, \lambda_2, \dots, \lambda_N)$ a cylinder set

$$C_s = \{t \in \mathcal{T} \colon N\text{-head}(t) = s\}$$

of infinite tableaux such that the first $N+1$ diagrams are the same as in s. By definition, a probability measure M on the space $\mathcal{T}$ is called *central*, if for any two Young tableaux $s, s' \in \mathcal{T}(\lambda)$ of the same shape λ the cylinders $C_s, C_{s'}$ have the same probabilities:

$$M(C_s) = M(C_{s'}) \quad \text{for all } s, s' \in \mathcal{T}(\lambda),\ \lambda \in \mathcal{Y}. \tag{2.4.1}$$

A central measure is completely determined by its one-dimensional distributions

$$\varphi_N(\lambda) = M\{t = (\lambda_0, \dots, \lambda_N, \dots) \in \mathcal{T} \colon \lambda_N = \lambda\}, \quad \lambda \in \mathcal{Y} \tag{2.4.2}$$

on the levels $\mathcal{Y}_N$ of the Young lattice.

Let us check that the Plancherel measure is central. To this end consider a quadruple $\lambda, \Lambda_1, \Lambda_2, \nu$ for which $\lambda \nearrow \Lambda_i \nearrow \nu$, $i = 1, 2$ (which means that the diagram ν is obtained from λ by attaching two boxes in two different orders). It suffices to prove that the Plancherel probabilities of transitions from λ to ν do not depend on the order in which the boxes were attached. Denote by x_k, x_m the abscissas of boxes to be attached, then it follows from the definition (2.2.3) of transition probabilities that both transitions $\lambda \nearrow \Lambda_i \nearrow \nu$ have the same probabilities

$$P = \frac{(x_m - x_k)^2}{(x_m - x_k)^2 - 1} \mu_k \mu_m.$$

It is well known that Young diagrams with N boxes index irreducible representations of the symmetric group $\mathfrak{S}_N$ of degree N. The dimension of representation π_λ associated to a diagram $\lambda \in \mathcal{Y}_N$, equals the number $\dim \lambda = |\mathcal{T}(\lambda)|$ of Young tableaux of shape λ. By the Burnside' formula, the sum of squares of dimensions of all irreducible representations of a finite group is equal to its order. In case of the symmetric group $\mathfrak{S}_N$ we obtain the formula

$$\sum_{\lambda \in \mathcal{Y}_N} \dim^2 \lambda = N! \tag{2.4.3}$$

The probability distribution with the weights $\varphi_N(\lambda) = \dim^2 \lambda / N!$, $\lambda \in \mathcal{Y}_N$ is called the Plancherel measure of finite symmetric group. One can check that this is exactly the one-dimensional distribution (2.4.2) of Plancherel Growth Process (which was our motivation for the choice of the term). The proof is based on the celebrated *hook formula* (see [10]). In our notations it can be written in the form

$$\dim \lambda = N! / \prod_{u<v} (v - u)^{q_\lambda(u) p_\lambda(v)}, \tag{2.4.4}$$

where $q_\lambda(u) = (1+\lambda'(u))/2$, $p_\lambda(v) = (1-\lambda'(v))/2$, and $u, v \in 1/2+\mathbb{Z}$ run through integer and semi-integer numbers.

The description of central probability measures on the space of Young tableaux $\mathcal{T}$ is equivalent to that of characters of the infinite symmetric group $\mathfrak{S}_\infty$. This problem had been completely solved in [2], [11]. We will indicate here only one particular property of the Plancherel measure.

Let us say that a sequence of Young diagrams $\lambda_N \in \mathcal{Y}_N$; $N = 1, 2, \dots$ *has zero frequencies*, if there is a numeric sequence $c_N = o(N)$ such that

$$\lambda_N(u) = |u| \quad \text{for } |u| \geq c_N.$$

For $\nu \in \mathcal{Y}_m$, we denote by $\dim_\nu(\lambda)$ the number of such Young tableaux $s = (\lambda_0, \lambda_1, \dots, \lambda_N = \lambda) \in \mathcal{T}(\lambda)$, that $\lambda_m = \nu$. The crucial fact in the classification of central measures for the Young lattice is the following

Lemma [2]. *Assume that a sequence* $\{\lambda_N\}_1^\infty$ *has zero frequencies. Then, for all diagrams* $\nu \in \mathcal{Y}_m$, $m = 1, 2, \dots$, *there exists the limit*

$$\lim_{N\to\infty} \frac{\dim_\nu(\lambda_N)}{\dim(\lambda_N)} = \frac{\dim^2 \nu}{m!}. \tag{2.4.5}$$

The limit (2.4.5) coincides with the distribution (2.4.2) of Plancherel Growth Process on the m-th level $\mathcal{Y}_m$ of Young lattice.

Informally speaking, the meaning of the Lemma is as follows. Consider two large natural numbers $1 \ll m \ll N$ and a Young diagram λ with N boxes inside a rectangle of size $c_N \times c_N$, $c_N/N \ll 1$. Choose a random tableau $s \in \mathcal{T}(\lambda)$ (with equal probabilities), and denote by $\nu = \lambda_m$ its m-th diagram. Then ν (after its area is normalized) will be close to the graph of the function (2.3.2), with no regards to the shape of the diagram λ (e.g. λ can have the shape of a triangle or that of a rectangle). This fact demonstrates the 'rigidity' of the function (2.3.2). In the Theorem 4.5 below we shall prove another result of this type.

3. Continuous diagrams and their transition probabilities

In this Section we extend the set of Young diagrams to obtain the space of continuous diagrams, and define the transition probabilities for these general diagrams. As a technical tool, we use the moments of diagrams and those of their transition distributions, and the corresponding generating functions.

3.1 Continuous diagrams. Let us complete the set $\mathcal{Y}$ of Young diagrams by considering rescaled Young diagrams $\omega_c(u) = c^{-1}\,\omega(cu)$ and uniform limits of the latter. It is fairly easy to describe the elements of the completion directly (we call them just *diagrams*).

Definition. A *diagram* is any function $\omega\colon \mathbb{R} \to \mathbb{R}$ such that

(i) $|\omega(u_1) - \omega(u_2)| \leq |u_1 - u_2|$,

(ii) $\omega(u) = |u - z|$ for some $z \in \mathbb{R}$ and sufficiently large $|u|$.

The number $z = z(\omega)$ is called the *center* of a diagram ω, and its *area* $A = A(\omega)$ is defined as the area of the region

$$D_\omega = \{(u, v)\colon |u| \leq v < \omega(u)\} \tag{3.1.1}$$

(the area of the square $\{|u| \le v < 2 - |u|\}$, i.e. one box of a true Young diagram, is taken as a unit). In what follows we are mostly interested in diagrams with zero center, $z = 0$.

A diagram ω will be called *rectangular*, if it is piecewise linear and $\omega'(u) = \pm 1$ for almost all (= all except finitely many) $u \in \mathbb{R}$. Rectangular diagrams are completely determined by their minima $x_1, \dots, x_n$ and the interlacing sequence of maxima $y_1, \dots, y_{n-1}$ (Fig. 4). We denote the set of such diagrams by $\mathcal{D}_0$. Young diagrams form a subset $\mathcal{Y} \subset \mathcal{D}_0$, and can be characterized by the following conditions:

1) the center is at zero, $z = 0$

2) all extrema are at integer points only.

For Young diagrams, the area equals the number of boxes.

By definition, a sequence of diagrams converges, $\omega_n \to \omega$, if $\omega_n(u) \to \omega(u)$ uniformly in $u \in \mathbb{R}$, and there exists such a $c > 0$ that $\omega_n(u) = |u|$ for $|u| > c$ and all $n = 1, 2, \dots$. The set $\mathcal{D}_0$ of rectangular diagrams is dense, in the corresponding topology, in the space $\mathcal{D}$ of all diagrams.

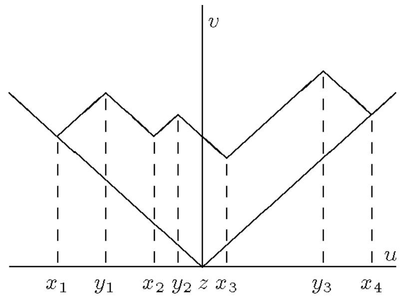

Fig. 4

3.2 The charge and the moments of a diagram. It is straightforward to check that the function $\sigma(u) = (\omega(u) - |u|)/2$ has the following properties:

$$|\sigma(u_1) - \sigma(u_2)| \le |u_1 - u_2|, \tag{3.2.1}$$

$$\sigma'(u) = \frac{1}{2}\begin{cases} \omega'(u) + 1 \ge 0 & \text{for } u < 0 \\ \omega'(u) - 1 \le 0 & \text{for } u > 0, \end{cases} \tag{3.2.2}$$

$$\max_u \sigma(u) = \omega(0)/2, \tag{3.2.3}$$

$$\sigma(-\infty) = z/2; \quad \sigma(\infty) = -z/2. \tag{3.2.4}$$

We call the function σ the *charge* of a diagram. The charge measures the discrepancy of a diagram from the modulus graph; its only purpose is to simplify notations. According to (3.2.1) and (3.2.2), the differential of the charge $d\sigma(u) = \sigma'(u)\, du$ is absolutely continuous, and its density $\sigma'(u)$ is positive for $u<0$ and negative for $u>0$. It follows from (3.2.3) and (3.2.4) that the total variation of the charge equals $|d\sigma| = \omega(0)$.

Let us define the *moments of a diagram* $\omega \in \mathcal{D}$ as

$$p_n = p_n(\omega) = -n \int_{\mathbb{R}} u^{n-1}\, d\sigma(u); \quad n = 1, 2, \ldots \tag{3.2.5}$$

The center and the area of a diagram have the following expressions in terms of moments:

$$z = p_1, \quad A = \frac{1}{2}(p_2 - p_1^2). \tag{3.2.6}$$

For z it follows from (3.2.4), and for A we use integration by parts:

$$A = \frac{1}{2}\int (\omega(u) - |u-z|)\, du = -\frac{1}{2}\int u\, d(\omega(u) - |u-z|) =$$
$$= -\int u\, d\sigma(u) - \frac{1}{2}\int u\, d(|u| - |u-z|) = \frac{p_2}{2} - \frac{z^2}{2}.$$

It will be convenient to take the function

$$S(x) = \sum_{n=1}^{\infty} \frac{p_n(\omega)}{n} x^{-n} = \int \frac{d\sigma(u)}{u-x}. \tag{3.2.7}$$

as the *moment generating function.* This function, as well as the sequence of moments $\{p_n(\omega)\}_{n=1}^{\infty}$, determine the charge σ and the diagram ω uniquely.

3.3 The moments of the diagram $\tilde{\Omega}$. The most important diagram considered in the present paper is the diagram

$$\tilde{\Omega}(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2. \end{cases}$$

This diagram has the center at zero and unit area.

Proposition. *The moments of the diagram $\tilde{\Omega}$ are equal to*

$$p_{2m-1} = 0, \quad p_{2m} = \binom{2m}{m}; \quad m = 1, 2, \ldots. \tag{3.3.1}$$

Proof. Let us write the definition of moments p_n in the form

$$p_n = -\int \sigma'(u)\, du^n.$$

Since the function $\tilde{\Omega}$ is even, all the odd moments vanish. Using the fact that $\tilde{\Omega}'(u) = (2/\pi)\arcsin(u/2)$ for $|u| \le 2$, we derive for an even $n = 2m$ that

$$p_{2m} = -2\int_{\mathbb{R}+} \sigma'(u)\, du^{2m} = \int_0^2 \left(1 - \frac{2}{\pi}\arcsin\frac{u}{2}\right) du^{2m}.$$

The substitution $u = 2\sin\varphi$ and integration by parts imply

$$p_{2m} = 2^{2m}\int_0^{\pi/2} (1 - 2\varphi/\pi)\, d\sin^{2m}\varphi = 2^{2m-1}\pi \int_0^{\pi/2} \sin^{2m}\varphi\, d\varphi =$$
$$= \frac{2^{2m}(2m-1)!!}{(2m)!!} = \frac{(2m)!}{m!\, m!},$$

and the Proposition follows. $\square$

Corollary. *The generating function (3.2.7) of the diagram $\tilde\Omega$ is*

$$S_{\tilde\Omega}(x) = \ln(x/2) + \ln(x - \sqrt{x^2-4}). \tag{3.3.2}$$

Proof. Using the binomial identity

$$(1-x^2)^{-1/2} = 1 + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!} x^{2m},$$

we infer that the derivative of the left hand side of (3.3.2) equals

$$S'_{\tilde\Omega}(x) = -\frac{1}{x}\sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!}\left(\frac{x}{2}\right)^{-2m} = \frac{1}{x} - \frac{1}{\sqrt{x^2-4}}.$$

Hence, it is identical with the derivative of the right hand side. One can easily check that the limits of both sides in (3.3.2), as $x \to \infty$, are both equal to zero. □

3.4 The transition distribution of a diagram. To every diagram $\omega \in \mathcal{D}$ we associate a probability distribution $\mu = \mu_\omega$ which we call the *transition distribution* of ω (cf. [5,6,7]). The measure μ can be characterized by the identity

$$\exp \int_{\mathbb{R}} \frac{d\sigma(u)}{u-x} = \int_{\mathbb{R}} \frac{d\mu(u)}{1-u/x}, \tag{3.4.1}$$

valid for sufficiently large x.

Consider the function

$$H(x) = \int \frac{d\mu(u)}{1-u/x} = \sum_{n=0}^{\infty} h_n(\mu) x^{-n} \tag{3.4.2}$$

as the generating function of moments

$$h_n = h_n(\mu) = \int u^n \, d\mu(u) \tag{3.4.3}$$

of the transition distribution μ. The characteristic identity (3.4.1) can be written in the form

$$\exp S(x) = \exp \sum_{n=1}^{\infty} \frac{p_n}{n} x^n = \sum_{n=0}^{\infty} h_n \, x^n = H(x). \tag{3.4.4}$$

It is well known (cf. [10]) that the coefficients in (3.4.4) are related by the identity

$$h_n = \sum_{\rho \vdash n} \prod_{k\ge 1} \left(\frac{p_k}{k}\right)^{\rho_k} \Big/ \rho_k!, \tag{3.4.5}$$

where ρ runs through the partitions $n = \rho_1 + 2\rho_2 + 3\rho_3 + \dots$ of a natural number n. Hence, if a diagram $\omega \in \mathcal{D}$ has the moment sequence $\{p_n\}_{n=1}^{\infty}$, then its transition measure can be defined as the unique distribution with compact support and with the moment sequence (3.4.5). The existence of the measure μ in this definition is by no means trivial. It relies on the analysis of the behavior of the function $H(x) = \exp S(x)$ in the complex domain, and an application of Riesz–Herglotz type theorems on the integral representations of resolvent functions (see [7,8]).

For a rectangular diagram $\omega \in \mathcal{D}_0$ the identity (3.4.1) takes the form

$$\frac{\prod(x-y_i)}{\prod(x-x_i)} = \sum_{k=1}^{n} \frac{\mu_k}{x-x_k},$$

where $x_1 < y_1 < x_2 < \ldots < y_{n-1} < x_n$ is the sequence of minima and maxima describing the diagram. In this case the transition measure has finite support, and its point masses μ_k at the points x_k coincide with the coefficients of partial fractions in the partial fraction decomposition of the rational function $\exp S(x)$. The coefficients can be written in the form

$$\mu_k = \prod_{i=1}^{k-1} \frac{x_k - y_i}{x_k - x_i} \prod_{i=k+1}^{n} \frac{x_k - y_{i-1}}{x_k - x_i}.$$

In case of a true Young diagram these are precisely the transition probabilities of the Plancherel measure (2.2.3).

Proposition. *For the semicircle distribution*

$$d\mu(u) = \frac{1}{2\pi}\sqrt{4-u^2}\,du, \quad |u| \le 2 \tag{3.4.6}$$

the moments have the form

$$h_{2m+1} = 0, \quad h_{2m} = \frac{1}{m+1}\binom{2m}{m}; \quad m = 0,1,2,\ldots. \tag{3.4.7}$$

The moment generating function of the semicircle distribution equals

$$H(x) = \frac{x}{2}\left(1 - \sqrt{1-(2/x)^2}\right), \quad x > 2. \tag{3.4.8}$$

Proof. Clearly, all odd moments vanish. The substitution $u = 2\sin\varphi$ implies

$$h_{2m} = \frac{1}{2\pi}\int_{-2}^{2} u^{2m}\sqrt{4-u^2}\,du = \frac{2^{2m+2}}{\pi}\int_0^{\pi/2} (\sin^{2m}\varphi - \sin^{2m+2}\varphi)\,d\varphi =$$
$$= \frac{2^{2m+2}}{\pi}\cdot\frac{\pi}{2}\left(\frac{(2m-1)!!}{(2m)!!} - \frac{(2m+1)!!}{(2m+2)!!}\right) = \frac{1}{m+1}\binom{2m}{m}.$$

It follows from the binomial identity that

$$\frac{1}{s}(1-\sqrt{1-s^2}) = \frac{s}{2}\sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!}\frac{s^{2m}}{m+1}.$$

Using the substitution $s = 2/x$ and the formula (3.4.7) which had been proved above, we derive

$$H(x) = \sum_{m=0}^{\infty} h_{2m}\, x^{-2m} = \frac{x^2}{2}\left(1 - \sqrt{1-(2/x)^2}\right). \quad \square$$

Corollary. *The transition measure of the diagram $\tilde{\Omega}$ coincides with the Semicircle Law (3.4.6):*

$$d\mu_{\tilde{\Omega}}(u) = \frac{1}{2\pi}\sqrt{4-u^2}\,du, \quad |u| \le 2. \tag{3.4.9}$$

Note that the numbers $c_m = \frac{1}{m+1}\binom{2m}{m}$, known as *Catalan numbers*, often arise in combinatorial problems.

In some cases it will be more convenient to use the function

$$R(x) = \sum_{n=0}^{\infty} h_n\, x^{-(n+1)} = \int \frac{d\mu(u)}{x-u}. \tag{3.4.10}$$

instead of the generating function (3.4.2). For the diagram $\tilde{\Omega}$ this function equals

$$R_{\tilde{\Omega}}(x) = \frac{1}{2}(x - \sqrt{x^2-4}). \tag{3.4.11}$$

4. The growth dynamics of continuous diagrams

In this Section we assemble the main results of the paper. We introduce and rewrite in several equivalent forms a differential equation describing the dynamics of diagrams. It is proved that the diagram $\tilde{\Omega}$ is the only fixed point, and that it is an attractor for all the solutions of the equation.

4.1 Continuous tableaux. General continuous diagrams, as well as true Young diagrams, can be ordered by the inclusion of subgraphs: $\omega_1 \prec \omega_2$ if and only if $D_{\omega_1} \subset D_{\omega_2}$. Let us call *tableau* every continuous map, with the values in $\mathcal{D}$, defined and strictly increasing in some interval $[t_0, \infty)$. In other words, a tableau is a family of diagrams $\omega(\cdot, t)$ increasing in t. All these diagrams have a common center $z(t) = z$; without loss of generality we assume that $z = 0$. The area $A(t)$ increases with t, and it will be convenient to take it as a parameter by assuming that $A(t) = t$.

Given a tableau $\omega(u,t)$, denote by $\sigma(u,t) = (\omega(u,t) - |u|)/2$ the corresponding family of charges. Taking into consideration the choice of parameter t and recalling the way we have normalized the area (see remark after (3.1.1)), we conclude that

$$\int \sigma(u,t)\,du = A(t) = t. \tag{4.1.1}$$

Definition. The function $T(u,t) = \sigma_t'(u,t)$ is called the *tangent density* of a tableau $\{\omega(\cdot,t)\}_{t\ge t_0}$ at the point $\omega(\cdot,t)$.

Taking the derivative in t we conclude that $T(u,t)$ is indeed the density of a probability distribution which we call *tangent distribution*:

$$\int T(u,t)\,du = A'(t) = 1. \tag{4.1.2}$$

Let us calculate the moments of the tangent distribution.

Lemma. *Let $p_n(t)$ be the moment of a diagram $\omega(u,t)$ in the tableau $\{\omega(\cdot,t)\}_{t\geq t_0}$. Then the moments of the tangent density can be written in the form*

$$\int_{\mathbb{R}} u^n T(u,t)\,du = \frac{p'_{n+2}(t)}{(n+1)(n+2)}. \tag{4.1.3}$$

Proof. By definition of the tangent density,

$$\int u^n\,T(u,t)\,du = \frac{d}{dt}\int u^n \sigma(u,t)\,du,$$

and the Lemma follows from the identities

$$\int u^n \sigma(u,t)\,du = -\int \frac{u^{n+1}}{n+1}\,d\sigma(u,t) = \frac{p_{n+2}(t)}{(n+1)(n+2)}. \quad \square$$

4.2 The basic dynamic equation. Consider a growing diagram, that is a tableau $\{\omega(\cdot,t)\}_{t\geq t_0}$, and let $\mu_t = \mu_{\omega(\cdot,t)}$ denote the transition measure of the diagram $\omega(u,t)$. We impose the following condition on the diagram $\omega(u,t)$: at every moment t it should grow in the direction of its transition distribution,

$$d\mu_t(u) = T(u,t)\,du. \tag{4.2.1}$$

The condition (4.2.1) represents an evolutionary equation in the infinite dimensional space of diagrams $\mathcal{D}$, where the correspondence $\omega \mapsto \mu_\omega$ determines the canonical vector field (more precisely, the field is defined on the subset of diagrams with an absolutely continuous transition measure).

Using the definition of the transition measure (3.4.1), one can write (4.2.1) in a closed form:

$$\int \frac{\sigma'_t(u,t)\,du}{1-u/x} = \exp\int \frac{\sigma'_u(u,t)\,du}{u-x} \qquad \text{for sufficiently large } x.$$

Let us give two more forms of the *basic dynamic equation* (4.2.1): one in terms of moments (3.2.5), and one in terms of the generating function

$$R(x,t) = \sum_{n=0}^{\infty} h_n(t)\,x^{-(n+1)} \tag{4.2.2}$$

(for a fixed t this function is almost identical with the generating function of (3.4.2): according to (3.4.10), $H(x,t) = xR(x,t)$).

Proposition. *The following forms of the basic dynamic equation are equivalent:*

$$\int \frac{\sigma'_t(u,t)\,du}{1-u/x} = \exp\int \frac{\sigma'_u(u,t)\,du}{u-x}, \tag{4.2.3}$$

$$\frac{p'_{n+2}(t)}{n+2} = (n+1)\,h_n(t); \quad n = 0,1,\dots, \tag{4.2.4}$$

$$R'_t(x,t) + R(x,t)\, R'_x(x,t) = 0. \tag{4.2.5}$$

Proof. The equivalence of (4.2.3) and (4.2.4) follows from Lemma 4.1. Consider the series

$$S(x,t) = \sum_{n \geq 0} \frac{p_{n+2}(t)}{n+2}\, x^{-(n+2)}.$$

Since $p_1(t) = z = 0$, we derive from (3.4.4) that $S(x,t) = \ln(xR(x,t))$ and

$$S'_t(x,t) = R^{-1}(x,t)\, R'_t(x,t).$$

Comparing the series

$$-R'_x(x,t) = \sum_{n=0}^{\infty} (n+1)\, h_n\, x^{-(n+2)},$$

$$S'_t(x,t) = \sum_{n=0}^{\infty} \frac{p'_{n+2}(t)}{n+2}\, x^{-(n+2)}$$

we conclude that (4.2.4) and (4.2.5) are equivalent, too. □

Note that (4.2.5) is one of the simplest examples of a quasilinear first order partial differential equation. It is commonly used as an illustration of non-linear phenomena, such as shock waves and breaking of wave crests. In one of a number of interpretations, the equation (4.2.5) describes the free motion of a medium of non-interacting particles. In this interpretation $R(x,t)$ denotes the velocity of a particle located at the point x at the time moment t.

4.3 Automodel solutions. For every diagram ω of unit area $A(\omega) = 1$ the formula

$$\omega(u,t) = \sqrt{t}\, \omega(u/\sqrt{t}), \quad t > 0 \tag{4.3.1}$$

determines a tableau. We shall refer to tableaux of type (4.3.1) as *automodel* tableaux. Let us show that the only automodel solution of the basic dynamic equation is generated by the diagram $\tilde{\Omega}$.

Theorem. *Let $\omega(u)$ denote a diagram of unit area $A(\omega) = 1$ and let $\omega(u,t) = \sqrt{t}\omega(u/\sqrt{t})$ be the corresponding automodel tableau. If its charge $\sigma(u,t)$ satisfies the equation (4.2.3), then $\omega(u) = \tilde{\Omega}(u)$.*

Proof. Consider the equation (4.2.3) in the equivalent form (4.2.5). One can easily check that the moments $p_n(t)$ of the diagram $\omega(u,t)$ can be written in terms of the moments p_n of the original diagram $\omega(u)$ as follows:

$$p_n(t) = t^{n/2} p_n.$$

Since the formulas (3.4.5) are homogeneous, we get the analogous relation for the moments of transition measures: $h_n(t) = t^{n/2} h_n$. It follows that the function

(4.2.2) equals $R(x,t) = R(x/\sqrt{t})/\sqrt{t}$. Using this expression in the equation (4.2.5) we obtain, after a simple calculation, the identity

$$(xR(x))' = \left(R^2(x)\right)'.$$

Hence, the functions $xR(x)$ and $R^2(x)$ differ by a constant. Comparing the constant terms we conclude that $R^2(x) - xR(x) + 1 = 0$. The only solution of this equation vanishing for $x \to +\infty$ coincides with the generating function (3.4.11) of the diagram $\tilde{\Omega}$. The Theorem is proved. $\square$

Informally speaking, we have shown that $\tilde{\Omega}(u)$ is the only diagram which remains self-similar (i. e. does not change its shape up to rescaling) in the course of growth in the direction of its own transition measure.

For automodel tableaux,

$$\omega'_t(u,t) = (\omega(u,t) - u\omega'_u(u,t))/2t. \tag{4.3.2}$$

The expression in the right hand side is determined entirely by the diagram $\omega(u) = \omega(u,t)$ for a fixed value of t and has a simple geometric interpretation discussed in the next Section.

4.4 The radial distribution of a diagram. Given a pair of real numbers $\alpha < \beta$, let us denote by M_α, M_β the corresponding points on the graph of the diagram ω, and let $A_{\alpha\beta}$ be the area of the curvilinear sector in between the rays OM_α, OM_β (where O denotes the origin) and the arc $M_\alpha M_\beta$ of the graph (see Fig. 5).

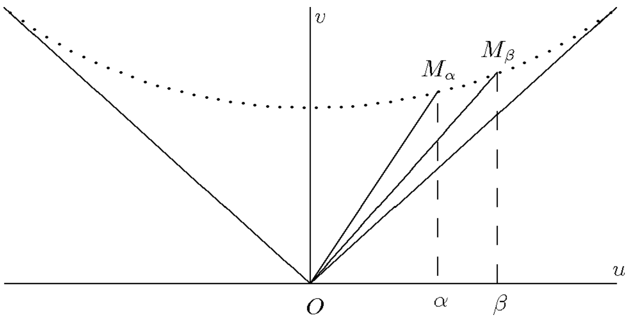

Fig. 5

Lemma.

$$A_{\alpha\beta} = \frac{1}{4}\int_\alpha^\beta (\omega(u) - u\omega'(u))\,du = \frac{1}{2}\int_\alpha^\beta (\sigma(u) - u\sigma'(u))\,du.$$

Proof. Consider the ray leaving the origin O by the angle θ and crossing the graph of the diagram at the point M. Let us denote by $r(\theta)$ the length of the segment OM and by u the abscissa of the point M, then u and θ determine each other uniquely, $\theta = \theta(u)$. Using integration in polar coordinates, we obtain the formula

$A_{\alpha\beta} = -1/4 \int_{\theta(\alpha)}^{\theta(\beta)} r^2(\theta)\, d\theta$. Using substitution $\theta = \theta(u)$, we have $\operatorname{ctg}\theta = u/\omega(u)$, and it follows that

$$[\omega(u) - u\omega'(u)]\, du = \omega^2(u)\, d(u/\omega(u)) = -\omega^2(u)\sin^{-2}\theta\, d\theta = -r^2(\theta)\, d\theta.$$

The second claim of the Lemma is straightforward from the definition of the charge of a diagram. □

Let us define the *radial density* of a diagram $\omega \in \mathcal{D}$ as

$$\rho(u) = \frac{1}{2A(\omega)}(\sigma(u) - u\sigma'(u)). \tag{4.4.1}$$

It follows from Lemma that

$$\int \rho(u)\, du = 1,$$

and the radial density provides a probability distribution. We use the term *radial* to refer to this distribution, too.

Proposition. *Let p_n denote the moment (3.2.5) of the diagram $\omega \in \mathcal{D}$. Then the moments of its radial distribution are equal to*

$$\int u^n\, \rho(u)\, du = \frac{p_{n+2}}{2(n+1)A(\omega)}; \quad n = 0, 1, 2, \dots. \tag{4.4.2}$$

Proof.

$$\int u^n(\sigma(u) - u\sigma'(u))\, du = -\int (u^{n+1} + \frac{u^{n+1}}{n+1})\, d\sigma(u) = \frac{p_{n+2}}{n+1}. \quad \square$$

We can take the following function as a generating function of moments of the radial distribution (assuming $p_1 = z = 0$):

$$xS'(x) = -\sum_{n=0}^{\infty} p_{n+2}x^{-(n+2)}. \tag{4.4.3}$$

Theorem. *Assume that the transition distribution of a diagram ω is equal to its radial distribution, $d\mu_\omega(u) = \rho_\omega(u)\, du$. Then the diagram ω coincides (up to similarity) with the diagram $\tilde{\Omega}$ for all $u \in \mathbb{R}$:*

$$\omega(u) = \sqrt{A}\tilde{\Omega}(u/\sqrt{A}), \tag{4.4.4}$$

where $A = A(\omega)$.

Proof. Let us write the equality of moments of the two distributions, $p_{n+2}/2(n+1)A = h_n$, in terms of the function $R(x) = \sum_{n\geq 0} h_n\, x^{-(n+1)}$. Using (4.4.3), we obtain

$$-\sum_{n=0}^{\infty} p_{n+2}\, x^{-(n+2)} = xS'(x) = (xR(x))'/R(x)$$

(since $xR(x) = \exp S(x)$ by (3.4.4)). Comparing the formula with the decomposition $R'(x) = -\sum_{n\geq 0}(n+1)h_n x^{-(n+2)}$, we get $(xR)'/R = 2AR'$ and $(xR)' = (AR^2)'$. Thus, $AR^2 - xR + C = 0$ for an appropriate constant C. Taking into consideration the lowest terms of decompositions

$$xR(x) = 1 + t/x^2 + \dots \quad \text{and} \quad AR^2(x) = A/x^2 + \dots$$

we find out that $C = 1$ and hence

$$AR^2(x) - xR(x) + 1 = 0.$$

Finally, $R(x) = (x - \sqrt{x^2 - 4A})/2A$ for $x > 2\sqrt{A}$ (since $R(x) \to 0$ as $x \to +\infty$). One only has to remark now that

$$R(x) = \frac{1}{2\sqrt{A}}(x/\sqrt{A} - \sqrt{x^2/A - 4}) = \frac{1}{\sqrt{A}} R_{\tilde{\Omega}}(x/\sqrt{A})$$

is identical to the analogous generating function in the right hand side of (4.4.4). $\square$

Let us remark that by (4.3.2) the tangent distributions of an automodel tableau at each diagram coincides with the radial distribution of the latter. It is easy to check that the condition characterizes the class of automodel tableau. One can consider the fact as the 'Second Kepler's Law for the Transverse Growth of Trees'. In fact, let us think of a diagram ω as if it was the profile of a cut of a tree trunk. The shape of the profile will be stable (up to similarity) iff the intensity of appearance of new cells within an arc is proportional to the area of the sector supported by this arc.

4.5 The asymptotics of the general solution. We have shown in Section 4.3 that the diagram $\tilde{\Omega}$ is, up to rescaling, the only fixed point of the basic dynamic equation (4.2.1). Let us check now that this solution is stable (attracting all other solutions). More precisely: every solution $\omega(u,t)$ of the equation (4.2.1) (corresponding to every choice of initial data), after normalizing its area converges uniformly, as $t \to \infty$, to the diagram $\tilde{\Omega}(u)$.

Theorem. *Assume that the charge $\sigma(u,t)$ of the tableau $\omega(u,t)$ satisfies the equation (4.2.3). Then*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \omega(u\sqrt{t}, t) = \tilde{\Omega}(u) \tag{4.5.1}$$

uniformly in $u \in \mathbb{R}$.

Proof. The moments $\tilde{p}_n(t)$ of the normalized diagram $\tilde{\omega}(u,t) = \omega(u\sqrt{t},t)/\sqrt{t}$ can be easily written in terms of the moments of the initial diagram $\omega(u,t)$:

$$\tilde{p}_n(t) = t^{-n/2} p_n(t).$$

Let us find the asymptotics of these moments, as $t \to \infty$, using the equation (4.2.4).

Lemma. *Assume that the functions $p_n(t)$ where $n = 1, 2, \dots$ satisfy the equations*

$$\frac{p'_{n+2}(t)}{n+2} = (n+1) \sum_{\rho \vdash n} \prod_{k \geq 1} \left(\frac{p_k(t)}{k} \right)^{\rho_k} \Big/ \rho_k!, \qquad n = 0, 1, 2, \dots \tag{4.5.2}$$

(here the right hand side is taken from (3.4.5)). Then there exist the constants $\{p_n\}_{n=1}^{\infty}$ such that

$$t^{-n/2} p_n(t) = p_n + o(t^{-1}) \tag{4.5.3}$$

for $t \to \infty$.

Proof. The first few equations of the system (4.5.2) are as follows:

$$\begin{aligned}
p'_3/3 &= 0 \\
p'_4/4 &= 3(p_1^2 + p_2)/2 \\
p'_5/5 &= 4(p_1^3 + 3p_1p_2 + 2p_3)/6 \\
p'_6/6 &= 5(p_1^4 + 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 + 6p_4)/24.
\end{aligned}$$

Solving the equations in the natural order we find out that

$$\begin{aligned}
p_3(t) &= c_3 = o(t^{1/2}) \\
p_4(t) &= 6t^2 + c_4 = 6t^2 + o(t) \\
p_5(t) &= \frac{4 \cdot 5}{3} c_3 t + c_5 = o(t^{3/2}) \\
p_6(t) &= 20t^3 + \frac{5 \cdot 6}{4} c_4 t + c_6 = 20t^3 + o(t^2),
\end{aligned}$$

where c_k are arbitrary constants. Since the right hand side of (4.5.2) is homogeneous, the Lemma follows by induction. $\square$

In order to complete the proof of the Theorem we have to show that the limiting values p_n in (4.5.3) are identical to the moments (3.3.1) of the diagram $\tilde{\Omega}$. It will be convenient to prove the fact by considering the equation (4.2.5).

If the function $\tilde{R}(x,t) = \sum_{n=0}^{\infty} \tilde{h}_n(t) x^{-(n+1)}$ corresponds to the normalized diagram $\tilde{\omega}(u,t) = \omega(u\sqrt{t}, t)/\sqrt{t}$, then

$$R(x,t) = \frac{1}{\sqrt{t}} \tilde{R}(\frac{x}{\sqrt{t}}, t).$$

The substitution of this expression in the equation (4.2.5) implies, after some simple calculations, the equation

$$\frac{\partial}{\partial x}(x\tilde{R} - \tilde{R}^2) = 2t \frac{\partial \tilde{R}}{\partial t}. \tag{4.5.4}$$

Let us write the function $\tilde{R}$ in the form

$$\tilde{R}(x,t) = \hat{R}(x) + \bar{R}(x,t), \tag{4.5.5}$$

where $\hat{R}(x) = \sum_{n=0}^{\infty} h_n x^{-(n+1)}$, and the coefficients h_n are related to p_n by the formula (3.4.5) Then, by Lemma,

$$\lim_{t\to\infty} \bar{R}(x,t) = 0, \quad \lim_{t\to\infty} t\bar{R}'_x(x,t) = 0.$$

Substituting (4.5.5) in the equation (4.5.4) and taking the limit as $t \to \infty$, we derive that $x\hat{R} - \hat{R}^2 = \text{const}$, and the proof can be completed in the same way as in Theorems (4.3), (4.4), namely using the quadratic equation $\hat{R}^2(x) - x\hat{R}(x) + 1 = 0$ and the formula (3.4.11) for the generating function of the transition measure of the diagram $\tilde{\Omega}$. $\square$

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