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The Boundary of Young Lattice and Random Young Tableaux

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ABSTRACT. The description of characters of the infinite symmetric group $\mathfrak{S}_\infty = \varinjlim \mathfrak{S}_n$ is considered in the paper as an asymptotical combinatorial problem. It is equivalent to the characterization problem of totally positive sequences. We derive the list of characters, first obtained by E. Thoma in 1964, using appropriate exact formulae for the number of skew Young tableaux. The character space is treated as a boundary of the Young lattice.

The character problem for a locally finite group and the classical de Finetti theorem on exchangeable random sequences are two particular cases of a more general problem in potential theory on a graph, namely that of the explicite description of Martin boundary of this graph. We show that Selberg type integrals arise naturally in this general setup in connection with Pieri formula for Jack symmetric polynomials.

Some remarkable connections between Plancherel measures of symmetric groups and topics in Analysis are indicated.

Contents

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1. Introduction

This paper is a brief introduction to some of the combinatorial and asymptotic problems arising in the theory of approximately finite-dimensional algebras. We focus on the distinguished particular example of the group algebra $\mathbb{C}[\mathfrak{S}_\infty]$ of the countable symmetric group $\mathfrak{S}_\infty$.

It will be shown that the Young lattice $\mathcal{Y}$ (the set of Young diagrams ordered by inclusion, see Fig. 1) has a natural compactification $\overline{\mathcal{Y}}$. The complement $\Delta = \overline{\mathcal{Y}} \setminus \mathcal{Y}$ will be referred to as the *boundary* of the Young lattice. We present below both the definition of the boundary Δ and its explicit description.

Several apparently unrelated problems lead to the concept of the boundary.

It is well known that Young diagrams with n boxes label the characters of irreducible representations of the symmetric group $\mathfrak{S}_n$. In the same way, the boundary Δ of the Young lattice provides the solution to the following problem.

Problem (i). *Find the characters of the infinite symmetric group $\mathfrak{S}_\infty$.* By definition, a character of $\mathfrak{S}_\infty$ is an extreme point in the compact convex set of positive definite, central, normalized functions on this group.

This problem in representation theory can be restated in terms of potential theory on the lattice $\mathcal{Y}$, and in probabilistic terms of random growth of Young diagrams.

Problem (ii). *Describe all non-negative harmonic functions on the Young lattice.* Here a function $\varphi : \mathcal{Y} \to \mathbb{R}$ is called *harmonic* if it satisfies the 'Mean Value Theorem' in the form

$$\varphi(\lambda) = \sum_{\mu} \varphi(\mu); \quad \lambda \in \mathcal{Y},$$

where μ runs through the set of diagrams covering λ in the lattice $\mathcal{Y}$.

Problem (iii). *Find all Markov processes of central growth of Young diagrams.* In this Problem we consider Markov chains on the state space $\mathcal{Y}$, with the empty diagram as the initial state. Only transitions to covering diagrams are allowed, hence the realizations of the chain are just Young tableaux. A process is said to be *central*, if the probability of a Young tableau depends but on its shape.

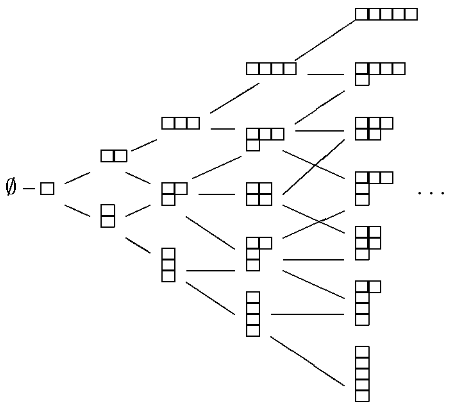

Figure 1. The Young lattice.

Yet another form of the Problem (i) is in terms of symmetric functions.

Problem (iv). *Describe all homomorphisms $\psi : \Lambda \to \mathbb{R}$ of the symmetric function ring Λ which are non-negative on Schur functions: $\psi(s_\lambda) \geq 0$ for all $\lambda \in \mathcal{Y}$.*

Problems (ii), (iii) make sense for more general infinite graphs, called Bratteli diagrams. To each graph of this type there corresponds canonically an increasing sequence

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$$

of finite-dimensional, semisimple algebras over $\mathbb{C}$. Problem (i), when restated for the limiting infinite-dimensional algebra $A_\infty = \varinjlim A_n$, is again equivalent to Problems (ii), (iii). Any one of the Problems (i) – (iii) leads to the general definition of the boundary of a Bratteli diagram, similar to the classical Martin boundary construction.

Given a Bratteli graph Γ, the computation of the boundary reduces itself to the following combinatorial Problem.

Problem (v). *Find the asymptotics of the number $d(\lambda, \mu)$ of saturated chains in an interval $[\lambda, \mu] \subset \Gamma$, when λ is fixed and μ goes to infinity along some path in Γ.*

In the particular case of Young lattice, $\Gamma = \mathcal{Y}$, there are useful exact formulae for the number of skew Young tableaux. As a consequence, the boundary Δ of the Young lattice is known. It can be identified with the space of pairs of non-negative sequences

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq 0; \quad \beta_1 \geq \beta_2 \geq \ldots \geq 0,$$

such that $\sum \alpha_k + \sum \beta_k \leq 1$. The parameters $\{\alpha_k\}_{k=1}^\infty$, $\{\beta_k\}_{k=1}^\infty$ can be interpreted as Frobenius coordinates of 'ideal Young diagrams' at the boundary of Young lattice. Let us remark that via solution of Problem (iv), the boundary Δ is in a natural one-to-one correspondence with the space of Polya frequency sequences.

Along with the Young lattice, we consider its one parameter deformation related to zonal (Jack) symmetric functions, and compute the boundary of its truncated version. As a particular case, Kingman's partition structures arise.

The character of the regular representation of $\mathfrak{S}_\infty$ corresponds to the point $\alpha_1 = \alpha_2 = \ldots = \beta_1 = \beta_2 = \ldots = 0$ of the boundary Δ. It is just the delta measure at the identity of the group. On the other hand, the associated central Markov chain (called here the Plancherel Growth Process) is highly non-trivial. Its transition probabilities can be written as

$$p(\lambda, \mu) = \frac{d(\emptyset, \mu)}{(n+1)\, d(\emptyset, \lambda)}; \quad \lambda \in \mathcal{Y}_n,$$

and the state distribution after n moves is the Plancherel measure of the group $\mathfrak{S}_n$:

$$M_n(\lambda) = \frac{d^2(\emptyset, \lambda)}{n!}; \quad \lambda \in \mathcal{Y}_n.$$

As $n \to \infty$, almost all Young diagrams $\lambda \in \mathcal{Y}_n$, with respect to the Plancherel measure M_n, have similar shapes uniformly close to a universal curve known explicitly. The same shape maximizes the number of Young tableaux for a large n.

Surprisingly, the Plancherel transition probabilities have a natural description in terms of partial fraction decompositions. This observation allows one to apply non-trivial combinatorics of Young diagrams to problems in Analysis. For instance, the Hook Walk algorithm has an application to Markov Moment Problem, and the asymptotics of root separation of classical orthogonal polynomials coincides with that of Young diagrams in the Plancherel Growth Process.

As an application of the boundary theory of Bratteli graphs, we derive Selberg type integral

$$\int\limits_{\substack{\alpha_1,\dots,\alpha_k\ge 0\\ \alpha_1+\dots+\alpha_k=1}}\cdots\int P_\lambda(\alpha_1,\dots,\alpha_k;\theta)\prod_{1\le i<j\le k}|\alpha_i-\alpha_j|^{2\theta}\prod_{j=1}^{k}\alpha_j^{A-1}\,d\alpha_1\dots d\alpha_{k-1}=$$

$$=\frac{P_\lambda(\overbrace{1,\dots,1}^{k\text{ times}};\theta)}{\Gamma(n+kA+k(k-1)\theta)}\prod_{j=1}^{k}\frac{\Gamma(\lambda_j+A+(k-j)\,\theta)\,\Gamma(j\theta+1)}{\Gamma(\theta+1)},$$

where $P_\lambda(\alpha;\theta)$ are zonal symmetric polynomials. To this end, we introduce a special class of *multiplicative* random Young diagrams, and we show that Selberg integral is a particular case of the general Poisson integral representation of associated harmonic functions.

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2. Pascal triangle

We start with a simple example of a Bratteli diagram (see the general definition in Section 3 below), for which Problems (ii), (iii) have easy solutions. Let $\Gamma=\bigcup\Gamma_n$ denote the Pascal triangle considered as an ordered graph. The vertices of the n-th level Γ_n are labelled by pairs (k,n), where $k=0,1,\dots,n$. For each vertex (k,n) there are two outgoing edges with the end vertices $(k,n+1)$ and $(k+1,n+1)$ (Fig. 2). Problem (ii) takes the following form.

Problem (ii) for the Pascal triangle. *Describe all functions $\varphi:\Gamma\to\mathbb{R}$ satisfying three conditions:*

$$\begin{gathered}\varphi(k,n)=\varphi(k,n+1)+\varphi(k+1,n+1);\quad 0\le k\le n,\\ \varphi(k,n)\ge 0;\quad 0\le k\le n,\\ \varphi(0,0)=1.\end{gathered}\tag{2.1}$$

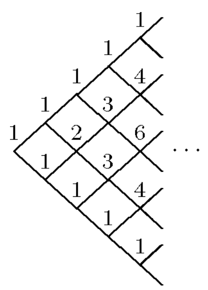

Figure 2. The Pascal triangle.

We say that a function with these properties is *harmonic.*

The solution to the Problem is straightforward. Consider a set of examples of harmonic functions:

$$\varphi_p(k,n)=p^k\,(1-p)^{n-k};\quad 0\le p\le 1.\tag{2.2}$$

Since the space of harmonic functions is convex, the integral

$$\varphi(k,n)=\int_0^1 p^k\,(1-p)^{n-k}\,\mu(dp) \tag{2.3}$$

also represents a harmonic function, for every choice of a probability measure μ. In fact, all harmonic functions of the Pascal triangle have a unique representation by the integral (2.3). This follows, for instance, from de Finetti theorem, cf. [**2**], Chapter VII, Section 4.

Problem (iii) in case of the Pascal triangle is related to the study of exchangeable random sequences. Recall that a sequence $X=(X_1,\dots,X_n)$ of binary random variables is called *exchangeable*, if the probability

$$\text{Prob}\,\{X_1=x_1,\dots,X_n=x_n\}=\varphi(k,n) \tag{2.4}$$

depends only on the number $k=x_1+\ldots+x_n$ of $1'$s and the number $n-k$ of $0'$s in the sequence x, but not on the order of these. A random binary sequence $X=(X_1,\dots,X_n,\dots)$ determines a random chain (path) in the Pascal triangle, and the latter process is central iff the sequence X is exchangeable. In this case (2.4) is a harmonic function on the Pascal triangle. Example (2.2) corresponds to independent coin tossings, with heads probability p. More general exchangeable sequences have unique representations as mixtures of independent, identically distributed sequences, with respect to appropriate probability distribution μ of the parameter $p\in[0,1]$. Since exchangeable sequences are in a natural one-to-one correspondence with central Markov chains on the Pascal triangle, we obtain the solution to Problem (iii).

The Polya urns model provides a concrete example of an exchangeable sequence. Initially, there are A white and B black balls in the urn. Each ball drawn from the urn returns back along with a new ball of the same color. In this example X_j is the number of white balls drawn at the j-th stage of the process ($X_j=0$ or $X_j=1$). The probability (2.4) does not depend on the order of $x_1,\dots,x_n$, and equals

$$\varphi(k,n)=\frac{(A)_k\,(B)_{n-k}}{(A+B)_n}, \tag{2.5}$$

where $(c)_n=c(c+1)\dots(c+n-1)$ is the Pochhammer symbol. The mixing measure μ in (2.3) is the Euler beta distribution:

$$\mu(dp)=\frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)}\,p^{A-1}\,(1-p)^{B-1}\,dp, \tag{2.6}$$

and (2.3) takes the form

$$\int_0^1 p^k\,(1-p)^{n-k}\,\mu(dp)=\frac{(A)_k\,(B)_{n-k}}{(A+B)_n}. \tag{2.7}$$

Note that the parameters $A,B\geq 0$ in (2.5) are not necessarily integers.

3. Bratteli diagrams and branchings

Pascal triangle enjoys the following properties:

(a) the end vertex of every edge belongs to the level set next to that of its source vertex;

(b) there is a single vertex $\emptyset\in\Gamma$ with no incoming edges;

(c) for every vertex there is at least one outgoing edge;
(d) all the levels Γ_n are finite.

An oriented graded graph $\Gamma = \bigcup \Gamma_n$ satisfying (a) – (d) is called *Bratteli diagram.* Sometimes, it is convenient to consider Γ as a partially ordered set with minimal element $\emptyset \in \Gamma_0$. We write $\lambda \nearrow \Lambda$ if (λ, Λ) is an edge of Γ, that is if Λ covers λ.

We will often associate a positive number $\varkappa(\lambda, \Lambda)$, called *edge multiplicity*, to every edge (λ, Λ) of a Bratteli diagram. A pair $(\Gamma, \varkappa)$ consisting of a Bratteli diagram and a multiplicity function $\varkappa$ will be referred to as a *branching.*

The terminology and the original host of examples comes from representation theory of locally finite groups and AF-algebras. Let G be a countable locally finite group, i.e. a union of an increasing sequence of finite subgroups

$$G_0 \subset G_1 \subset \ldots \subset G_n \subset \ldots. \tag{3.1}$$

We denote by $\Gamma_n = \hat{G}_n$ the finite set of (equivalence classes) of irreducible representations of G_n. For any two irreducibles $\lambda \in \Gamma_n$, $\Lambda \in \Gamma_{n+1}$ the coefficient $\varkappa(\lambda, \Lambda)$ of λ in the restriction

$$\operatorname{Res}^{G_{n+1}}_{G_n} \Lambda = \bigoplus_{\lambda : \lambda \nearrow \Lambda} \varkappa(\lambda, \Lambda) \cdot \lambda \tag{3.2}$$

coincides with the corresponding coefficient of Λ in the induced representation:

$$\operatorname{Ind}^{G_{n+1}}_{G_n} \lambda = \bigoplus_{\Lambda : \lambda \nearrow \Lambda} \varkappa(\lambda, \Lambda) \cdot \Lambda. \tag{3.3}$$

By definition, a pair (λ, Λ) for which $\varkappa(\lambda, \Lambda) \neq 0$ determines an edge of a branching, with multiplicity $\varkappa(\lambda, \Lambda)$, on the vertex set $\Gamma = \bigcup \Gamma_n$. (In this example we have integer multiplicities).

In particular, consider the infinite symmetric group $\mathfrak{S}_\infty$ as the direct limit of subgroups

$$\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \ldots \subset \mathfrak{S}_n \subset \ldots, \tag{3.4}$$

where $\mathfrak{S}_n$ acts on the set $\{1, 2, \ldots, n\}$. Then the associated branching is the Young lattice (Fig. 1). Its vertices are Young diagrams, and the edges are simple: $\varkappa(\lambda, \Lambda) = 1$. The paths starting at the empty diagram are called standard Young tableaux.

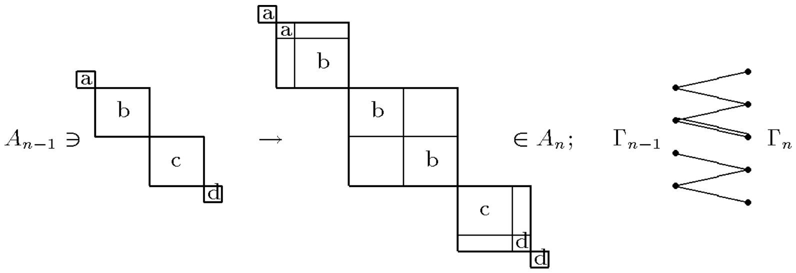

FIGURE 3. Matrix inclusions and bipartite graphs.

Let us remark that for every branching $(\Gamma,\varkappa)$ with integer multiplicities, there is a variant of Problem (i) equivalent to Problems (ii), (iii) for this branching. In fact, to such a branching one can canonically associate an increasing sequence

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots \tag{3.5}$$

of finite-dimensional, semisimple complex algebras. The n-th algebra

$$A_n = \bigoplus_{\lambda\in\Gamma_n} M_{d(\lambda)}(\mathbb{C}), \tag{3.6}$$

is isomorphic to a direct sum of algebras $M_k(\mathbb{C})$ of $k\times k$ complex matrices. The terms in (3.6) are indexed by the vertices of the n-th level Γ_n of the branching, and $d(\lambda)=d(\emptyset,\lambda)$ denotes the combinatorial dimension of the vertex $\lambda\in\Gamma_n$ (see Section 5 below).

The monomorphism $A_{n-1}\to A_n$ in the sequence is determined by the bipartite graph in between the levels Γ_{n-1}, Γ_n. The construction should be clear from an example in Fig. 3.

There is a limiting infinite-dimensional $*$-algebra $A_\infty=\varinjlim A_n$, and Problems (ii), (iii) are equivalent to the generalized Problem (i): describe characters of the algebra A_∞ associated to a branching $(\Gamma,\varkappa)$.

By definition, a *character* of the algebra A_∞ is a positive definite, central, normalized linear functional $\psi: A_\infty\to\mathbb{C}$, i.e.

(i) $\psi(a\,a^*)\ge 0$ for all $a\in A_\infty$;
(ii) $\psi(a\,b)=\psi(b\,a)$ for all $a,b\in A_\infty$;
(iii) $\psi(e)=1$, where e is the identity in A_∞.

It is straightforward to check that every character ψ uniquely determines a harmonic function φ on the corresponding branching via the formula

$$\psi\big|_{A_n} = \sum_{\lambda\in\Gamma_n}\varphi(\lambda)\,\chi_\lambda. \tag{3.7}$$

Here χ_λ denotes the normalized character of an irreducible representation of finite dimensional subalgebra A_n, labelled by $\lambda\in\Gamma_n$.

4. Multiplicative branchings

All the examples considered in the present paper belong to the special class of *multiplicative* branchings. The branchings in the class have some nice special features, and a specific techniques can be applied. In particular, we shall see that there is a variant of Problem (iv) which is equivalent to Problems (i) – (iii), for general multiplicative branchings.

Let us start with a graded $\mathbb{R}$-algebra

$$R=\bigoplus_{n=0}^{\infty} R_n, \tag{4.1}$$

and assume that a linear basis $\{s_\lambda\}_{\lambda\in\Gamma_n}$ is chosen in R_n for all $n\ge 0$. Suppose $s\in R_1$ is an element for which all the coefficients in the decomposition

$$s\cdot s_\lambda = \sum_{\Lambda:\lambda\nearrow\Lambda}\varkappa(\lambda,\Lambda)\,s_\Lambda \tag{4.2}$$

are nonnegative. Then the triple $(R, \{s_\lambda\}, s)$ determines a branching on the set $\Gamma = \bigcup \Gamma_n$. The edges are those pairs (λ, Λ) for which $\varkappa(\lambda, \Lambda) \neq 0$, and the coefficients are considered as multiplicities. The branchings of this type are called *multiplicative*.

In the examples below we use notations and terminology from [**15**].

Example A (Young lattice). R is the algebra of symmetric polynomials, s_λ is the Schur function indexed by a Young diagram $\lambda \in \mathcal{Y}$, and $s = x_1 + x_2 + \ldots = s_{(1)}$. By the Pieri formula,

$$s_{(1)} \cdot s_\lambda = \sum_{\Lambda : \lambda \nearrow \Lambda} s_\Lambda, \tag{4.3}$$

hence Young lattice is a multiplicative branching.

Example B (Kingman's branching). Consider the basis $\{m_\lambda\}$ of monomial symmetric functions in the symmetric function algebra R, and use the element s of Example A. Then the coefficients in

$$s_{(1)} \cdot m_\lambda = \sum_{\Lambda : \lambda \nearrow \Lambda} \varkappa(\lambda, \Lambda) \cdot m_\Lambda. \tag{4.4}$$

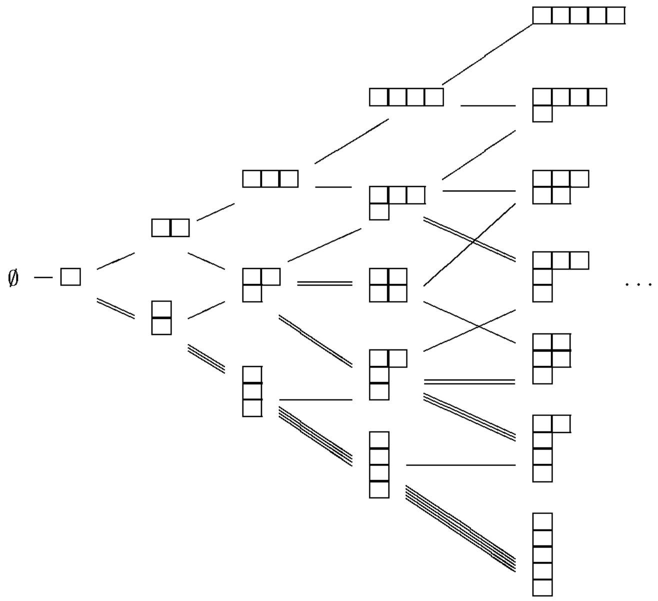

Figure 4. Kingman's branching.

are positive. The graph of the branching is the ordinary Young lattice, and the multiplicity

$$\varkappa(\lambda, \Lambda) = \rho_j(\Lambda) \tag{4.5}$$

equals the number of rows of length j in a Young diagram Λ, assuming the box $\Lambda \setminus \lambda$ is deleted from a row of length j in Λ. The branching $\mathcal{K} = (\mathcal{Y}, \varkappa)$ was studied in [**12**], and we call it *Kingman's branching* (Fig. 4).

EXAMPLE C (Jack's branching). The algebra R is the same as above. Let $P_\lambda(x;\theta)$ be the notation for zonal (Jack) symmetric polynomials with the parameter $\theta > 0$. We briefly recall the definition of Jack polynomials.

Define a scalar product on R by the formula

$$< p_\lambda, p_\mu > = \delta_{\lambda\mu}\, \theta^{-l(\lambda)}\, z_\lambda,$$

where p_λ is a power sum symmetric polynomial, $\lambda = (1^{\rho_1} 2^{\rho_2} \dots)$ is a Young diagram of length $l(\lambda)$, and $z_\lambda = 1^{\rho_1}\rho_1!\, 2^{\rho_2}\rho_2! \dots$. The zonal polynomials P_λ can be characterized by the following properties:

(P1) $< P_\lambda, P_\mu > = 0$ if $\lambda \neq \mu$;

(P2) if $P_\lambda = \sum_\mu v_{\lambda\mu}(\theta)\, m_\mu$, then $v_{\lambda\mu}(\theta) = 0$ unless $\mu \leq \lambda$ in the dominance (natural) ordering;

(P3) $v_{\lambda\lambda}(\theta) = 1$.

Note that Jack symmetric polynomials $J_\lambda(x,\theta)$ considered in [**19**] have a different normalization:

$$\theta^n\, J_\lambda(x,\theta) = P_\lambda(x,\theta) \prod_{b\in\lambda} (a(b) + (l(b)+1)\,\theta). \tag{4.6}$$

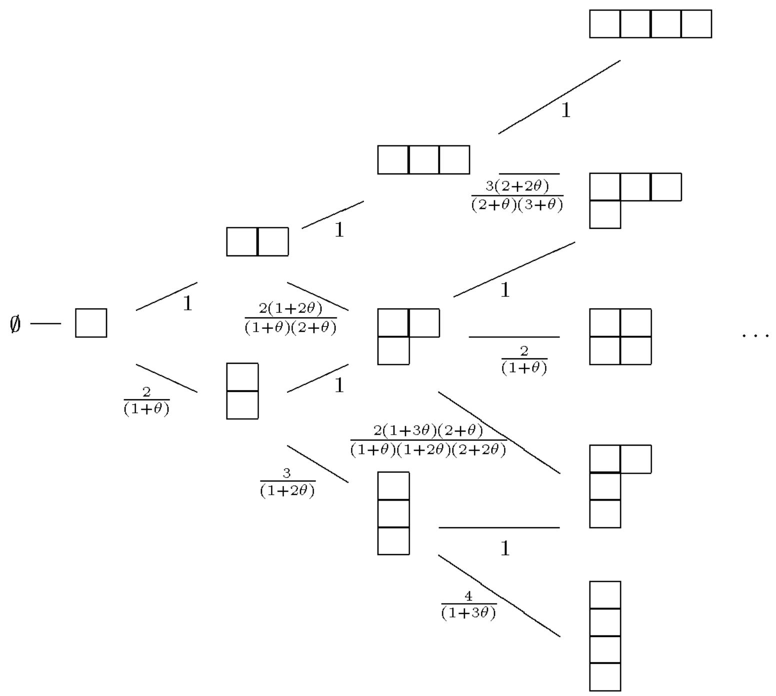

FIGURE 5. Multiplicities of Jack's branching.

Here $a(b) = \lambda_i - j$ is the arm length and $l(b) = \lambda'_j - i$ is the leg length of a box $b = (i,j) \in \lambda$. Pieri type formula

$$s_{(1)} \cdot P_\lambda(x;\theta) = \sum_{\lambda:\lambda\nearrow\Lambda} \varkappa_\theta(\lambda,\Lambda) \cdot P_\Lambda(x;\theta) \tag{4.7}$$

determines the multiplicities of edges in the Young lattice as follows:

$$\varkappa_\theta(\lambda,\Lambda)=\prod_{ver}\left(\frac{a(b)+(l(b)+2)\,\theta}{a(b)+(l(b)+1)\,\theta}\right)\left(\frac{a(b)+1+l(b)\,\theta}{a(b)+1+(l(b)+1)\,\theta}\right). \tag{4.8}$$

The product $\prod_{ver}$ runs over the boxes in the column of λ strictly above the new box $\Lambda\setminus\lambda$ (see [**16**]). We denote the Young lattice with edge multiplicities (4.8) by $\mathcal{J}^{(\theta)}$, and we call it *Jack branching* (Fig. 5). The family of Jack branchings provides a deformation of Example B ($\theta=0$) to the ordinary Young branching of Example A ($\theta=1$). In fact, $P_\lambda(x;0)=m_\lambda(x)$ and $P_\lambda(x;1)=s_\lambda(x)$.

Remark. All the branchings in examples above have *truncated* versions with a limited number of rows k in Young diagrams. The latter are multiplicative, too, and the construction of a corresponding triple $(R,\{s_\lambda\},s)$ is similar, except that polynomials in the algebra R depend on k variables only.

Note that the Pascal triangle of Section 2 is a multiplicative Bratteli diagram, too. In this example, $R=\mathbb{Z}[x,y]$ is the polynomial ring in two variables, the basis is formed of monomials $s_{(k,n)}=x^k\,y^{n-k}$, and the element $s=x+y$ determines the branching via (4.2).

5. Combinatorial dimensions

Let Γ be a Bratteli diagram. For any two vertices $\lambda,\mu\in\Gamma$, we denote by $d(\lambda,\mu)$ the number of chains from λ to μ. In case of nontrivial edge multiplicities, $\varkappa(\lambda,\Lambda)\neq 1$, the definition is slightly different. We assign a weight

$$w_u=\prod_{i=1}^{n}\varkappa(\lambda_{i-1},\lambda_i) \tag{5.1}$$

to a chain $u=(\lambda_0=\lambda,\lambda_1,\dots,\lambda_n=\mu)$, and define *dimension function* $d(\lambda,\mu)$ as the‘ sum

$$d(\lambda,\mu)=\sum_u w_u \tag{5.2}$$

of weights of all saturated chains in the interval $[\lambda,\mu]$. There are two useful ways to characterize combinatorial dimensions recurrently:

(a) $d(\lambda,\lambda)=1$ and

$$d(\lambda,\mu)=\sum_{\Lambda:\Lambda\nearrow\mu} d(\lambda,\Lambda)\cdot\varkappa(\Lambda,\mu), \tag{5.3}$$

or

(b) $d(\mu,\mu)=1$ and

$$d(\lambda,\mu)=\sum_{\Lambda:\lambda\nearrow\Lambda}\varkappa(\lambda,\Lambda)\cdot d(\Lambda,\mu). \tag{5.4}$$

We shall often use the abbreviation $d(\lambda)=d(\emptyset,\lambda)$. For the Pascal triangle,

$$d(k,n)=\binom{n}{k} \tag{5.5}$$

is the choose symbol.

EXAMPLE A. For Young lattice $\mathcal{Y}$, the combinatorial dimension is equal to that of the associated irreducible of the symmetric group $\mathfrak{S}_n$. The *hook formula* holds:

$$d(\lambda) = \frac{n!}{\prod_{b\in\lambda} h(b)}; \quad \lambda \in \mathcal{Y}_n, \tag{5.6}$$

where $h(b) = a(b) + l(b) + 1$ is the hook length of the box $b \in \lambda$.

EXAMPLE B. For Kingman's branching $\mathcal{K}$, the dimension $d(\lambda_1, \lambda_2, \dots)$ is equal to that of the representation of $\mathfrak{S}_n$ induced by the trivial representation of the Young subgroup $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \dots \subset \mathfrak{S}_{\lambda_n}$, namely

$$d(\lambda) = \frac{n!}{\lambda_1!\, \lambda_2! \dots}; \quad \lambda \in \mathcal{Y}_n. \tag{5.7}$$

EXAMPLE C. For the Jack branching $\mathcal{J}^{(\theta)}$, there is a hook formula (see [**19**]) generalizing both (5.6) and (5.7):

$$d(\lambda) = \frac{n!}{\prod_\lambda (a(b) + 1 + l(b)\,\theta)}. \tag{5.8}$$

6. Central Markov chains

We associate to each branching a class of *central* Markov chains. The definition is similar to that of Gibbs measures, the multiplicity function playing the role of a potential. For a branching of irreducible representations of a sequence of finite groups (3.1), central Markov chains are in one-to-one correspondence with characters of the limiting group $G = \lim G_n$.

Let Γ be a Bratteli diagram, and $\varkappa$ a multiplicity function. We denote by T the space of infinite paths (saturated chains) $t = (\lambda_0 = \emptyset, \lambda_1, \dots, \lambda_n, \dots)$ starting at the initial vertex $\emptyset$ of Γ. In the natural topology, T is compact and totally disconnected. In all examples of Section 4 the elements of T are infinite Young tableaux.

A Borel measure M on the path space T is determined by the measures $M(C_u)$ of cylindrical sets

$$C_u = \{t \in T :\ \lambda_1 = \nu_1, \dots, \lambda_n = \nu_n\}, \tag{6.1}$$

where $u = (\nu_1, \nu_2, \dots, \nu_n = \lambda)$ is a finite path with the last vertex $\omega(u) = \lambda$.

DEFINITION. A probability measure M on T is called *central*, with respect to the branching $(\Gamma, \varkappa)$, if

$$\frac{M(C_u)}{w_u} = \frac{M(C_v)}{w_v} \tag{6.2}$$

for any two finite paths u, v with equal last vertices $\omega(u) = \omega(v)$. (Recall the definition of weights w_u in (5.1)).

In particular, when the multiplicities are trivial, $\varkappa(\lambda, \Lambda) \equiv 1$, the definition reduces to this:

$$\omega(u) = \omega(v) \quad \text{implies} \quad M(C_u) = M(C_v). \tag{6.3}$$

It follows from the definition (6.2), that for a central measure the quotient

$$\frac{M(C_u)}{w_u} = \varphi(\lambda) \tag{6.4}$$

depends but on the last vertex λ of a path u. The function φ is *harmonic* in the sense that

$$\begin{gathered}\varphi(\lambda)=\sum_{\Lambda:\lambda\nearrow\Lambda}\varkappa(\lambda,\Lambda)\cdot\varphi(\Lambda);\quad \lambda\in\Gamma,\\ \varphi(\lambda)\geq 0;\quad \lambda\in\Gamma,\\ \varphi(\emptyset)=1.\end{gathered}\tag{6.5}$$

In the opposite direction, every harmonic function of a branching $(\Gamma,\varkappa)$ uniquely determines a central distribution with cylinder probabilities

$$M(C_u)=w_u\cdot\varphi(\omega(u)).\tag{6.6}$$

Let $u=(\lambda_1,\dots,\lambda_n=\lambda)$ be a path terminating at a vertex λ, and write $u\Lambda=(\lambda_1,\dots,\lambda_n,\Lambda)$ for the path u extended by a new edge (λ,Λ). Since the conditional probability

$$\text{Prob }\{C_{u\Lambda}|C_u\}=\frac{M(C_{u\Lambda})}{M(C_u)}=\frac{\varkappa(\lambda,\Lambda)\cdot\varphi(\Lambda)}{\varphi(\lambda)}\tag{6.7}$$

depends only on the last vertex λ of the path u, every central measure is a Markov chain, with transition probabilities

$$p(\lambda,\Lambda)=\frac{\varkappa(\lambda,\Lambda)\cdot\varphi(\Lambda)}{\varphi(\lambda)}.\tag{6.8}$$

A Markov chain is central iff

$$p(\lambda,\mu)\,p(\mu,\Lambda)=p(\lambda,\nu)\,p(\nu,\Lambda)\tag{6.9}$$

for every quadruple of vertices such that $\lambda\nearrow\mu\nearrow\Lambda$ and $\lambda\nearrow\nu\nearrow\Lambda$.

For any branching $(\Gamma,\varkappa)$, the natural generalizations of Problems (ii), (iii) of Section 1 are equivalent, since (6.4), (6.6) establish a one-to-one correspondence between harmonic functions and central measures. Another way of looking at the problem is in terms of coherent systems of probability distributions.

Consider two consecutive levels Γ_n and Γ_{n+1} and a vertex $\Lambda\in\Gamma_{n+1}$. Choose a random path $u=(\lambda_0=\emptyset,\dots,\lambda_n,\lambda_{n+1}=\Lambda)$ with probability proportional to its weight w_u. The conditional probability

$$q(\lambda,\Lambda)=\text{Prob }\{\lambda_n=\lambda\mid\lambda_{n+1}=\Lambda\}\tag{6.10}$$

to cross the level Γ_n at the vertex λ, terminating at Λ, will be called *co-transition probability*. In terms of combinatorial dimensions, it can be written as

$$q(\lambda,\Lambda)=\frac{d(\lambda)\,\varkappa(\lambda,\Lambda)}{d(\Lambda)}.\tag{6.11}$$

By (5.3), $\sum_\lambda q(\lambda,\Lambda)=1$ for all Λ.

Definition. Assume that there is a probability distribution M_n on the n-th level set Γ_n of a branching $(\Gamma,\varkappa)$, for all $n\geq 0$. The system $\{M_n\}_{n=0}^\infty$ is called *coherent*, if

$$M_n(\lambda)=\sum_{\Lambda:\lambda\nearrow\Lambda}q(\lambda,\Lambda)\,M_{n+1}(\Lambda)\tag{6.12}$$

for every $n\geq 0$ and $\lambda\in\Gamma_n$.

Let M be a central Markov chain for a branching $(\Gamma, \varkappa)$, and let

$$M_n(\lambda) = \text{Prob}\ \{t = (\lambda_1, \dots, \lambda_n, \dots) \in T:\ \lambda_n = \lambda\}; \quad \lambda \in \Gamma_n, \tag{6.13}$$

denote the one-dimensional distribution of M at the n-th level of Γ. Then the system $\{M_n\}$ is coherent. Vice versa, for every coherent system of probability distributions, the function $\varphi(\lambda) = M_n(\lambda)\,/\,d(\lambda)$, where $n \geq 0$ and $\lambda \in \Gamma_n$, is harmonic with respect to $(\Gamma, \varkappa)$ and determines via (6.6) a central Markov chain.

In the particular example of Kingman's branching considered in [**12**], the coherent systems were called *partition structures.*

7. The boundary of a branching

Let us start by recalling the classical integral representation of conventional harmonic functions in the disc. Denote by $u = u(z)$ a positive harmonic function in the unit disc $D = \{z:\ |z| < 1\}$, normalized by the condition $u(0) = 1$. It can be uniquely represented by the Poisson Integral

$$u(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\varphi}|^2}\, \mu(d\varphi), \tag{7.1}$$

where μ is a probability distribution on the boundary circle. The distribution μ is determined by u as a weak limit of probability measures

$$\mu_r(d\varphi) = u(re^{i\varphi})\ \frac{d\varphi}{2\pi} \tag{7.2}$$

when $r \to 1$ (see [**18**], Theorem 11.19 and its proof).

In order to apply similar terminology in the context of branchings, we shall define the analogues of the boundary and of the Poisson kernel

$$\Phi(z; e^{i\varphi}) = \frac{1 - |z|^2}{|z - e^{i\varphi}|^2}. \tag{7.3}$$

Given a branching $(\Gamma, \varkappa)$, assume that there is a compact topological space Δ, a map $i:\ \Gamma \to \Delta$, and a function $\Phi:\ \Gamma \times \Delta \to \mathbb{R}$ satisfying the following conditions:

(i) for every $\omega \in \Delta$ the function $\varphi_\omega(\lambda) = \Phi(\lambda, \omega)$ is harmonic with respect to the branching $(\Gamma, \varkappa)$;

(ii) the functions $\Phi_\lambda(\omega) = \Phi(\lambda; \omega)$; $\lambda \in \Gamma$ are continuous and span a dense linear subspace in the space $C(\Delta)$ of continuous functions on Δ;

(iii) for every $\omega \in \Delta$ consider the probability distributions

$$M_n(\lambda) = d(\lambda)\ \Phi(\lambda; \omega); \quad \lambda \in \Gamma_n. \tag{7.4}$$

Then the measures $i(M_n)$ weakly converge, as $n \to \infty$, to the degenerated measure δ_ω at the point $\omega \in \Delta$.

Definition. We say that the space Δ is the *boundary* of a branching $(\Gamma, \varkappa)$, and that $\Phi = \Phi(\lambda; \omega)$ is its *Poisson kernel.*

Theorem. *Let $(\Gamma, \varkappa)$ be a branching with the boundary Δ, an inclusion i : $\mathcal{Y} \to \Delta$, and the Poisson kernel $\Phi = \Phi(\lambda; \omega)$ satisfying conditions (i) – (iii) above. Then*

(1) every harmonic function φ can be uniquely represented by the Poisson integral

$$\varphi(\lambda) = \int_\Delta \Phi(\lambda;\omega)\,\mu(d\omega), \tag{7.5}$$

where μ is a probability measure on Δ;

(2) the representing measure μ in (7.5) is the weak limit of probability distributions $i(M_n)$, where

$$M_n(\lambda) = d(\lambda)\,\varphi(\lambda); \quad \lambda \in \Gamma_n. \tag{7.6}$$

The proof follows directly from the definition.

As a simple example, let us take for Γ the Pascal triangle. Then $\Delta = [0,1]$ is the unit interval, the imbedding $i : \Gamma \to \Delta$ is defined as $i(k,n) = k/n$, and the Poisson kernel is

$$\Phi(\lambda;p) = p^k\,(1-p)^{n-k}; \quad \lambda = (k,n) \in \Gamma. \tag{7.7}$$

The conditions (i), (ii) hold trivially, and (iii) is the simplest form of the Law of Large Numbers. Every harmonic function can be represented by the Poisson integral (2.3).

In the definition of the boundary, the space Δ can be identified with that of extreme points in the compact convex simplex of harmonic functions. It does not depend on any choices. On the contrary, the map $i : \Gamma \to \Delta$ is not canonical. In the next Section we sketch two general methods of computing the boundary Δ, and in Section 9 we consider a number of examples, indicating both the space Δ and a natural inclusion i.

8. Computing the boundary

Essentially, there are two methods of computing the boundary of a branching: the ergodic method and that of positive homomorphisms. The second one can only be applied to multiplicative branchings (see Section 4).

We start with the ergodic method.

Given a branching $(\Gamma, \varkappa)$, let us say that an infinite path $t = (\nu_1, \nu_2, \dots, \nu_n, \dots)$ in Γ is *regular*, if the limits

$$\varphi_t(\lambda) = \lim_{n\to\infty} \frac{d(\lambda,\nu_n)}{d(\emptyset,\nu_n)} \tag{8.1}$$

exist for all $\lambda \in \Gamma$.

The function $\varphi_n(\lambda) = d(\lambda,\nu_n)\,/\,d(\emptyset,\nu_n)$ is almost harmonic: by (5.4), the first identity in (6.5) is valid for all $\lambda \in \Gamma$, except for the vertex $\lambda = \nu_n$. It follows immediately that whenever the limit (8.1) exists, it represents a harmonic function. Hence, there is a harmonic function φ_t associated to every regular path t of the branching. In fact, every function which is an extreme point in the compact convex space of harmonic functions, can be obtained in this way.

Theorem ([**22**]). *Let M be an ergodic central measure on the path space T of a branching $(\Gamma, \varkappa)$. Then almost all paths $t \in T$, with respect to M, are regular and produce via (8.1) the harmonic function $\varphi = \varphi_t$ associated to M by (6.4).*

Note that M is said to be *ergodic* if every measurable subset $A \subset T$ saturated with respect to the tail equivalence relation on T, has M-measure 0 or 1.

Corollary 1. *For every extreme harmonic function φ there exists a regular path $t \in T$ such that $\varphi = \varphi_t$.*

Corollary 2. *Consider a locally finite group $G = \varinjlim G_n$ as a direct limit of finite subgroups G_n. Then every indecomposable character $\psi : G \to \mathbb{C}$ is a pointwise limit $\psi(g) = \lim \psi_n(g)$ of irreducible characters ψ_n of the groups G_n.*

By the Theorem above (which is a version of G.Birkhoff Ergodic Theorem) one can find all extreme harmonic functions of a branching using information on the asymptotic behavior of combinatorial dimensions $d(\lambda, \nu_n)$ for a fixed $\lambda \in \Gamma$ and ν_n going 'to infinity'. In [**22**] this idea had been applied to computing the boundary of the Young lattice. The crucial fact is the following Lemma on the asymptotics of the number of skew Young tableaux.

Let us first recall the definition of extended symmetric functions (see [**22**]) and that of symmetric polynomials of a Young diagram (cf. [**9**]).

The extended power sum symmetric functions p_n are defined as follows:

$$\begin{aligned} p_1(\alpha;\beta;\gamma) &= \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i + \gamma, \\ p_n(\alpha;\beta;\gamma) &= \sum_{i=1}^{\infty} \alpha_i^n + (-1)^{n+1} \sum_{i=1}^{\infty} \beta_i^n \quad \text{if} \quad n > 1. \end{aligned} \tag{8.2}$$

Other symmetric functions are polynomials in $p_1, p_2, \ldots$. For instance,

$$s_\lambda(\alpha;\beta;\gamma) = \sum_{\rho \in \mathcal{Y}_n} \chi_\rho^\lambda \prod_{k \geq 1} \frac{p_k^{\rho_k}(\alpha;\beta;\gamma)}{k^{\rho_k}\, \rho_k!} \tag{8.3}$$

is a formula for Schur functions. Here χ_ρ^λ is the value of a character of the symmetric group $\mathfrak{S}_n$, indexed by λ, on a conjugacy class indexed by $\rho = (1^{\rho_1}, 2^{\rho_2}, \ldots)$.

Denote by $(f_1, \ldots, f_d; g_1, \ldots, g_d)$ the *modified Frobenius parameters* of a Young diagram $\nu = (\nu_1, \nu_2, \ldots) \in \Gamma_n$ (that is, d is the diagonal length of ν, $f_k = \nu_k - k + 1/2$, and $g_k = \nu_k' - k + 1/2$ where ν_k' is the length of the k-th column of ν). Given a symmetric polynomial P, one can evaluate it on a Young diagram ν by applying the extended version of P to Frobenius parameters of ν:

$$P(\nu) = P(f_1, \ldots, f_d; g_1, \ldots, g_d; 0).$$

For instance, $s_\lambda(\nu)$ denotes the Schur symmetric function s_λ evaluated on the Young diagram ν.

Lemma ([**22, 9**]). *For every Young diagram $\lambda \in \mathcal{Y}$, there exists a symmetric polynomial Q_λ of degree $\deg Q_\lambda < |\lambda|$ such that*

$$\frac{\dim(\nu \setminus \lambda)}{\dim(\nu)} = \frac{s_\lambda(\nu) + Q_\lambda(\nu)}{|\nu|\,(|\nu| - 1) \ldots (|\nu| - |\lambda| + 1)} \tag{8.4}$$

for all $\nu \in \mathcal{Y}$.

Examples (we denote $|\nu|$ by n):

$$\frac{\dim(\nu \setminus (2))}{\dim(\nu)} = \frac{s_{(2)}(\nu) - 1/2\, p_1(\nu)}{n\,(n-1)} = \frac{1}{2}\left(1 + \frac{\sum f_i - \sum g_i}{n\,(n-1)}\right);$$

$$\frac{\dim(\nu \setminus (3))}{\dim(\nu)} = \frac{s_{(3)} - p_1^2 - p_2 + 3/4\, p_1}{n\,(n-1)\,(n-2)};$$
$$\frac{\dim(\nu \setminus (21))}{\dim(\nu)} = \frac{s_{(21)} - 1/2\, p_1^2 + 1/4\, p_1}{n\,(n-1)\,(n-2)};$$
$$\frac{\dim(\nu \setminus (1^3))}{\dim(\nu)} = \frac{s_{(1^3)} - p_1^2 + p_2 + 3/4\, p_1}{n\,(n-1)\,(n-2)}.$$

The second approach to computing the boundary can be used in case of a multiplicative branching. It relies on the following fact.

THEOREM ([**11**]). *Consider a multiplicative branching $(\Gamma, \varkappa)$ determined by the triple $(R, \{s_\lambda\}, s)$ (see Section 4). Then the formula $\varphi(\lambda) = \psi(s_\lambda)$ establishes a one-to-one correspondence between extreme harmonic functions $\varphi : \Gamma \to \mathbb{R}$, and ring homomorphisms $\psi : R \to \mathbb{R}$ such that $\psi(s) = 1$.*

Example. The boundary of the Young lattice is in one-to-one correspondence with the space of Polya Frequency (PF-) sequences $\{\varphi_n\}$ such that $\varphi_1 = 1$. By definition, $\{\varphi_n\}$ is called PF-sequence, if all the determinants

$$\det\left(\varphi_{\lambda_i - i + j}\right) \geq 0; \quad \lambda \in \mathcal{Y}, \tag{8.5}$$

are non-negative. The corresponding homomorphism $\varphi : R \to \mathbb{R}$ of the symmetric function ring R can be defined on complete homogeneous symmetric functions h_n as $\varphi(h_n) = \varphi_n$, $n = 1, 2, \ldots$. It follows from (8.5) and Jacobi – Trudy formula, that φ is non-negative on Schur functions. All PF-sequences were classified in [**1**], and independently in [**20**].

9. Poisson integral for Young lattice and related branchings

We claim that the triple (Δ, i, Φ) in the examples below satisfies the conditions (i) – (iii) of the Definition in Section 7.

EXAMPLE A (Young lattice $\mathcal{Y}$). The boundary Δ is the space of pairs $(\alpha; \beta)$ of sequences of non-negative real numbers $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$, $\beta = (\beta_1, \ldots, \beta_n, \ldots)$ such that

$$\begin{gathered} \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq \ldots \geq 0; \quad \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq \ldots \geq 0, \\ \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq 1. \end{gathered} \tag{9.1}$$

The topology is that of coordinatewise convergence.

In order to describe the imbedding $i : \mathcal{Y} \to \Delta$, we use the modified Frobenius parameters $(f_1, \ldots, f_d; g_1, \ldots, g_d)$ of a Young diagram $\lambda \in \Gamma_n$ introduced in Section 8. By definition,

$$i(\lambda) = \left(\frac{f_1}{n}, \frac{f_2}{n}, \ldots; \frac{g_1}{n}, \frac{g_2}{n}, \ldots\right) \in \Delta, \tag{9.2}$$

where both sequences are tailed by zeros. The Poisson kernel

$$\Phi(\lambda; \alpha, \beta) = s_\lambda(\alpha; \beta; \gamma), \tag{9.3}$$

where $\gamma = 1 - \sum \alpha_i - \sum \beta_i$, is determined by the extended Schur functions defined in Section 8.

The main result of [**22**] affirms that every harmonic function $\varphi : \mathcal{Y} \to \mathbb{R}_+$ can be uniquely represented by the Poisson integral

$$\varphi(\lambda) = \int_\Delta s_\lambda(\alpha;\beta;1-\Sigma\alpha_i-\Sigma\beta_i)\,d\mu(\alpha;\beta), \tag{9.4}$$

where μ is a probability distribution on the simplex Δ. The value of the corresponding character of the group $\mathfrak{S}_\infty$ on a permutation with cycle partition ρ can be written as

$$\chi_\rho = \int_\Delta p_\rho(\alpha;\beta;1-\Sigma\alpha_i-\Sigma\beta_i)\,d\mu(\alpha;\beta). \tag{9.5}$$

The proof relies on the combinatorial Lemma of Section 8 describing asymptotics of the number of skew Young tableaux. The characters of $\mathfrak{S}_\infty$ were first found in the pioneering paper [**20**].

Example B (Kingman's branching $\mathcal{K}$, see [**12**], [**3**]). The boundary Δ is the simplex of nonincreasing sequences $\alpha = (\alpha_1, ..., \alpha_n, ...)$ of nonnegative numbers, with a restricted sum, $\gamma = 1 - \Sigma\alpha_i \geq 0$. The imbedding $i : \mathcal{Y} \to \Delta$ is given by the formula

$$i(\lambda) = \left(\frac{\lambda_1}{n}, \frac{\lambda_2}{n}, \ldots\right), \quad \lambda \in \mathcal{Y}_n. \tag{9.6}$$

The Poisson kernel

$$\Phi(\lambda;\alpha) = m_\lambda(\alpha;0;\gamma) \tag{9.7}$$

is determined by extended monomial symmetric functions

$$m_{(1^{\rho_1},2^{\rho_2},\ldots)}(\alpha;0;\gamma) = \sum_{k=0}^{\rho_1} \frac{\gamma^k}{k!}\, m_{(1^{\rho_1-k},2^{\rho_2},\ldots)}(\alpha_1,\alpha_2,\ldots), \tag{9.8}$$

see [**3**]. The Poisson integral takes the form

$$\varphi(\rho) = \int_\Delta m_\rho(\alpha;0;1-\Sigma\alpha_i)\,d\mu(\alpha). \tag{9.9}$$

The corresponding coherent families $\{M_n\}_{n=0}^\infty$ of probability distributions

$$M_n(\lambda) = \frac{n!}{\lambda_1!\lambda_2!\ldots}\,\varphi(\lambda); \quad \lambda \in \mathcal{Y}_n, \tag{9.10}$$

were studied by Kingman [**12**] under the name of *partition structures.*

Example C (truncated Jack's branching). The precise description of the boundary and the Poisson kernel for the general Jack's branching is not available. According to the Conjecture of Section 7.3 in [**4**], for all $\theta > 0$ the boundary should be the same as for the Young lattice. We will confine ourselves with the truncated branching, i.e. one restricted to the set $\mathcal{Y}(k)$ of Young diagrams with k rows or less. This case can be easily handled by the method of positive homomorphisms in Section 8.

The boundary Δ_k of the *truncated Jack's branching* $(\mathcal{Y}(k), \varkappa_\theta)$ is the simplex of finite non-increasing sequences $\alpha = (\alpha_1, ..., \alpha_k)$ of non-negative numbers, with the sum $\Sigma\alpha_i = 1$. The imbedding $i : \mathcal{Y}(k) \to \Delta_k$ is given by the formula (9.6). The Poisson kernel

$$\Phi(\lambda;\alpha) = P_\lambda(\alpha;\theta) \tag{9.11}$$

is now determined by zonal (Jack) symmetric polynomials with the parameter θ.

Every function $\varphi : \mathcal{Y}(k) \to \mathbb{R}$, harmonic with respect to the truncated Jack branching, can be uniquely represented by the Poisson integral

$$\varphi(\lambda) = \int_{\Delta_k} P_\lambda(\alpha;\theta)\, d\mu(\alpha), \tag{9.12}$$

where μ is a probability distribution on the simplex Δ_k.

10. Plancherel Growth Process in Analysis

Of all central measures on the Young lattice, the most interesting is the Plancherel measure M corresponding to the point $\alpha = \beta = 0$ of the boundary $\Delta = \Delta(\mathcal{Y})$. The transition probabilities for this Markov chain are

$$p(\lambda, \Lambda) = \frac{d(\Lambda)}{|\Lambda|\, d(\lambda)}; \quad \lambda \nearrow \Lambda, \tag{10.1}$$

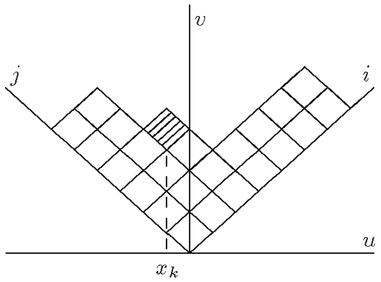

FIGURE 6. Growing Young diagram.

and its level distribution

$$M_n(\lambda) = \frac{d^2(\lambda)}{n!}; \quad \lambda \in \mathcal{Y}_n, \tag{10.2}$$

is the ordinary Plancherel measure of the symmetric group $\mathfrak{S}_n$.

The aim of this Section is to describe an amazing connection between the Plancherel measure of symmetric groups, and the partial fraction decomposition in Calculus. We shall give an entirely different presentation of transition probabilities (10.1).

To this end, it will be convenient to consider Young diagrams as piecewise linear continuous functions $v = \omega(u)$ such that

(a) $\omega'(u) = \pm 1$;

(b) $\omega(u) = |u|$ for large $|u|$.

Denote by $x_1, x_2, \dots, x_n$ the minima points of a diagram ω, and by $y_1, \dots, y_{n-1}$ the interlacing maxima points. For instance, we identify the one-box Young diagram with the function $\omega(u) = \max(|u|, 2 - |u|)$; in this example $x_1 = -1$, $x_2 = 1$ and $y_1 = 0$. We write $p(\lambda, \Lambda) = p_k$, if the new box $\Lambda \setminus \lambda$ is attached above the point x_k (Fig. 6). Thus, the transition measure (10.1) is a discrete one-dimensional probability distribution with finite support:

$$\operatorname{Prob}\{X = x_k\} = p_k; \quad k = 1, 2, \dots, n. \tag{10.3}$$

THEOREM ([**5**]). *Denote by*

$$x_1 < y_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < x_n \tag{10.4}$$

the interlacing sequences of minima and maxima of a Young diagram, and let (10.3) be the transition distribution of the Plancherel Growth Process. Then

$$\frac{\prod(x-y_j)}{\prod(x-x_j)} = \sum \frac{p_k}{x-x_k}, \tag{10.5}$$

or, equivalently,

$$p_k = \prod_{j<k} \frac{x_k - y_j}{x_k - x_j} \prod_{j>k} \frac{x_k - y_{j-1}}{x_k - x_j}.$$

Remark. For true Young diagrams the parameters x_k, y_k are integers, but we made no use of it in (10.5). For every pair of interlacing sequences (10.4), the coefficients p_k in (10.5) are positive and sum up to one. Hence, (10.5) can be used as a generalized *definition* of the Plancherel transition distribution.

COROLLARY ([**6**]). *Let $c(b) = j - i$ denote the content of a box $b \in \lambda$ in a Young diagram λ. Then the first two moments of the random variable c, with respect to the Plancherel measure, do not depend on the shape of the diagram λ:*

$$\begin{aligned} E(c) &= \sum_{\Lambda:\lambda\nearrow\Lambda} c(\Lambda \setminus \lambda)\, \frac{d(\Lambda)}{|\Lambda|\, d(\lambda)} = 0, \\ V(c) &= \sum_{\Lambda:\lambda\nearrow\Lambda} c^2(\Lambda \setminus \lambda)\, \frac{d(\Lambda)}{|\Lambda|\, d(\lambda)} = |\lambda|. \end{aligned} \tag{10.6}$$

Detailed information on the asymptotical behavior of the Plancherel measure (10.2) is available. The following result describes typical shapes of large random Young diagrams, with respect to this measure.

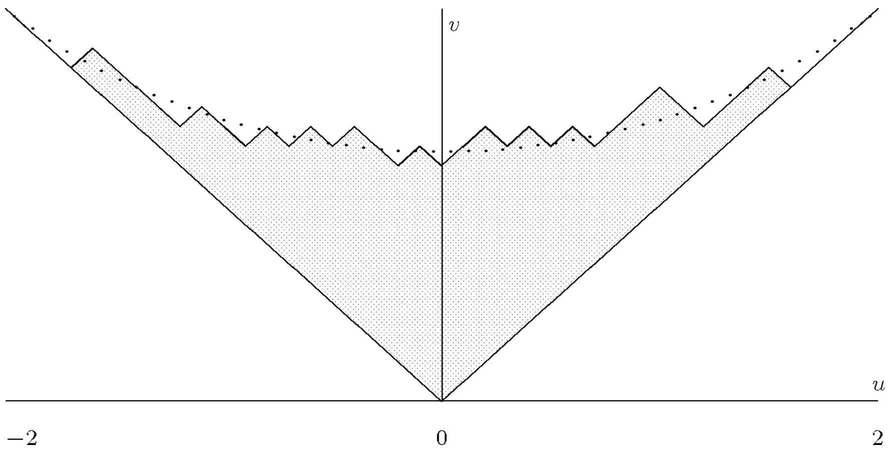

FIGURE 7. Typical shape of a Young diagram.

Theorem ([**14, 21**]). *Consider the function*

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2 \end{cases} \tag{10.7}$$

and an $\varepsilon > 0$, and let $\mathcal{Y}_n(\varepsilon) \subset \mathcal{Y}_n$ denote the set of Young diagrams with n boxes such that

$$\sup_u \left| \frac{1}{\sqrt{n}} \lambda(u\sqrt{n}) - \Omega(u) \right| < \varepsilon.$$

Then

$$\lim_{n\to\infty} M_n(\mathcal{Y}_n(\varepsilon)) = 1.$$

In Fig. 7, one can see a sample Young diagram with $n = 100$ boxes along with the graph of the function (10.7). The diagram was uniformly contracted in order to reduce its area to unity.

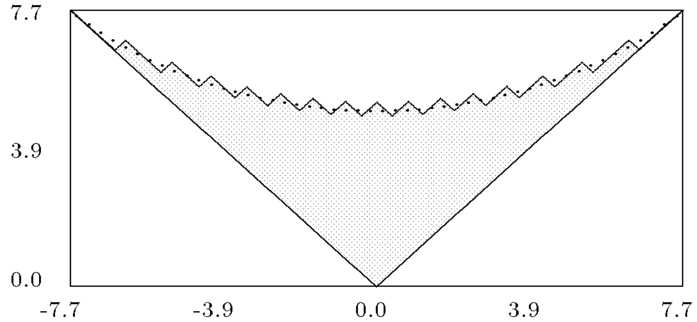

Figure 8. Root separation for Hermite polynomials.

It was shown in [**23**] that for any Young diagram $\lambda_n \in \mathcal{Y}_n$ which maximizes the number $d(\lambda)$ of Young tableaux over $\lambda \in \mathcal{Y}_n$, the shape of its normalized contraction is uniformly close to the graph of (10.7), too:

$$\lim_{n\to\infty} \sup_u \left| \frac{1}{\sqrt{n}} \lambda_n(u\sqrt{n}) - \Omega(u) \right| = 0.$$

The Theorem can be compared with the Law of Large Numbers. See [**7**] for the analog of Central Limit Theorem for large random Young diagrams.

In view of the direct relationship established by the formula (10.5) between Plancherel measure and the Partial Fraction Decomposition, there is no big surprise that the limiting shape of Young diagrams discovered in [**14, 21**] arises in various problems in Analysis, too. Let us mention just three examples: the asymptotics of root separation for generic orthogonal polynomials considered in [**5**], the asymptotics of solutions of Burgers equation

$$G_t(x,t) + G(x,t)\, G_x(x,t) = 0,$$

see [**8**], and the continuous version of Hook Walk algorithm with its application to Markov Moment Problem studied in [**6**].

For instance, the minima of a generalized diagram in Fig. 8 are located at zeros of Hermite polynomial $H_{16}(x)$ of degree 16, and the maxima at the roots of $H_{15}(x)$.

One can see that the diagram is uniformly close to the (scaled) graph of (10.7). The picture illustrates the main Theorem of [**5**].

It will be explained elsewhere that the Plancherel Growth Process of generalized Young diagrams coincides with the analog of Gaussian diffusion in Free Probability Theory of [**25**].

11. Multiplicative central measures

We introduce in this Section a class of *multiplicative* random Young diagrams. Their characteristic feature is that the probability of a Young diagram can be represented as a product of simple factors over all boxes in the diagram.

Example A (Young lattice). The Plancherel measure (10.2) of the symmetric group $\mathfrak{S}_n$ provides a simple example of such a multiplicative distribution. Using the Hook Formula (5.6) one can write (10.2) in the multiplicative form:

$$M_n(\lambda) = \frac{n!}{\prod_{b\in\lambda} h^2(b)}. \tag{11.1}$$

The Corollary of Section 10 implies that for any two constants u, v the following identity holds:

$$\sum_{\Lambda:\lambda\nearrow\Lambda} \frac{(c(\Lambda\setminus\lambda)+u)(c(\Lambda\setminus\lambda)+v)}{|\lambda|+uv}\cdot\frac{d(\Lambda)}{|\Lambda|d(\lambda)} = 1. \tag{11.2}$$

Assuming that the terms in the sum are nonnegative, we can declare them transition probabilities of a new Markov chain. Essentially, there are two such cases:

(a) u and v are complex conjugate numbers, $u = z$, $v = \bar{z}$. The transition probabilities are

$$p^{(z)}(\lambda,\Lambda) = \frac{|c(\Lambda\setminus\lambda)+z|^2}{|\lambda|+|z|^2}\cdot\frac{d(\Lambda)}{|\Lambda|\,d(\lambda)};\quad \lambda\nearrow\Lambda. \tag{11.3}$$

It is easy to see that these Markov chains are central on the ordinary Young lattice. Level distributions (6.13) can be written in the form

$$M_n^{(z)}(\lambda) = \frac{n!}{(|z|^2)_n}\prod_{b\in\lambda}\frac{|z+c(b)|^2}{h(b)^2}. \tag{11.4}$$

The family arises in harmonic analysis on the infinite symmetric group $\mathfrak{S}_\infty$. The Markov chains (11.3) are counterparts of spherical functions of a constant vector in pseudo-regular representations of that group, see [**10**] for details.

(b) $u = k$ is a positive integer, and $v = A + k - 1$, $A > 0$. The Markov chain with transition probabilities

$$p^{(k,A)}(\lambda,\Lambda) = \frac{(c(\Lambda\setminus\lambda)+k)\,(c(\Lambda\setminus\lambda)+A+k-1)}{|\lambda|+kA+k(k-1)}\cdot\frac{d(\Lambda)}{|\Lambda|\,d(\lambda)} \tag{11.5}$$

is supported by the set $\mathcal{Y}(k)$ of Young diagrams with k rows or less.

Again, the Markov chain (11.5) is central on the truncated Young lattice. It follows directly from (5.6) and (11.5) that its level distribution is multiplicative:

$$M_n^{(k,A)}(\lambda) = \frac{n!}{(kA+k(k-1))_n}\prod_{b\in\lambda}\frac{(k+c(b))\,(A+k-1+c(b))}{h^2(b)}. \tag{11.6}$$

Most of the multiplicative central measures on the Young lattice are decomposable, and in the next Section we will find the corresponding Poisson integrals. Meanwhile, let us describe multiplicative central measures for Kingman's branching and for the Jack branching.

EXAMPLE B (Kingman's branching). Consider the Young lattice $\mathcal{Y}$ with edge multiplicities (4.5). Using the dimension formula (5.7) and (6.11) we infer that co-transition probabilities are simply $q(\lambda, \Lambda) = j\,\rho_j(\Lambda)\,/\,|\Lambda|$, which is the probability that a random box in the Young diagram Λ will be drawn from a row of length j.

There is a well-known family of multiplicative probability distributions, given by the Ewens sampling formula (see [**12**]):

$$M_n^{(t)}(\lambda) = \frac{n!}{z_\lambda} \cdot \frac{t^{l(\lambda)}}{(t)_n}; \quad \lambda \in \mathcal{Y}. \tag{11.7}$$

The distribution depends on a positive parameter $t > 0$, and an important fact is that for each t the family (11.7) is coherent, i.e. the identity (6.12) holds. It is easy to describe the transition probabilities of the associated Markov chain. The probability that a new box will initiate a new row of a diagram λ is $t\,/\,(t+|\lambda|)$, and the probability that it will stick to an existing row of length j is $j\,\rho_j(\lambda)\,/\,(t+|\lambda|)$. See [**12**] for the history and applications of Ewens formula in genetics.

Consider now the *truncated* Kingman branching, where Young diagrams have at most k rows. The transition probabilities

$$p^{(k,A)}(\lambda, \Lambda) = \frac{(A+j)\rho_j(\lambda)}{kA + |\lambda|} \tag{11.8}$$

(where ρ_j denotes the number of rows of length j in λ) determine a central Markov chain. Its level distributions are multiplicative:

$$M_n^{(k,A)}(\lambda) = \frac{n!}{\lambda_1!\lambda_2!\dots} \cdot \frac{k!}{\rho_0!\rho_1!\rho_2!\dots} \cdot \frac{(A)_{\lambda_1}\dots(A)_{\lambda_k}}{(kA)_n}; \quad \lambda \in \mathcal{Y}_n. \tag{11.9}$$

Ewens formula (11.7) arises from (11.9) in the limit $k \to \infty$, $A \to 0$ and $kA \to t$.

EXAMPLE C (truncated Jack branching). The multiplicities for this branching were defined in (4.8). We can generalize transition probabilities (11.5) and (11.8) by introducing parameter θ:

$$\begin{aligned} p^{(k,A)}(\lambda, \Lambda) &= \frac{(k\,\theta + c_\theta(b))\,(A + (k-1)\theta + c_\theta(b))}{(kA + k(k-1)\theta + n)\,\theta} \times \\ &\times \prod_{hor} \left(\frac{a(b) + (l(b)+1)\,\theta}{a(b) + 1 + (l(b)+1)\,\theta} \right) \prod_{ver} \left(\frac{a(b) + 1 + l(b)\,\theta}{a(b) + 1 + (l(b)+1)\,\theta} \right). \end{aligned} \tag{11.10}$$

Here the product $\prod_{ver}$ runs over the boxes of the column of λ above the new box $\Lambda \setminus \lambda$, and $\prod_{hor}$ is a product along the row to the left of it. The level distributions will take the form

$$\begin{aligned} M_n^{(k,A)}(\lambda) &= \frac{n!}{(kA + k(k-1)\theta)_n} \times \\ &\times \prod_{\lambda} \frac{(k\theta + c_\theta(b))\,(A + (k-1)\theta + c_\theta(b))}{(a(b) + (l(b)+1)\,\theta)\,(a(b) + 1 + l(b)\,\theta)}. \end{aligned} \tag{11.11}$$

The formulae (11.10), (11.11) determine a family of central Markov chains on the truncated Jack branching $\mathcal{J}^{(\theta)}$. Note that for $\theta = 1$ they specialize to (11.5), (11.6), and for $\theta = 0$ to (11.8), (11.9).

By the construction,

$$\frac{n!}{(kA + k(k-1)\theta)_n} \sum_{\lambda \in \mathcal{Y}} \prod_{b \in \lambda} \frac{(k\theta + c_\theta(b))\,(A + (k-1)\theta + c_\theta(b))}{(a(b) + (l(b)+1)\,\theta)\,(a(b) + 1 + l(b)\,\theta)} = 1. \tag{11.12}$$

12. Selberg integrals

All of the multiplicative central measures considered in Section 11 (except for the Plancherel measure (11.1)) are decomposable. Hence, they have non-trivial representations by Poisson integrals (7.5). In this Section we find mixing measures and show that the corresponding Poisson integrals coincide with multidimensional integrals of Selberg type (see [**17**]).

Let us first show that the left hand side of (11.12) is an integral sum approximating Selberg integral.

LEMMA. *Let $\{\lambda^{(n)}\}_{n=1}^\infty$ be such a sequence of Young diagrams in $\mathcal{Y}_k$ that the limits*

$$\lim_{n\to\infty} \frac{\lambda_j^{(n)}}{n} = \alpha_j; \quad j = 1, 2, \dots, k, \tag{12.1}$$

exist. Then

$$\lim_{n\to\infty} n^{k-1}\, M_n^{(k,A)}(\lambda^{(n)}) = C_k(\theta) \prod_{1\le i<j\le k} |\alpha_i - \alpha_j|^{2\theta} \prod_{j=1}^k \alpha_j^{A-1}, \tag{12.2}$$

where the probabilities $M_n^{(k,A)}(\lambda)$ are given by (11.11) and

$$C_k(\theta) = \frac{\Gamma(kA + k(k-1)\,\theta)\,\Gamma^k(\theta)}{\prod_{j=1}^k \Gamma(A + (k-j)\,\theta)\,\Gamma(j\,\theta)}. \tag{12.3}$$

The proof is by a straightforward calculation. It follows that the weak limit of distributions $i(M_n^{(k,A)})$ is equal to the measure

$$\mu(d\alpha) = C_k(\theta) \prod_{1\le i<j\le k} (\alpha_i - \alpha_j)^{2\theta} \prod_{i=1}^k \alpha_i^{A-1}\, d\alpha_1 \dots d\alpha_{k-1} \tag{12.4}$$

on the simplex $\Delta_k = \{\alpha \in \mathbb{R}_+^k : \alpha_1 \ge \dots \ge \alpha_k, \sum \alpha_j = 1\}$. Since $\sum_\lambda M_n^{(k,A)}(\lambda) = 1$, it follows that $\mu(d\alpha)$ is a probability measure, and hence

$$\begin{aligned} &\int\!\!\cdots\!\!\int_{\tilde\Delta_k} \prod_{1\le i<j\le k} |\alpha_i - \alpha_j|^{2\theta} \prod_{j=1}^k \alpha_j^{A-1}\, d\alpha_1 \dots d\alpha_{k-1} = \\ &\qquad = \frac{\prod_{j=1}^k \Gamma(A + (k-j)\theta)\,\Gamma(j\theta + 1)}{\Gamma(kA + k(k-1)\theta)\,\Gamma^k(\theta + 1)}, \end{aligned} \tag{12.5}$$

where $\tilde\Delta_k = \{\alpha \in \mathbb{R}_+^k : \alpha_1 + \dots + \alpha_k = 1\}$ is the standard $(k-1)$-dimensional simplex.

Poisson integral (9.12) for the central Markov chain (11.10) coincides with the Selberg type integral

$$
\begin{aligned}
&\int\cdots\int_{\substack{\alpha_1,\dots,\alpha_k\ge 0\\ \alpha_1+\dots+\alpha_k=1}} P_\lambda(\alpha_1,\dots,\alpha_k;\theta)\prod_{1\le i<j\le k}|\alpha_i-\alpha_j|^{2\theta}\prod_{j=1}^{k}\alpha_j^{A-1}\,d\alpha_1\dots d\alpha_{k-1}=\\
&\qquad=\frac{P_\lambda(\overbrace{1,\dots,1}^{k\text{ times}};\theta)}{\Gamma(n+kA+k(k-1)\theta)}\prod_{j=1}^{k}\frac{\Gamma(\lambda_j+A+(k-j)\theta)\,\Gamma(j\theta+1)}{\Gamma(\theta+1)}.
\end{aligned}
\tag{12.6}
$$

This is to be compared with the integral (3.1) in [**17**]. Note that (12.6) is an easy consequence of the integral in [**17**], and implies, in its turn, the second Selberg integral (3.5) of [**26**].

Let us consider two specializations of (12.6).

For $\theta=1$, the formula for transition probabilities (11.10) takes the form (11.5), and the Poisson integral for the Markov chain can be written as

$$
\begin{aligned}
&\int\cdots\int_{\tilde\Delta_k} s_\lambda(\alpha_1,\dots,\alpha_k)\prod_{1\le i<j\le k}|\alpha_i-\alpha_j|^{2}\prod_{j=1}^{k}\alpha_j^{A-1}\,d\alpha_1\dots d\alpha_{k-1}=\\
&\qquad=\frac{s_\lambda(1,...,1)}{\Gamma(n+kA+k(k-1))}\prod_{j=1}^{k} j!\,\Gamma(\lambda_j+A+k-j).
\end{aligned}
\tag{12.7}
$$

In case $\theta=0$, transition probabilities (11.10) specialize to (11.8). The weak limit of distributions $i(M_n^{(k,A)})$ in Δ_k is

$$
\mu^{(k,A)}(d\alpha)=k!\,\frac{\Gamma(kA)}{\Gamma^k(A)}\prod_{j=1}^{k}\alpha_j^{A-1}\,d\alpha. \tag{12.8}
$$

This is the distribution of order statistics for the Dirichlet measure

$$
\tilde\mu^{(k,A)}(d\alpha)=\frac{\Gamma(kA)}{\Gamma^k(A)}\prod_{j=1}^{k}\alpha_j^{A-1}\,d\alpha \tag{12.9}
$$

on the simplex $\tilde\Delta_k$. The harmonic function associated to coherent distributions (11.9) is

$$
\varphi^{(k,A)}(\lambda)=\frac{k!}{\rho_0!\rho_1!\rho_2!\dots}\cdot\frac{(A)_{\lambda_1}\dots(A)_{\lambda_k}}{(kA)_n}, \tag{12.10}
$$

and its Poisson integral representation can be written in the form

$$
\int\cdots\int_{\tilde\Delta_k} m_\lambda(\alpha)\,\tilde\mu^{(k,A)}(d\alpha)=m_\lambda(\underbrace{1,\dots,1}_{k\text{ times}})\frac{(A)_{\lambda_1}\dots(A)_{\lambda_k}}{(kA)_n}. \tag{12.11}
$$

One can check that this is a special case $\theta=0$ of the general Selberg integral (12.6).

Ewens partition structures (11.7) can be obtained from (11.9) in the limit $k \to \infty$, $A \to 0$, $kA \to t$. The Poisson integral corresponding to the coherent family of Ewens distributions (11.7), can be written as

$$\frac{\lambda_1!\lambda_2!\dots}{z_\lambda} \cdot \frac{t^{l(\lambda)}}{(t)_n} = \int_\Delta m_\lambda(\alpha)\,dM_t(\alpha), \tag{12.12}$$

where M_t is the celebrated Poisson – Dirichlet measure on the infinite dimensional simplex of Example B, Section 9. See [**13**] for various descriptions of those measures and [**24**] for their fine structure.

Let us remark that no direct probabilistic description is known for the mixing measures of Markov chains (11.3). Apart from a few partial results in [**10**], the properties of Poisson integrals for this family remains far less understood than the Poisson – Dirichlet integrals (12.12). The study of these infinite dimensional integrals is a challenging open problem.

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