GAUSSIAN LIMIT FOR THE PLANCHEREL MEASURE OF THE SYMMETRIC GROUP

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We show in this paper that the Plancherel measure of the symmetric group $\mathfrak{S}_n$ converges weakly (as $n \to \infty$) to a Gaussian random process in the infinite dimensional vector space. With respect to the law of large numbers for this group, found earlier in [1,2] this assertion is analogous to the central limit theorem. The idea of such a result was mentioned in [3]. In the course of preparation of the article I had numerous valuable conversations with A.Vershik and G.Olshanski. In particular, it was Olshanski who suggested to use Lemma 2.3 below while discussing the preliminary version of the paper.

1. Main results. Let $\mathfrak{S}_n$ be the symmetric group of degree n and $\widehat{\mathfrak{S}}_n$ – the set (of equivalence classes) of its irreducible representations. The character of a representation $\lambda \in \widehat{\mathfrak{S}}_n$ will be denoted as χ^λ. If $\rho = (1^{r_1} 2^{r_2} \dots)$ is a partition of n, we consider the corresponding class of conjugate permutations in $\mathfrak{S}_n$ with r_k cycles of length k for $k = 1, 2, \dots$, and denote by χ^λ_ρ the value of the character χ^λ on this class.

Let $\dim \lambda = \chi^\lambda_{(1^n)}$ be the dimension of the representation $\lambda \in \widehat{\mathfrak{S}}_n$. The weights $M_n(\lambda) = \dim^2 \lambda / n!$ determine the probability distribution M_n on the set $\widehat{\mathfrak{S}}_n$ called *the Plancherel measure* of the group $\mathfrak{S}_n$. Define the random variable

$$\varphi_k(\lambda) = n^{k/2}\, \chi^\lambda_{(k,1^{n-k})} \,/\, \dim \lambda, \quad \lambda \in \widehat{\mathfrak{S}}_n, \tag{1.1}$$

as the (normalized) value of characters of representations λ on the conjugacy class of permutations with a single non-trivial cycle of length k. The central result of the paper is the following

Theorem 1. *For any $x_2, x_3, \dots, x_m \in \mathbb{R}$ there exists the limit*

$$\lim_{n\to\infty} M_n\{\lambda \in \widehat{\mathfrak{S}}_n : \varphi_k(\lambda) < x_k,\ \ 2 \le k \le m\} = \prod_{k=2}^{m} \frac{1}{\sqrt{2\pi k}} \int_{-\infty}^{x_k} \exp(-\frac{y_k^2}{2k})\, dy_k. \tag{1.2}$$

In other words, the functionals $\varphi_k(\lambda)$ are asymptotically independent and have the Gaussian limiting distributions with zero mean value and with the variance k.

Let us indicate a few alternative statements of the Theorem 1 dealing with other natural choices of main functionals. We shall use the standard identification of the set $\widehat{\mathfrak{S}}_n$ and the set $\mathcal{Y}_n$ of Young diagrams with n boxes.

We introduce the following notations for the unital Tchebychef polynomials of the first and of the second kind on the interval $[-2, 2]$:

$$t_r(2\cos\theta) = 2\cos r\theta, \quad u_r(2\cos\theta) = \sin(r+1)\theta \,/\, \sin\theta. \tag{1.3}$$

Let d be the length of the diagonal of a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{Y}_n$ and let $\tilde{a}_i = (\lambda_i - i)/\sqrt{n}$, $\tilde{b}_i = (\lambda'_i - i)/\sqrt{n}$; $i = 1, \dots, d$ be its normalized Frobenius co-ordinates. For any $r = 2, 3, \dots$ we set

$$t_r(\lambda) = \sum_i (t_r(\tilde{a}_i) + (-1)^{r+1}\, t_r(\tilde{b}_i)). \tag{1.4}$$

We also denote by $< f >_n = \sum_{\lambda\in\mathcal{Y}_n} f(\lambda)\, M_n(\lambda)$ the mean value of a function f according to the Plancherel measure M_n.

Theorem 2. *The Plancherel' average of the random variable (1.4) is zero for even r and is equal to*

$$< t_{2m+1} >_n = \frac{(-1)^m (2m+1)}{m(m+1)} \sqrt{n} + o(\sqrt{n}) \tag{1.5}$$

for $r = 2m+1$. As far as $n \to \infty$, the centralized variables $\overset{\circ}{t}_r(\lambda) = t_r(\lambda) - < t_r >_n$, $\lambda \in \mathcal{Y}_n$ are asymptotically independent and normal. The limiting variance of $t_r(\lambda)$ is equal to $\lim < \overset{\circ}{t}{}_r^2 >_n = r$.

Let $c(i,j) = (j-i)/\sqrt{n}$ denote the so called *contents* of a box (i,j) in the i-th row and j-th column of a Young diagram λ. For $r = 1, 2, \dots$ we set

$$u_r(\lambda) = \frac{1}{\sqrt{n}} \sum_{(i,j)\in\lambda} u_r(\frac{j-i}{\sqrt{n}}). \tag{1.6}$$

Theorem 3. *The Plancherel' average of the random variable (1.6) is zero for odd r and is equal to*

$$< u_{2m} >_n = \frac{(-1)^m}{m(m+1)} \sqrt{n} + o(\sqrt{n}) \tag{1.7}$$

for $r = 2m$. As far as $n \to \infty$, the centralized variables $\overset{\circ}{u}_r(\lambda) = u_r(\lambda) - < u_r >_n$, $\lambda \in \mathcal{Y}_n$ are asymptotically independent and normal. The limiting variance of $u_r(\lambda)$ is equal to $\lim < \overset{\circ}{u}{}_r^2 >_n = 1/(r+1)$.

Let us now consider Young diagrams as piecewise linear functions ω with the derivative ± 1 (this viewpoint was essential in [1], see also [3 – 7]). For instance, the one-box diagram $\omega \in \mathcal{Y}_1$ is identified with the function $\omega(x) = \max(|x|, 2-|x|)$. It follows from the law of large numbers established in [1] that the Plancherel average $1/\sqrt{n} \sum_{\omega\in\mathcal{Y}_n} \omega(x\sqrt{n})\, M_n(\omega)$ of a random diagram with n boxes (reduced by $\sqrt{n}$ along both axes) converges uniformly to the function

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2. \end{cases} \tag{1.8}$$

Let us embed the set $\mathcal{Y}_n$ of Young diagrams with n boxes in a function space, attaching to a diagram $\omega \in \mathcal{Y}_n$ its deviation $\Delta_\omega(x)$ from the curve (1.8). More precisely,

$$\Delta_\omega(x) = \omega(x\sqrt{n}) - \sqrt{n}\, \Omega(x). \tag{1.9}$$

We identify the Plancherel measure M_n with its image under this imbedding.

Theorem 4. *The Plancherel measure M_n converges weakly (as $n \to \infty$) to Gaussian random function on the interval $[-2, 2]$. The limiting process has zero average and the covariance function*

$$B(2\cos\varphi, 2\cos\psi) = \frac{1}{2\sin\varphi\sin\psi} \ln\left(\frac{\sin|\varphi+\psi|/2}{\sin|\varphi-\psi|/2}\right) - 1. \tag{1.10}$$

It can be represented by the series $\Delta(x) = \sum_{n\geq 1} \xi_n\, u_n(x)/\sqrt{n+1}$, where $\{\xi_n\}$ are independent standard normal random variables.

2. The sketch of the proof of Theorem 1. Let $\mathcal{Y} = \bigcup_{n\geq 0} \mathcal{Y}_n$ be the set of all Young diagrams (or, equivalently, the set of partitions). To any partition $\rho \in \mathcal{Y}$ we associate the family of central elements $c_{\rho;n} \in \mathbb{Q}[\mathfrak{S}_n]$, $n \geq 0$ defined by the formula

$$c_{\rho;n} = (n)_r\, \tilde{C}_{(\rho,1^{n-r})}. \tag{2.1}$$

Here $\rho = (1^{r_1}, 2^{r_2}, \dots)$ is the partition of $r = |\rho| = \sum k\, r_k$, $\tilde{C}_{(\rho,1^{n-r})}$ is the uniform probability distribution on the conjugacy class $C_{(\rho,1^{n-r})}$ of the group $\mathfrak{S}_n$, and $(n)_r = n(n-1)\dots(n-r+1)$. If $n < r$, we set $c_{\rho;n} = 0$. Remarkably, the convolution of classes, normalized as in (2.1), is stable in n:

Proposition 2.1. *For any pair of partitions σ, τ there exists the unique family of integer nonnegative coefficients $f^{\rho}_{\sigma,\tau}$, such that for all $n \geq 0$*

$$c_{\sigma;n} * c_{\tau;n} = \sum_{\rho} f^{\rho}_{\sigma,\tau}\, c_{\rho;n}. \tag{2.2}$$

We omit the proof of this Proposition and restrict ourselves by an example:

$$c_{(2);n} * c_{(2);n} = c_{(2^2);n} + 4c_{(3);n} + 2c_{(1^2);n}.$$

Note, that in the basis of true conjugacy classes the coefficients depend on n:

$$C_{(2,1^{n-2})} * C_{(2,1^{n-2})} = 2C_{(2^2,1^{n-4})} + 3C_{(3,1^{n-3})} + \binom{n}{2} C_{(1^n)}.$$

Let us define the *class ring* $\mathcal{C}$ as the free abelian group with generators $\{c_\rho\}$, $\rho \in \mathcal{Y}$, and the multiplication determined by structural constants $f^{\rho}_{\sigma,\tau}$, see (2.2). The elements $c_\rho \in \mathcal{C}$ determine the functionals on $\widehat{\mathfrak{S}} = \bigcup_{n\geq 0} \widehat{\mathfrak{S}}_n$ by the formula $c_\rho(\lambda) = \chi^\lambda(c_{\rho;n})/\dim\lambda$, and one can easily see that $(c_\sigma * c_\tau)(\lambda) = c_\sigma(\lambda)\, c_\tau(\lambda), \quad \lambda \in \widehat{\mathfrak{S}}$.

If $\deg(c_\rho) = r_1 + |\rho|$, then $f^{\rho}_{\sigma,\tau} \neq 0$ in (2.2) implies $\deg(c_\rho) \leq \deg(c_\sigma) + \deg(c_\tau)$. Hence, the function deg is a filtration of the class ring and we describe the corresponding graded ring $\operatorname{gr}\mathcal{C}$. By definition, this ring has the same additive generators $\{c_\rho\}$, and the multiplication is defined by the highest degree terms:

$$c_\sigma\, c_\tau = \sum_{\deg\rho = \deg\sigma + \deg\tau} f^{\rho}_{\sigma,\tau}\, c_\rho. \tag{2.3}$$

In the formulae below the products and powers of the elements c_ρ are taken in the ring $\operatorname{gr}\mathcal{C}$, i.e. up to the terms of lower degree. We refer to the elements $c_k = c_{(k)}$, corresponding to partitions $\rho = (k)$ with a single part, as *fundamental classes.*

Proposition 2.2. *The elements $c_1, c_2, \ldots$ are algebraically independent and generate the ring $\mathrm{gr}\,\mathcal{C} \simeq \mathbb{Z}[c_1, c_2, \ldots]$. If the partitions σ, τ have no parts of equal length greater than one, then $c_\sigma\, c_\tau = c_{\sigma\cup\tau}$ where the partition $\sigma \cup \tau$ is the union of parts of σ and τ. If all the parts of a partition are equal to k, then*

$$c_{(k^r)} = s_k^r\, H_r(c_k/s_k), \tag{2.4}$$

where $H_r(x) = (-1)^r\, e^{x^2/2}\,(e^{-x^2/2})^{(r)}$ is the Hermite polynomial and $s_k = \sqrt{k c_1^k}$.

The proof is based on the simple identity $c_{(\sigma,k^r)}\; c_k = c_{(\sigma,k^{r+1})} + k\, r\, c_{(\sigma,k^{r-1},1^k)}$, where we assume that σ has no parts of length k. Using induction in r one can see that

$$c_k^r = \sum_{s\geq 0} \binom{r}{2s}\,(2s-1)!!\, k^s\, c_{(k^{r-2s},1^{ks})} \tag{2.5}$$

in the ring $\mathrm{gr}\,\mathcal{C}$. The formula (2.4) is just the inversion of (2.5).

It is important that Plancherel averages $< f >_n$ for $f \in \mathcal{C}$ are polynomials in n.

Lemma 2.3. $< \sum_\rho f_\rho\, c_\rho >_n = \sum_k f_{(1^k)}\, (n)_k$.

The proof follows from the orthogonality relations for the characters: $< c_\rho >_n = (n)_r\,/\,n!\, \sum_\lambda \chi^\lambda_{(\rho,1^{n-r})} \chi^\lambda_{(1^n)} = (n)_r\, \delta_{\rho,(1^r)}$. It follows from this Lemma and (2.5) that

$$< c_k^{2r} >_n = k^r\,(2r-1)!!\, n^{kr}\,(1+o(1)) \tag{2.6}$$

and $< c_k^r >_n = 0$ for odd r. Moreover,

$$< c_2^{r_2} c_3^{r_3} \ldots >_n = \prod_{k\geq 2} < c_k^{r_k} >_n + o(n^{\deg(c_\rho)/2}). \tag{2.7}$$

We set $\varphi_\rho(\lambda) = n^{-\deg(c_\rho)/2}\, c_\rho(\lambda)$, $\rho \in \mathcal{Y}$. In particular, the variables $\varphi_k = \varphi_{(k)}$ are equivalent to those defined in (1.1). Denote by $< \varphi_2^{r_2}\varphi_3^{r_3} \ldots >_\infty = \lim < \varphi_2^{r_2}\varphi_3^{r_3} \ldots >_n$ the limits of moments of the Plancherel measure. One can see from (2.6), (2.7) that

$$< \varphi_k^r >_\infty = \begin{cases} k^m\,(2m-1)!!, & \text{if } r = 2m \\ 0, & \text{if } r = 2m-1 \end{cases} \tag{2.8}$$

and

$$< \varphi_2^{r_2}\varphi_3^{r_3} \ldots >_\infty = \prod_{k\geq 2} < \varphi_k^{r_k} >_\infty . \tag{2.9}$$

Hence, the moments (2.8) coincide with those of normal distribution with variance k, and (2.9) implies that $\varphi_2, \varphi_3, \ldots$ are independent with respect to the limiting measure.

3. On the proofs of Theorems 2 − 4. Consider the modified Frobenius co-ordinates $a_i = \lambda_i - i + 1/2$, $b_i = \lambda'_i - i + 1/2$. According to Lemma 1 of [8], the supersymmetric Newton sums $p_r(\lambda) = \sum_i (a_i^r + (-1)^{r+1} b_i^r)$ of the parameters a_i, b_i of a Young diagram λ belong to $\mathbb{Q}[c_1, c_2, \ldots]$, and generate it as a ring. It is easy to check that the conventional Newton sums $\pi_n(\lambda) = \sum_{(i,j)\in\lambda} (j-i)^n$ of contents of the boxes in a diagram λ generate the same ring.

Proposition 3.1. *The normalized fundamental classes (1.1) can be expressed in terms of the Newton sums $p_n(\lambda)$ in the following way:*

$$\varphi_r(\lambda) = \sum_{2j\le r} (-1)^j \frac{r}{r-j} \binom{r-j}{j} \frac{p_{r-2j}(\lambda)}{n^{(r-2j)/2}} - r\,\delta_{r-1}\sqrt{n} + o(1), \tag{3.1}$$

where $\delta_{2m-1} = 0$, $\delta_{2m} = (-1)^m/m(m+1)$ and $o(1)$ refers to L_2-norm.

Comparing (3.1) with the explicit formula $t_r(x) = \sum_{2j\le r}(-1)^j \frac{r}{r-j}\binom{r-j}{j} x^{r-2j}$ for Tchebychef polynomials of the first kind, we deduce Theorem 2 as a consequence of Theorem 1. Theorem 3 is equivalent to Theorem 2, since $p_{r+1} = (r+1)\,\pi_r$ up to the terms of lower degree deg.

In order to prove Theorem 4 we note that the expression

$$\frac{1}{\sqrt{n}}\, u_r(\omega) = \frac{1}{n} \sum_{(i,j)} u_r\left(\frac{j-i}{\sqrt{n}}\right)$$

is the integral sum for the double integral

$$\frac{1}{2} \iint_{D_{\tilde{\omega}}} u_r(x)\, dx\, dy = \frac{1}{2} \int u_r(x)\,(\tilde{\omega}(x) - |x|)\, dx,$$

where $D_{\tilde{\omega}} = \{(x,y) : |x| \le y \le \tilde{\omega}(x)\}$ and $\tilde{\omega}(x) = \omega(x\sqrt{n})/\sqrt{n}$. An easy calculation shows that $\frac{1}{2}\int u_r(x)\,(\Omega(x) - |x|)\, dx = \delta_r$. Hence,

$$u_r(\omega) - < u_r >_n = \frac{1}{2} \int u_r(x)\, \Delta_\omega(x)\, dx + o(1); \quad \omega \in \mathcal{Y}_n, \tag{3.2}$$

where $\Delta_\omega(x)$ is defined in (1.9). Theorem 4 follows now from Theorem 3.

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