ON AN INFINITE-DIMENSIONAL GROUP OVER A FINITE FIELD

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Introduction. The asymptotic representation theory studies the behavior of representations of large classical groups and their infinite-dimensional analogs. One of the key examples is the series of linear groups $GL_n(\mathbb{F}_q)$ over a finite field $k = \mathbb{F}_q$, and its inductive limit $GL_\infty(\mathbb{F}_q)$, the group of infinite invertible matrices over the field k with only finitely many non-zero entries.

In this note we introduce a new locally compact group GLB, and we study its structure, characters and continuous unitary representations. The group contains $GL_\infty(k)$ as a countable dense subgroup. The importance of the group GLB relies on its close connections with the theory of parabolic induction – the core of representation theory of the groups $GL_n(k)$.

The description of irreducible representations of the groups $GL_n(k)$ was first obtained by Green [6]. The main tool of his theory, later on developed in the papers [5, 10], is the operation of parabolic induction of representations. It is this operation that determines the inclusions of group algebras $\mathbb{C}(GL_n(k))$ leading to the group algebra A of the group GLB to be studied in this paper. The algebra A can be represented as an inductive limit of the algebras $\mathbb{C}(GL_n(k))$, though the inclusions of the group algebras are not generated by the group inclusions but rather are determined by the operators of taking average over the conjugacy classes with respect to a subgroup.

The group GLB has been defined by the present authors together with A. V. Zelevinsky (see the translation editor's addendum in the book [3]) in 1981 in the course of discussion of the connections of his paper [10] with the theory of representations and characters of locally finite groups (similar to $GL_\infty(k)$).

Our approach relies on the general theory of inductive limits of finite dimensional semisimple algebras. Note that in case of $q = 1$ we obtain the representation theory of the infinite symmetric group studied in a number of author's papers (see, e.g., the references in [1]). The remarkable fact that our infinite dimensional non-discrete group GLB is locally compact gives way to the full power of the classical methods of representation theory, the method of induced representations in particular. If the forthcoming papers we are going to study realizations of factor-representations of the groups GLB and $GL_\infty(k)$.

1. The group GLB. The objects to be defined below depend on the cardinality q of the field but for sake of the notational convenience we do not always make this dependence explicit.

Consider the k-linear space $V = k_\infty$ of all finite vectors with the coordinates in a finite field $k = \mathbb{F}_q$. Fix a basis of vectors $\{e_i, i = 1, 2 \dots\}$ in the space V, and denote by V_n the subspace generated by the first n vectors.

Definition. The group GLB consists of k-linear transformations of the space V preserving all but finitely many subspaces V_n.

Let us phrase the definition in terms of matrices of the operators $g \in GLB$ with respect to the distinguished basis. An infinite matrix $g = \bigl(g_{ij}\bigr)$, $i, j = 1, 2 \dots$ is called *almost triangular* if the number of its non-zero subdiagonal elements $a_{ij} \neq 0$, $i > j$ is finite. The group GLB consists exactly of operators with almost triangular matrices.

2. The Borel subgroup. Let $G(m, \infty) \subset GLB$ be the subgroup of operators in the space V, preserving the subspaces V_n for all $m \leq n$. If $m \leq n$, we denote by $G(m, n) \subset GL_N(k)$ the parabolic subgroup of operators in V_n respecting the subspaces V_s for $m \leq s \leq n$. The group $G(m, \infty) \cong \varprojlim G(m, n)$ is profinite, hence compact and totally disconnected in the projective limit topology. The system of subgroups $N(n, \infty)$, $n \geq m$ forms a basis of neigborhoods in $G(m, \infty)$. Here we denote by $N(n, \infty)$ the group of operators in V respecting the subspaces $V_n, V_{n+1}, \dots$ and trivial on V_n. In matrix terms, $G(m, \infty)$ can be described as the group of infinite matrices $g = (g_{ij})$ with zero subdiagonal elements below the m-th row, i.e., $g_{ij} = 0$ for $i > \max(j, m)$.

If $m < n$, the group $G(m, \infty)$ forms a subgroup of finite index in $G(n, \infty)$. The group $GLB = \varinjlim G(m, \infty)$ is the union (inductive limit) of the compact groups $G(m, \infty)$. The topology on the group GLB, as well as on the subgroups $G(m, \infty)$, is determined by the basis of neigborhoods of identity consisting of the subgroups $N(n, \infty)$. In this topology the group GLB is totally disconnected, locally compact, and is actually an infinite dimensional Lie group over the field k. Clearly, $GL_\infty(k)$ is a dense subgroup of GLB. The closure of every finitely generated subgroup in GLB is compact, and the quotient group $PGLB \cong GLB/k^*$ over the center of GLB is topologically simple.

The group $B(n) = G(1, n)$ of upper triangular matrices is a Borel subgroup in $GL_n(k)$. Therefore, one can naturally consider the profinite group $B = G(1, \infty)$ as a Borel subgroup in GLB. The group B consists of those operators in the space V that respect all the subspaces V_n, $n = 1, 2, \dots$. The matrices of the operators in B are exactly all invertible upper triangular matrices. As a topological space, the group B is an infinite product of subsets in the finite field k, hence it is compact and totally disconnected. The Haar measure on B is the product measure with uniform distributions of the factors. Directly from the definitions we obtain

Theorem 1. *The group GLB with the above topology is a locally compact unimodular amenable group.*

We fix a Haar measure μ on GLB normalized by $\mu(B) = 1$.

3. Bruhat-Schwartz group algebra. The important subalgebra A of the group algebra $L^1(GLB, \mu)$ which we define in this Section is the analog of the Bruhat-Schwartz function space in the theory of p-adic linear groups, and we retain the same term. In the algebra of all complex valued continuous functions with a compact support on the group GLB, with usual operations of convolution and involution, we define a $*$-subalgebra A consisting of functions with finitely many values. Note that the algebra A is dense in $L^1(GLB, \mu)$. We shall show that A is *locally semisimple*, that is, representable as a union of increasing chain of finite dimensional semisimple subalgebras over the field $\mathbb{C}$.

Theorem 2. *The algebra A is a locally semisimple $*$-algebra without identity.*

Let $A_n \subset A$ be the subalgebra of functions supported by a compact subgroup $G(n,\infty)$ and depending on the restriction of an operator $g \in G(n,\infty)$ to the subspace V_n only. The algebra A_n is finite dimensional, semisimple, and isomorphic to the group algebra $\mathbb{C}(G_n)$ of the finite group $G_n = GL_n(k)$. Given $g \in G_n$, we denote by $\tilde{a}_g \in A_n$ the characteristic function of the set of operators $h \in G(n,\infty)$ in the space V preserving the subspaces $V_n, V_{n+1}, \dots$ and coinciding with g when restricted to V_n. Set $a_g = |B/N(n,\infty)|\,\tilde{a}_g = (q-1)^n\, q^{\binom{n}{2}}\tilde{a}_g$. Then the functions a_g, $g \in G_n$ form a linear basis in the algebra A_n, such that $a_{gh} = a_g\, a_h$ and $a_g^* = a_{g^{-1}}$. The element a_e (where e is the identity operator in V_n) is the identity of the algebra A_n.

The algebra $A = \bigcup A_n$ is the union of imbedded subalgebras A_n. Since a_e is not the identity in the bigger algebra A_{n+1}, the algebra A does not have the identity.

Though the algebra A is represented as a union of group algebras, their inclusions are not generated by the group inclusions. More precisely, the inclusion $i : A_n \to A_{n+1}$ is determined by the operation of taking average over a subgroup,

$$i(a_g) = \frac{1}{|N_n|} \sum_{h \in N_n} a_{gh}; \qquad g \in G_n,$$

where $N_n = G_{n+1} \bigcap N(n,\infty)$ is the subgroup of operators in V_{n+1} identical on the subspace V_n. In the basis $\{\tilde{a}_g\}$ the inclusions are given by the formula $i(\tilde{a}_g) = \sum_{h \in N_n} \tilde{a}_{gh}$.

The increasing family $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ of finite dimensional semisimple algebras, and hence the limiting algebra $A = \bigcup A_n$, are entirely determined by its *branching graph* (Bratteli diagram) describing the branching of irreducible representations of the algebra A_{n+1} when restricted to a smaller subalgebra A_n, and *by dimensions of irreducible representations* (cf., e.g., [1]). The vertices of the branching graph are the equivalence classes of irreducible representations of all algebras A_n or, equivalently, of all groups $GL_n(k)$. Since the algebra inclusions do not respect the identities, the restriction of a non-zero representation of the algebra A_{n+1} onto the subalgebra A_n may contain the zero representation of the latter algebra.

4. The branching diagram of the family of algebras A_n. We start with a more general construction introducing the description of the Bratteli diagram of the inductive family of algebras A_n.

Let $C = \bigcup_{d=1}^{\infty} C_d$ be a graded family of finite sets and $e \in C_1$ a distinguished point. The number $d = d(c)$ will be called the *degree* of the element $c \in C_d$. Denote by $\mathbb{Y} = \bigcup_{n=0}^{\infty} \mathbb{Y}_n$ the set of Young diagrams, and let $|\lambda| = n$ be the number of boxes in a diagram $\lambda \in \mathbb{Y}_n$. We call a *family of Young diagrams* over C every function $\varphi : C \to \mathbb{Y}$ with a finite number of nonvoid values $\varphi(c) \neq \varnothing$. We define a grading by declaring the degree of a family φ to be $|\varphi| \equiv \sum_{c\in} |\varphi(c)|\, d(c)$. Let Φ_n be the set of families of degree $|\varphi| = n$, and $\Phi = \bigcup_{n=0}^{\infty} \Phi_n$ the set of all families. We take Φ for the vertex set of a graded graph to be defined below. Its edges are defined as follows: two families $\varphi \in \Phi_n$, $\psi \in \Phi_{n+1}$ are joined by an edge (in this case we write $\varphi \nearrow \psi$) if $\varphi(c) = \psi(c)$ for all $c \neq e$, and the Young diagram $\psi(e)$ is obtained from $\varphi(e)$ by attaching a single box. Thus, we have defined a branching graph (Bratteli diagram) Φ. The elements of the set C can be naturally identified with such families that their only nonvoid diagram contains just one box. We say that such families are *caspidal.*

Note that the diagram Φ is not connected and splits into a union of connected components isomorphic to the Young graph. The components are labelled by their *initial families* φ, such that the diagram $\varphi(e) = \varnothing$ is empty. We denote the set of initial families by $\widetilde{\Phi}$.

The branching graph of the inductive family of algebras $\{A_n\}$ that we are looking for, can be obtained upon a special choice of the set $C = C(q)$ depending on q. For our purposes the following combinatorial description of the set C will suffice. Fix a positive integer q. For $d \geq 2$, let C_d be the set of Lyndon words of length d in the alphabet $\{0, 1, \ldots, q-1\}$ of q letters. Recall, that a word is said to be a *Lyndon word*, if it is lexicographically strictly smaller than all of its cyclic shifts. It is known that $|C_d| = \sum_{m|d} \mu(d/m)\, q^m/d$ where μ is the classical Mobius function. For $d = 1$ we set $C_1(q) = \{1, \ldots, q-1\}$ (that is, the word 0 is deleted), and we take $e = 1$ for the distinguished element. Denote by $\Phi = \Phi(q)$ the branching graph of the families of Young diagrams constructed using the graded set $C = \bigcup C_d(q)$.

Theorem 3. *Let q be a power of a prime number. Then the branching graph Φ determines the Bratteli diagram of the family of subalgebras $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ approximating the group algebra A of the group GLB. This means that the characters χ_φ of irreducible representations π_φ of the algebra A_n are labelled by the families $\varphi \in \Phi_n$ of degree n, and that the restriction of character of an irreducible representation π_ψ of the algebra A_{n+1} to the subalgebra A_n decomposes into a multiplicity free sum*

$$\frac{1}{|N_n|} \sum_{h \in N_n} \chi_\psi(gh) = \sum_{\varphi:\, \varphi \nearrow \psi} \chi_\varphi(g), \qquad g \in G_n. \tag{1}$$

Recall the parameterization of irreducible representations of linear groups $G_n = GL_n(k)$ over a finite field (see [5,6,10]). The key role is plaid by the following operation of *parabolic induction*. Given representations π_1, π_2 of the groups G_m, G_{n-m} we denote by $\pi = \pi_1 \circ \pi_2$ the representation of the group G_n induced by the trivial extension of the representation $\pi_1 \times \pi_2$ of the subgroup $G_m \times G_{n-m} \subset P$ to the parabolic subgroup $P \subset G_n$. Here P consists of operators in V_n respecting the subspace V_m. In a similar way one can define the product of several representations $\pi_1 \circ \ldots \circ \pi_s$.

An irreducible representation of the group G_d is said to be *caspidal*, if it is not a subrepresentation of any nontrivial parabolic product of representations of groups of lower degree. The caspidal representations of the group $GL_d(\mathbb{F}_q)$ can be labelled by the points of the set C_d. Note that all characters of the group $G_1 \cong \mathbb{F}_q^*$ are caspidal, in particular the identity character e which plays a distinguished role in representation theory of the group GLB.

An irreducible representation of the group G_n is called c-primary, if it is a component of some power $c \circ c \circ \ldots \circ c$ of a caspidal representation c. It is known that primary representations are parameterised (given $c \in C_d$) by the set $\mathbb{Y}_m$ of Young diagrams with $m = n/d$ boxes. Denote by π^c_λ the irreducible primary representation of the group G_{md} contained in the product $c \circ c \circ \ldots \circ c$ and corresponding to a Young diagram $\lambda \in \mathbb{Y}_m$. Every irreducible representation of the group G_n can be uniquely (up to the order of the factors) represented as a parabolic product $\pi_\varphi = \prod_{c \in C} \pi^c_{\varphi(c)}$ of disjoint (i.e., contained in powers of distinct caspidal representations) primary representations $\pi^c_{\varphi(c)}$, where $n = \sum_c |\varphi(c)|\, d(c)$.

Therefore, the irreducible representations π_φ of the algebra A_n are parameterised by families $\varphi \in \Phi$ of Young diagrams of degree n, and the vertices of the branching graph of the algebras A_n coincide with those of the graph Φ. In particular, caspidal representations correspond to caspidal families, primary representations correspond to the families with a single nonvoid diagram, and the representations disjoint with the identity representation correspond to the arbitrary initial families. Our goal now is to check that the branching of representations of the algebras A_n coincide with that of the graph Φ.

Let $N_n \subset G_{n+1}$ be the group of operators in V_{n+1}, preserving V_n and coinciding on V_n with the identity operator. Let P_n be the group of operators in V_{n+1} preserving V_n. The group P_n is the semidirect product of the subgroup $G_n \subset G_{n+1}$ and the normal subgroup N_n. Consider the homomorphism $i : \mathbb{C}(G_n) \to \mathbb{C}(G_{n+1})$ determined by the formula $i(g) = |N_n|^{-1} \sum_{h \in N_n} gh$ as a composition of the inclusion $i : \mathbb{C}(G_n) \to \mathbb{C}(P_n)$ determined by the same formula, and the homomorphism $\mathbb{C}(P_n) \to \mathbb{C}(G_{n+1})$ determined by the group inclusion $P_n \to G_{n+1}$.

If π is an irreducible representation of the group G_n, we denote by $\widetilde{\pi}$ its extension to P_n trivial on the subgroup N_n. It is clear that $\widetilde{\pi}(i(g)) = \pi(g)$ for all $g \in G_n$. If an irreducible representation σ of the group P_n is non-trivial on N_n, then

$$\sigma\big(i(g)\big) = \frac{\sigma(g)}{|N_n|} \sum_{h \in N_n} \sigma(h) = 0$$

for all $g \in G_n$. By the Frobenius Reciprocity Theorem an irreducible representation $\widetilde{\pi}_\varphi$ of the group P_n is contained in the restriction to P_n of an irreducible representation π_ψ of the group G_{n+1} with exactly the same multiplicity as π_ψ is contained in the representation $\pi_\varphi \circ 1 = \bigoplus_{\psi : \varphi \nearrow \psi} \pi_\psi$, the latter decomposition being well known.

Denote by $\bar{A}$ the algebra A with the identity attached. This algebra, as well as the algebra A, is locally semisimple. The two-sided ideals of the algebras A and $\bar{A}$ are identical. Recall that in case of the algebras without identity the Bratteli diagram does not entirely determine the algebra: one also has to indicate the dimensions of simple modules. The Bratteli diagram of the algebra $\bar{A}$ with the attached identity can be obtained from the graph Φ by attaching a new vertex δ_n on each level Φ_n, $n = 1, 2, \ldots$. The new vertices of neighboring levels are connected by multiplicity free edges. Also, from every vertex δ_n we draw edges of appropriate multiplicities to all other vertices of the next level. Note that the edges of the diagram of the algebra A are multiplicity free, and the Bratteli diagram of the family $\{A_n\}$ is obtained from that of the family $GL_\infty(k)$ by erasing some edges.

5. The structure of the algebra A. There is a general theory (cf. [1,9]) describing the structure of a locally semisimple algebra, its characters, representations, and the Grothendieck group in terms of the Bratteli diagram of this algebra. Relying on the Theorem 3, we apply this theory to obtain the basic facts on the group GLB.

If $\varphi \in \widetilde{\Phi}$ is an initial family and λ is a Young diagram, we denote by $\psi = \varphi \circ \lambda$ the family of Young diagrams, such that $\psi(e) = \lambda$ and $\psi(c) = \varphi(c)$ for $c \neq e$. By definition, $\pi_{\varphi \circ \lambda} = \pi_\varphi \circ \pi^e_\lambda$ where π^e_λ is the e-primary representation corresponding to the Young diagram λ. Recall that by e we denote the distinguished point of the set C_1 labeling the identity representation of the group $GL_1(k)$.

Denote by $A_n(\varphi)$ the two-sided ideal of the algebra $A_n \cong \mathbb{C}(GL_n(k))$ defined as the intersection of kernels of irreducible representations π_ψ, $\psi \in \Phi_n$, distinct from π_φ. Given $\varphi \in \widetilde{\Phi}_d$, we set $I_n(\varphi) = \bigoplus_\lambda A_n(\varphi \circ \lambda)$ (the sum runs over all $\lambda \in \mathbb{Y}_{n-d}$), and we denote by $I(\varphi) = \bigcup_n I_n(\varphi)$ the corresponding two-sided ideal of the algebra A.

The Hecke algebra H_∞ of the pair (GLB, B) is defined as the subalgebra of those functions $a \in A$ which are constant on the double cosets with respect to the Borel subgroup B. This algebra is isomorphic to the group algebra $\mathbb{C}(\mathfrak{S}_\infty)$ of the infinite symmetric group. See [2] for the details on the characters of the algebra H_∞.

The above definition of the Hecke algebra is parallel to the classical definition of the finite dimensional Hecke algebra H_n as the double coset algebra with respect to the Borel subgroup in the group $GL_n(k)$. It is equivalent to the modern definition in terms of the infinite number of generators and well known relations.

Theorem 4. *The algebra A is the sum of countably many of two-sided ideals $I(\varphi)$, $\varphi \in \widetilde{\Phi}$. Each of these ideals is stably isomorphic to the algebra of finitely supported functions on the infinite symmetric group $\mathfrak{S}_\infty$. The ideal $I(\varnothing)$ contains the Hecke algebra $\mathcal{H}$, and is stably isomorphic to this algebra.*

6. The traces of the algebra A. We shall denote by χ_φ the trace of the representation of the algebra $A_n \cong \mathbb{C}(GL_n(k))$ corresponding to the family $\varphi \in \Phi_n$ normalized by the condition $\chi_\varphi(a_1) = 1$, where a_1 is the identity element of the algebra A_1. The branching rule for these traces is determined by the formula (1).

We define the *Thoma simplex* Δ as the space of pairs of real weakly decreasing sequences $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0)$, such that $\gamma \equiv 1 - \sum \alpha_i - \sum \beta_i \geq 0$. Recall [1] that the points of the Thoma simplex label the indecomposable harmonic functions on the Young graph.

Given a Young diagram λ, we denote by $s_\lambda(\alpha; \beta)$ the Schur function defined on the Thoma simplex Δ as indicated in [1]. If $g \in G_n$ and $\varphi \in \widetilde{\Phi}_d$ for all $d \leq n$, we set

$$\chi_{\alpha,\beta,\varphi}(a_g) = \sum_{\lambda \vdash (n-d)} \chi_{\varphi\circ\lambda}(g)\, s_\lambda(\alpha; \beta), \tag{2}$$

where $\chi_{\varphi\circ\lambda}$ is the character of the irreducible representation $\pi_{\varphi\circ\lambda} = \pi_\varphi \circ \pi^e_\lambda$ of the group G_n. By virtue of the branching rule and the harmonicity of the function $\lambda \mapsto s_\lambda(\alpha; \beta)$ on the Young graph, the function $\chi_{\alpha,\beta,\varphi}$ is correctly defined on the algebra A. One can also write

$$\chi_{\alpha,\beta,\varphi}(a_g) = \sum_{\mu \vdash (n-d)} \big(\chi_\varphi \circ \chi^e_{(\mu_1)} \circ \ldots \circ \chi^e_{(\mu_l)}\big)(g)\, m_\mu(\alpha; \beta), \tag{3}$$

where $\chi^e_{(\mu_1)} \circ \ldots \circ \chi^e_{(\mu_l)}$ is the product of the identity characters $\chi^e_{(\mu_j)}$ of the groups G_{μ_j} corresponding to the rows of $\mu = (\mu_1, \ldots, \mu_l)$ considered as new Young diagrams, and $m_\mu(\alpha; \beta)$ are the monomial supersymmetric functions in Thoma parameters.

Theorem 5. *The function $\chi_{\alpha,\beta,\varphi}$ is an indecomposable finite trace on the algebra A, normalized by the condition $\chi(a_1) = 1$. All of such traces have this form.*

All of the traces of the algebra A, except for the trivial one, are not bounded on the approximate identity of this algebra, hence determine via the well known GNS construction the representations of the algebras A, $\bar{A}$ of infinite type. We refer to the traces supported by the ideal generated by the Hecke algebra, as well as to the corresponding representations, as to *unipotent* traces (respectively, factor representations). The latter can be defined as representations containing a nonzero vector invariant with respect to the Borel subgroup.

7. The K_0-functor of the algebra A and multiplicativity of traces. In the theory of locally semisimple algebras the K_0-functor (Grothendieck group) plays the role of the major invariant (see [1]). Let $R_n = K_0(G_n)$ the Grothendieck group for $G_n = GL_n(k)$ and $R = \bigoplus_{n=0}^{\infty} R_n$. The parabolic induction determines a multiplication providing R with the structure of an associative algebra. Moreover, there is a Hopf algebra structure in R, and R is isomorphic, as Hopf algebra, to the tensor product of the algebras Λ_c (see §9 in[10]). Here $c \in C$ runs over the set of parameters of caspidal representations C, and Λ_c is the Hopf algebra of symmetric polynomials in the variables $x_1^c, x_2^c, \dots$. All of these variables are assigned the degree $d(c)$. Our goal is to relate R with the K_0-functor of the algebra A.

Denote by $\Lambda_\varnothing$ the subalgebra generated, as a linear space, by unipotent representations of the groups G_n. Let $\Lambda_\varphi = \Lambda_\varnothing \pi_\varphi$ be the $\Lambda_\varnothing$-submodule in R generated by the element $\pi_\varphi \in R$. The algebra R splits as a direct sum of $\Lambda_\varnothing$-submodules Λ_φ. The multiplication of a basis element π_φ by the identity representation $e \in R_1$ determines the branching graph Φ. Hence, this graph is *multiplicative* in the sense of [1].

The traces of the algebra A can be naturally identified with such additive functions $f : R \to \mathbb{R}$ that $f(\pi_\varphi) \geq 0$ for all $\varphi \in \Phi$, and $f(\pi \circ e) = f(\pi)$ for all $\pi \in R$. Clearly, every such function is the sum of its restrictions to the submodules Λ_φ. Therefore, every *indecomposable* function is nonzero at only one element π_φ, $\varphi \in \bar{\Phi}_d$. Up to normalization, all the functions $f_{\alpha,\beta,\varphi}$ with this condition are given by the formula

$$f_{\alpha,\beta,\varphi}(\pi_{\varphi\circ\lambda}) = s_\lambda(\alpha;\beta),$$

where $\pi_\varphi \in R_d$. They correspond to the trace $\chi_{\alpha,\beta,\varphi}$ of the algebra A determined by the formula (2).

Theorem 6. *Let χ be a unipotent character of a factor representation of the algebra A. Let $g \in G_m$ and $h \in G_{n-m}$ be two operators, such that no eigenvalue of g is an eigenvalue of h (the eigenvalues exist in the algebraic closure $\bar{k}$ of the field $k = \mathbb{F}_q$). Denote by $g \oplus h$ the corresponding block diagonal operator in G_n. Then the formula $\chi(a_{g\oplus h}) = \chi(a_g)\chi(a_h)$ holds true.*

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