INTERLACING MEASURES

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Abstract. We introduce the notion of interlacing measures generalizing that of interlacing sequences. Let $\tau = \tau_+ - \tau_-$ be the Jordan decomposition of a signed measure of bounded variation on the real line. Then the measures τ_+ and τ_- interlace if, and only if, there exists a probability distribution μ, such that the ordinary Cauchy-Stieltjes transform of μ equals the multiplicative version of this transform for τ:

$$\int \frac{\mu(du)}{z-u} = \exp \int \ln \frac{1}{z-u}\, \tau(du), \qquad \Im z \neq 0.$$

We survey various applications of this identity, including: (1) distributions of mean values of Poisson-Dirichlet random measures; (2) Markov moment problem; (3) exponential representations of functions of negative imaginary type; (4) spectral shift function of a self-adjoint operator; (5) the random growth of Young diagrams with respect to the Plancherel measures of the symmetric groups.

Assuming that the measures μ, τ have finite moments h_n, p_n for all $n = 0, 1, 2, \ldots$, the above identity implies that $\sum_{n=0}^{\infty} h_n z^n = \exp \sum_{n=1}^{\infty} p_n z^n / n$. We emphasize the combinatorial approach based on the study of moments of the measures μ, τ.

Résumé. On introduit une notion de mesures entrelacées qui généralise la notion de suites entrelacées. Soit $\tau = \tau_+ - \tau_-$ la décomposition de Jordan d'une mesure signée à variation bornée sur l'axe réel. Alors, les mesures τ_+ et τ_- sont entrelacées si et seulement si il existe une distribution de probabilité μ telle que la transformation ordinaire de Cauchy–Stieltjes de μ soit égale à la version multiplicative de la transformation de τ :

$$\int \frac{\mu(du)}{z-u} = \exp \int \ln \frac{1}{z-u}\, \tau(du), \qquad \Im z \neq 0.$$

On étudie plusieurs applications de cette identité, parmi lesquelles : (1) la distribution des moyennes des mesures aléatoires de Poisson–Dirichlet ; (2) le problème des moments de Markov ; (3) la représentation exponentielle des fonctions de type imaginaire négatif ; (4) la fonction spectrale décalée d'un opérateur auto-adjoint ; (5) la croissance aléatoire des diagrammes de Young selon la distribution de Plancherel des groupes symmetriques.

En supposant que les mesures μ et τ possèdent des moments finis h_n et p_n pour $n = 0, 1, 2, \ldots$, l'identité ci-dessus implique que $\sum_{n=0}^{\infty} h_n z^n = \exp \sum_{n=1}^{\infty} p_n z^n / n$. On insiste sur une approche combinatoire basée sur l'étude des moments des mesures μ et τ.

Key words and phrases. Interlacing sequences, Young diagrams, Markov moment problem, functions of negative imaginary type, Poisson-Dirichlet random measures, continued fractions, spectral shift functions, partition structures.

Contents

Introduction

The goal of the present paper is to study the relationship between a probability distribution μ and a bounded signed measure τ on the real line $\mathbf{R}$, satisfying the identity

$$\int_{-\infty}^{\infty} \frac{\mu(du)}{z-u} = \exp \int_{-\infty}^{\infty} \ln \frac{1}{z-u}\, \tau(du), \qquad z \in \mathbf{C} \setminus \mathbf{R}. \tag{1}$$

In other words, the ordinary Cauchy-Stieltjes transform of μ equals the multiplicative version of this transform for τ. By virtue of the Stieltjes inversion formula the measures μ, τ determine each other uniquely.

The identity (1) arises in quite a number of apparently unrelated contexts, such as:

(1) distributions of mean values of Poisson-Dirichlet random measures, see [10];
(2) the Markov moment problem, see [28];
(3) exponential representations of functions of negative imaginary type, see [3, 20];
(4) the spectral shift function of a self-adjoint operator, see [7, 29];
(5) the Plancherel growth of Young diagrams, see [23, 24].

As a basic introductory example motivating the study of equation (1) consider a rational function $R(z)$ written in two different ways:

$$\sum_{k=1}^{n} \frac{\mu_k}{z-x_k} = \frac{(z-y_1)\dots(z-y_{n-1})}{(z-x_1)(z-x_2)\dots(z-x_n)} \tag{2}$$

It is well known that the following properties of the fraction (2) are equivalent:

$$\mu_1, \mu_2, \dots, \mu_n > 0; \quad \sum_{k=1}^{n} \mu_k = 1 \quad \Longleftrightarrow \quad x_1 < y_1 < x_2 < \dots < x_{n-1} < y_{n-1} < x_n.$$

Assuming that these conditions hold, denote by μ the discrete distribution with the weights μ_k at the points x_k. Let τ be the signed measure with the weights $+1$ at the points x_k, and the weights -1 at the points y_k. Then the identity (1) specializes to (2).

In order to generalize this example we introduce the notion of interlacing measures. By definition, two finite positive measures τ_+, τ_- on the real line *interlace*, if

$$\tau_+\{(-\infty,u)\} \geq \tau_-\{(-\infty,u)\}; \qquad \tau_+\{(u,\infty)\} \geq \tau_-\{(u,\infty)\}$$

for all $u \in \mathbf{R}$, and $\tau_+\{\mathbf{R}\} - \tau_-\{\mathbf{R}\} = 1$. As an obvious example of interlacing measures one can take the uniform discrete measures τ_+, τ_- with unit weights at the points of two interlacing sequences. Another simple example is provided by any probability distribution $\tau_+ = \tau$, and a zero measure $\tau_- \equiv 0$.

Consider a bounded signed measure τ with the Jordan decomposition $\tau = \tau_+ - \tau_-$. Then a probability distribution μ satisfying (1) exists if, and only if, the Jordan components of τ interlace.

Assume that the measures μ, τ in the equation (1) have finite moments

$$h_n = \int_{-\infty}^{\infty} u^n \mu(du); \qquad p_n = \int_{-\infty}^{\infty} u^n \tau(du)$$

of all orders $n = 1, 2, \dots$, and that they are uniquely determined by the moments. Then the identity (1) can be written in the form

$$\sum_{n=0}^{\infty} h_n z^n = \exp \sum_{n=1}^{\infty} \frac{p_n}{n} z^n. \tag{3}$$

This equality of formal power series is equivalent to a series of polynomial equations

$$h_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{k \geq 1} p_k^{r_k(\pi)}; \qquad n = 0, 1, 2, \dots ,$$

where $r_k(\pi)$ denotes the number of cycles of length k in a permutation π of n objects.

A solid foundation for the study of identity (1) is provided by the theory of functions of negative imaginary type (i.e., the functions $w = R(z)$ analytic in the half-plane $\Im z > 0$ and having there non-positive imaginary part, $\Im R(z) \leq 0$), see [3, 20, 28]. The Cauchy-Stieltjes transforms of probability distributions,

$$R(z) = \int_{-\infty}^{\infty} \frac{\mu(du)}{z-u}, \tag{4}$$

form a part $\mathcal{N}$ in the space of functions of negative imaginary type, characterized by the condition $\lim_{y \to +\infty} iy\, R(iy) = 1$. Every function $R \in \mathcal{N}$ admits an *exponential representation* of the form

$$R(z) = \frac{1}{z} \exp\Big(- \int_{-\infty}^{0} \frac{F(u)du}{z-u} + \int_{0}^{\infty} \frac{1-F(u)}{z-u} du \Big), \tag{5}$$

where an integrable function F satisfies the conditions $0 \leq F(u) \leq 1$ and

$$\int_{-\infty}^{-1} F(u) \frac{du}{|u|} < \infty; \qquad \int_{1}^{\infty} \big(1 - F(u)\big) \frac{du}{u} < \infty.$$

In the special case of a function F of finite variation the equation (5) can be written as

$$R(z) = \exp \int_{-\infty}^{\infty} \ln \frac{1}{z-u}\, dF(u). \tag{6}$$

The distribution μ and the function F in the canonical representations (4), (5) of a function $R \in \mathcal{N}$ determine each other uniquely. We refer to the non-linear transformation $F \mapsto \mu$ as to the *Markov transform.*

It was probably A. A. Markov who initiated the study of equation (1). In his paper [34], published a century ago, he proved (see equation (8) in [34]) that for every bounded density $0 \le f(u) \le 1$, the coefficients $c_1, c_2, \dots$ of the continued fraction expansion

$$\frac{1}{z} \exp \int_0^1 \frac{f(u)}{z-u} du = \cfrac{1}{z - \cfrac{c_1}{1 - \cfrac{c_2}{z - \cfrac{c_3}{1 - \dots}}}}$$

are all positive. It follows from the celebrated memoir [37] by Stieltjes that such a fraction converges to an integral $\int \frac{\mu(du)}{z-u}$, with a positive measure μ.

A number of fundamental ideas were introduced in the field by M. G. Krein and his colleagues (see [20, 28, 29]). In particular, the Markov moment problem on a finite interval was reduced to the well-studied Hausdorff moment problem. M. G. Krein also developed to a much greater generality the idea of a spectral shift function suggested by I. M. Lifshits in [30]. As a powerful tool in these investigations, Krein used R. Nevanlinna's theory of integral representations of functions of negative imaginary type which he extended and generalized considerably. Important general results on the exponential representations of functions of negative imaginary type, and their relationship to the corresponding additive representations, were obtained by N. Aronszajn and W. F. Donoghue in [3]. See the equation (2.3.5) below for an example of fascinating properties of exponential representations found in [3].

Both Markov and Krein were mostly dealing with compactly supported measures which can be adequately studied in terms of their moments. On the contrary, in the approach to equation (1) based on the study of distributions of mean values of Poisson-Dirichlet random measures (see [10, 43]) one works with almost general probability distributions on the real line. We specialize and simplify the Aronszajn-Donoghue theory in such a way that it generalizes and unifies the two above approaches.

An important new idea concerning the representation (5), (6) is to treat a function F as a generalized Young diagram. More precisely, a function F in the exponential representation (5) can be written, under mild assumptions, in the form

$$F(x) = \frac{1}{2}\big(1 + \omega'(x)\big); \qquad 1 - F(x) = \frac{1}{2}\big(1 - \omega'(x)\big),$$
$$\omega(x) = \int_{-\infty}^{x} F(u)\, du + \int_{x}^{\infty} \big(1 - F(u)\big) du.$$

If a measure μ is the Markov transform of F, we say that the function ω is the *diagram* of μ. In the particular case where ω is the graph of a true Young diagram, the measure μ describes the transition probabilities of the Plancherel growth process, see [23, 24]. This analogy was used in [24] to obtain a new probabilistic procedure providing the Markov transform μ of a general pair of interlacing measures, the *interval shrinkage process*. In [23], the same idea led to a convergence theorem for interlacing roots of orthogonal polynomials. See also [25] for the equivalence between the Plancherel growth of generalized Young diagrams, and the Burgers-Hopf equation $\frac{\partial}{\partial t}R(z,t)+R(z,t)\frac{\partial}{\partial z}R(z,t)=0$.

The present paper is to a great extent expository and intended to exhibit a variety of applications of the identity (1). Still, the author believes the following results may be new:

(1) a characterization (3.5.2) of moments of interlacing sequences and measures in terms of characters of symmetric groups;
(2) the formula (3.7.4) for the difference derivatives of the polynomials (3.7.3) and the related generalization of the basic identity (3) to the general *partition structures* in the sense of Kingman [26];
(3) the combinatorial description (5.2.6) for the asymptotic series of the logarithmic derivative of a real J-type continued fraction;
(4) a direct combinatorial description for the moments of the spectral shift function of a bounded self-adjoint operator, see section 6;
(5) an explicit exponential representation (8.1.4) for the Cauchy-Stieltjes transform of the standard Gaussian measure.

The contents of the paper is as follows. In Section 1 we introduce the basic definitions of interlacing measures and Rayleigh functions, and define topology in the corresponding spaces. In Section 2 we establish a homeomorphism between the space of Rayleigh functions and that of probability distributions, based on the theory of functions of negative imaginary type. The moments of interlacing measures are defined in Section 3. We prove an identity for the difference derivatives of the moments of the measures μ, τ related by the identity (1) and provide a characterization of the moment sequences of interlacing measures. The connection with Poisson-Dirichlet random measures is sketched in Section 4. In Section 5 we present a combinatorial description of moments of interlacing measures similar to Flajolet theorem. A version of this result in the context of operator theory is outlined in Section 6. In Section 7 we briefly mention the important connection between interlacing measures and Young diagrams. In the final Section 8 we provide an explicit exponential representation for the Cauchy-Stieltjes transform of the Gaussian measure, and for its deformation to the semi-circle distribution related to the "associated Hermite polynomials" of Askey and Wimp.

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1. Limits of interlacing sequences

1.1 Interlacing sequences. Let $x = \{x_k\}_{k=1}^n$, $y = \{y_k\}_{k=1}^m$ be two finite sets[1] of real numbers. They are said to be *interlacing* if there is a point of x in between of any two points of y, and a point of y in between of any two points of x. As a matter of fact, there is exactly one point of one sequence in between of any two *neighboring* points of the other.

The numbers of elements of interlacing sequences can differ at most by one, $|n-m| \le 1$. In this paper we focus on the particular case when $m = n-1$, so that the sequences x, y interlace if, and only if,

$$x_1 < y_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < x_n. \tag{1.1.1}$$

The interlacing sequences of this type arise in quite a number of various contexts. We mention a few examples which are intended to motivate our further considerations.

(1.1.2) Example. Rayleigh theorem. Let $x_1 < x_2 < \ldots < x_n$ be the lengths of semi-axes of an ellipsoid X in $\mathbf{R}^n$. Denote by $Y = X \cap H$ the intersection of X with some hyperplane $H \subset \mathbf{R}^n$, and let $y_1 < \ldots < y_{n-1}$ be the lengths of semi-axes of the ellipsoid Y. Then the sequences x, y interlace.

(1.1.3) Example. Orthogonal polynomials. Denote by $\{P_n(x)\}_{n=0}^\infty$ a sequence of polynomials orthonormal with respect to a probability measure μ. Then the roots of two consecutive polynomials $P_n(x)$, $P_{n-1}(x)$ interlace.

(1.1.4) Example. Young diagrams. Let λ be a Young diagram with n boxes considered in the usual way as a subset of the plane (see Fig. 1). Call the number $c(b) = j - i$ the *content* of a point $b = (i,j)$ at the crossing of row i and column j. Then the contents of all corner points at the border of the diagram λ form a pair of interlacing sequences. For instance, Fig. 1 represents the Young diagram of an integer partition $7 = 4+2+1$, and the corresponding sequences are $x = (-3,-1,1,4)$ and $y = (-2,0,3)$. Interlacing sequences which arise from Young diagrams have two specific features: their elements are *integer* numbers, and $\sum_{k=1}^n x_k = \sum_{k=1}^{n-1} y_k$.

1.2 Convergence of interlacing sequences. In the above examples it is quite natural to look for the *limits* of interlacing sequences as $n \to \infty$. In order to define topology on the set $\mathcal{F}_0$ of interlacing sequences (1.1.1), we shall embed this set in a space of measures.

Note that the function

$$F_{x,y}(u) = \#\{k : x_k < u\} - \#\{k : y_k < u\}, \qquad u \in \mathbf{R} \tag{1.2.1}$$

determines a pair of interlacing sequences uniquely. We say that $F_{x,y}(u)$ is the *distribution function* of $(x,y) \in \mathcal{F}_0$ (see Fig. 2). This function is bounded, $0 \le F_{x,y}(u) \le 1$ for all

[1] We regard a finite set of reals as a sequence by enumerating its elements in the increasing order.

$u \in \mathbf{R}$, hence integrable on every finite interval in $\mathbf{R}$. We consider $F_{x,y}(u)$ as the *density* of an infinite absolutely continuous measure $F_{x,y}(u)du$ in $\mathbf{R}$, and endow the set $\mathcal{F}_0$ with the topology of proper weak convergence of such measures.

(1.2.2) Definition. Given a sequence $\{(x^{(n)}, y^{(n)})\}_{n=1}^{\infty}$ in $\mathcal{F}_0$, denote by F_n the distribution function of the pair $(x^{(n)}, y^{(n)})$. The sequence is said to *converge*, if the following two conditions hold:

(1) there exists an integrable function F, such that

$$\lim_{n\to\infty} \int_a^b F_n(u)du = \int_a^b F(u)du$$

for every finite interval $(a, b) \subset \mathbf{R}$;

(2) for every $\varepsilon > 0$ there exists $A > 0$, such that

$$\int_{-\infty}^{-A} \frac{F_n(u)du}{1+|u|} < \varepsilon, \qquad \int_A^{\infty} \frac{1-F_n(u)}{1+u} du < \varepsilon$$

for all $n = 1, 2, \ldots$.

The second condition is equivalent to the assumption

$$\int_{-\infty}^{0} \frac{F(u)du}{1+|u|} < \infty, \qquad \int_0^{\infty} \frac{1-F(u)}{1+u} du < \infty.$$

The purpose of imposing this condition in the definition of convergence will be made clear in Section 2.3. In a few words, our definition of convergence corresponds to the *proper* weak convergence of probability distributions to be associated to interlacing sequences.

(1.2.3) Example. For $n = 1, 2, \ldots$ define a pair $(x^{(n)}, y^{(n)})$ of interlacing sequences as

$$x_k^{(n)} = 2(k-n), \quad k = 0, 1, \ldots, 2n; \qquad y_k^{(n)} = 2(k-n) - 1, \quad k = 1, \ldots, 2n.$$

Then the functions $F_n(u)$ converge pointwise, $\lim_{n\to\infty} F_n(u) = F(u)$, and the condition (1) holds. Still, the family does not converge since the condition (2) fails.

If there exists an interval $[\alpha, \beta]$, such that $\alpha \le x_1^{(n)}$ and $x_n^{(n)} \le \beta$ for all $n = 1, 2, \ldots$, the condition (2) is redundant.

(1.2.4) Example. Consider a family of interlacing sequences

$$x_k^{(n)} = \frac{2k}{2^n} - 1, \quad k = 0, 1, \ldots, 2^n; \qquad y_k^{(n)} = \frac{2k-1}{2^n} - 1, \quad k = 1, \ldots, 2^n.$$

This family converges, the limiting function F being

$$F(u) = \begin{cases} 0, & \text{if } u < -1 \\ 1/2, & \text{if } -1 < u < 1 \\ 1, & \text{if } 1 < u. \end{cases}$$

Note that F is *not* a pointwise limit of F_n, see Fig. 3.

The function F in the example 2 is the distribution function of a discrete probability measure τ with two equal point masses $1/2$ at the points $x = -1$ and $x = 1$. As this example suggests, distribution functions of interlacing sequences may converge to a distribution function of a probability measure. We introduce the notion of interlacing measures which includes both probability measures and interlacing sequences.

1.3 Interlacing measures. Given a sequence $x = \{x_k\}_{k=1}^n$ and a real number $u \in \mathbf{R}$, denote by $L_x(u) = \#\{k : x_k < u\}$, $R_x(u) = \#\{k : x_k > u\}$ the numbers of points in x to the left and to the right of u. Note that the sequences $x = \{x_k\}_{k=1}^n$ and $y = \{y_k\}_{k=1}^{n-1}$ interlace if, and only if, $L_x(u) \geq L_y(u)$ and $R_x(u) \geq R_y(u)$ for all $u \in \mathbf{R}$. In this form the definition can be generalized to measures in $\mathbf{R}$.

Let τ be a finite positive measure in $\mathbf{R}$. We denote by $L_\tau(u) = \tau\{(-\infty, u)\}$ the distribution function of the measure τ, and by $R_\tau(u) = \tau\{(u, \infty)\}$ its hazard function.

(1.3.1) Definition. We say that two finite positive measures τ_+, τ_- *interlace* if the following four conditions hold:

$$L_{\tau_+}(u) \geq L_{\tau_-}(u), \qquad R_{\tau_+}(u) \geq R_{\tau_-}(u); \qquad \forall u \in \mathbf{R}; \tag{1.3.2}$$

$$\tau_+\{(-\infty, \infty)\} - \tau_-\{(-\infty, \infty)\} = 1; \tag{1.3.3}$$

$$\int_{-\infty}^{\infty} \ln(1 + |u|)\tau_+(du) < \infty, \qquad \int_{-\infty}^{\infty} \ln(1 + |u|)\tau_-(du) < \infty; \tag{1.3.4}$$

$$\text{the measures } \tau_\pm \text{ are mutually singular.} \tag{1.3.5}$$

(1.3.6) Example. Let τ_+, τ_- be uniform discrete measures with unit weights at the points of interlacing sequences x, y correspondingly. Then the measures $\tau_\pm$ interlace.

(1.3.7) Example. Let $\tau_+ = \tau$ be any probability distribution for which the first integral in (1.3.4) converges, and let τ_- be the zero measure. Then τ_+ and τ_- interlace.

Denote by $\tau = \tau_+ - \tau_-$ the difference of interlacing measures $\tau_\pm$. By the condition (1.3.5), $\tau = \tau_+ - \tau_-$ is the Jordan decomposition of the signed measure τ, hence τ determines τ_+ and τ_- uniquely. One can restate the definition of the interlacing measures in terms of their difference.

(1.3.8) Definition. Let τ be a bounded measure in $\mathbf{R}$, not necessarily positive. We say that τ is a *Rayleigh measure*, if the following three conditions hold:

$$0 \leq \tau\{(-\infty, u)\} \leq 1, \qquad \forall u \in \mathbf{R}; \tag{1.3.9}$$

$$\tau\{(-\infty, \infty)\} = 1; \tag{1.3.10}$$

$$\int_{-\infty}^{\infty} \ln(1 + |u|)\, |\tau(du)| < \infty. \tag{1.3.11}$$

The distribution function of a sample Rayleigh measure is shown in Fig. 4. Two positive measures interlace if, and only if, their difference is a Rayleigh measure. For each Rayleigh measure τ its Jordan components τ_+, τ_- interlace.

1.4 Rayleigh functions. The set of Rayleigh measures is only a part of a larger space $\mathcal{F}$ which arises as a completion of the space $\mathcal{F}_0$ of interlacing sequences. We proceed with the description of $\mathcal{F}$.

(1.4.1) Definition. Any measurable function $F : \mathbf{R} \to \mathbf{R}$, such that

$$0 \leq F(u) \leq 1 \qquad \forall u \in \mathbf{R}, \tag{1.4.2}$$

$$\int_{-\infty}^{0} \frac{F(u)du}{1 + |u|} < \infty, \qquad \int_{0}^{\infty} \frac{1 - F(u)}{1 + u} du < \infty, \tag{1.4.3}$$

will be referred to as a *Rayleigh function.* If the difference of two Rayleigh functions vanishes a.e., we consider such functions as identical.

(1.4.4) Example. Choose two *infinite* sequences x, y, such that $x_1 < y_1 < x_2 < y_2 < \dots$ and $\lim y_n = \infty$, and define their distribution function $F(u) = F_{x,y}(u)$ by the equation (1.2.1). Then the condition (1.4.3) is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \frac{x_{k+1} - y_k}{y_k} < \infty. \tag{1.4.5}$$

It implies that the infinite product

$$R(z) = \prod_{k=1}^{\infty} \frac{z - y_k}{z - x_k} \tag{1.4.6}$$

converges for $z \neq x_k$, $k = 1, 2, \dots$, and that the sum of the coefficients of the partial fractions in the decomposition

$$R(z) = \sum_{k=1}^{\infty} \frac{\mu_k}{z - x_k} \tag{1.4.7}$$

equals one, $\sum \mu_k = 1$. The latter condition yields, in its turn, the assumption (1.4.5).

By definition, the limit F of a converging family of interlacing sequences is always a Rayleigh function. Every Rayleigh function can be approximated by a family of interlacing sequences.

A Rayleigh function $F(u)$ determines two finite absolutely continuous positive measures on left and right semi-axes, with the densities

$$F^-(u) = \frac{F(u)}{1 + |u|}, \quad u < 0; \qquad F^+(u) = \frac{1 - F(u)}{1 + u}, \quad u > 0.$$

We define the convergence of Rayleigh functions as the proper weak convergence of the measures $F^-(u)du$, $F^+(u)du$.

(1.4.8) Definition. A sequence $\{F_n\}_{n=1}^{\infty}$ of Rayleigh functions *converges* to a function F, if

$$\lim_{n \to \infty} \int_{-\infty}^{x} \frac{F_n(u)du}{1 + |u|} = \int_{-\infty}^{x} \frac{F(u)du}{1 + |u|}, \quad \lim_{n \to \infty} \int_{x}^{\infty} \frac{1 - F_n(u)}{1 + |u|} du = \int_{x}^{\infty} \frac{1 - F(u)}{1 + |u|} du \tag{1.4.9}$$

for all $x \in \mathbf{R}$.

The definition conforms to that of convergence of interlacing sequences in Section 1.2. The limiting function F is a Rayleigh function, too.

We denote the space of Rayleigh functions by $\mathcal{F}$. The convergence $F_n \to F$ to a function $F \in \mathcal{F}$ simply means that

$$\lim_{n \to \infty} \int_a^b F_n(u)du = \int_a^b F(u)du$$

for all finite intervals (a, b). The subset $\mathcal{F}_0$ of interlacing sequences is dense in $\mathcal{F}$.

If τ is a Rayleigh measure, its distribution function $F(u) = L_\tau(u)$ (see Fig. 4) is of bounded variation, hence Lebesgue integrable in every finite interval, and the condition (1.4.2) holds. The condition (1.4.3) is equivalent to the assumption (1.3.11). Thus, we have identified the Rayleigh measures with Rayleigh functions of bounded variation. We denote by $\mathcal{F}_b$ the subspace of such functions. Fig. 5 represents a Rayleigh function of infinite variation.

2. Functions of negative imaginary type

2.1 Integral representations. Let $w = R(z)$ denote a function of a complex variable z, analytic in the upper half-plane $\Im z > 0$. Recall that R is said to be of *negative imaginary type* (see [6]) if

$$\Im z > 0 \qquad \Longrightarrow \qquad \Im R(z) \le 0. \tag{2.1.1}$$

We denote the space of such functions by $\widetilde{\mathcal{N}}$, and we endow it with the topology of uniform convergence on compact subsets in the open upper half-plane.

Assume that a function $R \in \widetilde{\mathcal{N}}$ is rational, and that

$$\lim_{z\to\infty} z\, R(z) = 1. \tag{2.1.2}$$

Then R is a ratio of unimodal polynomials with interlacing real simple roots. Conversely, for each pair $(x, y) \in \mathcal{F}_0$ of interlacing sequences (1.1.1) the function

$$R(z) = \frac{(z - y_1)\dots(z - y_{n-1})}{(z - x_1)(z - x_2)\dots(z - x_n)} \tag{2.1.3}$$

is of negative imaginary type and the condition (2.1.2) holds.

A proper rational fraction can also be represented by its partial fraction decomposition,

$$R(z) = \sum_{k=1}^{n} \frac{\mu_k}{z - x_k}. \tag{2.1.4}$$

Note that $R \in \widetilde{\mathcal{N}}$ if, and only if, $\mu_k > 0$ for all $k = 1, 2, \dots, n$. The condition (2.1.2) is equivalent to the identity $\sum \mu_k = 1$.

The formula (2.1.4) is a particular case of the classical integral representation valid for all functions $R \in \widetilde{\mathcal{N}}$.

(2.1.5) Theorem (R. Nevanlinna, see [28], Theorem A.2). *Every function R of negative imaginary type can be uniquely represented in the form*

$$R(z) = -\alpha z + \beta + \int_{-\infty}^{\infty} \Big(\frac{1}{z - t} + \frac{t}{1 + t^2}\Big)\nu(dt), \tag{2.1.6}$$

where $\alpha > 0$, $\beta \in \mathbf{R}$ and ν is a positive measure, such that $\int_{-\infty}^{\infty} \frac{\nu(dt)}{1+t^2} < \infty$.

(2.1.7) Corollary (see [36], Lemma II.2.2). *A function $w = R(z)$ can be represented as a Cauchy-Stieltjes transform*

$$R(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t} \tag{2.1.8}$$

of a probability distribution μ if, and only if, it has negative imaginary type, and

$$\lim_{y\to+\infty} iy\, R(iy) = 1. \tag{2.1.9}$$

The measure μ in the representation (2.1.8) is determined by the function R uniquely, and can be restored by the Stieltjes-Perron inversion formula

$$\int_a^b \mu(du) = -\frac{1}{\pi} \lim_{y\to+\infty} \int_a^b \Im R(x+iy)dx. \tag{2.1.10}$$

(2.1.11) Example. Consider the rational fraction (2.1.3) whose poles and zeroes are given by the sequences $x = (-3,-1,1,4)$, $y = (-2,0,3)$ of Example 1.1.4 (these sequences correspond to the Young diagram in Fig. 1). Fig. 6 represents the curve $w = R(u+0.1i)$ in the lower half-plane, $-\infty < u < \infty$. In the Fig. 7 one can see the density $-\frac{1}{\pi}\Im R(u+0.1\,i)$ approximating the discrete measure with the weights $9/28, 1/5, 1/4, 8/35$ at the points $-3,-1,1,4$ correspondingly. The latter is a transition distribution of the Markov chain on the Young lattice, called *Plancherel growth process*, see equation (7.1.1) below. The graph of the function $1+\frac{1}{\pi}\arg R(u+0.1\,i)$ is shown in Fig. 8; it approaches the Rayleigh function of Fig. 2.

We denote by $\mathcal{N}$ the set of all functions $R \in \widetilde{\mathcal{N}}$ satisfying the condition (2.1.9). A sequence of functions $R_n \in \mathcal{N}$ is said to converge *properly* if it converges in $\widetilde{\mathcal{N}}$, and for any $\varepsilon > 0$ there exists $A > 0$, such that $|iy\, R_n(iy) - 1| < \varepsilon$ for all $y > A$ and all $n = 1, 2, \ldots$. The limit of such a sequence belongs to $\mathcal{N}$. If a sequence R_n converges in $\widetilde{\mathcal{N}}$ to a function $R \in \mathcal{N}$, the convergence is the proper one.

Let $\mathcal{M}$ be the space of all probability distributions in $\mathbf{R}$ endowed with the topology of proper weak convergence (see [13]). Then the representation (2.1.8) establishes a homeomorphism between the spaces $\mathcal{M}$ and $\mathcal{N}$. It is clear from (2.1.4) that the subset $\mathcal{M}_0$ of discrete distributions $\mu \in \mathcal{M}$ with finite support corresponds to the subset $\mathcal{N}_0 \subset \mathcal{N}$ of rational functions.

2.2 Exponential representations. For every function $R \in \widetilde{\mathcal{N}}$ its logarithm $\ln R$ also belongs to $\widetilde{\mathcal{N}}$, hence admits an integral representation of type (2.1.6). Since the imaginary part of the logarithm is bounded, $-\pi \le \Im \ln R(z) = \arg R(z) \le 0$, its integral representation has specific features.

(2.2.1) Theorem (see [28], Theorem A.3 or [3], 1.10). *Every function $R \in \widetilde{\mathcal{N}}$ admits a unique representation*

$$R(z) = -C \exp \int_{-\infty}^{\infty} \Big(\frac{1}{u-z} - \frac{u}{1+u^2}\Big) F(u)du, \tag{2.2.2}$$

where $C > 0$ and $0 \le F(u) \le 1$ for all $u \in \mathbf{R}$.

When specialized for the subclass $\mathcal{N} \subset \widetilde{\mathcal{N}}$, the *exponential representation* (2.2.2) can be written as follows.

(2.2.3) Theorem. *Every function $R \in \mathcal{N}$ admits a unique representation of the form*

$$R(z) = \frac{1}{z} \exp\Big(- \int_{-\infty}^{0} \frac{F(u)du}{z-u} + \int_{0}^{\infty} \frac{1-F(u)}{z-u} du \Big), \tag{2.2.4}$$

where $F \in \mathcal{F}$ is a Rayleigh function. Given a Rayleigh function F, the integrals in the right hand side of (2.2.4) converge for all non-real z, and $R \in \mathcal{N}$.

Proof. Assume that $\Im z > 0$. Since $\Im R(z) \le 0$, we have $\ln R(z) = -i\pi + \ln(-R(z))$, and it follows from (2.2.2) that

$$\ln R(z) = -i\pi + \ln C + \int_{-\infty}^{\infty} \Big(\frac{1}{u-z} - \frac{u}{1+u^2} \Big) F(u) du.$$

One can easily check (see [3], eq.(1.9i)) that

$$\ln\Big(-\frac{1}{z} \Big) = \int_{0}^{\infty} \Big(\frac{1}{u-z} - \frac{u}{1+u^2} \Big) du$$

and hence

$$0 = -i\pi - \ln\frac{1}{z} + \int_{0}^{\infty} \Big(\frac{1}{u-z} - \frac{u}{1+u^2} \Big) du.$$

It follows that

$$\ln R(z) = \ln\frac{C}{z} + \int_{-\infty}^{0} \Big(\frac{1}{u-z} - \frac{u}{1+u^2} \Big) F(u) du - \int_{0}^{\infty} \Big(\frac{1}{u-z} - \frac{u}{1+u^2} \Big) (1-F(u)) du.$$

Since F is a Rayleigh function, the convergence of the integrals (1.4.3) implies that the integrals

$$\int_{-\infty}^{0} \frac{uF(u)du}{1+u^2} < \infty, \qquad \int_{0}^{\infty} \frac{u(1-F(u))du}{1+u^2} < \infty$$

converge, too. Set $C_1 = C_2 C$, where

$$C_2 = \exp\Big(- \int_{-\infty}^{0} \frac{u}{1+u^2} F(u) du + \int_{0}^{\infty} \frac{u}{1+u^2} (1-F(u)) du \Big).$$

Then $R(z)$ may be written in the form

$$R(z) = \frac{C_1}{z} \exp\Big(- \int_{-\infty}^{0} \frac{F(u)du}{z-u} + \int_{0}^{\infty} \frac{1-F(u)}{z-u} du \Big),$$

and it is easy to see that the condition (2.1.9) is equivalent to $C_1 = 1$. □

(2.2.5) Example. Let $F(u) = F_{x,y}(u)$ be the distribution function of a pair of interlacing sequences defined by the equation (1.2.1). Then the exponential representation (2.2.4) simplifies to (2.1.3).

The topology of the space $\mathcal{F}$ of Rayleigh functions was designed in Section 1.4 in such a way that the representation (2.2.4) establishes a homeomorphism of the spaces $\mathcal{F}$ and $\mathcal{N}$. The subset $\mathcal{F}_0$ of interlacing sequences corresponds to the subset $\mathcal{N}_0$ of rational functions in $\mathcal{N}$.

The exponential representation (2.2.4) takes a particularly simple form if F is a Rayleigh function of *bounded variation*, i.e., if it is the distribution function $F(u) = \tau\{(-\infty, u)\}$ of a Rayleigh measure τ, see Theorem 2.4.1 below.

2.3 The Markov transform. An immediate consequence of the existence of the canonical representations (2.2.4) and (2.1.8) is the following fact which is central for the present paper.

(2.3.1) Theorem. *For every Rayleigh function $F \in \mathcal{F}$ there exists a unique probability distribution $\mu \in \mathcal{M}$, such that*

$$\int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t} = \frac{1}{z} \exp\Big(- \int_{-\infty}^{0} \frac{F(u)du}{z-u} + \int_{0}^{\infty} \frac{1-F(u)}{z-u} du \Big) \tag{2.3.2}$$

for all $z \in \mathbf{C} \setminus \mathbf{R}$. The transformation $F \mapsto \mu$ establishes a homeomorphism between the space $\mathcal{F}$ of Rayleigh functions and the space $\mathcal{M}$ of probability distributions.

We refer to (2.3.2) as to the *master identity*, and we call μ the *Markov transform* of F. The relationship between the two basic integral representations (2.1.6), (2.2.2) of a general function R of negative imaginary type had been studied in great detail (see [20, 28] and very informative papers [3, 4]). We derive a number of basic properties of the Markov transform just as simplified versions of the more general results of [3].

We start with a simple remark concerning the supports of F and μ. If the Rayleigh function F in the master identity is identically zero in some interval, $F(u) \equiv 0$ for $a < u < b$, then $\mu\{(a,b)\} = 0$, and the same conclusion holds if $F(u) \equiv 1$ for $a < u < b$. We say that a Rayleigh function F is *supported* by an interval $[a,b]$, if $F(u) \equiv 0$ for $u < a$ and $F(u) \equiv 1$ for $u > b$. The smallest interval supporting a Rayleigh function F coincides with the smallest interval supporting its Markov transform μ.

Another nice feature of the transform $F \mapsto \mu$ is that it commutes with the affine transformations of the real line: if F and μ are related by (2.3.2), then the measure $\widetilde{\mu}$ with the distribution function $\widetilde{\mu}\{(-\infty,u)\} = \mu\{(-\infty, au+b)\}$ is the Markov transform of the Rayleigh function $\widetilde{F}(u) = F(au+b)$, for any $a > 0$, $b \in \mathbf{R}$.

We turn now to a characterization of the discrete and singular parts of the measure μ in terms of the corresponding Rayleigh function F. Recall that a probability measure μ is said to be *singular*, if $\mu(A) = 1$ for a measurable set $A \subset \mathbf{R}$ of zero Lebesgue measure.

(2.3.3) Theorem (Aronszajn and Donoghue [3], Theorem IX). *Let $\mu \in \mathcal{M}$ and $F \in \mathcal{F}$ be related by the master identity (2.3.2). Then μ is singular if, and only if, $F = \chi_A$ is (* mod 0 *equivalent to) a characteristic function of a measurable set $A \subset \mathbf{R}$. In particular, if A is a union of a finite number of intervals in $\mathbf{R}$, the measure μ is discrete and finitely supported.*

This beautiful result is worth specializing it to a still simpler case of compactly supported measures.

(2.3.4) Corollary. *The identity*

$$\int_{0}^{1} \frac{\mu(du)}{z-u} = \frac{1}{z} \exp \int_{B} \frac{du}{z-u} \qquad \forall z \in \mathbf{C} \setminus [0,1] \tag{2.3.5}$$

establishes a bijective correspondence between the singular probability distributions μ in the interval $[0,1]$, and the (classes mod 0 *of) measurable subsets $B \in [0,1]$.*

One can easily describe the *discrete* part of a measure μ in terms of the corresponding Rayleigh function F.

(2.3.6) Theorem (Aronszajn and Donoghue [3], Theorem IV). *Let μ be the Markov transform of a Rayleigh function F (i.e., the master identity (2.3.2) holds). Then the weight $\mu\{c\}$ of μ at a point $c \in \mathbf{R}$ equals*

$$\mu\{c\} = \exp\Big(- \int_{-\infty}^{c} \frac{F(u)du}{|u-c|} - \int_{-\infty}^{c} \frac{1-F(u)}{|u-c|} du \Big). \tag{2.3.7}$$

In particular, $\mu\{c\} = 0$ unless both integrals in (2.3.7) converge.

2.4 The master identity for interlacing measures. If a Rayleigh function F is of bounded variation, $F \in \mathcal{F}_b$, it can be considered as the distribution function of some Rayleigh measure τ. In this case the master identity can be written in a more attractive form.

(2.4.1) Corollary. *Assume that $F(u) = L_\tau(u)$ is the distribution function of a Rayleigh measure τ. Then*

$$\frac{1}{z} \exp\Big(- \int_{-\infty}^{0} \frac{F(u)du}{z-u} + \int_{0}^{\infty} \frac{1-F(u)}{z-u} du \Big) = \exp\Big(- \int_{-\infty}^{\infty} \ln(z-u)\tau(du) \Big),$$

and the master identity (2.3.2) takes the form

$$\int_{-\infty}^{\infty} \frac{\mu(du)}{z-u} = \exp \int_{-\infty}^{\infty} \ln \frac{1}{z-u} \tau(du), \qquad \Im z > 0. \tag{2.4.2}$$

Proof. Direct integration by parts. $\square$

Note that the right hand side of (2.4.2) is nothing else than the multiplicative version of the Cauchy-Stieltjes transform. Since

$$\frac{d}{dz} \int_{-\infty}^{\infty} \ln(z-u) d\tau(u) = \int_{-\infty}^{\infty} \frac{d\tau(u)}{z-u},$$

the master identity (2.4.2) can be written in the equivalent form

$$-\frac{d}{dz} \ln \int_{-\infty}^{\infty} \frac{\mu(du)}{z-u} = \int_{-\infty}^{\infty} \frac{\tau(du)}{z-u}. \tag{2.4.3}$$

Assume that τ is a bounded signed measure on the real line, such that the integral $\int_{-\infty}^{\infty} \ln(1+|u|)|\tau(du)|$ converges. Then the right hand side of (2.4.2) determines a function $R(z)$ of negative imaginary type if, and only if, τ is a Rayleigh measure.

(2.4.4) Example. Take for τ the Cauchy distribution $\tau(du) = \frac{1}{\pi} \frac{du}{1+u^2}$. One can easily check that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{du}{(z-u)(1+u^2)} = \frac{1}{z+i}, \qquad \Im z > 0$$

$$\exp\Big(- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln(z-u)}{(1+u^2)} du \Big) = \frac{1}{z+i}, \qquad \Im z > 0$$

so that the additive and the multiplicative Cauchy-Stieltjes transforms of τ coincide. It follows that in this case the Markov transform μ is identical with τ. As it was shown in [44], the Cauchy distributions are the only fixed points of the Markov transform.

Let μ denote the Markov transform of a Rayleigh measure τ. It follows from a result of Aronszajn and Donoghue ([3], Theorem VIII) that μ is absolutely continuous, except possibly for a finite set of point masses. Cifarelly and Regazzini have found in [8] the following explicit formula for the density of μ in the particular case of a *positive* measure τ (the same formula was independently conjectured in [24] for *all* Rayleigh measures).

(2.4.5) Theorem (Cifarelly and Regazzini [8]). *Let μ be the Markov transform of a probability distribution τ in a finite interval $[a, b]$. Then the measure μ is absolutely continuous, and its density equals*

$$\frac{\mu(dx)}{dx} = \frac{\sin \pi L_\tau(x)}{\pi} \exp \int_a^b \ln \frac{1}{|x-u|} \tau(du), \tag{2.4.6}$$

where $L_\tau(x) = \tau\{(-\infty, x)\}$ is the distribution function of τ.

Recall that the point masses of μ were described by the equation (2.3.7).

3. The moments of interlacing measures

3.1 The definition of moments. It follows from a result by Aronszajn and Donoghue ([3], Theorem A-d) that a Rayleigh measure τ has the moment

$$p_n = \int_{-\infty}^{\infty} u^n \tau(du) \tag{3.1.1}$$

of some order n if, and only if, its Markov transform μ has the moment

$$h_n = \int_{-\infty}^{\infty} u^n \mu(du) \tag{3.1.2}$$

of the same order. Because of the tight connection between interlacing measures $\tau_\pm$ and the Rayleigh measure $\tau = \tau_+ - \tau_-$, we shall sometimes call p_n the nth *moment of interlacing measures* τ_+, τ_-.

(3.1.3) Example. The mth moment of interlacing sequences (1.1.1) equals

$$p_m = \sum_{k=1}^{n} x_k^m - \sum_{k=1}^{n-1} y_k^m; \qquad m = 0, 1, 2, \ldots .$$

The mth moment of the distribution μ determined by the partial fraction decomposition (2.1.4) can be written as

$$h_m = \sum_{k=0}^{m} (-1)^k \sum y_{i_1} \ldots y_{i_k} x_{j_1} \ldots x_{j_{m-k}},$$

where the indices $i_1 < \ldots < i_k$ in the second sum increase strictly, while the indices $j_1 \leq \ldots \leq j_{m-k}$ increase weakly.

3.2 The Markov transform in terms of moments. Assume that a probability distribution $\mu \in \mathcal{M}$ has finite moments of all orders. Then its Cauchy-Stieltjes transform can be considered as the moment generating function,

$$\int_{-\infty}^{\infty} \frac{\mu(du)}{z-u} = \sum_{n=0}^{\infty} \frac{h_n}{z^{n+1}}. \tag{3.2.1}$$

The series in the right hand side may very well diverge for all $z \in \mathbf{C}$, but it is always the asymptotic series of the function in the left hand side, as $z \to \infty$, with respect to every domain $\Im z \geq \varepsilon > 0$. This means that

$$R(z) = \sum_{n=0}^{N-1} \frac{h_n}{z^{n+1}} + o\Big(\frac{1}{z^N}\Big); \qquad N = 1, 2, \dots$$

if $|z| \to \infty, |\operatorname{Im} z| > \varepsilon > 0$ (see [36], Theorem II.2.1 or [18], chapter 9). This implies that

$$h_n = \lim_{z\to\infty} z^{n+1}\Big(R(z) - \sum_{k=0}^{n-1} \frac{h_k}{z^{k+1}}\Big),$$

so that the moments are truly determined by the function $R(z)$. For a compactly supported measure μ the series in (3.2.1) converges for sufficiently large $|z|$, and its sum equals $R(z)$.

In a similar way,

$$\ln z - \int_{-\infty}^{\infty} \ln(z-u)\tau(du) = \sum_{n=1}^{\infty} \frac{p_n}{nz^n} \tag{3.2.2}$$

is the asymptotic series of the function in the left hand side. One can readily derive the relations between the moments of the measures τ and μ.

(3.2.3) Theorem. *Assume that a Rayleigh measure τ and its Markov transform μ have moments of all orders. Then the master identity (2.4.2) implies the identity*

$$\sum_{n=0}^{\infty} h_n z^n = \exp \sum_{n=1}^{\infty} \frac{p_n}{n} z^n \tag{3.2.4}$$

of moment generating functions (to be understood as the equality of formal power series).

In many situations the moments (3.1.1) determine the measure μ uniquely, though this is by no means true in the general case. They say (see, e.g., [36]) that the moment problem for the measure μ is *determined* or *indeterminate* accordingly. Note that the moment problem is determined for every compactly supported measure.

(3.2.5) Corollary. *The moment problem for a Rayleigh measure τ is determined if, and only if, the moment problem for its Markov transform μ is determined. If the measures τ, μ are determined by their moments uniquely, the master identity (2.4.2) is equivalent to the identity (3.2.4).*

(3.2.6) Remark. Let μ be a probability distribution with finite moments $\{h_n\}_{n=0}^{\infty}$. Recall that *cumulants* (or *semi-invariants*) $\{s_n\}_{n=1}^{\infty}$ of the measure μ can be defined by their exponential generating series

$$\sum_{n=1}^{\infty} \frac{s_n}{n!} z^n = \ln \sum_{n=0}^{\infty} \frac{h_n}{n!} z^n. \tag{3.2.7}$$

The series in the right hand side of (3.2.7) is simply the Laplace transform $\int e^{zu}\mu(du)$ of the measure μ. It follows that $s_n(\mu * \nu) = s_n(\mu) + s_n(\nu)$ for all $n = 1, 2, \ldots,$ where $\mu * \nu$ is the convolution of the measures μ, ν. The same property holds for the numbers $\beta_n = s_n/(n-1)!$. The equation (3.2.7) takes the form

$$\ln \sum_{n=0}^{\infty} \frac{h_n}{n!} z^n = \sum_{n=1}^{\infty} \frac{\beta_n}{n} z^n,$$

similar to the equation (3.2.4). Nevertheless, there seem to be no simple relations between the numbers β_n and p_n.

3.3 Explicit formulae. The identity (3.2.4) of formal power series can be written as a series of polynomial identities between their coefficients. For instance, one can easily check that

$$h_n = \sum_{\rho \vdash n} \frac{p_1^{r_1} p_2^{r_2} \ldots}{1^{r_1} r_1! 2^{r_2} r_2! \ldots}; \qquad n = 1, 2, \ldots \tag{3.3.1}$$

where the sum runs over all integer partitions $\rho = (1^{r_1}, 2^{r_2}, \ldots)$ of $n = r_1 + 2r_2 + \ldots$. Since $n! \,/\, (1^{r_1} r_1! 2^{r_2} r_2! \ldots)$ equals the number of permutations $\pi \in \mathfrak{S}_n$ with exactly $r_k = r_k(\pi)$ cycles of length k, one can rewrite (3.3.1) in the form

$$h_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{k \geq 1} p_k^{r_k(\pi)}. \tag{3.3.2}$$

Taking logarithmic derivatives of both sides of the equation (3.2.4), one readily derives a useful recurrence relation (see [32], I.2.11)

$$n\, h_n = \sum_{k=1}^{n} h_{n-k}\, p_k. \tag{3.3.3}$$

The identities relating the coefficients h_n, p_n of the series in (3.2.4) may also be written as the following determinantal identities (see [32], I.2, Ex.8):

$$h_n = \frac{1}{n!}\begin{vmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \dots & -n+1 \\ p_n & p_{n-1} & p_{n-2} & \dots & p_1 \end{vmatrix}. \tag{3.3.4}$$

The first few equations are as follows:

$$\begin{aligned} h_1 &= p_1; \\ 2!\,h_2 &= p_1^2 + p_2; \\ 3!\,h_3 &= p_1^3 + 3\,p_1p_2 + 2\,p_3; \\ 4!\,h_4 &= p_1^4 + 6\,p_1^2p_2 + 3\,p_2^2 + 8\,p_1p_3 + 6\,p_4. \end{aligned} \tag{3.3.5}$$

Conversely,

$$p_n = (-1)^{n-1}\begin{vmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2\,h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n\,h_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix}. \tag{3.3.6}$$

In particular,

$$\begin{aligned} p_1 &= h_1; \\ p_2 &= 2\,h_2 - h_1^2; \\ p_3 &= 3\,h_3 - 3\,h_1h_2 + h_1^3; \\ p_4 &= 4\,h_4 - 4\,h_1h_3 - 2\,h_2^2 + 4\,h_1^2h_2 - h_1^4. \end{aligned} \tag{3.3.7}$$

3.4 The moments of Rayleigh functions. Using the integration by parts, the definition of the moment p_n in (3.1.1) can be stated in the form

$$p_n = -\int_{-\infty}^{0} nu^{n-1}F(u)du + \int_0^{\infty} nu^{n-1}\big(1-F(u)\big)du, \tag{3.4.1}$$

where $F(u) = \tau\{(-\infty, u)\}$ is the distribution function of τ. The right hand side makes sense even if the function F has infinite variation.

(3.4.2) Definition. Given a Rayleigh function $F \in \mathcal{F}$ and an integer $n = 1, 2, \dots$, assume that the integrals in the right hand side of (3.4.1) converge absolutely. Then the real number p_n determined by this formula will be called the nth *moment of the Rayleigh function* F. We also set $p_0 \equiv 1$ for all $F \in \mathcal{F}$. The linear functional

$$\mathcal{M}_F(Q) = Q(0) - \int_{-\infty}^{0} Q'(u)F(u)du + \int_0^{\infty} Q'(u)\big(1-F(u)\big)du, \tag{3.4.3}$$

$Q \in \mathbf{C}[u]$, will be referred to as the *moment functional* of F.

As a matter of fact, one can also write the moment functional as

$$\mathcal{M}_F(Q) = Q(c) - \int_{-\infty}^{c} Q'(u)F(u)du + \int_{c}^{\infty} Q'(u)\big(1 - F(u)\big)du, \tag{3.4.4}$$

since the right hand side does not depend on the choice of c.

The definition is consistent with that of the moments of a Rayleigh measure, and the equations (3.2.4), (3.3.2), (3.3.4) hold for all $F \in \mathcal{F}$.

(3.4.5) Example. Let $F = F_{x,y}$ be the distribution function (1.2.1) of a pair of interlacing sequences $x_1 < y_1 < x_2 < \ldots < y_{n-1} < x_n$. Then the moment functional of F equals

$$\mathcal{M}_F(Q) = \sum_{k=1}^{n} Q(x_k) - \sum_{k=1}^{n-1} Q(y_k). \tag{3.4.6}$$

3.5 Interlacing measures and symmetric group characters. According to a well-known Hamburger theorem, a sequence $\{h_n\}_0^\infty$ is a moment sequence of a positive measure if, and only if, the inequalities $\det(h_{i+j})_{i,j=0}^{n} \geq 0$ hold for all $n = 0, 1, \ldots$. Using (3.3.2), this result can be restated as a characterization of moment sequences of Rayleigh functions. Remarkably enough, the irreducible characters of symmetric groups are involved in this characterization.

(3.5.1) Theorem. *For $n = 1, 2, \ldots$ let χ_n denote the irreducible character of the symmetric group $\mathfrak{S}_N$ of order $N = n(n+1)$ corresponding to the Young diagram $\lambda^{(n)} = (n^{n+1})$ with $n+1$ rows of equal length n. A sequence $\{p_n\}_{n=1}^\infty$ of real numbers is a moment sequence of some Rayleigh function $F \in \mathcal{F}$ if, and only if,*

$$(-1)^{n(n+1)/2} \sum_{\pi \in \mathfrak{S}_{n(n+1)}} \chi_n(\pi) \prod_{k \geq 1} \frac{1}{r_k(\pi)!} \left(\frac{p_k}{k}\right)^{r_k(\pi)} \geq 0 \qquad \forall n = 1, 2, \ldots. \tag{3.5.2}$$

Proof. Consider the Schur symmetric function s_λ corresponding to the Young diagram $\lambda = (n^{n+1})$. By the Jacobi-Trudi identity (see [32], I.3, Ex.8),

$$s_\lambda = \begin{vmatrix} h_n & h_{n+1} & \ldots & h_{2n} \\ h_{n-1} & h_n & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_0 & h_1 & \ldots & h_n \end{vmatrix} = (-1)^{n(n+1)/2} \begin{vmatrix} h_0 & h_1 & \ldots & h_n \\ h_1 & h_2 & \ldots & h_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ h_n & h_{n+1} & \ldots & h_{2n} \end{vmatrix}.$$

It is well known that a sequence $\{h_n\}_{n=0}^\infty$ is a moment sequence of a positive measure μ iff the last determinant is positive, for each $n = 0, 1, 2, \ldots$. It remains to recall the formula expressing the Schur function s_λ in terms of the power sum symmetric polynomials p_n,

$$s_\lambda = \sum_{\pi \in \mathfrak{S}_{n(n+1)}} \chi_n(\pi) \prod_{k \geq 1} \frac{1}{r_k(\pi)!} \left(\frac{p_k}{k}\right)^{r_k(\pi)},$$

see [32], I.7.10. □

As it follows from the character tables in [17], the first inequalities (3.5.2) are $p_2 - p_1^2 \geq 0$ and $18p_2p_4 - 18p_1^2p_4 - 16p_3^2 + 24p_1p_2p_3 + 8p_1^3p_3 - 9p_2^3 - 9p_1^2p_2^2 + 3p_1^4p_2 - p_1^6 \geq 0$.

3.6 An identity for the difference derivatives. Let τ be a probability distribution in the interval $[0,1]$, and denote by p_n its moment (3.1.1) of order n. Define a sequence h_n, $n=0,1,\dots$, by the equation (3.3.2). It follows from the definition of Markov transform that $\{h_n\}_{n=0}^{\infty}$ is also the moment sequence of a probability measure. We give a direct proof of this simple fact. An interesting feature of the proof is that it links the Markov transform with the Kingman's theory of partition structures, see [26].

By the classical Hausdorff theorem, a sequence $\{h_n\}_{n=0}^{\infty}$ is a moment sequence of some positive measure in the interval $[0,1]$ if, and only if, it is *completely monotonic*. By definition, this means that

$$(\Delta^k h)_n \equiv \sum_{j=0}^{k}(-1)^j \binom{k}{j} h_{n+j} \geq 0 \tag{3.6.1}$$

for all $k,n=0,1,\dots$. Here Δ is the difference derivative operator $(\Delta h)_n = h_n - h_{n+1}$.

(3.6.2) Theorem. *Assume that a sequence of real numbers $\{p_n\}_{n=0}^{\infty}$ is completely monotonic, and define a new sequence $\{h_n\}_{n=0}^{\infty}$ by the equation (3.3.2). Then $\{h_n\}_{n=0}^{\infty}$ is completely monotonic, too.*

This result is an immediate consequence of the following explicit formula for the difference derivatives of the sequence $\{h_n\}_{n=0}^{\infty}$.

(3.6.3) Lemma. *Given a sequence $\{p_0=1,p_1,p_2,\dots\}$, let $\{h_0=1,h_1,h_2,\dots\}$ denote the sequence defined by the equations*

$$h_n = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\prod_{k\geq 1} p_k^{r_k(w)}, \tag{3.6.4}$$

$n=0,1,2,\dots$. Then

$$(\Delta^k h)_n = \frac{1}{(n+k)!}\sum_{\pi\in\mathfrak{S}_{n+k}}\prod_{j=1}^{l(\pi)}(\Delta^{k_j(\pi)}p)_{m_j(\pi)}, \tag{3.6.5}$$

where $l(\pi)$ is the number of cycles in a permutation $\pi\in\mathfrak{S}_{n+k}$, $m_j(\pi)+k_j(\pi)$ is the number of elements in the jth cycle, and $m_j(\pi)$, $k_j(\pi)$ denote the amounts of those elements in the cycle which are $\leq n$ or $> n$ correspondingly.

(3.6.6) Example. If one takes $n=1$ and $k=2$, the identity simplifies to

$$\begin{aligned} h_1 - 2h_2 + h_3 = \frac{1}{6}\big(&p_1(1-p_1)^2 + (1-p_1)(p_1-p_2) + p_1(1-2p_1+p_2) + \\ &+ (1-p_1)(p_1-p_2) + (p_1-2p_2+p_3) + (p_1-2p_2+p_3)\big). \end{aligned}$$

The terms of the sum in the right hand side correspond to permutations (1)(2)(3), (12)(3), (1)(23), (12)(3), (123), (132) $\in \mathfrak{S}_3$.

Proof. In order to prove the formula (3.6.5), we introduce appropriate notations. For any $j = 1, \dots, n$ and a permutation $\pi \in \mathfrak{S}_n$ we denote by $d_j(\pi) \in \mathfrak{S}_{n-1}$ the permutation obtained from π by removing the element j from its cycle. Clearly, the operators d_i, d_j commute. The operator $D_k = (1 - d_{m+1})(1 - d_{m+2}) \dots (1 - d_{m+k})$, when applied to a permutation $\pi \in \mathfrak{S}_{m+k}$, returns a formal sum of permutations

$$D_k \pi = \sum_I (-1)^{k-|I|} d_I \pi,$$

where I runs over all subsets $I \subset \{m+1, \dots, m+k\}$, and $d_I = \prod_{i \in I} d_i$ denotes the operation of removing all the elements in I from the cycles of π. We shall say that the elements of the set $\{1, 2, \dots, m\}$ are *small*, and the elements of $\{m+1, \dots, m+k\}$ are *big*.

To simplify notations, we shall write simply p_π instead of $p_{\rho(\pi)}$, and $p_g = \sum g_\pi p_\pi$ for an element $g = \sum g_\pi \pi$ of the group algebra $\mathbf{R}[\mathfrak{S}_n]$. The proof of (3.6.5) is a result of three simple steps.

If $\pi \in \mathfrak{S}_{m+k}$ is a permutation with a single cycle of length $n = m + k$, then the permutation $d_J \pi$ is cyclic, too. Obviously, $p_\pi = p_n$ and $p_{d_J \pi} = p_{n-j}$ for $j = |J|$, so that

$$p_{D_k \pi} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} p_{m+k-j} = (\Delta^k p)_m.$$

For a generic permutation $\pi \in \mathfrak{S}_{m+k}$ with $l = l(\pi)$ cycles, denote by m_i the number of "small" elements in the ith cycle, and by k_i the number of "big" elements in this cycle. The choice of a set J of big elements is equivalent to independent choices of subsets $J_1, \dots, J_l$ of big elements in the cycles of permutation π. Hence,

$$p_{D_k \pi} = \sum_{j_1=0}^{k_1} \dots \sum_{j_l=0}^{k_l} (-1)^{k-j} \binom{k_1}{j_1} \dots \binom{k_l}{j_l} p_{m_1+k_1-j_1} \dots p_{m_l+k_l-j_l} = \prod_{i=1}^{l(\pi)} (\Delta^{k_i} p)_{m_i}.$$

Let us fix a subset J of big elements, and let $\mathfrak{S}_{\bar{J}}$ denote the group of permutations of the remaining set $\bar{J} = \{1, 2, \dots, m+k\} \setminus J$. One can easily see that for each permutation $\sigma \in \mathfrak{S}_{\bar{J}}$ there are exactly $(m+k-j+1)(m+k-j+2) \dots (m+k)$ permutations $\pi \in \mathfrak{S}_{m+k}$ such that $d_J \pi = \sigma$. It follows that

$$\frac{1}{(m+k)!} \sum_\pi p_{d_J \pi} = \frac{1}{(m+k-j)!} \sum_\sigma p_\sigma = h_{m+k-j}, \tag{3.6.6}$$

hence

$$\frac{1}{(m+k)!} \sum_\pi p_{D_k \pi} = \sum_{j=0}^{k} \sum_{J:|J|=j} \frac{1}{(m+k)!} \sum_\pi p_{d_J \pi} = (\Delta^k h)_m,$$

and the identity (3.6.5) follows. $\square$

3.7 General partition structures. An advantage of the Theorem 3.6.2, apart of its simplicity, is that it suggests a generalization of the Markov transform.

Choose a probability distribution M_n on the symmetric group $\mathfrak{S}_n$ for all $n = 1, 2, \dots$ in such a way that the probability $M_n(\pi)$ only depends on the cycle lengths of a permutation $\pi \in \mathfrak{S}_n$. Recall (see [26]) that the family $\{M_n\}_{n=1}^{\infty}$ is called[2] a *partition structure*, if

$$M_n(\pi) = \sum_{\sigma:\sigma'=\pi} M_{n+1}(\sigma) \qquad \forall \pi \in \mathfrak{S}_n, \quad \forall n = 1, 2, \dots . \tag{3.7.1}$$

Here σ' denotes the permutation in $\mathfrak{S}_n$ obtained from $\sigma \in \mathfrak{S}_{n+1}$ by deleting the element $n+1$ from its cycle.

The formula (3.6.5) can be generalized as follows.

(3.7.2) Theorem. *Let $\{M_n\}_{n=1}^{\infty}$ be a partition structure. Given a sequence of real numbers $p_0 = 1, p_1, p_2, \dots,$ define a new sequence $h_0 = 1, h_1, h_2, \dots$ by the formula*

$$h_n = \sum_{\pi \in \mathfrak{S}_n} M_n(\pi) p_1^{r_1(\pi)} p_2^{r_2(\pi)} \dots ; \qquad n = 1, 2, \dots . \tag{3.7.3}$$

Then the difference derivatives satisfy the equation

$$(\Delta^k h)_n = \sum_{\pi \in \mathfrak{S}_{n+k}} M_{n+k}(\sigma) \prod_{j=1}^{l(\pi)} (\Delta^{k_j(\pi)} p)_{m_j(\pi)}, \tag{3.7.4}$$

with the same notations as in the formula (3.6.5).

Proof. One can argue exactly as in the proof of Lemma (3.6.3), except for the formula (3.6.6). In the present more general setup it should read as follows:

$$\sum_{\pi \in \mathfrak{S}_{m+k}} p_{d_J \pi}\, M_{m+k}(\pi) = \sum_{\sigma \in \mathfrak{S}_{m+k-j}} p_\sigma\, M_{m+k-j}(\sigma) = h_{m+k-j}. \tag{3.7.5}$$

(3.7.6) Corollary. *For every partition structure $\{M_n\}_{n=1}^{\infty}$, and every probability distribution τ in the interval $[0,1]$, there exists a unique probability distribution μ in $[0,1]$, such that the moments (3.1.1), (3.1.2) are related by the formula (3.7.3).*

(3.7.7) Example. A distinguished family of partition structures is provided by the *Ewens sampling formula*

$$M_n(\pi) = \frac{\theta^{l(\pi)}}{\theta(\theta+1)\dots(\theta+n-1)}; \qquad \pi \in \mathfrak{S}_n, \tag{3.7.8}$$

where $\theta > 0$ is a parameter. The relation between the measures τ and μ can be stated in this case in the form

$$\int_0^1 \frac{\mu(du)}{(z-u)^\theta} = \exp\Big(-\theta \int_0^1 \ln(z-u)\ \tau(du)\Big). \tag{3.7.9}$$

If $\theta = 1$, we return to the master identity (2.4.2).

[2] Our terminology at this point is slightly different from the original one.