ROOKS ON FERRERS BOARDS AND MATRIX INTEGRALS

Sergei Kerov

Steklov Mathematical Institute, St. Petersburg
and LaBRI, Bordeaux

Abstract. Let $C(n,N)=\int_{H_N}\operatorname{tr} Z^{2n}\ \mu(dZ)$ denote a matrix integral by a $U(N)$-invariant gaussian measure μ on the space H_N of hermitian $N\times N$ matrices. The integral is known to be always a positive integer. We derive a simple combinatorial interpretation of this integral in terms of rook configurations on Ferrers boards. The formula

$$C(n,N)=(2n-1)!!\sum_{k=0}^{n}\binom{N}{k+1}\binom{n}{k}2^k$$

found by J. Harer and D. Zagier follows from our interpretation immediately.

1. Introduction. Consider a matrix integral

$$C(n,N)=\int_{H_N}\operatorname{tr} Z^{2n}\ \mu_N(dZ). \tag{1.1}$$

taken over the space H_N of $N\times N$ hermitian matrices. We denote by $z_{jk}=x_{jk}+iy_{jk}$ the matrix elements of a matrix $Z=(z_{jk})\in H_N$, and by μ_N the gaussian measure

$$\mu_N(dZ)=\prod_{1\le j<k\le N}\frac{1}{\pi}e^{-|z_{jk}|^2}dx_{jk}\,dy_{jk}\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi}}e^{-x_{kk}^2/2}dx_{kk}. \tag{1.2}$$

in the space H_N.

The integral (1.1) was computed in [2] in two different ways. The first evaluation results in the formula

$$C(n,N)=\sum_{0\le g\le n/2}\varepsilon_g(n)N^{n+1-2g} \tag{1.3}$$

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where $\varepsilon_g(n)$ denotes the number of ways to obtain an orientable surface of genus g by identifying in pairs the sides of a $2n$-gone. See [7] for a comprehensive account of this calculation based on the Wick formula.

The second evaluation in [2], based on manipulations with generating functions, provides the formula

$$C(n,N) = (2n-1)!! \sum_{k=0}^{n} \binom{N}{k+1} \binom{n}{k} 2^k. \tag{1.4}$$

In this paper we give a simple combinatorial interpretation of the numbers $C(n,N)$ in terms of rook configurations on Ferrers boards or, equivalently, in terms of appropriate involutions. There are two main ingredients in our approach: (1) an explicit formula for the one-particle correlation function of the gaussian unitary ensemble (i.e., the ensemble of eigenvalues of matrices $Z \in H_N$ chosen from the distribution (1.2)) and (2) the Flajolet – Viennot combinatorial theory of orthogonal polynomials.

2. Rook configurations. Let $Y(n)$ be the set of Young diagrams[1] $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ with n rows of lengths not exceeding n, $0 \le \nu_n \le \dots \le \nu_1 \le n$. Given a diagram $\nu \in Y(n)$ and a non-negative integer s, we denote by

$$\nu[s] = (\nu_1 + s, \nu_2 + s, \dots, \nu_n + s) \tag{2.1}$$

the Young diagram obtained from ν by increasing all of its rows by s (or, equivalently, by adding s new columns of length n). Let $Y(n,s) = \{\nu[s] : \nu \in Y(n)\}$ be the set of such diagrams.

Definition 1. A subset $c \subset \mu$ of a Young diagram μ is called a *rook configuration on the board* μ, if each row and each column of the diagram μ contains at most one element (rook) in c. Let $RC_k(\mu)$ denote the set of all configurations of $k = |c|$ rooks on the board μ. We also write

$$RC(n,s) = \{(\mu, c) : \mu \in Y(n,s), c \in RC_n(\mu)\} \tag{2.2}$$

for the set of pairs (μ, c) where c is a configuration of n rooks on a board $\mu \in Y(n,s)$, and

$$RC^*(n,N) = \bigcup_{s=0}^{N-1} RC(n,s) \tag{2.3}$$

for the disjoint union of the sets $RC(n,s)$. By definition, $RC^*(n,N)$ is the set of triples (s, μ, c) where $s = 0, 1, \dots, N-1$ is a number, $\mu = \nu[s] \in Y(n,s)$ is a Young diagram, such that $s \le \mu_n \le \dots \le \mu_1 \le n+s$, and $c \in RC_n(\mu)$ is a configuration of n rooks on the board μ.

We are in a position now to present the main results of this paper.

[1] We make no difference between a Young diagram and a Ferrers diagram.

Theorem 1. *The integral (1.1) equals the number of elements in the set $RC^*(n,N)$,*

$$C(n,N) = \sum_{s=0}^{N-1} |RC(n,s)|. \tag{2.4}$$

Theorem 2. *The number of elements in the set $RC(n,s)$ equals*

$$|RC(n,s)| = (2n-1)!! \sum_{k=0}^{n} \binom{s}{k}\binom{n}{k} 2^k. \tag{2.5}$$

The theorems are proved in sections 4, 5. Presently, we derive the equation (1.4) as a corollary of these theorems.

Let us use the symbol $(s)_k = s(s-1)\dots(s-k+1)$ for the lower factorial powers. Since $(s+1)_{k+1} - (s)_{k+1} = (k+1)\,(s)_k$, it follows that

$$\sum_{s=0}^{N-1} \binom{s}{k} = \binom{N}{k+1},$$

so that the formula (1.4) is a direct corollary of (2.4) and (2.5).

3. The correlation function. If a matrix $Z \in H_N$ is chosen at random from the distribution (3.1), then its eigenvalues $x_1, x_2, \dots, x_N$ are distributed with the density

$$P_N(x_1, x_2, \dots, x_N) = \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{k=1}^{N} \frac{1}{k!\sqrt{2\pi}} e^{-x_k^2/2}, \tag{3.1}$$

and one can write (see [2], §4)

$$C(n,N) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \sum_{k=1}^{N} x_k^{2n}\; P_N(x_1, \dots, x_N)\, dx_1 \dots dx_N. \tag{3.2}$$

Following [3] we introduce the one-particle *correlation function* by the equation

$$\sigma_N(x) = N \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P_N(x, x_2, \dots, x_N)\, dx_2 \dots dx_N. \tag{3.3}$$

By definition, $\frac{1}{N}\sigma_N(x)\,dx$ is the probability to find an eigenvalue of a random matrix $Z \in H_N$ in the interval $(x, x+dx)$. Since the density (3.1) is symmetric in its variables, it follows from (3.2) that

$$C(n,N) = \int_{-\infty}^{\infty} x^{2n}\, \sigma_N(x)\, dx \tag{3.4}$$

is a moment of the measure $\sigma_N(x)\,dx$.

The following fact is well known (see [3], pp. 74–76). For the reader's convenience, we reproduce a sketch of the proof, too.

Proposition 1 ([3]). *The correlation function (3.3) can be written in the form*

$$\sigma_N(x) = \sum_{s=0}^{N-1} D_s^2(x), \tag{3.5}$$

where $D_s(x) = (2\pi)^{-1/4}(s!)^{-1/2}e^{-x^2/4}H_s(x)$ is a parabolic cylinder function and $H_s(x)$ is the monic Hermite polynomial of degree s.

Proof. Consider a kernel $K(x,y) = \sum_{s=0}^{N-1} D_s(x)\,D_s(y)$ and a matrix $\Phi = \big(D_{i-1}(x_j)\big)_{i,j=1}^{N}$. One can easily check that

$$\det\Phi = (2\pi)^{-N/4}\prod_{i=0}^{N-1}(i!)^{-1/2}\prod_{j=1}^{N} e^{-x_j^2/4}\prod_{i<j}(x_j - x_i).$$

Since $\big(K(x_i,x_j)\big)_{i,j=1}^{N} = \Phi^T\Phi$, one obtains a determinantal formula

$$P_N(x_1,\,\dots\,,x_N) = \frac{1}{N!}\det\Big(K(x_i,x_j)\Big)_{i,j=1}^{N} \tag{3.6}$$

for the density (3.1).

The orthogonality relations $\int D_i(x)D_j(x)\,dx = \delta_{ij}$ for the functions $D_s(x)$ imply that

$$\int_{-\infty}^{\infty} K(x,y)K(y,z)\,dy = K(x,z); \qquad \int_{-\infty}^{\infty} K(x,x)\,dx = N. \tag{3.7}$$

It can be derived from (3.7) that for every positive integer n

$$\int_{-\infty}^{\infty}\det\big(K(x_i,x_j)\big)_{i,j=1}^{n}\,dx_n = (N-n+1)\det\big(K(x_i,x_j)\big)_{i,j=1}^{n-1}.$$

This formula and the equation (3.6) yield the Proposition. $\square$

4. The proof of Theorem 1. It follows from (3.5) that the moments of the correlation function (3.3) can be written as

$$C(n,N) = \sum_{s=0}^{N-1}\mathcal{M}_n(s) \tag{4.1}$$

where

$$\mathcal{M}_n(s) = \frac{1}{s!\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2n}\,H_s^2(x)\,e^{-x^2/2}\,dx. \tag{4.2}$$

Lemma 1. *The integral (4.2) equals the number of points in the set $RC(n,s)$,*

$$\mathcal{M}_n(s) = |RC(n,s)|. \tag{4.3}$$

Proof. Let us recall a result from the Flajolet–Viennot combinatorial theory of orthogonal polynomials, see [2, 5].

Assume that $P_0(x) = 1, P_1(x) = x, \dots$ is a family of monic polynomials orthogonal with respect to a measure M and satisfying the recurrence relations

$$P_{n+1}(x) = x\,P_n(x) - \lambda_n\,P_{n-1}(x); \qquad n = 1, 2, \dots .$$

Then according to the Proposition 17 in [5],

$$\int_{-\infty}^{\infty} x^{2n} P_s^2(x)\,M(dx) = \lambda_1\lambda_2\dots\lambda_s \sum_{\omega} V(\omega), \tag{4.4}$$

where ω in the sum runs over the set $L(n,s)$ of all lattice paths of length $2n$ in the integer upper half-plane, with the endpoints at the level s. The valuation $V(\omega)$ of a path ω is defined as a product $V(\omega) = \lambda_{k_1}\lambda_{k_2}\dots\lambda_{k_n}$ taken over all n descending edges of the path. The numbers $k_1, k_2, \dots, k_n$ indicate the starting levels of these edges. Obviously, the set $L(n,s)$ can be identified with the set $Y(n,s)$ of Young diagrams (see Fig. 1). The descending edges of the path ω correspond to the ends of rows of the associated Young diagram $\mu \in Y(n,s)$, and the starting level of the edge at the end of the kth row equals $s + \mu_k - n + k$. We prefer to use Young diagrams instead of lattice paths, so we write $V(\mu)$ instead of $V(\omega)$.

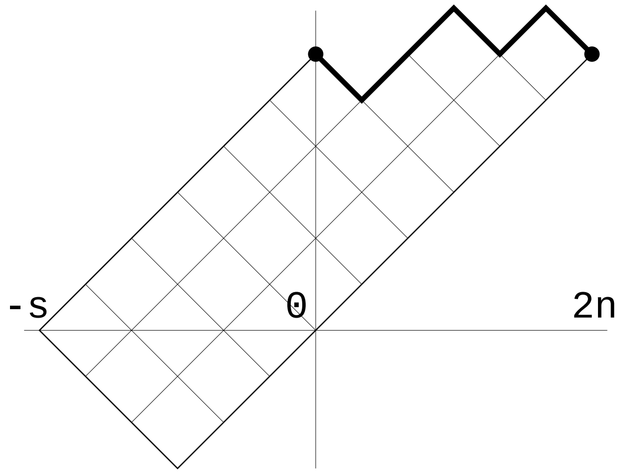

Fig. 1. Lattice paths and Young diagrams are identical.

We apply the equation (4.4) to Hermite polynomials $P_s(x) = H_s(x)$, so that $\lambda_s = s$ for all $s = 1, 2, \dots$. Hence,

$$V(\mu) = \prod_{k=1}^{n} (s + \mu_k - n + k). \tag{4.5}$$

Note that $\lambda_0 = 0$ and $V(\omega) = 0$ if a path ω visits the lower half-plain. Hence, the sum in (4.4) can be taken over *all* Young diagrams $\nu[s]$, $\nu \in Y(n)$.

It is obvious (cf. [4], Theorem 2.4.1) that (4.5) can be interpreted as the number $V(\nu[s]) = |RC_n(\nu[s])|$ of configurations of n rooks on the board $\nu[s] = (\nu_1 + s, \nu_2 + s, \dots, \nu_n + s)$ where $\nu \in Y(n)$. Since $\lambda_1 \lambda_2 \dots \lambda_s = s!$ in (4.4), Lemma 1 follows. $\square$

Theorem 1 is a direct corollary of equations (4.1) and (4.3).

5. The proof of Theorem 2. Let $RC_k(n, s)$ denote the subset of pairs $(\mu, c) \in RC(n, s)$, such that exactly k rooks are contained in the first s columns of the diagram $\mu = \nu[s]$. We prove that

$$|RC_k(n, s)| = (2n-1)!! \binom{s}{k} \binom{n}{k} 2^k. \tag{5.1}$$

To this end, we first introduce a set of partial involutions $I_k(n, s)$ and prove that the number of its elements $|I_k(n, s)|$ equals the right hand side of (5.1). Then we establish a bijection between the sets $RC_k(n, s)$ and $I_k(n, s)$.

Definition 2. Let $[s, 2n]$ denote a totally ordered set of $s + 2n$ points. We distinguish between the first s *negative points* and the remaining $2n$ *positive points*. Call an *arc* any unordered pair of points (vertices) in the set $[s, 2n]$. A set A of n pairwise disjoint arcs will be referred to as a *partial involution*, if each of its arcs has at least one positive vertex. Denote by $I_k(n, s)$ the set of partial involutions, such that exactly k arcs have a negative vertex.

Lemma 2. *The number of partial involutions in the set $I_k(n, s)$ equals*

$$|I_k(n, s)| = (2n-1)!! \binom{s}{k} \binom{n}{k} 2^k.$$

Proof. In order to choose a partial involution $A \in I_k(n, s)$, one should

(1) choose a subset of k negative vertices for the left ends of k arcs in A;
(2) find a sequence of k positive vertices corresponding to the arcs with a negative vertex;
(3) choose a subset of k positive vertices which belong to no arc of the involution A;
(4) join in pairs all $2n - 2k$ remaining positive points.

The number of such choices is obviously

$$|I_k(n, s)| = \binom{s}{k} \frac{2n(2n-1)\dots(2n-2k+1)}{k!} (2n-2k-1)!! = (2n-1)!! \binom{s}{k} \binom{n}{k} 2^k,$$

and Lemma 2 follows. $\square$

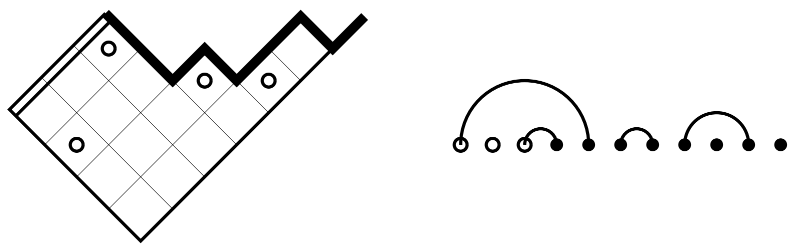

Fig. 2. The bijection between the sets $RC_k(n,s)$ and $I_k(n,s)$.

Lemma 3. *There exists a natural bijection between the set $I_k(n,s)$ of partial involutions and the set $RC_k(n,s)$ of rook configurations.*

Proof. Let c be a configuration of k rooks on the board $\nu[s]$, where $\nu \in Y(n)$. Note that the upper boundary $[s,2n]$ of the Young diagram $\nu[s]$ consists of $s+2n$ elementary edges (see Fig. 2), and that the set of edges is totally ordered. The first s edges, called "negative", are all ascending.

Every rook in the configuration c determines a hook of the diagram, hence a pair of edges on the boundary $[s,2n]$, the first one of which is ascending and the second one descending. Thus, each rook determines an arc in $[s,2n]$. Since different rooks belong to different rows and columns, the arcs are pairwise disjoint. The negative edges are always ascending and can not terminate an arc. As a result, we obtain a partial involution $A = F(\nu[s], c)$ on the set $[s,2n]$.

Given a partial involution $A \in I_k(n,s)$ on the set $[s,2n]$, we define a Young diagram μ and a rook configuration $c \in RC_n(\mu)$ as follows. If a vertex $j \in [s,2n]$ is a *right* end (with respect to the total order of the set $[s,2n]$) of an arc in A, we declare the corresponding edge *descending*; otherwise, if the vertex is a *left* vertex of an arc, or if it does not belong to an arc at all, we declare such edge *ascending*. The Young diagram μ thus obtained has the form $\mu = \nu[s]$ for some diagram $\nu \in Y(n)$. Put rooks on the board μ in such a way that their hooks correspond to the arcs of the involution A. We denote by $(\mu, c) = G(A)$ the resulting pair in $RC_k(n,s)$.

It is easy to see that the maps $F : RC_k(n,s) \to I_k(n,s)$ and $G : I_k(n,s) \to RC_k(n,s)$ constructed above are mutually inverse bijections (see Fig. 2). $\square$

6. A probabilistic problem. In conclusion, we discuss an elementary probabilistic problem which interprets and generalizes the formula (5.1). First of all, note that the formula can be written in a slightly different form,

$$|RC_k(n,s)| = (2n-1)!!\binom{s}{k}\binom{n}{k}2^k = \frac{1}{2^{n-k}}\binom{2n}{n}(n-k)!\binom{n}{k}^2 (s)_k,$$

where $(s)_m = s(s-1)\dots(s-m+1)$ denotes decreasing factorial power. Denote by $\widetilde{RC}_k(n)$ the set of all pairs (ν, c), where ν is a Young diagram in the rectangle $\Pi = (n^n)$, and $c \subset \nu$ is a configuration of k rooks on the diagram ν. Since the number $(s)_k$ equals the number of configurations of k rooks in a rectangle with k rows of length s, we obtain (replacing k

with $n-k$) that

$$|\widetilde{RC}_k(n)| = \frac{1}{2^k}\binom{2n}{n} k! \binom{n}{k}^2. \tag{6.1}$$

Note that the factor $\binom{2n}{n} = |Y(n)|$ enumerates the set of Young diagrams in the rectangle Π, and $k!\binom{n}{k}^2 = |\Omega_k(n)|$ can be regarded as the number of elements in the set $\Omega_k(n)$ of all configurations of k rooks in Π. Therefore, if we choose independently a random Young diagram $\nu \subset \Pi$ and a random configuration of k rooks in the rectangle Π, the probability to have *all* rooks inside the diagram ν will be equal to

$$\frac{|\widetilde{RC}_k(n)|}{|Y(n)|\,|\Omega_k(n)|} = \frac{1}{2^k}. \tag{6.2}$$

The formula (6.2) is essentially equivalent to that of (5.1) from which we derived Theorem 2. It can be generalized as follows.

Proposition 2. *Denote by $r(\nu, c) = |c \cap \nu|$ the number of rooks in a configuration $c \in \Omega_k(n)$ which are inside a diagram $\nu \subset \Pi$, and assume that ν and c are uniformly distributed and independent. Then*

$$\mathrm{Prob}\{(\nu, c) \in Y(n) \times \Omega_k(n) : r(\nu, c) = j\} = \binom{k}{j}\frac{1}{2^k} \tag{6.3}$$

for all $j = 0, 1, \dots, k$, i.e., the distribution of the random variable r is binomial, with the success probability $1/2$.

After this piece of work was presented in St. Petersburg seminar on representation theory, Professor S. V. Fomin has kindly informed me that the formula (6.2) coincides with Lemma 3.8.6 in his paper [8]. Moreover, one can easily see that the clever argument of [8] actually proves the more general Proposition 2, too.

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Steklov Mathematical Institute, SPb branch (POMI), Fontanka 27, St. Petersburg, 191011, Russia

E-mail address: kerov@pdmi.ras.ru