THE ASYMPTOTICS OF ROOT SEPARATION FOR ORTHOGONAL POLYNOMIALS

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The aim of this paper is to show that for a wide class of orthogonal polynomials (including the classical polynomials of Hermite, Laguerre and Jacobi) there is a common universal asymptotics of root separation. We emphasize that we discuss below the asymptotic character of root separation, not the limiting distribution of roots.

Let us consider a family of orthogonal polynomials $\{P_n(u)\}_{n=0}^{\infty}$ given by a linear recurrence relation

$$P_n(u) + (b_n - u)P_{n-1}(u) + c_n^2 P_{n-2}(u) = 0. \tag{1}$$

It is well known that the roots of adjacent polynomials

$$P_{n-1}(u) = \prod_{i=1}^{n-1}(u - y_i), \quad P_n(u) = \prod_{i=1}^{n}(u - x_i) \tag{2}$$

interlace: $x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n$. We associate to the pair of polynomials (2) a piecewise linear function $v = \omega_n(u)$ (we call it a diagram) characterized by

$$\omega_n'(u) = \operatorname{sign}\frac{P_{n-1}(u)}{P_n(u)}; \qquad \omega_n(u) = |u - z_n| \quad \text{for sufficiently large } |u|, \tag{3}$$

where $z_n = \sum x_i - \sum y_i$. The roots of polynomials P_n are the minima for ω_n, and the roots of polynomial P_{n-1} are its maxima (Figure 1).

The central result of the paper is

Theorem 1.5. *Assume that*

$$\lim_{n\to\infty} \frac{c_{n-1}}{c_n} = 1, \quad \lim_{n\to\infty} \frac{b_n - b_{n-1}}{c_n} = 0. \tag{4}$$

in the recurrence relation (1). Then the limit

$$\lim_{n\to\infty} \frac{1}{c_n}\omega_n(uc_{n-1} + b_n) = \tilde{\Omega}(u), \tag{5}$$

where

$$\tilde{\Omega}(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2, \end{cases} \tag{6}$$

exists uniformly w.r.t. $u \in \mathbf{R}$.

A weaker result had been established in [7]. Instead of the conditions (4) we were assuming there that

$$\lim_{n\to\infty} c_n = c > 0, \qquad \lim_{n\to\infty} b_n = 0. \tag{7}$$

Let $d\mu$ be a measure for which the family $\{P_n\}$ is orthogonal. The conditions (7) will hold, if the interval $[-1,1]$ is the support of $d\mu(u)$ and the function $\ln \mu'(\cos\theta)$ is integrable on the interval $[0,\pi]$. On the other hand, the conclusion of the Theorem is definitely false, if the support of the measure $d\mu$ is not connected. The case of classical Hermite and Laguerre polynomials sits in between: the conditions (4) are valid, while the conditions (7) are not.

It is well known [12] that if $c = 1$ in (7), then there exists a common limiting density of root distribution

$$\lim_{n\to\infty} \frac{1}{n} \#\{i \colon x_i < u\} = \frac{1}{\pi} \int_{-1}^{u} \frac{dx}{\sqrt{1-x^2}}, \qquad |u| \le 1.$$

For the Hermite polynomials the limiting density will be different:

$$\lim_{n\to\infty} \frac{1}{n} \#\{i \colon \frac{x_i}{\sqrt{2n}} < u\} = \frac{2}{\pi} \int_{-1}^{u} \sqrt{1-x^2}\, dx, \qquad |u| \le 1,$$

though the asymptotics of root separation is the same in both cases. Thus, our result is more universal (and also more robust) than the theorems on limiting root distributions for orthogonal polynomials.

The function $\tilde{\Omega}(u)$ had first appeared in the paper [2] in the apparently unrelated context of the asymptotic representation theory of the symmetric group. Let us recall the result thereof.

Irreducible representations of the symmetric group $\mathfrak{S}_n$ of order n can be parametrized by Young diagrams with n boxes. Let us denote the set of such diagrams by $\mathcal{Y}_n$, and let $\dim\lambda$ denote the dimension of the representation corresponding to the diagram $\lambda \in \mathcal{Y}_n$. According to the well-known Burnside formula,

$$\sum_{\lambda\in\mathcal{Y}_n} \dim^2 \lambda = n!$$

so that the weights $(\dim^2\lambda)/n!$; $\lambda \in \mathcal{Y}_n$ determine the probability measure μ_n on $\mathcal{Y}_n$ (called the Plancherel measure).

By definition, a Young diagram is a compact subset in $\mathbf{R}^+ \times \mathbf{R}^+$ containing with each point (X, Y) all the points (X', Y') for which $[X'] < X$, $[Y'] < Y$ (here the brackets denote the integer part of a number). Let us introduce co-ordinates $u = Y - X$, $v = Y + X$ and define a piecewise linear function $v = \omega_\lambda(u)$ by $\omega_\lambda(u) = \min(X + Y)$, where the minimum is taken over the set $d_u = \{(X, Y) \in \mathbf{R}^2 \setminus \lambda : Y - X = u\}$.

In [2, 9] the following Law of Large Numbers for the Plancherel measure had been obtained:

Theorem. *For any $\epsilon > 0$,*

$$\lim_{n\to\infty} \mu_n\{\lambda \in \mathcal{Y}_n : \sup_u |\frac{1}{\sqrt{n}}\omega_\lambda(u\sqrt{n}) - \tilde{\Omega}(u)| < \epsilon\} = 1,$$

where the function $\tilde{\Omega}$ is determined by the formula (6).

Thus, for large n, the border of a typical diagram (w.r.t. the Plancherel measure) turns out to be uniformly close to the graph of the function $\tilde{\Omega}$. It was also shown in [4] that if a Young diagram λ_n maximizes the dimension over the set of all diagrams $\lambda \in \mathcal{Y}_n$, then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\omega_{\lambda_n}(u\sqrt{n}) = \tilde{\Omega}(u)$$

uniformly in $u \in \mathbf{R}$.

The reasons for the appearance of one and the same function (6) in two different contexts above are not quite clear to the author.

Another context, where the curve $\tilde{\Omega}$ arises, is related to the spectral theory of random matrices and Wigner's Semicircle Law.

Let us consider a hyperplane $h_n \subset \mathbf{R}^n$ and a symmetric matrix $A^{(n)}$ of order n with the characteristic polynomial $P_n(u) = \det(uE - A^{(n)})$. By p we denote the projection onto h_n, and by $Q_{n-1}(u) = \det(uE - pA^{(n)}p)$ the characteristic polynomial for the restriction of $A^{(n)}$ to the hyperplane h_n. It is well known that the roots $y_1, \ldots, y_{n-1}$ of polynomial Q_{n-1} and the roots $x_1, \ldots, x_{n-1}$ of P_n interlace. We define the diagram $v = \omega_n(u)$ by the conditions similar to (3), using the roots of polynomials Q_{n-1} and P_n.

Let us now choose a matrix $A^{(n)}$ and a hyperplane h_n at random. Then the diagram ω_n is also random, and one can ask about the mean value $E(\omega_n)$ of this random function. We derive from Wigner's Semicircle Law

Theorem 3.6. *Let the normal vector to the hyperplane h_n be uniformly distributed on the unit sphere in $\mathbf{R}^n$, and let the matrix elements $a_{ij}^{(n)}$, $i \le j$ of the matrix $A^{(n)}$ be independent of h_n and of each other, and identically distributed with the mean $E(a_{ij}^{(n)}) = 0$ and the variance $D(a_{ij}^{(n)}) = 1$. Then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} E\omega_n(u\sqrt{n}) = \tilde{\Omega}(u)$$

uniformly w.r.t. $u \in \mathbf{R}$.

This statement can be improved considerably, and the author hopes to deal with it later on.

The relationship between the curve $\tilde{\Omega}$ and the semi-circle distribution $1/2\pi \times\sqrt{4-u^2}\,du$ is an example of the non-trivial correspondence between continuous diagrams similar to $\tilde{\Omega}$ (see the precise definition in Section 2.1), and probability distributions on the reals. Again, there is an enigmatic connection between two subjects which seem to have very little in common: combinatorial analysis of permutations (the Hook Formula for dimensions of irreducible representations) and classical questions of the function theory (the Markov's moment problem), see [13].

The appearance of the function $\tilde{\Omega}$ as a standard asymptotic behavior in some problems of mathematical physics is not limited by the cases described in Theorems 1.5, 3.6. Another example arises in connection with the differential model of growth of Young diagrams obtained in [15], and leads to the quasilinear equation of the first order

$$R'_t + RR'_x = 0,$$

describing the free motion of a continuum of non-interacting particles.

The plan of this article is as follows: in Section 1 we make precise the formulation of the main result which is proved in Section 2. In Section 3 some parallel results are discussed: the Wigner' Theorem and the asymptotics for the Plancherel measure of the symmetric group.

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§1. Statement of the main theorem

1.1. Rectangular diagrams and interlacing sequences. Let $v = \omega(u)$ be a continuous piecewise-linear function satisfying

(i) $\omega'(u) = \pm 1$

(ii) $\omega(u) = |u - z|$ for some $z \in \mathbf{R}$ and sufficiently large $|\,u\,|$.

We will call such functions *rectangular diagrams*, and the number $z = z(\omega)$ the *center* of the diagram ω. After a shift and rotation by 45°, the subgraph

$$D_\omega = \{(u, v)\colon |u - z| \le v < \omega(u)\} \tag{1.1.1}$$

(see Figure 1) looks like a Young diagram; this fact will be used later on and explains the choice of the term. By definition, the area $A = A(\omega)$ of the subgraph (1.1.1) is called the *area of the diagram.*

We shall use rectangular diagrams to visualize relative positions of two interlacing sequences. For a given diagram ω, we denote by $x_1, \ldots, x_n$ (and

$y_1, \ldots, y_{(n-1)}$ respectively) the locations of its local minimum (maximum) values. These sequences interlace:

$$x_1 < y_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < x_n. \tag{1.1.2}$$

In the opposite direction, for any pair of interlacing sequences (1.1.2) there exists a unique rectangular diagram ω with these two sequences as maxima and minima. In fact, let us take

$$z = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} y_i \tag{1.1.3}$$

for the center and define the values of ω at the break points by $\omega(x_1) = z - x_1$ and the recurrence relations

$$\omega(y_k) - \omega(x_k) = y_k - x_k, \quad \omega(x_{k+1}) - \omega(y_k) = y_k - x_{k+1} \tag{1.1.4}$$

for $k = 1, \ldots, n-1$. Then $\omega(x_n) = x_n - z$, and both conditions (i), (ii) in the definition of rectangular diagram hold. The uniqueness of the diagram with the given extreme values is guaranteed by the condition (ii).

Apart from the visual clarity, the diagram representation of interlacing sequences has another, and more important, advantage: the conception of limiting diagrams characterizing asymptotic mutual separation is immediate. We shall return to this subject in Section 2.1.

1.2. Separation of roots of orthogonal polynomials. Let us consider the polynomials

$$P_n(x) = \prod_{k=1}^{n} (x - x_k), \quad P_{n-1}(x) = \prod_{k=1}^{n-1} (x - y_k),$$

belonging to a family of polynomials orthogonal w.r.t. a certain measure $d\mu(x)$. It is well known ([12]) that for their roots the mutual separation condition (1.1.2) holds. For example, Table 1 exhibits the roots (for $n = 15, 16$) of Tchebychef polynomials of the second kind

$$U_n(x) = \frac{\sin{(n+1)}\mathrm{arccos}\, x}{\sin \mathrm{arccos}\, x},$$

of the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2}\right),$$

and of the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(x^n e^{-x}\right).$$

The Figures 2 – 4 exhibit the corresponding rectangular diagrams.

1.3. The universal diagram Ω. One can easily see that the diagrams of root separation represented in Figures 2 – 4 are similar (up to the scale parameters). We emphasize the fact by superimposing (in a suitable scale) the graph of the function

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u \arcsin u + \sqrt{1-u^2}), & \text{if } |u| \le 1 \\ |u|, & \text{if } |u| \ge 1. \end{cases} \tag{1.3.1}$$

The function Ω first appeared in [2], [9] in connection with a certain problem of asymptotic representation theory of symmetric groups (the description of the limiting shape of large Young diagrams, see Section 3 below). We shall show (Theorem 1.5) that this very function represents the asymptotic behavior of interlacing roots for a wide class of families of orthogonal polynomials.

Some formulas will simplify, if we use the function $\tilde{\Omega}(u) = 2\Omega(u/2)$ instead of Ω, so that

$$\tilde{\Omega}(u) = \begin{cases} \frac{2}{\pi}(u \arcsin \frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \le 2 \\ |u|, & \text{if } |u| \ge 2. \end{cases} \tag{1.3.2}$$

We remark that the area $A = A(\Omega)$ of the subgraph $\{(u, v) : | u | \le v < \tilde{\Omega}(u)\}$ for this function equals 1.

1.4. Linear recurrence relations. In what follows it will be convenient to use another standard approach to the theory of orthogonal polynomials, in which they arise as solutions of difference equations (see, for example, [1]).

Let us consider a linear difference equation of the second order

$$\phi_{n+1} + b_{n+1}\,\phi_n + c_n^2\,\phi_{n-1} = u\,\phi_n; \quad n \ge 1 \tag{1.4.1}$$

with the spectral parameter u, and let $\phi_n = P_n(u)$, $n = 0, 1, \ldots$ be its unique solution satisfying the initial conditions

$$P_0(u) = 1, \quad P_1(u) = u - b_1. \tag{1.4.2}$$

Manifestly, $P_n(u)$ is a unital polynomial of degree n. Assuming that $c_n^2 > 0$ for all $n \ge 1$, it is easy to see that the roots of polynomials $P_n(u)$ are real and simple, and that the roots of adjacent polynomials $P_{n-1}(u)$, $P_n(u)$ interlace (see [1], Ch.4). Favard's theorem (see [11], Ch. 8.6) guarantees that there exists a measure μ for which the polynomials are orthogonal,

$$\int_{-\infty}^{\infty} P_m(u)\,P_n(u)\,d\mu(u) = 0 \quad \text{for } m \ne n. \tag{1.4.3}$$

On the other hand, the polynomials orthogonal w.r.t. a certain measure satisfy the linear recurrence relation (1.4.1).

1.5. Theorem. *Let the orthogonal polynomials $P_n(u)$ be defined by the equation (1.4.1) and the initial conditions (1.4.2), and let ω_n be a rectangular diagram which determines the mutual separation of roots of the polynomials $P_{n-1}(u)$, $P_n(u)$. We assume that the following limits exist for the coefficients of (1.4.1):*

$$\lim_{n\to\infty} \frac{c_{n-1}}{c_n} = 1, \qquad \lim_{n\to\infty} \frac{b_{n+1}-b_n}{c_n} = 0. \tag{1.5.1}$$

Then

$$\lim_{n\to\infty} \frac{1}{c_n}\, \omega_n(u\, c_{n-1} + b_n) = \tilde{\Omega}(u) \tag{1.5.2}$$

uniformly in $u \in \mathbf{R}$.

Let us note that this result is more rough (but also more universal) than the known theorems concerning the limiting distributions of roots of orthogonal polynomials (see [12]). For instance, the limiting density of roots is proportional to $(1-u^2)^{1/2}$ for Hermite polynomials, and to $(1-u^2)^{-1/2}$ for those of Tchebychef. Nevertheless, the character of root separation in both cases is the same.

§2. Proof of the Theorem

2.1. Continuous diagrams. A continuous function $v = \omega(u)$ will be called *diagram*, if the following two conditions are satisfied:

(i) $|\omega(u_1) - \omega(u_2)| \le |u_1 - u_2|$

(ii) $\omega(u) = |u - z|$ for some $z \in \mathbf{R}$ and sufficiently large $|\, u \,|$.

For rectangular diagrams these conditions do hold. The functions Ω and $\tilde{\Omega}$ defined in (1.3.1–2) may illustrate the more general concept of a diagram. The number $z = z(\omega)$ in the condition (ii) will be called the *center of a diagram* ω, and by its *area* we shall mean the area $A = A(\omega)$ of the region (1.1.1).

Continuous diagrams arise as limiting objects for rectangular ones: the space of diagrams $\mathcal{D}$ is the closure of the subset $\mathcal{D}_0$ of rectangular diagrams in the topology of bounded uniform convergence.

2.2. The R-function of a diagram. For a given diagram $\omega \in \mathcal{D}$ with the center z, let us consider the function

$$\sigma(u) = \frac{1}{2}(\omega(u) - |u - z|). \tag{2.2.1}$$

This function is compactly supported, nonnegative and unimodal (Figure 5); by virtue of Lipschitz condition (i) the differential $d\sigma(u) = \sigma'(u)du$ determines the absolutely continuous *charge of diagram.*

Definition. The R-*function* of a diagram ω is given by the formula

$$R(u) = \frac{1}{u-z} \exp \int \frac{d\sigma(t)}{t-u} \tag{2.2.2}$$

(here and below the integrals with no explicitly indicated limits are to be taken over the whole axis). For diagrams with bounded support, the R-function is correctly defined and analytic for sufficiently large $\mid u \mid$.

Example. For the diagram Ω we have

$$R_\Omega(u) = 2(u - \sqrt{u^2-1}). \tag{2.2.3}$$

In fact, $d\sigma_\Omega(u) = (1/\pi \arcsin u - 1/2 \operatorname{sign} u)du$ and the integral

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\arcsin t\, dt}{u-t} = \ln(u + \sqrt{u^2-1}) - \ln\sqrt{(u^2-1)}$$

(for $u > 1$) may be reduced to the known integral

$$\frac{1}{\pi}\int_{-1}^{1} \frac{dt}{(u-t)\sqrt{1-t^2}} = \frac{1}{\sqrt{u^2-1}}$$

by means of integration by parts and differentiation w.r.t. the parameter.

2.3. The R-function of a rectangular diagram. Let ω be a rectangular diagram with the center z, constructed by the pair of interlacing sequences (1.1.2). For any smooth function f the following version of the Newton-Leibnitz formula holds:

$$\int f'(t)\, d\sigma(t) = f(z) + \sum f(y_k) - \sum f(x_k). \tag{2.3.1}$$

Indeed, $x_1 < z < x_n$ implies that for some k, either $y_{k-1} \le z \le x_k$ or $x_k \le z \le y_k$. In the first case both parts of (2.3.1) coincide with

$$\sum_{i=1}^{k-1} \int_{x_i}^{y_i} f'(t)\, dt - \int_{z}^{x_k} f'(t)\, dt - \sum_{k+1}^{n} \int_{y_{i-1}}^{x_i} f'(t)\, dt;$$

the second case is similar.

Let us find the R-function of a rectangular diagram.

Lemma. *Let a rectangular diagram be constructed by the interlacing roots of polynomials*

$$P(u) = \prod_{k=1}^{n} (u - x_k), \quad Q(u) = \prod_{k=1}^{n-1} (u - y_k).$$

Then $R(u) = Q(u)/P(u)$.

Proof. It is sufficient to apply (2.3.1) to the function $f(t) = \ln(u-t)$.

2.4. Translations and homothetic transformations. If the rectangular diagram ω is constructed by the sequences (1.1.2), then (for $c > 0$) the function

$$\tilde{\omega}(u) = \frac{1}{c}\omega(cu+b) \tag{2.4.1}$$

is also a rectangular diagram and corresponds to the sequences

$$\tilde{x}_k = \frac{x_k - b}{c}, \quad \tilde{y}_k = \frac{y_k - b}{c}. \tag{2.4.2}$$

Its center is shifted to the point $\tilde{z} = (z-b)/c$, and its area $\tilde{A} = A/c^2$ is reduced by the factor c^2. The charge and the R-function are given by the formulas

$$\begin{aligned} d\tilde{\sigma}(u) &= \frac{1}{c} d\sigma(cu+b), \\ \tilde{R}(u) &= c\, R(cu+b). \end{aligned} \tag{2.4.3}$$

All these formulae (excluding (2.4.2), of course) are valid for arbitrary (continuous) diagrams.

For example, for the diagram $\tilde{\Omega} = 2\Omega(u/2)$ we obtain $R_{\tilde{\Omega}}(u) = \frac{1}{2}(u - \sqrt{u^2-4})$.

2.5. An equation for the limiting R-function. Denote by $R_n(u) = P_{n-1}(u)/P_n(u)$ the R-function of the diagram ω_n in Theorem 1.5. After having divided both parts of the identity

$$P_{n+1}(u) = (u - b_{n+1})P_n(u) - c_n^2 P_{n-1}(u)$$

by $P_n(u)$, we obtain the following recurrence relation for R-functions:

$$\frac{1}{R_{n+1}(u)} = u - b_{n+1} - c_n^2\, R_n(u). \tag{2.5.1}$$

Let us pass to normalized functions, i.e. to the R-functions

$$\tilde{R}_n(u) = c_{n-1}\, R_n(uc_{n-1} + b_n), \quad \tilde{R}_{n+1}(u) = c_n\, R_{n+1}(uc_n + b_{n+1})$$

of diagrams $\tilde{\omega_n}(u) = \omega_n(uc_{n-1} + b_n)/c_{n-1}$; $\omega_{\tilde{n+1}}(u) = \omega_{n+1}(uc_n + b_{n+1})/c_n$. After the change of variable u by $(u - b_{n+1})/c_n$ the formula (2.5.1) takes the form

$$\frac{1}{\tilde{R}_{n+1}(u)} = u - \frac{c_n}{c_{n-1}}\tilde{R}_n\left(\frac{uc_n + \Delta_n}{c_{n-1}}\right),$$

where $\Delta_n = b_{n+1} - b_n$.

Let us assume that the limit

$$\tilde{R}(u) = \lim_{n\to\infty} \tilde{R}_n(u) \tag{2.5.2}$$

exists for sufficiently large $|\, u \,|$. Then, by virtue of the conditions (1.5.1) on the coefficients, we obtain the equation

$$\tilde{R}^2(u) - u\,\tilde{R}(u) + 1 = 0, \tag{2.5.3}$$

hence $\tilde{R}(u) = \frac{1}{2}(u - \sqrt{u^2-4})$ (the sign of the square root is chosen from the condition $\tilde{R}(\infty) = \lim \tilde{R}_n(\infty) = 0$).

In order to prove that the limit (2.5.2) exists, let us investigate the behavior of Taylor's coefficients at infinity.

2.6. Taylor's coefficients. Let us rewrite the relation (2.5.1) in the form

$$\frac{1}{R_{n+1}(u-\Delta_n+b_{n+1})} = u - \Delta_n - c_n^2\, R_n(u+b_n),$$

and consider the power series

$$R_n(u+b_n) = \sum_{k\geq 0} \frac{r_k^{(n)}}{u^{k+1}}.$$

Since

$$R_{n+1}(u-\Delta_n+b_{n+1}) = \sum_{k\geq 0} \frac{r_k^{(n+1)}}{(u-\Delta_n)^{k+1}} = \sum_{k\geq 0} \frac{\rho_k^{(n+1)}}{u^{k+1}},$$

where

$$r_k^{(n+1)} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \rho_{k-i}^{(n+1)} \Delta^i, \tag{2.6.1}$$

we infer the following recurrence relations for coefficients:

$$\rho_{k+1}^{(n+1)} = \rho_k^{(n+1)}\,\Delta_n + c_n^2 \left(\rho_{k-1}^{(n+1)}\, r_0^{(n)} + \ldots + \rho_0^{(n+1)}\, r_{k-1}^{(n)}\right) \tag{2.6.2}$$

for $k = 0, 1, 2, \ldots$.

Examples:

$$\begin{aligned}
\rho_0^{(n+1)} &= 1, & r_0^{(n+1)} &= 1,\\
\rho_1^{(n+1)} &= \Delta_n, & r_1^{(n+1)} &= 0,\\
\rho_2^{(n+1)} &= c_n^2 + \Delta_n^2, & r_2^{(n+1)} &= c_n^2,\\
\rho_3^{(n+1)} &= 2\Delta_n c_n^2 + \Delta_n^3, & r_3^{(n+1)} &= -\Delta_n c_n^2,\\
\rho_4^{(n+1)} &= c_n^4 + c_n^2 c_{n-1}^2 + 3\Delta_n^2 c_n^2 + \Delta_n^4, & r_4^{(n+1)} &= c_n^4 + c_n^2 c_{n-1}^2 + \Delta_n^2 c_n^2.
\end{aligned}$$

More generally, (2.6.1), (2.6.2) imply

Lemma. *The coefficients $r_k^{(n-1)}$ are homogeneous polynomials of degree k in the first $k-1$ variables of the sequence c_n, Δ_n, c_{n-1}, $\Delta_{n-1}, \dots$.*

Corollary. *Under the assumptions of Theorem 1 the limits $\lim_{n\to\infty} \tilde r_k^{(n)} = C_k$ exist, where $\tilde r_k^{(n)} = r_k^{(n)}/c_n^k$ are the Taylor coefficients of the normalized R-function*

$$\tilde R_n(u) = \sum_{k\geq 0} \tilde r_k^{(n)}/u^{k+1}.$$

Let us remark that the limiting coefficients $C_{2k} = \frac{1}{k+1}\binom{2k}{k}$ are the Catalan's numbers. These very numbers also arise as the coefficients of the expansion $R_{\tilde\Omega}(u) = C_0/u + C_2/u^3 + \dots$ of the R-function of the diagram $\tilde\Omega$. It follows from the convergence of R-functions, $\lim \tilde R_n(u) = R_{\tilde\Omega}(u)$, that the diagrams converge uniformly (see [13]), $\lim \tilde{\omega_n}(u) = \tilde\Omega(u)$, and the theorem is proved.

§3. The diagram Ω in other contexts

3.1. The growth of Young diagrams. It is convenient to define here a Young diagram as a rectangular diagram for which

(i) all extrema $x_1 < y_1 < \dots < y_{n-1} < x_n$ are integer and

(ii) $z(\omega) = 0$.

The straight lines $v - u = 2X$, $v + u = 2Y$, where $X, Y \in \mathbf{Z}$, divide the set

$$D_\omega = \{(u,v) \colon |u| \leq v < \omega(u)\} \tag{3.1.1}$$

into *boxes* (Figure 6); the number of boxes will be denoted by $n = | D_\omega |$. After rotation by 45°, the diagram takes a more usual form. It represents the partition of the natural number n where a number $\sum_{j>k}(x_j - y_{j-1})$ occurs $(y_k - x_k)$ times, $k = 1, 2, \dots, n-1$. Let us denote by $\mathcal{Y}_n$ the (finite) set of Young diagrams with n boxes. The set of all Young diagrams $\mathcal{Y} = \cup_{n=0}^{\infty} \mathcal{Y}_n$ ordered by inclusion of subgraphs (3.1.1) is called the *Young lattice*. A Young diagram Λ immediately follows a diagram λ in the Young lattice, $\lambda \nearrow \Lambda$, iff $D_\lambda \subset D_\Lambda$, and D_Λ differs from D_λ by exactly one box. By a *Young tableau* of shape λ we mean an oriented path

$$t = (\emptyset \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \dots \nearrow \lambda_n = \lambda)$$

in the Young lattice leading from the null diagram $\emptyset$ to λ. In other words, a Young tableau is a way of growth for a Young diagram.

The Young lattice plays a central role in representation theory of symmetric groups $\mathfrak{S}_n$ (see [6]). The irreducible representations π_λ of the group $\mathfrak{S}_n$ are parametrized by Young diagrams $\lambda \in \mathcal{Y}_n$, the dimension of representation $\dim \lambda = \dim \pi_\lambda$ being the number of tableaux of the shape λ. The ordering in the Young lattice determines the branching of representations when restricted to subgroups:

$$\pi_\Lambda|_{\mathfrak{S}_n} = \bigoplus_{\lambda:\lambda\nearrow\Lambda} \pi_\lambda.$$

3.2. The Plancherel measure. Let $\mathcal{T}$ be a (totally disconnected, compact) space of infinite Young tableaux. The Plancherel measure M was defined in [3] as the Markov probability measure on $\mathcal{T}$ determined by its transition probabilities

$$p_{\lambda\Lambda} = \frac{\dim \Lambda}{(n+1)\dim\lambda}; \quad \lambda \in \mathcal{Y}_n.$$

The conditional distribution at the n-th level of Young lattice $\mathcal{Y}_n$ coincides with the Plancherel measure of a finite group $\mathfrak{S}_n$,

$$P\{t \in \mathcal{T}\colon \lambda_n = \lambda\} = \frac{\dim^2\lambda}{n!}. \tag{3.2.1}$$

One can see (cf. [3]) that the measure M is related to the regular representation of the infinite group of finite permutations $\mathfrak{S}_\infty$; this is another reason for the choice of its name.

It was established in the papers [2], [9], [4] that in the course of random growth almost all of Young diagrams become uniformly close (after the area is normalized) to the unique universal shape, namely the graph of the diagram Ω:

Theorem ([2], [9]). *For almost all (w.r.t. the Plancherel measure M) random Young tableaux $t = (\omega_1, \dots, \omega_n, \dots) \in \mathcal{T}$,*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\omega_n(u\sqrt{n}) = \Omega(u)$$

uniformly in $u \in \mathbf{R}$.

It is shown in [4] that in case of a large number of boxes n the shape of a Young diagram which maximizes the dimension function is also close to the graph of Ω.

The appearance of the same function in Theorems 1.5 and 3.2 is even more astonishing since not only the contexts, but also the characteristic properties of the diagram Ω used in two proofs are completely different. In the proof of Theorem 5.1 it was the equation (2.5.3) for R-function, and in the proof of Theorem 3.2 it was important that Ω minimizes a certain functional — the hook integral.

3.3. The Hook integral. The famous Hook Formula (see [6, 10])

$$\dim \lambda = \frac{n!}{\prod_{ij} h_{ij}},$$

where h_{ij} is the hook length for the j-th box of the i-th line of a Young diagram $\lambda \in \mathcal{Y}_n$, allows one to rewrite (3.2.1) in the form

$$-\frac{\ln P\{\lambda_n = \lambda\}}{\sqrt{n}} = J_\lambda \sqrt{n} + o(1) \tag{3.3.1}$$

(here $n \to \infty$, and the estimate is uniform in $\lambda \in \mathcal{Y}_n$), where

$$J_\lambda = 1 + \frac{2}{n} \sum_{ij} \ln \frac{h_{ij}}{\sqrt{n}}. \tag{3.3.2}$$

The expression (3.3.2) is an integral sum for the *hook integral*

$$\theta(\omega) = 1 + 2 \int\int \ln h_\omega(x, y)\, dx\, dy,$$

taken over the subgraph (3.1.1).

It follows from the asymptotic formula of Hardy-Ramanujan,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp \frac{2\pi}{\sqrt{6}} \sqrt{n},$$

that the total number $p(n)$ of Young diagrams with n boxes grows subexponentially. Hence, the set of diagrams whose probabilities decrease more rapidly than $\exp(-c\sqrt{n})$, $c = 2\pi/\sqrt{6}$ is asymptotically a null set. Comparing this with (3.3.1-2) we infer that the limiting shape of large Young diagrams, typical for the Plancherel measure, makes the hook integral zero.

The key property of the function Ω used in Theorem 3.2 is the following

Lemma. *Let ω be a diagram with the area $A(\omega) = 1$. Then $\theta(\omega) > 0$ for $\omega \neq \tilde{\Omega}$, and $\theta(\tilde{\Omega}) = 0$.*

The change of variables $2x = \omega(v) - v$, $2y = \omega(u) + u$ (after which we have $h_\omega(x, y) = v - u$) allows one to rewrite the hook integral in the form

$$\theta(\omega) = 1 + \frac{1}{2} \iint_{u<v} \ln(v - u)\, d(u + \omega(u))\, d(v - \omega(v)).$$

One can not yet see in this form that it is nonnegative. But, it was shown in [4] that $A(\omega) = 1$ implies

$$\theta(\omega) - \theta(\tilde{\Omega}) = \frac{1}{4} ||\omega - \tilde{\Omega}||^2 + \int_{|u|>2} (\omega(u) - |u|)\, \mathrm{arch} \frac{|u|}{2}\, du, \tag{3.3.3}$$

where

$$||f||^2 = \iint \left(\frac{f(u)-f(v)}{u-v}\right)^2 du\, dv$$

is the squared Sobolev norm and $\operatorname{arch} u = \ln(u + \sqrt{u^2-1})$. It is clear from (3.3.3) that $\theta(\tilde{\Omega})$ is the minimum value of the hook integral on the set of diagrams of area 1. One can check by a direct calculation that $\theta(\tilde{\Omega}) = 0$.

3.4. Wigner's Semicircle Law. Let $A^{(n)}$ be a symmetric matrix of order n with the characteristic polynomial

$$P_n(x) = \prod_k (x - x_k) = \det(xE - A^{(n)}).$$

We denote by $\mu^{(n)}$ the discrete distribution with equal weights $\mu_k^{(n)} = 1/n$ located at the characteristic roots x_k; $k = 1, \ldots, n$. Denote by

$$F_n(x) = \frac{1}{n}\#\{k\colon x_k < x\}$$

the distribution function of the measure $\mu^{(n)}$, and by $F(x)$ the distribution function of the measure $d\mu(u) = (2\pi)^{-1}\sqrt{4-u^2}$ on the interval $[-2, 2]$. We are interested in the case of a random matrix $A^{(n)}$; as a result, the polynomial P_n, its roots, the measure $\mu^{(n)}$ and the function F_n are random, too.

Theorem (cf. [5]). *Assume that the matrix elements $a_{ij}^{(n)}$ for $i \le j$ be independent, identically distributed and that $E(a_{ij}^{(n)}) = 0$, $D(a_{ij}^{(n)}) = 1$. Then for any $\epsilon > 0$*

$$P\{\sup_x |F_n(x\sqrt{n}) - F(x)| < \epsilon\} \to 1 \tag{3.4.1}$$

as far as $n \to \infty$.

This result can be restated in terms of diagrams. As above, let us denote by $x_1, \ldots, x_n$ the characteristic roots of the matrix $A^{(n)}$, and by $y_1, \ldots, y_{n-1}$ the extreme values of the characteristic polynomial $P_n(x)$ of matrix $A^{(n)}$ which interlace with the roots. Let $\overline{\omega}_n$ be the rectangle diagram constructed by the pair of sequences $x_1 < y_1 < \cdots < y_{n-1} < x_n$.

Proposition. *Under the assumptions of the theorem,*

$$P\{\sup_u |\frac{1}{\sqrt{n}}\bar{\omega}_n(u\sqrt{n}) - \tilde{\Omega}(u)| < \epsilon\} \to 1 \tag{3.4.2}$$

for any $\epsilon > 0$, as far as $n \to \infty$.

The equivalence of the latter result to the Wigner's Theorem is a corollary of the remarkable correspondence between diagrams and probability measures. We present here only the main statements; for details see [7], [8], [13].

3.5. Diagrams and measures. Let $\mathcal{M}[a,b]$ be the space of probability distributions on a certain interval $[a,b]$, with the topology of weak convergence. Note that the subset $\mathcal{M}_0$ of distributions with finite support is dense in $\mathcal{M}$. Let us define the *R-function of a measure* $\mu \in \mathcal{M}$ by the formula

$$R_\mu(u) = \int \frac{d\mu(t)}{u-t}. \tag{3.5.1}$$

Theorem ([7], [13], cf. [8]). *There exists a well-defined bijective and bicontinuous correspondence between probability measures μ and diagrams ω, preserving R-functions:*

$$R_\omega(u) = \frac{1}{u-z} \exp \frac{1}{2} \int \frac{d(\omega(t) - |t-z|)}{t-u} = \int \frac{d\mu(t)}{u-t} = R_\mu(u). \tag{3.5.2}$$

Example. By the results of Section 2.4 the semi-circle distribution, with its R-function $R_\mu(u) = \frac{1}{2}(u - \sqrt{u^2-4})$ (for $u > 2$), corresponds to the diagram $\tilde{\Omega}$.

The formula (3.5.2) may be interpreted as a generalized partial fraction decomposition. In fact, in the case of a rectangular diagram $\omega \in \mathcal{D}_0$ the formula takes the form

$$\frac{\prod(u-y_k)}{\prod(u-x_k)} = \sum_k \frac{\mu_k}{u-x_k}, \tag{3.5.3}$$

and relates two standard representations of a rational function: as a quotient of polynomials and as a sum of partial fractions. The fact that the weights μ_k are positive is equivalent to the mutual separation condition (1.1.2). The theorem claims that the above mentioned correspondence between $\mathcal{M}_0$ and $\mathcal{D}_0$ may be extended continuously to a homeomorphism between the spaces $\mathcal{M}$ and $\mathcal{D}$.

Since

$$\frac{P_n'(x)}{P_n(x)} = \sum_{k=1}^n \frac{1}{x-x_k}, \tag{3.5.4}$$

the diagram ϖ constructed using the interlacing roots and extreme values of a polynomial, corresponds to the discrete uniform distribution located at the roots of the polynomial, so that (3.4.2) is equivalent to (3.4.1).

3.6. Small oscillations and rigidity diagram. Consider a stable linear mechanical system with n degrees of freedom oscillating near the equilibrium. Its potential energy is quadratic and is determined by a positively definite symmetric matrix A. Its squared frequencies coincide with the eigenvalues $x_1, \ldots, x_n$ of this matrix.

If a linear constraint is imposed upon the system, the new frequencies interlace with the initial ones, i.e.

$$x_1 < y_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < x_n, \tag{3.6.1}$$

where $y_1, \ldots, y_{n-1}$ are the eigenvalues for the restriction of matrix A to the corresponding hyperplane. We consider the rectangular diagram ω_n describing (see Section 1.1) the separation character for squared frequencies of a linear system (3.6.1) under a linear constraint, and call it the *rigidity diagram.*

Now let the system (i.e. the matrix A) and the linear constraint (the hyperplane $h \subset \mathbf{R}^n$) be chosen at random. It turns out that, under mild conditions, the property of 'self-amenability' (see [5]) takes place: as far as the number of degrees of freedom grows, the random rigidity diagram approaches the non-random one (in a suitable scale). Let us show that this limiting diagram turns out to be exactly the diagram $\tilde{\Omega}$.

Theorem. *Let ω_n be the rigidity diagram for a random symmetric matrix $A^{(n)}$ of order n when restricted to a random hyperplane $h \subset \mathbf{R}^n$. Suppose that the normal to h is uniformly distributed on a unit sphere in $\mathbf{R}^n$, the matrix elements $a_{ij}^{(n)}$, $i \leq j$, are independent of h and among themselves, and that these elements are identically distributed with the mean value $E(a_{ij}^{(n)}) = 0$ and with the variance $D(a_{ij}^{(n)}) = 1$. Then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} E\omega_n(u\sqrt{n}) = \tilde{\Omega}(u) \tag{3.6.2}$$

uniformly in $u \in \mathbf{R}$.

Proof. Let us first consider the average w.r.t. the random constraint h.

Lemma. *Let $P_n(x) = \prod(x - x_i)$ be the characteristic polynomial of a matrix A of order n, let $h \subset \mathbf{R}^n$ be a random hyperplane with the normal uniformly distributed on the unit sphere, and let $Q_n(x) = \prod(x - y_i)$ be the characteristic polynomial of the matrix $A_h = pAp$ (here p is the orthogonal projection on h). Then*

$$E(Q_n) = \frac{1}{n} P_n'. \tag{3.6.3}$$

Proof. Represent the quotient Q_n/P_n by the sum of partial fractions:

$$\frac{Q_n(x)}{P_n(x)} = \sum_{k=1}^{n} \frac{\mu_k}{x - x_k}.$$

It is not difficult to see that $\mu_k = \xi_k^2$, where $\xi = (\xi_1, \ldots, \xi_n)$ is the normal vector of h. Using the assumption that the distribution of ξ is uniform, we derive that

$$E(\mu_1) = \ldots = E(\mu_n) = 1/n,$$

since $\sum \mu_k = 1$. It remains only to apply (3.5.4).

The theorem follows now from the variant (3.4.2) of the Semicircle Law.

§4. Separation of roots for Tchebychef polynomials

In this section we present the direct proof of Theorem 1.5 in the particular case of Tchebychef polynomials of the second kind. It contains a simple estimate for the rate of convergence in (1.5.2).

Proposition. *Let $x_k = -\cos(\frac{\pi k}{n+1})$; $k = 1, \dots, n$ be the roots of Tchebychef polynomials of the second kind $U_n(x)$, and let $y_k = -\cos(\frac{\pi k}{n})$, $k = 1, \dots, n-1$ be the roots of $U_{n-1}(x)$. Denote by ω_n the rectangular diagram constructed according to the formulas of Section 1.1. Then*

$$|\omega_n(u) - \Omega(u)| = O(1/n) \tag{4.1}$$

uniformly in $u \in \mathbf{R}$.

Proof. It suffices to prove (4.1) for $u = x_k$ and for $u = y_k$. It is clear from the definition (1.3.1) that

$$\Omega(y_k) = -y_k + \frac{2}{\pi}\left(\sin\frac{\pi k}{n} - \frac{\pi k}{n}\cos\frac{\pi k}{n}\right). \tag{4.2}$$

According to the description (1.1.4) of the diagram ω_n, we obtain

$$\omega(y_k) = 2z_k - (y_k - z) = -y_k + 2z_k; \quad \text{where } z_k = \sum_{i=1}^{k}(y_i - x_i).$$

Simplifying the sum of cosines by the formula

$$1 + 2\sum_{j=1}^{k}\cos jv = \sin(k+\frac{1}{2})v \Big/ \sin\frac{1}{2}v,$$

we find out that

$$2z_k = \frac{\sin(2k+1)\alpha_{n+1}}{\sin\alpha_{n+1}} - \frac{\sin(2k+1)\alpha_n}{\sin\alpha_n},$$

where $\alpha_n = \pi/2n$. Using the standard formula for the increment, we derive that (up to an inaccuracy of order $O(1/n^2)$)

$$2z_k = \frac{2}{\pi}\left[\sin(\frac{\pi k}{n} + \alpha_n) - (\frac{\pi k}{n} + \alpha_n)\cos(\frac{\pi k}{n} + \alpha_n)\right].$$

The comparison with (4.2) completes the proof (in case $u = x_k$ the proof is similar).

Figure 1. A rectangular diagram

Table 1
The roots of polynomials (of degree 15 and 16)

Tchebychef		Hermite		Laguerre	
x_n	y_n	x_n	y_n	x_n	y_n
−0.98297	−0.98079	−6.63088	−6.36395	0.08765	0.09331
−0.93247	−0.92388	−5.47223	−5.19009	0.46270	0.49269
−0.85022	−0.83147	−4.49296	−4.19621	1.14106	1.21560
−0.73901	−0.70711	−3.60087	−3.28908	2.12928	2.26995
−0.60263	−0.55557	−2.76025	−2.43244	3.43709	3.66762
−0.44574	−0.38268	−1.95198	−1.60671	5.07802	5.42534
−0.27366	−0.19509	−1.16383	−0.79913	7.07034	7.56592
−0.09227	0.00000	−0.38676	0.00000	9.43831	10.12023
0.09227	0.19509	0.38676	0.79913	12.21422	13.13028
0.27366	0.38268	1.16383	1.60671	15.44153	16.65441
0.44574	0.55557	1.95198	2.43244	19.18016	20.77648
0.60263	0.70711	2.76025	3.28908	23.51591	25.62389
0.73901	0.83147	3.60087	4.19621	28.57873	31.40752
0.85022	0.92388	4.49296	5.19009	34.58340	38.53068
0.93247	0.98079	5.47223	6.36395	41.94045	48.02608
0.98297		6.63088		51.70116	

References

1. F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York - London, 1964.
2. A. M. Vershik, S. V. Kerov, *Asymptotics of the Plancherel measure of the symmetric group, and the limiting shape of Young tablaux*, Soviet Math. Dokl. **18** (1977), 527 - 531.
3. A. M. Vershik, S. V. Kerov, *Asymptotic character theory of the symmetric group*, Funct. Analysis and its Applications **15** (1981), 15 - 27.
4. A. M. Vershik, S. V. Kerov, *The asymptotics of maximal and typical dimension of irreducible representations of the symmetric group*, Funct. Analysis and its Applications **19** (1985), 25 - 36.
5. L. A. Pastur, *Spectra of random self-adjoint operators*, Russian Math. Surveys **28** (1973), 1 - 67.
6. G. D. James, *The representation theory of the symmetric groups*, Lect. Notes in Mathematics **682** (1978), Springer - Verlag, Berlin - Heidelberg - New York.
7. S. Kerov, *Separation of roots of orthogonal polynomials and the limiting shape of generic large Young diagrams*, Preprint, **8** (1992), University of Trondheim.
8. M. G. Krein, A. A. Nudelman, *The Markov Moment Problem, and Extremum Problems*, Nauka, Moscow, 1973.
9. B. F. Logan, L. A. Shepp, *A variational problem for random Young tableaux*, Adv. Math. **26** (1977), 206 - 222.
10. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.
11. I. P. Natanson, *Constructive Function Theory*, Moscow, 1949.
12. G. Szegö, *Orthogonal Polynomials*, AMS, New York, 1959.
13. S. V. Kerov, *Transition Probabilities of continuousYoung diagrams, and the Markov Moment Problem*, Funct. Analysis and its Applications **27** (1993), 32 - 49.
14. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Nauka, Moscow, 1974.
15. S. V. Kerov, *The Differential Growth Model of Young Diagrams*, Proceedings of St. Petersburg Math. Society (1994).
16. S. V. Kerov, *Gaussian limit for the Plancherel measure of the symmetric group*, C. R. Acad. Sci. Paris **316** (1993), 303 - 308.