TRANSITION PROBABILITIES FOR CONTINUAL YOUNG DIAGRAMS AND MARKOV MOMENT PROBLEM

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1. Introduction

Hausdorff moment problem is that of describing the set of moment sequences

$$h_n = \int_0^1 t^n \, d\mu(t), \quad n = 0, 1, 2, \dots ,$$

where μ is an arbitrary Borel measure on the interval (there is also the question of restoring the measure in terms of its momenta). The solution for this problem is well known. In the less known Markov moment problem one has to describe the set of moment sequences

$$p_n = \int_0^1 t^n \, dr(t)$$

provided the measure $dr(t) = r'(t)\, dt$ is absolutely continuous and its density is restricted: $0 \le r'(t) \le 1$.

We are most interested in the relation between the problems of Hausdorff and Markov: the second problem reduces non-trivially to the first one [10]. In order to recall this connection we introduce the moment generating function

$$S(u) = \int_0^1 \frac{dr(t)}{t-u} = \sum_{n=1}^{\infty} p_{n-1} \, u^{-n}.$$

Note that the restrictions imposed on the measure $dr(t)$ imply that $-\pi < \operatorname{Im} S(u) < 0$, hence $H(u) = \exp S(u)$ is a resolvent function, i.e. it is holomorphic in the upper half-plane and $\operatorname{Im} H(u) \le 0$ for $\operatorname{Im} u > 0$. It follows from the known integral representation of the functions of this type that there exists the unique probability measure μ for which

$$\exp \int_0^1 \frac{dr(t)}{t-u} = \int_0^1 \frac{d\mu(t)}{1-t/u}. \tag{1.1}$$

In order to express the momenta of Markov problem in terms of the momenta of the measure μ we rewrite (1.1) as

$$\exp \sum_{n=1}^{\infty} p_{n-1} \, u^{-n} = \sum_{n=0}^{\infty} h_n \, u^{-n}. \tag{1.2}$$

Thus, the description of Markov moment sequences reduces to the well studied Hausdorff moment problem.

The connection between the measures dr and $d\mu$ is especially simple if the density $r'(t)$ equals unity on subintervals $(0, x_1)$, ..., (y_{n-1}, x_n), and is zero on the complementary intervals $(x_1, y_1), \dots, (x_n, 1)$. In this case the function $H(u)$ is rational, and the measure μ is determined by the weights

$$\mu_k = \prod_i (x_k - y_i) / \prod_{i \neq k} (x_k - x_i) \tag{1.3}$$

at the points x_k, $k = 1, \dots, n$. The defining relation (1.1) reduces itself to the usual decomposition of $H(u)/u$ as a sum of partial fractions:

$$\frac{\prod u - y_i}{\prod u - x_i} = \sum_k \frac{\mu_k}{u - x_k}. \tag{1.4}$$

The initial observation of the present paper is that the equation (1.3) has a natural interpretation in representation theory of the symmetric groups: the weights μ_k coincide with the transition probabilities of the Plancherel measure for the infinite symmetric group $\mathfrak{S}_\infty$. We consider a Young diagram as a piecewise linear function ω with the derivative $\omega'(u) = \pm 1$, and the function r in Markov' problem is equal to $r(u) = (u - \omega(u))/2$.

This connection proves to be useful in both directions: one can extend the combinatorial results on permutations and partitions and, on the other hand, the transformation (1.1) can be applied to the asymptotic problems in representation theory of the symmetric groups.

There are two main results.

It is shown in Section 3.3 that the transition probabilities for the Plancherel measure of the group $\mathfrak{S}_\infty$ approach the distribution with semi-circle density in the limit of infinitely growing Young diagrams. To this end we make essential use of the theorem of [3,11] on the limiting shape of large random diagrams.

Section 4 contains an algorithm for the random division of an interval, controlled by the diagram ω (or, equivalently, by the function r). The algorithm produces a random variable with the distribution μ related to the function r by the equation (1.1). In the course of the proof there arises an integral equation (4.3.5) for the transition probabilities. The algorithm itself is a generalization of the *hook walk* process introduced in [1,2] (see also [9]). It was used there to provide a probabilistic proof of the celebrated hook formula for the dimensions of irreducible representations of the symmetric group.

In the present article we use diagrams instead of the densities $r'(u)$ in the Markov moment problem. The bijection between diagrams and distributions is established below by the identity (2.3.3) analogous to (1.1); thus diagrams play the role of 'logarithms' for probability distributions. Though the result is probably well known to the experts, we include the complete proof, emphasizing the idea of partial fraction decomposition. This proof is compared to that of [10]. Another idea, leading to the notion of diagrams is considered in [7].

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2. Diagrams and measures

In this Section we define the diagrams, and establish the correspondence between diagrams and probability measures on the reals.

2.1 Diagrams. Following [8] (see also [3]) define *diagrams* to be the continuous functions $v = \omega(u)$ with two properties (Fig. 1):

(i) $|\omega(u_1) - \omega(u_2)| \le |u_1 - u_2|$

(ii) $\omega(u) = |u - z|$ for some $z \in \mathbb{R}$ and sufficiently large $|u|$.

The number $z = z(\omega)$ in (ii) is called the *center* of the diagram ω; the *area* of the diagram $A = A(\omega)$ is defined to be the area of the region

$$D_\omega = \{(u, v): |u| \le v < \omega(u)\} \tag{2.1.1}$$

(we assume that the square $\{|u| \le v < 2 - |u|\}$ has unit area).

In what follows we fix the interval $[a, b]$, and denote by $\mathcal{D} = \mathcal{D}[a, b]$ the set of diagrams with the property $\omega(u) = |u - z|$ for $u \notin [a, b]$. Let us endow the space $\mathcal{D}$ with the topology of uniform convergence.

A diagram ω is called *rectangular* if it is piecewise linear and $\omega'(u) = \pm 1$ for (almost) all $u \in \mathbb{R}$ (Fig.2). The rectangular diagram is completely determined by the co-ordinates of its minima $x_1, \dots, x_n$ and those of its maxima $y_1, \dots, y_{n-1}$. The two sequences interlace:

$$x_1 < y_1 < x_2 < \dots < x_{n-1} < y_{n-1} < x_n. \tag{2.1.2}$$

On the other hand, any pair of interlacing sequences (2.1.2) uniquely determines the rectangular diagram ω with the corresponding extrema and with the center at the point $z = \sum x_k - \sum y_k$. The area of the rectangular diagram is

$$A = \sum_{1 \le i < j \le n} (y_i - x_i)(x_j - y_{j-1}).$$

Remark that the subset of rectangular diagrams $\mathcal{D}_0$ is dense in $\mathcal{D}$.

2.2 R-function of a diagram. To any diagram $\omega \in \mathcal{D}[a, b]$ we associate the function $\sigma(u) = \frac{1}{2}(\omega(u) - |u|)$. This function has the following properties:

$$|\sigma(u_1) - \sigma(u_2)| \le |u_1 - u_2|,$$

$$\sigma'(u) = \frac{1}{2} \begin{cases} \omega'(u) + 1 \ge 0 & \text{for } u < 0 \\ \omega'(u) - 1 \le 0 & \text{for } u > 0, \end{cases}$$

$$\max_u \sigma(u) = \omega(0)/2,$$

$$\sigma(-\infty) = z/2; \quad \sigma(\infty) = -z/2.$$

The differential $d\sigma(u) = \sigma'(u)\, du$ determines the absolutely continuous *charge of a diagram* with the support in the interval $[a, b]$; the total variation of the charge is equal to $|d\sigma| = \omega(0)$.

Definition. *R-function of a diagram* $\omega \in \mathcal{D}[a,b]$ is defined by the formula

$$R_\omega(u) = \frac{1}{u}\exp\int_a^b \frac{d\sigma(t)}{t-u} = \frac{1}{u}\exp\frac{1}{2}\int_a^b \frac{d(\omega(t)-|t|)}{t-u}. \tag{2.2.1}$$

This function is holomorphic outside the interval $[a,b]$.

Example 1. Consider $\omega = \Omega$ with

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u\arcsin\frac{u}{2} + \sqrt{4-u^2}) & \text{if } |u| \le 2, \\ |u| & \text{if } |u| \ge 2. \end{cases} \tag{2.2.2}$$

Then $R_\Omega(u) = (u - \sqrt{u^2-4})/2$ for $u \ge 2$.

Example 2. Let $\omega = \omega_p$ be the diagram with the center $z = p - q$ (where $0 < p < 1$ and $q = 1-p$), linear on the interval $[-1,1]$, i.e.

$$\omega_p(u) = \begin{cases} 1 - uz & \text{for } |u| \le 1 \\ |u - z| & \text{for } |u| \ge 1. \end{cases} \tag{2.2.3}$$

Then

$$d\sigma(t) = \begin{cases} q\,du & \text{for } -1 < u < 0 \\ -p\,du & \text{for } 0 < u < 1 \end{cases}$$

and $R_\omega(u) = (u+1)^{-q}(u-1)^{-p}$ for $u \ge 1$.

Example 3. Let ω be the rectangular diagram with the extrema (2.1.2). Using polynomials $P(u) = \prod_{k=1}^n (u - x_k)$ and $Q(u) = \prod_{k=1}^{n-1}(u - y_k)$ one can write down the R-function as

$$R_\omega(u) = \frac{Q(u)}{P(u)}. \tag{2.2.4}$$

This follows from the formula

$$\int_a^b f'(t)\,d\sigma(t) = f(0) + \sum f(y_k) - \sum f(x_k), \tag{2.2.5}$$

applied to the function $f(t) = \ln(u-t)$. The formula (2.2.5) is just a variant of that of Newton-Leibnitz.

2.3 The correspondence between measures and diagrams. Denote by $\mathcal{M} = \mathcal{M}[a,b]$ the space of probability measures on the interval $[a,b]$ with the topology of weak convergence, and let $\mathcal{M}_0 \subset \mathcal{M}$ be the dense subset of the measures with finite support.

Definition. We define an *R-function of the measure* $\mu \in \mathcal{M}[a,b]$ as

$$R_\mu(u) = \int_a^b \frac{d\mu(t)}{u-t}. \tag{2.3.1}$$

Similarly to the R-function of a diagram, it is holomorphic outside the interval $[a,b]$.

Example: for the discrete measure $\mu \in \mathcal{M}_0$ with the weights μ_k at the points x_k (where $k = 1, \dots, n$) we have

$$R_\mu(u) = \sum_{k=1}^{n} \frac{\mu_k}{u - x_k}. \tag{2.3.2}$$

Definition. We call μ the *transition measure* of a diagram ω if their R-functions coincide:

$$R_\mu(u) = \int_a^b \frac{d\mu(t)}{u - t} = \frac{1}{u} \exp \frac{1}{2} \int_a^b \frac{d(\omega(t) - |t|)}{t - u} = R_\omega(u). \tag{2.3.3}$$

Theorem ([8,10]). *For any diagram $\omega \in \mathcal{D}[a,b]$ its transition measure $\mu = \mu^{(\omega)}$ exists, is unique and is supported by the interval $[a,b]$. The map $\omega \mapsto \mu^{(\omega)}$ determines the homeomorphism of the space $\mathcal{D}[a,b]$ on $\mathcal{M}[a,b]$.*

Example 1. The distribution $d\mu(u) = (2\pi)^{-1}\sqrt{4 - u^2}\, du$ with semicircle density on the interval $[-2, 2]$ is the transition distribution of the diagram Ω (Example 1, Section 2.2), since for $u \geq 2$

$$R_\mu(u) = \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - t^2}\, dt}{u - t} = (u - \sqrt{u^2 - 4})/2.$$

Example 2. The transition measure for 'triangular' diagram of Example 2, Section 2.2 is the beta-distribution with the density

$$\phi_p(u) = \frac{\sin \pi p}{\pi} (1 + u)^{-q} (1 - u)^{-p}$$

on the interval $[-1, 1]$. In fact, it follows from the integral representation of the hypergeometric function and from the identity

$$\Gamma(p)\Gamma(q) = \pi / \sin \pi p = \pi / \sin \pi q$$

that

$$\int_{-1}^{1} \frac{\phi_p(t)\, dt}{u - t} = \frac{1}{u + 1} F(1, p; 1; \frac{2}{u + 1}) = (u + 1)^{-q} (u - 1)^{-p}.$$

2.4 The correspondence between $\mathcal{D}_0$ and $\mathcal{M}_0$. Comparing (2.2.4) and (2.3.2) we conclude that the transition measure μ of the rectangular diagram ω is discrete, and the defining relation (2.3.3) can be written in the form

$$\sum_{k=1}^{n} \frac{\mu_k}{u - x_k} = \frac{\prod (u - y_k)}{\prod (u - x_k)}. \tag{2.4.1}$$

The weights μ_k of the measure μ are localized in the minima x_k of the diagram ω, these weights coincide with the coefficients in the decomposition of the rational fraction (2.4.1) as a sum of partial fractions. Both sides of (2.4.1) determine each other uniquely.

The positivity of the coefficients μ_k is equivalent to the interlacement property (2.1.2). Let us write the transition formulae explicitly:

$$\mu_k = \frac{\prod_i (x_k - y_i)}{\prod_{i \neq k} (x_k - x_i)} \tag{2.4.2}$$

and

$$\prod_{k=1}^{n-1} (u - y_k) = \sum_{k=1}^{n} \mu_k \prod_{i \neq k} (u - x_i). \tag{2.4.3}$$

We shall show that the bijection between $\mathcal{D}_0$ and $\mathcal{M}_0$ established in (2.4.2-3) can be extended by continuity to the homeomorphism of $\mathcal{D}[a,b]$ to $\mathcal{M}[a,b]$. To this end it will be convenient to use moments.

2.5 Moments. A finite measure μ on an interval $[a,b]$ is uniquely determined by the sequence of its momenta $\{h_n\}_{n=0}^{\infty}$ where

$$h_n = h_n(\mu) = \int_a^b t^n \, d\mu(t) \tag{2.5.1}$$

(one says that the Hausdorff Moment Problem is determinate). Weak convergence of measures $\mu_n \to \mu$ is equivalent to the convergence of momenta: $h_k(\mu_n) \to h_k(\mu)$ for all $k \in \mathbb{N}$. Let $\mathbb{R}^\infty$ be the space of sequences with the topology of coordinatewise convergence. Then (2.5.1) determines the homeomorphism of the space $\mathcal{M}[a,b]$ onto the closed subset $\tilde{\mathcal{M}} \subset \mathbb{R}^\infty$ of moment sequences. The sequences in $\tilde{\mathcal{M}}$ can be characterized by a series of algebraic inequalities which express the positivity of integrals $\int_a^b (b-t)^k (t-a)^m \, d\mu(t)$ in terms of momenta, but we shall not need the details.

The moment generating function of the measure μ is given by the series

$$H(u) = \sum_{n=0}^{\infty} h_n(\mu) \, u^{-n}, \tag{2.5.2}$$

convergent for sufficiently large $|u|$. Since the generating function and the R-function of the measure μ are connected by the equality $H(u) = u\,R(u)$, it follows that

$$H(u) = \int_a^b \frac{d\mu(t)}{1 - t/u}. \tag{2.5.3}$$

Let us now describe an imbedding of the diagram space $\mathcal{D}$ into $\mathbb{R}^\infty$, based on the momenta. By definition, the *momenta of the diagram* $\omega \in \mathcal{D}[a,b]$ are the numbers $\{p_n\}_{n=1}^{\infty}$, where

$$p_n = p_n(\omega) = -\frac{n}{2} \int_a^b t^{n-1} \, d(\omega(t) - |t|). \tag{2.5.4}$$

We shall call the series

$$S(u) = \sum_{n=1}^{\infty} \frac{p_n(\omega)}{n} u^{-n} \tag{2.5.5}$$

the *moment generating function of the diagram* ω. The series converges for sufficiently large $|u|$ and

$$S(u) = \frac{1}{2}\int_a^b \frac{d(\omega(t) - |t|)}{t-u}. \tag{2.5.6}$$

The map $\omega \mapsto \{p_n(\omega)\}_{n=1}^{\infty}$ determines the homeomorphism of the space $\mathcal{D}[a,b]$ onto the closed subset $\tilde{\mathcal{D}} \subset \mathbb{R}^{\infty}$. In fact, since the total variation of the charge $|d\sigma| = \omega(0)$ is uniformly bounded for $\omega \in \mathcal{D}[a,b]$, the convergence of momenta is equivalent to the weak convergence of charges. Taking into account the Lipschitz condition (i) in the definition of diagrams, we get the uniform convergence of diagrams on the interval $[a,b]$ (and hence on the reals).

The identity

$$H(u) = \sum_{n=0}^{\infty} h_n\, u^{-n} = \exp \sum_{n=1}^{\infty} \frac{p_n}{n} u^{-n} = \exp S(u) \tag{2.5.7}$$

determines the homeomorphism $\mathrm{Exp}\colon \mathbb{R}^{\infty} \to \mathbb{R}^{\infty}$ mapping the sequence $\{p_n\}$ to $\{h_n\}$ (h_n is a polynomial in p_n's and vice versa). It follows from (2.5.3), (2.5.6) and (2.4.1) that $\mathrm{Exp}(\tilde{\mathcal{D}}_0) = \tilde{\mathcal{M}}_0$. Moreover, the functor Exp maps the momenta of the diagram ω to the momenta of the distribution $\mu = \mu^{(\omega)}$. Since the set $\tilde{\mathcal{D}}_0$ of moment sequences of rectangular diagrams is dense in $\tilde{\mathcal{D}}$, and the corresponding set $\tilde{\mathcal{M}}_0$ of measures with finite support is dense in $\tilde{\mathcal{M}}$, we conclude that $\mathrm{Exp}(\tilde{\mathcal{D}}) = \tilde{\mathcal{M}}$ and the Theorem 2.3 is proved. We denote the homeomorphism constructed above by $\mathrm{Exp}_{a,b}\colon \mathcal{D}[a,b] \to \mathcal{M}[a,b]$.

2.6 The symmetric functions. The identity (2.5.7) is of importance in the symmetric function theory (see [12]). In this context

$$h_n = h_n(x_1, \dots, x_n) = \sum_{i_1 \le \dots \le i_n} x_{i_1} \dots x_{i_n}$$

are the complete homogeneous symmetric functions and

$$p_n = p_n(x_1, \dots, x_n) = \sum_i x_i^n$$

are called the Newton power sums. The generating function for the polynomials h_n can be written as

$$H(u) = \sum_{n=0}^{\infty} h_n\, u^{-n} = \prod_i (1 - \frac{x_i}{u})^{-1}.$$

Any symmetric polynomial in the variables $x = \{x_i\}$ has the natural extension to the polynomial depending on two sequences of arguments $x = \{x_i\}$, $y = \{y_i\}$. The polynomials arising in this way are called *the extended symmetric* (or else *supersymmetric*) functions. They are useful in various situations: λ-rings, characters of the symmetric group $\mathfrak{S}_{\infty}$, cf. [6]. We shall use the supersymmetric polynomials to describe the correspondence between rectangular diagrams and distributions with finite support. Let us recall the definitions (see [12], ch.1, Example 5.27).

The *extended power sums* are defined as

$$p_n(x;-y) = \sum_i x_i^n - \sum_i y_i^n, \tag{2.6.1}$$

and the extended *complete homogeneous symmetric functions* can be expressed in terms of these sums in the usual way ([12], ch.1, (2.14′)):

$$h_n(x;-y) = \sum_\rho \prod_{k\ge 1} \frac{p_k^{r_k}(x;-y)}{k^{r_k}\, r_k!}, \tag{2.6.2}$$

where $\rho = (1^{r_1}, 2^{r_2}, \dots)$ runs over all the partitions of the number $n = \sum_{k\ge1}(k-1)r_k$. The generating function for these polynomials is

$$H(u) = \sum_{n=0}^{\infty} h_n(x;-y)\, u^{-n} = \frac{\prod(1-y_i/u)}{\prod(1-x_i/u)}.$$

Comparing this to the formula (2.4.1) written in the form

$$\sum_k \frac{\mu_k}{1-x_k/u} = \frac{\prod(1-y_i/u)}{\prod(1-x_i/u)},$$

we get the following result.

Proposition. *Let $\omega \in \mathcal{D}_0$ be the rectangular diagram associated to the interlacing sequences $x = \{x_k\}$, $y = \{y_k\}$, and let $\mu = \mathrm{Exp}_{a,b}(\omega)$ be the corresponding transition measure. Then the moments of the diagram and those of the measure are the extended symmetric polynomials in $x, -y$:*

$$p_n = p_n(x;-y); \quad h_n = h_n(x;-y). \tag{2.6.3}$$

For instance, $h_1 = p_1 = \sum x_i - \sum y_i = z$ is the center of the diagram ω and the mean value of the measure μ; also, $h_2 - h_1^2 = (p_2 - p_1^2)/2 = A$ is the area of the diagram and the variance of the measure.

2.7 Another proof (see [10]). This proof is based on the theorems concerning integral representations of resolvent functions. One says that the function $R(u)$ belongs to the class $\mathcal{R}[a,b]$ if

1) it is holomorphic for $\mathrm{Im}\, u > 0$
2) $\mathrm{Im}\, R(u) \ge 0$ for $\mathrm{Im}\, u > 0$
3) $R(u) \ge 0$ if $u < a$ and $R(u) \le 0$ if $u > b$.

It is known ([10], Theorem Π6) that such functions can be represented in the form

$$R(u) = \int_a^b \frac{d\mu(t)}{t-u},$$

where μ is a nonnegative measure on the interval $[a,b]$; the representation is unique.

The function $H(u)$ belongs to the class $\mathcal{S}[a,b]$ if

1) it is holomorphic for $\operatorname{Im} u > 0$

2) $\operatorname{Im} H(u) \geq 0$ for $\operatorname{Im} u > 0$

3) $H(u) \geq 0$ for $u < a$ and for $u > b$.

It follows from [10], Theorem Π7, that this class coincides with that of functions

$$H(u) = (b-u) \int_a^b \frac{d\mu(t)}{t-u},$$

where μ is a uniquely defined nonnegative measure on $[a, b]$.

Hence, the formula

$$H(u) = (b-u)\, R(u) \tag{2.7.1}$$

establishes the bijection between the classes $\mathcal{S}[a, b]$ and $\mathcal{R}[a, b]$.

Let us rewrite the right hand side of (2.3.3) in the form

$$\frac{1}{u} \exp \frac{1}{2} \int_a^b \frac{d(\omega(t) - |t|)}{t-u} = \frac{1}{u-b} \exp \int_a^b \frac{dr(t)}{t-u}$$

where $r(t) = \frac{1}{2}(\omega(t) + t)$. Since $\operatorname{Im} (t-u)^{-1} = \eta/((t-\xi)^2 + \eta^2)$ for $u = \xi + i\eta$ and

$$0 < \operatorname{Im} \int_a^b \frac{dr(t)}{t-u} < \int_{-\infty}^{\infty} \frac{\eta\, dt}{(t-\xi)^2 + \eta^2} = \pi$$

for $\eta > 0$, the function $H(u) = \exp S(u)$, where $S(u) = \int (t-u)^{-1}\, dr(t)$, belongs to the class $\mathcal{S}[a, b]$ and (2.3.3) follows from (2.7.1).

2.8 Some explicit formulae. For generic diagrams $\omega \in \mathcal{D}$ the transition measure $\mu = \mu^{(\omega)}$ is defined by (2.3.3) implicitly. One would like to have the explicit formulae generalizing (2.4.2) in case of the rectangular diagram.

Using approximation of the diagram $\omega \in \mathcal{D}$ by the rectangular diagrams, and taking the limit in (2.4.2) one can prove the following expression for the point masses of the transition measure:

$$\mu^{(\omega)}\{t\} = \exp \int_a^b \frac{d\sigma_t(u)}{u-t}, \tag{2.8.1}$$

where $\sigma_t(u) = (\omega(u) - |u-t|)/2$.

Also, we conjecture the formula for the density $\varphi^{(\omega)}$ of the absolutely continuous part of the transition measure $d\mu^{(\omega)(t)} = \varphi^{(\omega)}(t)\, dt$ of the diagram ω:

$$\varphi^{(\omega)}(t) = \frac{1}{\pi} \cos \frac{\pi \omega'(t)}{2}\, \frac{\exp \frac{1}{2} \int_a^b \frac{d\omega(u)}{u-t}}{\sqrt{(b-t)(t-a)}}. \tag{2.8.2}$$

3. The asymptotics of transition probabilities for the Plancherel measure

In Section 2 we have described the bijection between diagrams and measures. Perhaps, this correspondence had been first introduced (in an equivalent form) in function theory in connection with the so called $(0, L)$ Markov moment problem. Another interpretation, arising in the representation theory of the countable symmetric group $\mathfrak{S}_\infty$, follows.

3.1 The growth of Young diagrams. Let $\mathfrak{S}_\infty$ be the group of finite permutations of the set of natural numbers. This group is the union of the increasing sequence of finite symmetric subgroups

$$\mathfrak{S}_0 \to \mathfrak{S}_1 \to \cdots \to \mathfrak{S}_N \to \cdots ;$$

this observation underlies the representation theory and character theory of the group $\mathfrak{S}_\infty$ (cf. [6]). We recall a few results concerning representation theory of the symmetric groups.

The irreducible representations of the symmetric group $\mathfrak{S}_N$ are parametrized by the set of partitions, i.e. by representations of N as a sum of positive integers: $N = \lambda_1 + \lambda_2 + \dots$. It is convenient to visualize the partitions by means of Young diagrams. A *Young diagram* can be defined in this context as a rectangular diagram $\lambda \in \mathcal{D}_0$ (see Section 2.1), such that

(i) all the extrema $x_1 < y_1 < \ldots < y_{n-1} < x_n$ are integers

(ii) $z = \sum x_i - \sum y_i = 0$.

For such a diagram the lines $v - u = 2X, v + u = 2Y$, where $X, Y \in \mathbb{Z}$, cut the region

$$D_\lambda = \{(u,v): |u| \le v < \lambda(u)\} \tag{3.1.1}$$

into *boxes*; the number of boxes is equal to $N = A(\lambda)$. The boxes are assembled in *rows* (see Fig. 3); there are $(y_i - x_i)$ rows of length $\sum_{j=i+1}^{n}(x_j - y_{j-1})$ for $i = 1, \dots, n-1$. Thus, a Young diagram with N boxes determines a partition of N. Denote by $\mathcal{Y}_N$ the set of Young diagrams with N boxes. The dimension of irreducible representation π_λ of the group $\mathfrak{S}_N$, corresponding to the diagram $\lambda \in \mathcal{Y}_N$ will be denoted by $\dim \lambda$. The set of all Young diagrams $\mathcal{Y} = \cup_{N=0}^{\infty}\mathcal{Y}_N$ is partially ordered by inclusion of regions (3.1.1); it is called the *Young lattice*. A Young diagram Λ follows immediately the diagram λ, $\lambda \nearrow \Lambda$, if $D_\lambda \subset D_\Lambda$ and if D_Λ differs from D_λ in exactly one box. Note that this box is attached to one of the minima x_k of the diagram λ (Fig. 3). On the other hand, for any x_k there is a diagram Λ immediately following λ. Recall that a *Young tableau* of the shape λ is an oriented path

$$t = (\emptyset \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \ldots \nearrow \lambda_N = \lambda)$$

in the Young lattice connecting the empty diagram $\emptyset$ to λ. In other words, Young tableau is a growth history of Young diagram. The order in Young lattice determines the branching of representations when restricted to subgroups:

$$\pi_\Lambda|_{\mathfrak{S}_N} = \bigoplus_{\lambda:\lambda\nearrow\Lambda} \pi_\lambda.$$

In particular, the number of Young tableaux of the shape λ equals the dimension $\dim \lambda$. The dimension can be calculated by the *hook formula*

$$\dim \lambda = N!/\prod h_{ij}, \tag{3.1.2}$$

where the product runs over the boxes of λ, and h_{ij} is the *hook length* of the j-th box in the i-th row (cf. [12], ch.1, Example 1.1).

3.2 Plancherel' measure. Let $\mathcal{T}$ be the (compact, totally disconnected) space of infinite Young tableaux. The *Plancherel measure* M of the infinite symmetric group $\mathfrak{S}_\infty$ is defined in [4] as the Markov probability measure on $\mathcal{T}$ with the transition probabilities

$$p_{\lambda\Lambda} = \frac{\dim \Lambda}{(N+1)\dim \lambda}; \qquad \lambda \in \mathcal{Y}_N, \quad \lambda \nearrow \Lambda. \tag{3.2.1}$$

The distribution of M on the N-th level $\mathcal{Y}_N$ of Young lattice coincides with the ordinary Plancherel measure of the group $\mathfrak{S}_N$:

$$P\{t \in \mathcal{T} : \lambda_N = \lambda\} = \frac{\dim^2 \lambda}{N!}. \tag{3.2.2}$$

It is known (see [4]) that the measure M is related to the regular representation of the group $\mathfrak{S}_\infty$, providing another motivation to the choice of the term.

Let us write p_k instead of $p_{\lambda\Lambda}$ if the box, distinguishing Λ from λ is attached to the minima x_k of Young diagram λ (Fig. 3).

Proposition. *The transition distribution of the Plancherel measure at a diagram λ has the weights*

$$p_k = \prod_{i=1}^{k-1} \frac{x_k - y_i}{x_k - x_i} \prod_{i=k+1}^{n} \frac{x_k - y_{i-1}}{x_k - x_i}. \tag{3.2.3}$$

Hence, it coincides with the transition measure of the diagram λ in the sense of definition (2.3.3).

The proof is based on the hook formula (3.1.2) which implies that

$$p_{\lambda\Lambda} = \prod h_{ij}(\lambda) / \prod h_{ij}(\Lambda). \tag{3.2.4}$$

We shall calculate this fraction. Let r be the row number and c be the column number of the new box in the diagram Λ. Since the hook lengths of the boxes outside the r-th row and c-th column of the diagrams λ, Λ do not differ, we can write (3.2.4) as

$$p_k = \prod_{i=1}^{c-1} \frac{h_{ri}}{h_{ri}+1} \prod_{j=1}^{r-1} \frac{h_{jc}}{h_{jc}+1}.$$

The factor $\prod h_{ri}/(h_{ri}+1)$ decomposes in a product

$$\frac{x_k - y_m}{x_k - y_m + 1} \cdot \frac{x_k - y_m + 1}{x_k - y_m + 2} \cdots \frac{x_k - x_m - 1}{x_k - x_m} = \frac{x_k - y_m}{x_k - x_m},$$

it follows that

$$\prod_{i=1}^{c-1} \frac{h_{ri}}{h_{ri}+1} = \prod_{m=1}^{k-1} \frac{x_k - y_m}{x_k - x_m}.$$

In the same way,

$$\prod_{j=1}^{r-1} \frac{h_{jc}}{h_{jc}+1} = \prod_{m=k+1}^{n} \frac{x_k - y_{m-1}}{x_k - x_m}.$$

3.3 The asymptotics for large Young diagrams. It was proved in [3,11,5] that in the course of random growth almost all Young diagrams become close (after suitable renormalization of the area) to the universal shape; actually to that of the diagram Ω of Example 1, formula (2.2.2). More precisely, the following result holds:

Theorem [3,11]. *For almost all Young tableaux $t = (\lambda_1, \dots, \lambda_N, \dots) \in \mathcal{T}$, according to the Plancherel measure M, the limit*

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \lambda_N(u\sqrt{N}) = \tilde{\Omega}(u) \tag{3.3.1}$$

exists uniformly in $u \in \mathbb{R}$.

It had been shown in [5], that for the large number of boxes N the shape of Young diagram, maximizing the dimension, is also close to the same diagram Ω.

Using the continuity of the correspondence between diagrams and transition measures, it is possible to restate this result as the 'semicircle law' for transition probabilities of the Plancherel measure.

Corollary. *Let μ_N be the transition distribution corresponding to the Young diagram λ_N. Then for almost all, according to the Plancherel measure, random tableaux $t = (\lambda_1, \dots, \lambda_N, \dots) \in \mathcal{T}$ the limit*

$$\lim_{N\to\infty} \mu_N\{\tau : \tau < u\sqrt{N}\} = \frac{1}{2\pi} \int_{-2}^{u} \sqrt{4-\tau^2}\, d\tau \tag{3.3.2}$$

exists for all $|u| \le 2$.

The proof follows immediately from the Theorem 3.3, taking into account Theorem 2.3, Proposition 3.2 and Example 1 of Section 2.3.

4. Random interval splitting

In this Section we describe the probabilistic algorithm of interval shrinkage, controlled by a diagram $\omega \in \mathcal{D}[a,b]$ on some interval $I = [a,b]$. This is the Markov process with the subintervals of I as its states. At each stage it reduces the current interval $[\alpha,\beta] \subset I$ to a smaller subinterval; any particular realization of the process is a sequence of diminishing intervals $\{[\alpha_n,\beta_n]\}_{n=0}^{\infty}$. With unit probability the intersection $\bigcap[\alpha_n,\beta_n]$ contains the unique point X. The main result claims that the distribution of the random variable X is exactly the transition measure $\mu^{(\omega)} = \mathrm{Exp}(\omega)$ defined in Section 2.

We start with the simplest case of rectangular diagram ω.

4.1 The discrete case. Consider a pair of interlacing sequences

$$x_1 < y_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < x_n \tag{4.1.1}$$

and define the (positive) weights of the measure μ at the points x_k as

$$\mu_k = \prod_i (x_k - y_i) / \prod_{i\neq k} (x_k - x_i), \tag{4.1.2}$$

where $k = 1, \dots, n$. These numbers are the coefficients of the decomposition

$$\frac{\prod(u - y_i)}{\prod(u - x_i)} = \sum_{k=1}^{n} \frac{\mu_k}{u - x_k} \tag{4.1.3}$$

in a sum of partial fractions. Multiplying both sides by u and taking the limit $u \to \infty$, one can check that $\sum \mu_k = 1$.

There is a nice Markov chain, the *shrinkage process*, with final distribution (4.1.2). Its state space is finite and consists of the intervals $[x_i, x_j]$ for $1 \le i \le j \le n$. For the initial state we take the largest interval $[x_1, x_n]$.

Assume that the current state is the interval $[x_i, x_j]$. In order to switch to the next state we choose a random point γ, uniformly distributed on this interval. The point γ belongs to one of the intervals

$$\gamma \in [x_k, y_k] \quad \text{or} \quad \gamma \in [y_{k-1}, x_k] \tag{4.1.4}$$

for some k. By definition, in the first case the new state is the interval $[x_i, x_k]$, in the second case it is the interval $[x_k, x_j]$. At any stage of the process the interval shrinks; after at most n stages the current interval degenerates in a point, and the process terminates.

Proposition. *The probability that the shrinkage process will terminate at the final state $[x_k, x_k]$ equals μ_k (see (4.1.2)).*

Proof. For any triple $i \le k \le j$ let $\mu_k^{(ij)}$ be the probability to terminate the process at the state $[x_k, x_k]$, assuming that the process starts at the interval $[x_i, x_j]$, and denote by $R_k^{(ij)}$, $L_k^{(ij)}$ the probabilities to terminate at $[x_k, x_k]$ starting from the intervals $[y_{i-1}, x_j]$, $[x_i, y_j]$ respectively. By the total probability formula,

$$\mu_k^{(ij)} = \frac{z_{ij} - x_i}{x_j - x_i} R_k^{(i+1,j)} + \frac{x_j - z_{ij}}{x_j - x_i} L_k^{(i,j-1)},$$

where $z_{ij} = \sum_{s=i}^{j} x_s - \sum_{s=i}^{j-1} y_s$. Applying the total probability formula ones more, one gets

$$R_k^{(ij)} = \frac{x_i - y_{i-1}}{z_{ij} - y_{i-1}} \mu_k^{(ij)} + \frac{z_{ij} - x_i}{z_{ij} - y_{i-1}} R_k^{(i+1,j)},$$

$$L_k^{(ij)} = \frac{x_j - z_{ij}}{y_j - z_{ij}} L_k^{(i,j-1)} + \frac{y_j - x_j}{y_j - z_{ij}} \mu_k^{(ij)}.$$

These formulae express the numbers $\mu_k^{(ij)}$, $R_k^{(ij)}$, $L_k^{(ij)}$ in terms of the corresponding quantities with smaller values of $j - i$. Starting with the trivial values $\mu_k^{(kk)} = R_k^{(kk)} = L_k^{(kk)} = 1$, we get by induction argument in $j - i$ that

$$\mu_k^{(ij)} = \prod_{s=i}^{k-1} \frac{x_k - y_s}{x_k - x_s} \prod_{t=k+1}^{j} \frac{x_k - y_{t-1}}{x_k - x_t},$$

$$R_k^{(ij)} = \frac{x_k - y_{i-1}}{z_{ij} - y_{i-1}}\,\mu_k^{(ij)} = \frac{x_k - x_{i-1}}{z_{i-1,j} - x_{i-1}}\mu_k^{(i-1,j)},$$

$$L_k^{(ij)} = \frac{y_j - x_k}{y_j - z_{ij}}\,\mu_k^{(ij)} = \frac{x_{j+1} - x_k}{x_{j+1} - z_{i,j+1}}\,\mu_k^{(i,j+1)}.$$

In case $i = 1, j = n$ the first formula gives (4.1.2) and we are done.

The shrinkage process is a variant of the *hook walk* procedure introduced in [1,2]. The present form of the algorithm and its q-analog were considered in [9].

4.2 Random shrinking in the general case. In the Section above we have constructed, in terms of the parameters (4.1.1) of a rectangular diagram, the random variable distributed according to the transition measure of the diagram. In case of a general diagram $\omega \in \mathcal{D}[a,b]$, there is a similar construction, but the process has a continual state space, and the random variable X with the distribution $\mu^{(\omega)}$ arises in the limit of infinite number of iterations. We proceed with the precise description of the *general shrinkage process.*

By definition, we declare the states of the process to be any closed subintervals $[\alpha,\beta] \subset I$ of the main interval $I = [a,b]$. For the initial state we take the interval $[a,b]$. Any step of the process is the result of two independent random events: 1) the splitting of the current interval $[\alpha,\beta]$ by a random interior point γ and 2) the choice of one of two parts $[\alpha,\gamma]$, $[\gamma,\beta]$ as a new state.

The distribution of the random point γ is always uniform on the corresponding interval $[\alpha,\beta]$. The diagram $\omega \in \mathcal{D}[a,b]$, controlling the process, determines only the probabilities q,p to choose the left or the right one of the intervals $[\alpha,\gamma]$, $[\gamma,\beta]$. By definition, these probabilities are taken to be

$$q = (1 + \omega'(\gamma))/2, \quad p = (1 - \omega'(\gamma))/2. \tag{4.2.1}$$

In the course of iterations of the procedure a sequence of nested intervals arises:

$$[a,b] \supset [\alpha_1,\beta_1] \supset \ldots \supset [\alpha_n,\beta_n] \supset \ldots . \tag{4.2.2}$$

The endpoints of the intervals converge, with unit probability, to the joint limit

$$X = \lim \alpha_n = \lim \beta_n. \tag{4.2.3}$$

Theorem. *Let X be the random variable (4.2.3) defined by the shrinkage process described above. If the process is controlled (see (4.2.1)) by the diagram ω, then the distribution of X coincides with transition measure $\mu^{(\omega)}$ of this diagram (Theorem 2.3).*

In the particular case of triangular diagram ω_p of Example 2, Section 2.2, the probabilities (4.2.1) are constant and do not depend on the current state $[\alpha,\beta]$. The random variable X has the beta-distribution of Example 2, Section 2.3. The simplest case $p = q = 1/2$ is worth mentioning separately:

Corollary. *Consider the random sequence of intervals*

$$[-1,1] \supset [\alpha_1,\beta_1] \supset \ldots \supset [\alpha_n,\beta_n] \supset \ldots ,$$

defined by the following rules: the interval $[\alpha_{n+1}, \beta_{n+1}]$ *arises from the previous interval* $[\alpha_n, \beta_n]$ *by splitting it with the uniformly distributed random point, and by independent choice, with equal probabilities, of one of the two parts. Then the limit* $X = \lim \alpha_n = \lim \beta_n$ *exists a.e., and is distributed according to the semicircle law with the density*

$$\phi(u) = \frac{1}{\pi} \frac{1}{\sqrt{1-u^2}}$$

on the interval $[-1, 1]$.

We present also another description of the shrinkage process, similar to that of the hook walk algorithm (cf. [9]).

For a point O situated over the graph of the diagram $v = \omega(u)$ we draw through it the straight lines with the slope ± 1; denote by M_α, M_β the points where these lines intersect with the graph, and let $\alpha < \beta$ be their abscissas. The union $H_{\alpha,\beta}$ of the intervals $M_\alpha O$ (the *leg of the hook*) and OM_β (the *arm of the hook*) is called the *hook* with the vertex at the point O. The hook is completely determined by the corresponding interval $[\alpha, \beta]$ (Fig. 4).

The hook walk process starts with the hook $H_{a,b}$ over the initial interval $[a, b]$. We choose a uniformly defined random point of the hook, and consider the new hook H_{α_1,β_1} with the vertex at this point. Then the procedure repeats itself with respect to the new hook to produce another hook H_{α_2,β_2}, and so on. The equivalence of the hook walk and of the shrinkage process is clear from the following remark.

Let M be the point on the graph of $v = \omega(u)$ with the abscissa $u \in [\alpha, \beta]$. Denote by B the point on the arm of the hook $H_{\alpha,\beta}$ at the crossing with the line $V = \omega(u) + (U - u)$, and let A be the intersection point of the leg of the same hook with the line $V = \omega(u) - (U - u)$ (see Fig. 4). We shall use the notation

$$r(u) = \frac{u - \omega(u)}{2}, \quad l(u) = \frac{u + \omega(u)}{2}, \tag{4.2.4}$$

then the abscissas of the points A, B are $r(\alpha)+l(u)$ and $r(u)+l(\beta)$ correspondingly. If the point A (correspondingly, B) is uniformly distributed on the leg (arm) of the hook $H_{\alpha,\beta}$, then u has the density $l'(u)$ (correspondingly, $r'(u)$). Since $l'(u) = (1 + \omega'(u))/2 = q$ and $r'(u) = (1 - \omega'(u))/2 = p$, the equivalence of the shrinkage process and of the hook walk process follows.

Informally, one can say that $dr(u)$ is the illumination measure of the graph of the diagram from the north-east direction, and $dl(u)$ is the illumination measure by the north-western light.

4.3 The proof of the Theorem. Denote by $\mu_{\alpha,\beta}$ the distribution, arising as a result of shrinkage process starting with the initial interval $[\alpha, \beta]$. Let also R (respectively, L) be the distribution of the random variable (4.2.3) under the condition that the left (respectively, the right) subinterval is taken in the first division of the interval $[a, b]$ in the shrinkage process. The distributions L, R can be written as averages of the measures $\mu_{\alpha,\beta}$:

$$L = \frac{1}{b - z} \int_a^b \mu^{a,x}\, dl(x), \tag{4.3.1}$$

$$R = \frac{1}{z-a}\int_a^b \mu^{x,b}\, dr(x). \tag{4.3.2}$$

The characteristic property of the distribution $\mu = \mu_{a,b}$, reflecting the recurrence in the definition of the shrinkage process, can be expressed as follows:

$$\mu = \frac{b-z}{b-a}\, L + \frac{z-a}{b-a}\, R. \tag{4.3.3}$$

In order to prove Theorem 4.2 we shall check that (4.3.3) holds also for the transition distributions $\mu^{\alpha,\beta}$, corresponding to the restrictions $\omega^{\alpha,\beta}$ of the diagram ω onto the subintervals $[\alpha,\beta] \subset I$.

Thus, let $\mu^{\alpha,\beta} = \mathrm{Exp}_{\alpha,\beta}(\omega^{\alpha,\beta})$, and replace (4.3.1) by the integral

$$L = \frac{1}{b-z}\int_a^b \mu^{a,x}\, dl(x). \tag{4.3.4}$$

Lemma 1. *The measure L is absolutely continuous with respect to μ, and its density is linear:*

$$\frac{dL(u)}{d\mu(u)} = \frac{b-u}{b-z}.$$

Proof. Consider the function

$$\phi_t(x) = \int_a^x \frac{t-x}{t-u}\, d\mu^{a,x}(u) = \exp\int_a^x \frac{dl(u)}{u-t};$$

its differential is equal to

$$d\phi_t(x) = \frac{\phi_t(x)}{x-t}\, dl(x) = \left(\int_a^x \frac{d\mu^{a,x}(u)}{u-t}\right) dl(x).$$

By the definition of the integral (4.3.4) the Stieltjes transform of the measure $(b-z)dL(u)$ equals

$$(b-z)\int_a^b \frac{dL(u)}{t-u} = \int_a^b \left(\int_a^x \frac{d\mu^{a,x}(u)}{t-u}\right) dl(x) = -\int_a^b d\phi_t(x) = 1-\phi_t(b).$$

It coincides with the Stieltjes transform of the measure $(b-u)\, d\mu(u)$:

$$\int_a^b \frac{b-u}{t-u}\, d\mu(u) = 1 - \int_a^b \frac{t-b}{t-u}\, d\mu(u) = 1-\phi_t(b)$$

and the Lemma follows from the Stieltjes inversion theorem.

Lemma 2. *Define the measure R as the integral of the distributions $\mu^{x,b}$:*

$$R = \frac{1}{z-a}\int_a^b \mu^{x,b}\, dr(x).$$

Then R is absolutely continuous with respect to $\mu = \mu^{a,b}$, and its density is

$$\frac{dR(u)}{d\mu(u)} = \frac{u-a}{z-a}.$$

The proof of this Lemma is similar to that of Lemma 1, but the function $\phi_t(x)$ must be replaced by

$$\psi_t(x) = \int_x^b \frac{x-t}{u-t}\, d\mu^{x,b}(u) = \exp \int_x^b \frac{dr(u)}{t-u}.$$

It follows from Lemmas 1,2 that the transition distributions $\mu^{\alpha,\beta}$, determined by the restrictions of the diagram ω to various subintervals, are subjected to the relation

$$\mu^{a,b} = \frac{b-u}{b-a} \int_a^b \mu^{a,x}\, dl(x) + \frac{u-a}{b-a} \int_a^b \mu^{x,b}\, dr(x), \tag{4.3.5}$$

which is analogous to the defining equation (4.3.3). The Theorem is proved.

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