HARMONIC ANALYSIS ON THE INFINITE SYMMETRIC GROUP. A DEFORMATION OF THE REGULAR REPRESENTATION

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We investigate in this paper a family of representations T_z of the infinite symmetric group $S(\infty)$, which is a deformation, with complex parameter z, of the regular representation T_∞. The main results are the calculation of characters of representations T_z, and the decomposition of these representations, in case of integer values of $z \in \mathbb{Z}$, in a direct integral of factor-representations. The terminology of factor-representations is equivalent to that of irreducible representations of the Gelfand pair $(S(\infty) \times S(\infty), S(\infty))$, and we use the latter one.

1. Virtual permutations. Let $S(n)$ be the symmetric group of degree n and denote by $S = S(\infty) = \bigcup S(n)$ the group of finite permutations of the set $\{1, 2, \dots\}$. Let K be the diagonal subgroup in $G = S \times S$, and let $G(n) = S(n) \times S(n)$, $K(n) = G(n) \cap K$ denote the corresponding finite subgroups. For any $k \geq n$ we consider $S(k)$ as a space of two-sided action of the group $S(n)$.

Theorem 1.1. *There exists the unique map p_n: $S(n+1) \to S(n)$ commuting with the two-sided action of $S(n)$. More explicitly, $x' = p_n(x)$ is the induced permutation of x on the subset $I_n = \{1, \dots, n\}$, i.e. $x'(i) = x(i)$ if $x(i) \leq n$ and $x'(i) = x(n+1)$ otherwise.*

We consider the projective limit $X = \operatorname{lim\,proj} S(n)$ according to the *canonical projections* p_n and we refer to the elements of X as *virtual permutations*. Since $p_n|_{S(n)} = \mathrm{id}$, the compact space X naturally includes S as the dense subset. The right action of G on the homogeneous space $S = K \setminus G$ can be extended to an action $X \times G \to X$ by homeomorphisms. Thus X is a compactification of S in the category of G-spaces.

Let μ_t^n, $t > 0$ be the measure on $S(n)$ defined as $\mu_t^n(\{x\}) = t^{[x]}/(t)_n$ where $(t)_n = t(t+1)\dots(t+n-1)$, and let $[x]$ denote the number of cycles in $x \in S(n)$. This is the probability measure; for $t = 1$ it coincides with the normalized Haar measure on $S(n)$.

Theorem 1.2. *For a fixed $t > 0$, the measures μ_t^n are coherent under the canonical projections: $p_n(\mu_t^{n+1}) = \mu_t^n$. The probability measure $\mu_t = \operatorname{lim\,proj} \mu_t^n$ on X is K-invariant and G-quasi-invariant; the measure μ_1 is in fact G-invariant.*

For the action of G on X, there exists a distinguished additive 1-cocycle $c : X \times G \to \mathbb{Z}$ such that $c(x, g) = [p_n(xg)] - [p_n(x)]$ for sufficiently large n. The cocycle c describes the derivative of the measure μ_t under the action of the group G:

$$\mu_t(d(xg)) = t^{c(x,g)} \mu_t(dx); \quad t > 0,\, x \in X,\, g \in G. \tag{1}$$

The space of virtual permutations X admits a number of interesting interpretations: as a path space of a certain branching diagram, as the space of growing forests, as the direct product $I = I_1 \times I_2 \times \dots$.

Theorem 1.3. *In the latter realization μ_t is the product measure $\tilde{\mu}_t = \tilde{\mu}_t^1 \times \tilde{\mu}_t^2 \times \ldots$ where the distribution $\tilde{\mu}_t^n$ on I_n is defined as $\tilde{\mu}_t^n(\{n\}) = t\,/\,(t+n-1)$ and $\tilde{\mu}_t^n(\{i\}) = 1\,/\,(t+n-1)$ for $i<n$. The measures μ_t, $t>0$ are pairwise singular.*

Note that for a.e. virtual permutation the normalized cycle lengths $\{\sigma_k\}$, $k \geq 1$ do exist. The projection of a K-invariant measure on X onto the simplex Σ of such sequences is called its *radial part*, and it determines the initial measure completely. The radial parts of the measures μ_t are known as *Poisson – Dirichlet distributions*, they were studied in [1,3,9,10].

2. The generalized regular representations. Let $z \in \mathbb{C} \setminus \{0\}$ and $t = |z|^2$. Using (1), one can define a unitary representation T_z of the group G in the Hilbert space $\mathcal{H} = L^2(X, \mu_t)$ by the formula

$$(T_z(g)f)(x) = f(xg)z^{c(x,g)}; \quad g \in G,\ f \in \mathcal{H}. \tag{2}$$

We refer to T_z as the *generalized regular representation of G*. The reason is that the definition (2) can be naturally extended to provide the representations T_0 and T_∞, and that the latter one is the conventional biregular representation of the group S.

Denote by $\mathcal{H}^\infty = \bigcup \mathcal{H}^n$ the dense subspace of $\mathcal{H}$, consisting of cylindrical functions. Here $\mathcal{H}^n$ is obtained by lifting functions via the natural projection $X \to S(n)$; we identify $\mathcal{H}^n$ with $L^2(S(n), \mu_t^n)$. If T_z^n denotes the unitary representation of $G(n)$ in the $G(n)$-invariant subspace $\mathcal{H}^n$, then $T_z = \lim\operatorname{ind} T_z^n$. Let $\mathrm{Reg}^n = T_1^n$ denote the representation of $G(n)$ in $H^n = L^2(K(n) \setminus G(n))$. The multiplication by the function $F_z^n(x) = (n!/(t)_n)^{1/2} z^{[x]}$ on $S(n)$ determines an isometry $\mathcal{H}^n \to H^n$ intertwining the representations T_z^n and Reg^n. The natural embeddings $\mathcal{H}^n \to \mathcal{H}^{n+1}$ generate certain embeddings $L_z^n : H^n \to H^{n+1}$, and one can write $T_z = \lim\operatorname{ind} \mathrm{Reg}^n$ where the limit is taken with respect to L_z^n. The embeddings L_z^n and the limit T_z are correctly defined by continuity for all $z \in \mathbb{C} \cup \{\infty\}$.

Any T_z is an admissible representation of (G, K) in the sense of [4]. By Theorem 4.1 of [4], it generates a type 1 von Neumann algebra.

Theorem 2.1. *The representations T_z, $z \in \mathbb{C}$ are reducible and decompose (in the essentially unique way) in a multiplicity free direct integral of irreducible admissible representations. Recall, that T_∞ is irreducible.*

3. Spherical functions. We use in this paper the terminology of irreducible spherical representations of the pair (G, K) which is equivalent to that of factor-representations of the group S, used in the initial publications [7,8]. Let Φ denote the convex set of all positive definite functions on G which are K-biinvariant and normalized at the unity. The *extreme spherical functions* of (G, K) (i.e. the extreme points of Φ) are parametrized by the Thoma' set Δ consisting of the triples $\delta = (\alpha\beta\gamma)$ where

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0),\ \beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0), \quad \gamma \geq 0, \quad \sum \alpha_i + \sum \beta_i + \gamma = 1,$$

see [7,8]. For $\delta \in \Delta$ let $\phi_\delta \in \Phi$ denote the corresponding extreme spherical function, and let $T(\delta)$ be the cyclic unitary representation of G generated by ϕ_δ. Then $T(\delta)$ are exactly the irreducible spherical representations of the Gelfand pair (G, K), and any $\phi \in \Phi$ decomposes uniquely as $\phi = \int \phi_\delta m\,(d\delta)$ where m is a probability measure on Δ, called the *spectral measure* for ϕ. The same measure also describes the decomposition of the cyclic unitary representation generated by ϕ.

We study the spherical functions $\phi \in \Phi$ by decomposing the restriction $\phi|_{G(n)}$, $n = 1, 2, \dots$ as a sum of extreme spherical functions of the finite Gelfand pair $(G(n), K(n))$ (which essentially are irreducible normalized characters of the group $S(n)$).

Let $\mathcal{Y} = \bigcup \mathcal{Y}_n$ denote the set of all Young diagrams, where $\mathcal{Y}_n$ is the set of diagrams with $n = |\lambda|$ nodes. Let χ^λ be the irreducible character of $S(n)$ corresponding to $\lambda \in \mathcal{Y}_n$ and denote by $\dim \lambda$ its dimension. To any $\phi \in \Phi$, we attach a function $M : \mathcal{Y} \to \mathbb{R}_+$ defined as

$$\phi|_{(S(n)\times\{e\})} = \sum_{\lambda \in \mathcal{Y}_n} M(\lambda)\, \chi^\lambda / \dim \lambda, \quad n = 1, 2 \dots \tag{3}$$

Note that $M_n := M|_{\mathcal{Y}_n}$ is a probability measure on $\mathcal{Y}_n$. All the information on ϕ is implicit in M. In particular, the spectral measure m can be obtained as a limit of $\{M^n\}$ if we approximate Δ by the $\mathcal{Y}_n$'s using the Frobenius coordinates of Young diagrams as defined in [8], formula (10).

Consider now the generalized regular representations T_z, and the distinguished unit K-invariant vector $f_0 \equiv 1$ in the space $\mathcal{H}$ of the representation T_z. Denote by $\phi \in \Phi$ the corresponding matrix coefficient. Let M_z be the function on $\mathcal{Y}$ determined by (3), and let m_z be the spectral measure of ϕ_z.

Theorem 3.1. *Let $z \in \mathbb{C} \setminus \{0\}$, $t = |z|^2$, $\lambda \in \mathcal{Y}$, and $n = |\lambda|$. Then*

$$M_z(\lambda) = ((t)_n)^{-1} \prod_{(ij)\in\lambda} |z + j - i|^2 \cdot \dim^2 \lambda / n! \tag{4}$$

To derive (4), we represent F_z^n as a sum of characters χ^λ, $\lambda \in \mathcal{Y}_n$ (for another purpose, this decomposition has been calculated independently by Bernhard Keller; see also [6]). The formula (4) plays a fundamental role in our investigations. We extend the definition of ϕ_z and M_z to the values $z = 0, \infty$ by continuity.

Theorem 3.2. *If $z \notin \mathbb{Z}$, then f_0 is the cyclic vector for T_z. Hence, the decomposition of T_z involves only spherical representations.*

This follows from the Theorem 3.1 since M_z does not vanish when $z \notin \mathbb{Z}$. Since $\phi_z = \phi_{\bar z}$, Theorem 3.2 implies that $T_z \simeq T_{\bar z}$.

4. The commutant of T_z. Given $\lambda \in \mathcal{Y}$, let λ^+ denote the set of $\Lambda \in \mathcal{Y}_{n+1}$ such that $\Lambda \setminus \lambda$ consists of a unique node (ij), and set $c_{\lambda\Lambda} = j - i$. For $z \in \mathbb{C} \cup \{\infty\}$, $\lambda \in \mathcal{Y}_n$ and $\Lambda \in \lambda^+$, we set

$$p_z(\lambda, \Lambda) = \left(\frac{n!}{(t)_n}\right)^{1/2} |z + c_{\lambda\Lambda}|^2\, p_\infty(\lambda, \Lambda), \quad p_\infty(\lambda, \Lambda) = \frac{\dim \Lambda}{(n+1)\dim \lambda}.$$

Then $p_z(\lambda, \Lambda)$ is just the transition distribution for the central measure on the space of paths in $\mathcal{Y}$ (called Young tableaux) corresponding to M_z (see [8] for definitions). The same expression arises also in another way:

Theorem 4.1. *Let $z \in \mathbb{C} \cup \{\infty\}$, denote by $(T_z)'$ the commutant of T_z, and let $A_z \subset \mathcal{C}(\mathcal{Y})$ be the space of all bounded functions $a : \mathcal{Y} \to \mathbb{C}$, satisfying $a(\lambda) = \sum p_z(\lambda, \Lambda)\, a(\Lambda)$ (sum over $\Lambda \in \lambda^+$) for any $\lambda \in \mathcal{Y}$. Then there exists a natural Banach space isometry $(T_z)' \to A_z$.*

5. The structure of representations T_z. Suppose first that $z = k \in \mathbb{Z}$. For $\lambda \in \mathcal{Y}$ the *k-level* $\mathrm{lev}_k(\lambda)$ is defined as the number of nodes $(ij) \in \lambda$ with $j - i = k$. We regard T_z as $\lim\mathrm{ind}\,\mathrm{Reg}^n$ with respect to the embeddings L_z^n. Remark, that the irreducible components of Reg^n are labelled by the set $\lambda \in \mathcal{Y}_n$, and consider the decomposition $\mathrm{Reg}^n = \bigoplus \mathrm{Reg}_{kl}^n$ (sum over $l \in \mathbb{Z}_+$) where Reg_{kl}^n denotes the sum of components with $\mathrm{lev}_k(\lambda) = l$.

Theorem 5.1. *For any $l \in \mathbb{Z}_+$, the embedding L_k^n maps Reg_{kl}^n into Reg_{kl}^{n+1}, and the representation $T_{kl} = \lim\mathrm{ind}\,\mathrm{Reg}_{kl}^n$ is correctly defined. The representation $T_k = \bigoplus T_{kl}$ decomposes as a direct sum over $l \in \mathbb{Z}^+$. In terms of Theorem 4.1, the projector onto $T_{kl} \subset T_k$ corresponds to the characteristic function of the subset $\{\lambda \in \mathcal{Y} : \mathrm{lev}_k(\lambda) = l\} \subset \mathcal{Y}$.*

Along with the pair k, l consider the pair $p, q \in \mathbb{Z}_+$ where $p - q = k$ and $\min(p, q) = l$. We denote by Δ_{pq} the simplex

$$\Delta_{pq} = \{\delta = (\alpha\beta\gamma) \in \Delta : \alpha_{p+1} = \alpha_{p+2} = \ldots = \beta_{q+1} = \beta_{q+2} = \ldots = \gamma = 0\}.$$

Theorem 5.2. *There is a direct integral decomposition $T_{kl} \simeq \int_{\Delta_{pq}} T(\delta)\tilde{m}_{pq}(d\delta)$, with the measure $\tilde{m}_{pq}$ on Δ_{pq} equivalent to the Lebesgue measure.*

To prove the Theorem, we choose, for any T_{kl}, a distinguished K-invariant vector f_{kl} and we calculate the corresponding spectral measure which turns out to be equivalent to the Lebesgue measure on Δ_{pq}. Next, we check that f_{kl} is cyclic in T_{kl}; to this end we use Theorem 4.1 and verify that the state on the algebra $(T_{kl})'$, corresponding to f_{kl}, is faithful.

Thus, we get a complete description of the representations T_k for $k \in \mathbb{Z}$.

Theorem 5.3. *Assume that $z \neq w$, $z \neq \bar{w}$. Then the representations T_z and T_w are disjoint.*

For $z, w \in \mathbb{Z}$ this follows from Theorem 5.2. For $z \notin \mathbb{Z}$, we prove that $m_z(\Delta_{pq}) = 0$ for all p, q. Next we verify the Theorem for $z, w \in \mathbb{C} \setminus \mathbb{Z}$ by analyzing the behavior of $M_z(\lambda) \,/\, M_w(\lambda)$ as $|\lambda| \to \infty$.

Theorem 5.4. *For any $z \in \mathbb{C}$, the spectral measure m_z is supported by the subset $\{(\alpha\beta\gamma) \in \Delta : \gamma = 0\}$.*

An interesting open problem is to describe the spectral measure m_z on Δ for $z \in \mathbb{C} \setminus \mathbb{Z}$. Also, we would like to obtain the construction of representations T_z, similar to that of [11] in case of irreducibles of the group G.

There are very similar one-parameter families of reducible representations for other (G, K)-pairs of classical infinite dimensional symmetric spaces of compact type (see the definitions in [5]). One can define in the same way the virtual unitary, orthogonal and symmetric matrices, the measures which are K-invariant and G-quasi-invariant, the cocycles and so on. We hope to deal with harmonic analysis for these pairs if further papers.

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