Low-dimensional Problems — 2002

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Problem 1 (V. I. Arnold) *Sturm–Hurwitz theorem in dimension 2.*

For functions of one variable this theorem affirms that a periodic function without $n-1$ lower harmonics

$$f(x) = \sum_{k=n}^{N} (a_k \cos(kx) + b_k \sin(kx))$$

has at least $2n$ zeroes on the period.

Find a 2-dimensional analogue: estimate from below the number of solutions of the system $f = g = 0$ where f and g are twice periodic functions on $\mathbb{R}^2$ without lower harmonics.

Problem 2 (V. I. Arnold) *Doubling the integer residues.*

Let n be an odd integer. Denote by Γ_n the set of residues modulo n mutually prime with n. The cardinality of Γ_n is the Euler function $\varphi(n)$. Let $A : \Gamma_n \to \Gamma_n$ be the mapping that takes each x into $2x$. It is easy to see that the Young diagram of the permutation A is rectangular: the permutation consists of a certain number of cycles, each of length $l(n)$. The sequence $l(n)$ is very far from being monotonous; however, if we denote by $\lambda(n)$ the average of $l(k)$ for $k \leq n$, then $\lambda(n)$ is almost monotonous. The problem is to find the asymptotics of $\lambda(n)$.

Conjecture. As $n \to \infty$, we have: $\lambda(n) \sim Cn$.

Problem 3 (I. Bogaevsky) *Manifold of 5-tuples.*

Let M be manifold of unordered 5-tuples of points in $\mathbb{R}^3$, generic in the sense that no 4 points belong to a common plane and all 5 points do not lie on a common sphere. How many connected components are there in M? (This is the simplest case where the author does not know the answer; higher-dimensional generalizations are possible.)

Problem 4 (M. Gromov) *Parallel tangents to a plane curve.*

Let γ be an immersed plane curve. For every line $l \subset \mathbb{R}^2$ compute the number of points $x \in \gamma$ where the tangent is parallel to l. Take the maximum of these numbers as l rotates, then take the minimum of the results as the curve γ varies within its topological class:

$$M(\gamma) = \min_{\gamma' \sim \gamma} \max_{l \in \mathbb{R}P^1} \#\{p \in \gamma : T_x(\gamma) \parallel l\}$$

There is an evident estimate $M(\gamma) \geq 2w(\gamma)$ where w is the rotation number of the curve. The problem is to find a better estimate.

Problem 5 (P. Pushkar') *Ovals of piecewise linear curves.*

Develop the analog of real plane algebraic geometry for piecewise linear, instead of algebraic, curves.

More exactly, consider the class of functions

$$f(x, y) = \sum_{i=1}^{n} \pm |l_i(x, y)| + l_0(x, y),$$

where $l_i(x, y) = a_i x + b_i y + c_i$ are linear inhomogeneous functions. The *degree* of such function f is by definition $n + 1$.

Question 1. Describe the combinatorics of the zero set $f(x, y) = 0$ for functions of a given degree. Find the possible number of connected components ("ovals"), restrictions on their mutual position, extremal examples etc.

Question 2. Find the analogue of the Bézout theorem. Is it true that if the number of solutions to the system of equations $f = 0$, $g = 0$ is finite, then it cannot exceed mn with $m = \deg f$, $n = \deg g$? A known upper bound is $mn + (m-1)(m-2)/2$.

Question 3. Complex version of the previous questions, where the modulus is replaced by the function $z^2/|z|$.

Problem 6 (B. Shapiro) *Inflexion points of plane curves.*

Let γ be an immersed plane curve. Denote by $I(\gamma)$ the minimum number of inflexion points for all curves topologically equivalent to γ.

Problem: Estimate $I(\gamma)$ in terms of other combinatorial invariants of γ, e.g. rotation number $w(\gamma)$ and Arnold's invariants J^+, J^-, St. For example, this is interesting for the special case of tree-like curves (curves that split into two parts if any self-intersection point is deleted). Find a criterion for $I(\gamma) = 0$.

Problem 7 (B. Shapiro) *Zeroes of n-th Wronskian.*

The n-th Wronskian of a function $f(x)$ is, by definition, the determinant of order n formed by the derivatives of f:

$$W_n(f) = \begin{vmatrix} f & f' & f'' & \cdots \\ f' & f'' & f''' & \cdots \\ f'' & f''' & f^{\mathrm{iv}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

A conjecture of Gauss asserts that

$$\#\{\text{real zeroes of } W_2(f)\} \leq \#\{\text{non-real zeroes of } f\}$$

Problems: (1) prove the Gauss conjecture, (2) find similar estimates for $n > 2$.

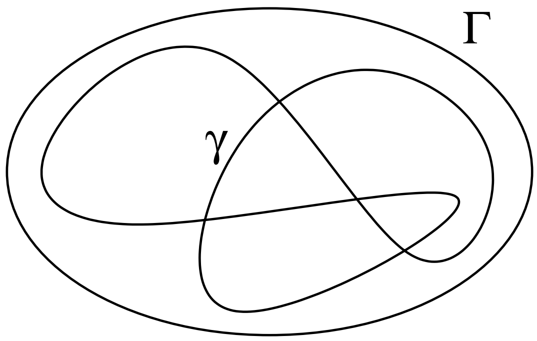
Problem 8 (B. Shapiro) *Critical points of the sum of potentials.*

Estimate from above the number of critical points of the function $\sum_{i=1}^{k} f(r(x, a_i))$, where $a_1, ... a_k$ are fixed points in $\mathbb{R}^n$, r is the Euclidean distance and f is one of the functions $1/r^m$, $\ln(r)$ describing the physical potential in Euclidean space (possibly of dimension different from n).

According to B. Shapiro, the problem is open even in the simplest case of the function $1/r(x, a_1) + 1/r(x, a_2) + 1/r(x, a_3)$ in the plane. In Maxwell's collected works there is an assertion that the number of critical points in this case cannot be greater than 2. For the regular triangle, however, the answer is 4 (one minimum and 3 saddle points).

Problem 9 (S. Tabachnikov) *DNA in a cell.*

Let Γ be a smooth closed convex curve in the plane (the *cell*) and γ a smooth closed curve lying inside Γ (the *DNA*):



The *total curvature* of a closed curve is the integral of the absolute value of the curvature with respect to the arc length parameter. The *average curvature* is the total curvature divided by the length of the curve.

Theorem ([1]). *The average curvature of the DNA is greater than or equal to the average curvature of the cell.*

Questions:

1. The proof given in [1] is extremely involved. Is there a simpler proof?

2. Prove that the equality holds if and only if the DNA coincides with the cell, possibly traversed more than once.

3. Find a higher-dimensional generalizations of this theorem.

Tabachnikov's preprint [2] fom which I take these problems, contains four enlightening proofs of the theorem in the case when Γ is a circle.

References.

[1] J. Lagarias, T. Richardson. *Convexity and the average curvature of plane curves.* Geom. Dedicata 67 (1997), 1–30.

[2] S. Tabachnikov. *The tale of a geometric inequality.*

Problem 10 (V. Vassiliev) *Interpolating families.*

A vector space Φ of real functions on $\mathbb{R}^2$ is said to be *n-interpolating*, if for any set of n points $p_1, \ldots, p_n \in \mathbb{R}^2$ and any set of real numbers $c_1, \ldots, c_n \in \mathbb{R}$ there is a function $\varphi \in \Phi$ the prescribed values at the given points: $\varphi(p_i) = c_i$, $i = 1, ..., n$. Let $d(n)$ be the minimal dimension of an n-interpolating family.

Theorem (V. V.)
(1) $d(n) \leq 2n - 1$,
(2) $d(n) \geq 2n - b_n$, where b_n is the number of 1's in the binary notation of n.

The theorem implies that $d(2) = 3$ and $d(4) = 7$. It is also known that $d(3) = 4$. The first case where the exact value of $d(n)$ is unknown is $n = 5$.

Problem 11 (S. V. Duzhin) *Pairs and splittings.*

Given a set H of cardinality 6, there are 15 ways to pick up a pair of its elements:

```
A = {12,13,14,15,16,23,24,25,26,34,35,36,45,46,56}
```

Also, there are 15 ways to split H into 3 pairs:

```
B = {(12,34,56), (12,35,46), (12,36,45), (13,24,56), (13,25,46),
     (13,26,45), (14,23,56), (14,25,36), (14,26,35), (15,23,46),
     (15,24,36), (15,26,34), (16,23,45), (16,24,35), (16,25,34)}.
```

Problem: Find a natural (in some sense) one-to-one correspondence between these two sets.

To make the question more precise, I need to make a few explanations.

The splittings (elements of B) can also be understood as involutions of H without fixed points, i.e. as permutations on 6 elements with cyclic structure $(2,2,2)$.

Given a one-to-one mapping $f : A \to B$, one can construct a graph with 42 vertices G which is a 6-fold covering of the complete graph on 7 vertices K_7. To do this, take 7 copies of the set $\{1,2,3,4,5,6\}$, one per each vertex of K_7. Label the vertices of K_7 by 0, 1, ..., 6 and the corresponding copies of H by H_0,...,H_6. In the covering, each fibre H_i is connected with each fibre H_j by 6 edges, according to a certain permutation. Now, we connect H_0 to H_i by an identical permutation, and we connect H_i to H_j by the permutation $f(\{i,j\})$. For a clever choice of the mapping f, the graph covering obtained may have the following amazing property:

For any closed path γ in K_7 of length $r = 2,3,4,5,6,7$ the monodromy of the covering over γ is a permutation of order $r-1$. More precisely, these permutations have the following cyclic ctructure: $(1,1,1,1,1,1)$, $(2,2,2)$, $(3,3)$, $(4,1,1)$, $(5,1)$, $(3,2,1)$.

A direct computer search shows that among 15! one-to-one mappings $A \to B$ there are 501 mappings f with the above property. One example is

```
{1,2} -> [216543]    {1,3} -> [341265]    {1,4} -> [465132]
{1,5} -> [532614]    {1,6} -> [654321]    {2,3} -> [564312]
{2,4} -> [351624]    {2,5} -> [645231]    {2,6} -> [432165]
{3,4} -> [632541]    {3,5} -> [456123]    {3,6} -> [215634]
{4,5} -> [214365]    {4,6} -> [546213]    {5,6} -> [361542]
```

Problem: find a natural (in an appropriate sense, e.g. in terms of finite geometry etc.) construction of such a mapping f.

Problem 12 (S. V. Duzhin) *3 tetrahedra in 3-space.*

Let A_1,...,A_6 be six points in 3-space, no four of which lie in a common plane. Let $T_1 = A_1A_2A_3A_4$, $T_2 = A_1A_2A_5A_6$, $T_3 = A_3A_4A_5A_6$ be three tetrahedra with specified vertices. Is it possible that the intersections $T_1 \cap T_2$, $T_1 \cap T_3$, $T_2 \cap T_3$ have non-empty interiors, while the interior of $T_1 \cap T_2 \cap T_3$ is empty?

These problems were announced by S. V. Duzhin on November 1, 2002, at the seminar on low-dimensional mathematics at St. Petersburg Mathematical Institute. The PostScript file is available at the home-page of the seminar: `http://www.pdmi.ras.ru/~lowdimma`.

Notice: Names in parentheses refer to mathematicians from whom, directly or indirectly, I learned about the corresponding problems. The problems are stated as I understand them. The reader should blame the undersigned, not the indicated authors, for all discrepancies present in this text.

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