Compactification and volume of bounded symmetric domains

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January 30, 2004

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3. Compactification and volume of bounded symmetric domains

The unit disc in $\mathbb{C}$

Let $\Delta \subset \mathbb{C}$ be the unit disc, oriented by the volume form

$$\alpha = \frac{\mathrm{i}}{2\pi} \,\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} = \frac{\mathrm{i}}{2\pi} \partial \overline{\partial} \,(z\overline{z})\,.$$

Then

$$\int_\Delta (1 - z\overline{z})^s \,\alpha = \frac{1}{s+1} \qquad (\mathrm{Re}\, s > -1)\,.$$

In particular,

$$\int_\Delta \alpha = 1.$$

Proof. In polar coordinates, $\phi : (r, \theta) \mapsto z = r\,\mathrm{e}^{\mathrm{i}\,\theta}$,

$$\phi^* \alpha = \frac{1}{\pi} r \,\mathrm{d}\, r \wedge \mathrm{d}\, \theta$$

and

$$\int_\Delta (1 - z\overline{z})^s \,\alpha = 2 \int_0^1 \left(1 - r^2\right)^s r \,\mathrm{d}\, r.$$

The Fubini-Study metric on $\mathbb{P}_1(\mathbb{C})$

Denote by $\mathbb{P}_1(\mathbb{C})$ the complex projective space of dimension 1, by

$$\begin{aligned} \pi : \mathbb{C}^2 \setminus \{0\} &\to \mathbb{P}_1(\mathbb{C}) \\ \left(z^0, z^1\right) &\mapsto \left[z^0, z^1\right] \end{aligned}$$

the canonical projection. The *Fubini-Study metric* on $\mathbb{P}_1(\mathbb{C})$ is the $(1,1)$-form β defined by

$$\pi^*\beta = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\ln\left(z^0\overline{z^0} + z^1\overline{z^1}\right).$$

Denote by $\widetilde{\beta}$ the pull-back of β by the inclusion $\mathbb{C} \subset \mathbb{P}_1(\mathbb{C})$, $z \mapsto [1, z]$:

$$\widetilde{\beta} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\ln\left(1 + z\overline{z}\right).$$

Then

$$\widetilde{\beta} = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}}{(1 + z\overline{z})^2}.$$

The Fubini-Study metric: existence

If β exists, it is unique as π is a submersion.

If U is open in $\mathbb{P}_1(\mathbb{C})$ and $f = (f_0, f_1)$ is a section of π defined on U, then

$$\beta = f^*\pi^*\beta = \frac{\mathrm{i}}{2\pi} f^*\partial\overline{\partial} \ln\left(z^0\overline{z^0} + z^1\overline{z^1}\right).$$

As

$$\partial\overline{\partial}\ln\left(z^0\overline{z^0} + z^1\overline{z^1}\right) = \mathrm{d}\left(\frac{z^0\,\mathrm{d}\,\overline{z^0} + z^1\,\mathrm{d}\,\overline{z^1}}{z^0\overline{z^0} + z^1\overline{z^1}}\right),$$

we have

$$\beta = \frac{\mathrm{i}}{2\pi}\,\mathrm{d}\left(\frac{f^0\,\mathrm{d}\,\overline{f^0} + f^1\,\mathrm{d}\,\overline{f^1}}{f^0\overline{f^0} + f^1\overline{f^1}}\right).$$

If $g = (g_0, g_1)$ is another section, then $f = \lambda g$ where $\lambda : U \to \mathbb{C}$ is a non-vanishing function on U; we have

$$\begin{aligned}
\mathrm{d}\left(\frac{f^0\,\mathrm{d}\,\overline{f^0} + f^1\,\mathrm{d}\,\overline{f^1}}{f^0\overline{f^0} + f^1\overline{f^1}}\right) &= \mathrm{d}\left(\frac{g^0\,\mathrm{d}\,\overline{g^0} + g^1\,\mathrm{d}\,\overline{g^1}}{g^0\overline{g^0} + g^1\overline{g^1}} + \mathrm{d}\,\overline{\lambda}\right) \\
&= \mathrm{d}\left(\frac{g^0\,\mathrm{d}\,\overline{g^0} + g^1\,\mathrm{d}\,\overline{g^1}}{g^0\overline{g^0} + g^1\overline{g^1}}\right),
\end{aligned}$$

which shows the existence of β.

The Fubini-Study metric on $\mathbb{C}$

On $U_0 \subset \mathbb{P}_1(\mathbb{C})$, $U_0 = \{[1, z] \mid z \in \mathbb{C}\} \simeq \mathbb{C}$, we take the section $f_0 = 1$, $f_1 = z$, which gives

$$\begin{aligned}
\widetilde{\beta} &= \frac{\mathrm{i}}{2\pi} \mathrm{d} \left(\frac{z \, \mathrm{d}\, \overline{z}}{1 + z\overline{z}} \right) \\
&= \frac{\mathrm{i}}{2\pi} \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{1 + z\overline{z}} - \frac{\mathrm{i}}{2\pi} \frac{\overline{z}\, \mathrm{d}\, z \wedge z \, \mathrm{d}\, \overline{z}}{(1 + z\overline{z})^2} \\
&= \frac{\mathrm{i}}{2\pi} \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{(1 + z\overline{z})^2}.
\end{aligned}$$

From $\mathbb{C}$ to $\varDelta$ (1)

Consider the real-analytic map

$$\begin{aligned} \psi : \varDelta &\to \mathbb{C} \\ z &\mapsto \frac{z}{(1 - z\overline{z})^{1/2}}. \end{aligned}$$

The map ψ is a diffeomorphism and its inverse is

$$\psi^{-1} : u \mapsto \frac{u}{(1 + u\overline{u})^{1/2}}.$$

Proposition.

$$\psi^* \left((1 + z\overline{z})^s \, \widetilde{\beta} \right) = (1 - z\overline{z})^{-s} \, \alpha.$$

From $\mathbb{C}$ to Δ (2)

Proof. For $u = \psi(z) = \frac{z}{(1-z\overline{z})^{1/2}}$,we have

$$\begin{aligned}
1 + u\overline{u} &= (1 - z\overline{z})^{-1}, \\
\mathrm{d}\,u &= \frac{\mathrm{d}\,z}{(1 - z\overline{z})^{1/2}} - \frac{z}{2}\frac{-\overline{z}\,\mathrm{d}\,z - z\,\mathrm{d}\,\overline{z}}{(1 - z\overline{z})^{3/2}} \\
&= \frac{1 - \frac{z\overline{z}}{2}}{(1 - z\overline{z})^{3/2}}\,\mathrm{d}\,z + \frac{z^2}{2}\frac{\mathrm{d}\,\overline{z}}{(1 - z\overline{z})^{3/2}}, \\
\mathrm{d}\,\overline{u} &= \frac{\overline{z}^2}{2}\frac{\mathrm{d}\,z}{(1 - z\overline{z})^{3/2}} + \frac{1 - \frac{z\overline{z}}{2}}{(1 - z\overline{z})^{3/2}}\,\mathrm{d}\,\overline{z}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}\,u \wedge \mathrm{d}\,\overline{u} &= \left(\left(1 - \frac{z\overline{z}}{2}\right)^2 - \frac{z^2\overline{z}^2}{4}\right)\frac{\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}}{(1 - z\overline{z})^3} \\
&= \frac{\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}}{(1 - z\overline{z})^2}.
\end{aligned}$$

This means

$$\begin{aligned}
\psi^*\left((1 + z\overline{z})^s\right) &= (1 - z\overline{z})^{-s}, \\
\psi^*\alpha &= (1 - z\overline{z})^{-2}\,\alpha, \\
\psi^*\widetilde{\beta} &= \psi^*\left((1 + z\overline{z})^{-2}\,\alpha\right) = \alpha.
\end{aligned}$$

From $\mathbb{C}$ to Δ (3)

The map ψ and the above proof are better understood using the polar coordinates map

$$\begin{aligned} \Phi :]0, +\infty[\times S^1 &\to \mathbb{C} \setminus \{0\} \\ (\lambda, c) &\mapsto \lambda c. \end{aligned}$$

Transposed by Φ, the map ψ is $\Psi = \Phi^{-1} \circ \psi \circ \Phi$,

$$\begin{aligned} \Psi :]0, 1[\times S^1 &\to]0, +\infty[\times S^1 \\ (\lambda, c) &\mapsto \left(\frac{\lambda}{(1 - \lambda^2)^{1/2}}, c \right). \end{aligned}$$

The pull-backs of α and $\widetilde{\beta}$ by Φ are

$$\begin{aligned} \Phi^* \alpha &= 2\lambda \, \mathrm{d}\, \lambda \wedge \Theta_1, \\ \Phi^* \widetilde{\beta} &= \frac{2\lambda}{(1 + \lambda^2)^2} \, \mathrm{d}\, \lambda \wedge \Theta_1, \end{aligned}$$

where Θ_1 is the rotation-invariant form on S^1 such that $\int_{S^1} \Theta_1 = 1$. If $\mu = \frac{\lambda}{(1-\lambda^2)^{1/2}}$, then $1 + \mu^2 = \left(1 - \lambda^2\right)^{-1}$ and

$$2\mu \, \mathrm{d}\, \mu = \mathrm{d} \left(\mu^2\right) = \mathrm{d} \left(\frac{\lambda^2}{1 - \lambda^2} \right) = \mathrm{d} \left(\frac{1}{1 - \lambda^2} \right) = \frac{2\lambda \, \mathrm{d}\, \lambda}{(1 - \lambda^2)^2},$$

which proves again $\psi^* \alpha = (1 - z\overline{z})^{-2} \, \alpha$.

Volume of $\mathbb{P}_1(\mathbb{C})$

Proposition.

$$\int_{\mathbb{P}_1(\mathbb{C})} \beta = 1.$$

Proof.

$$\int_{\mathbb{P}_1(\mathbb{C})} \beta = \int_{\mathbb{C}} \widetilde{\beta} = \int_{\Delta} \alpha = 1,$$

as $\psi^* \widetilde{\beta} = \alpha$.

Remark. More generally, for $\mathrm{Re}\, s > -1$,

$$\int_{\mathbb{C}} (1 + z\overline{z})^s \, \widetilde{\beta} = \int_{\Delta} (1 - z\overline{z})^{-s} \, \alpha = \frac{1}{s+1}$$

follows from $\psi^* \left((1 + z\overline{z})^s \, \widetilde{\beta} \right) = (1 - z\overline{z})^{-s} \, \alpha$.

Exercise. Compute $\int_{\Delta} \widetilde{\beta}$, $\int_{\Delta} (1 + z\overline{z})^s \, \widetilde{\beta}$.

The Hermitian ball in $\mathbb{C}^n$

Consider the standard Hermitian space $V = \mathbb{C}^n$ of dimension n, with the Hermitian scalar product

$$(z \mid t) = \sum_{j=1}^{n} z^j \overline{t^j}$$

and the Hermitian norm $\| \ \|$ defined by $\|z\|^2 = (z \mid z)$. The *canonical* $(1,1)$-*form* on V is

$$\alpha = \frac{\mathrm{i}}{2\pi} \partial\overline{\partial}\,(z \mid z) = \frac{\mathrm{i}}{2\pi} \sum_{j=1}^{n} \mathrm{d}\, z^j \wedge \mathrm{d}\, \overline{z^j}.$$

The *volume form* on V is α^n. The *Hermitian unit ball* is

$$B_n = \{ z \in V \mid \|z\| < 1 \}\,.$$

The volume of B_n is then

$$\int_{B_n} \alpha^n = 1.$$

The Fubini-Study metric on $\mathbb{P}_n(\mathbb{C})$

Denote by $\mathbb{P}_n(\mathbb{C})$ the complex projective space of dimension n, by

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n(\mathbb{C}),$$
$$\left(z^0, z^1, \dots, z^n\right) \mapsto \left[z^0, z^1, \dots, z^n\right]$$

the canonical projection. Let $(z \mid t) = \sum_{j=0}^{n} z^j \overline{t^j}$ the standard Hermitian form on $\mathbb{C}^{n+1}$. The *Fubini-Study metric* on $\mathbb{P}_n(\mathbb{C})$ is the $(1,1)$-form β defined by

$$\pi^* \beta = \frac{\mathrm{i}}{2\pi} \partial \overline{\partial} \ln (z \mid z) \, .$$

Denote by $\widetilde{\beta}$ the pull-back of β by the inclusion $\mathbb{C}^n \subset \mathbb{P}_n(\mathbb{C})$, $\left(z^1, \dots, z^n\right) \mapsto \left[1, z^1, \dots, z^n\right]$:

$$\widetilde{\beta} = \frac{\mathrm{i}}{2\pi} \partial \overline{\partial} \ln \left(1 + \sum_{j=1}^{n} z^j \overline{z^j} \right) .$$

The Fubini-Study metric on $\mathbb{P}_n(\mathbb{C})$: existence

If β exists, it is unique as π is a submersion.

If U is open in $\mathbb{P}_n(\mathbb{C})$ and $f = (f_0, \ldots, f_n)$ is a section of π defined on U, then

$$\beta = f^*\pi^*\beta = \frac{\mathrm{i}}{2\pi} f^*\partial\overline{\partial} \ln{(z \mid z)}\,.$$

As

$$\partial\overline{\partial} \ln{(z \mid z)} = \mathrm{d}\left(\frac{(z \mid \mathrm{d}\, z)}{(z \mid z)}\right),$$

we have

$$\beta = \frac{\mathrm{i}}{2\pi}\,\mathrm{d}\left(\frac{(f \mid \mathrm{d}\, f)}{(f \mid f)}\right),$$

where $(f \mid \mathrm{d}\, f) = \sum_{j=0}^{n} f^j\, \mathrm{d}\, \overline{f^j}$. If $g = (g_0, g_1)$ is another section, then $f = \lambda g$ where $\lambda : U \to \mathbb{C}$ is a non-vanishing function on U; we have

$$\begin{aligned}\mathrm{d}\left(\frac{(f \mid \mathrm{d}\, f)}{(f \mid f)}\right) &= \mathrm{d}\left(\frac{(g \mid \mathrm{d}\, g)}{(g \mid g)} + \mathrm{d}\,\overline{\lambda}\right)\\ &= \mathrm{d}\left(\frac{(g \mid \mathrm{d}\, g)}{(g \mid g)}\right),\end{aligned}$$

which shows the existence of β.

The Fubini-Study metric on $\mathbb{C}^n$

On $U_0 \subset \mathbb{P}_n(\mathbb{C})$, $U_0 = \{[1, z] \mid z \in \mathbb{C}^n\} \simeq \mathbb{C}^n$, we take the section $f = (1, z)$, which gives

$$\begin{aligned}\widetilde{\beta} &= \frac{\mathrm{i}}{2\pi} \mathrm{d} \left(\frac{(z \mid \mathrm{d}\, z)}{1 + (z \mid z)} \right) \\ &= \frac{\mathrm{i}}{2\pi} \frac{(\mathrm{d}\, z \mid \mathrm{d}\, z)}{1 + (z \mid z)} - \frac{\mathrm{i}}{2\pi} \frac{(\mathrm{d}\, z \mid z) \wedge (z \mid \mathrm{d}\, z)}{(1 + (z \mid z))^2},\end{aligned}$$

where

$$(z \mid \mathrm{d}\, z) = \sum_{j=1}^{n} z^j \, \mathrm{d}\, \overline{z^j},$$

$$(\mathrm{d}\, z \mid z) = \sum_{j=1}^{n} \overline{z^j} \, \mathrm{d}\, z^j,$$

$$(\mathrm{d}\, z \mid \mathrm{d}\, z) = \sum_{j=1}^{n} \mathrm{d}\, z^j \wedge \mathrm{d}\, \overline{z^j}.$$

The projective volume form on $\mathbb{C}^n$

The canonical volume form on $\mathbb{P}_n(\mathbb{C})$ is

$$\omega = \beta^n.$$

Its pull-back $\widetilde{\beta}^n$ to $\mathbb{C}^n$ is then

$$\begin{aligned}\widetilde{\beta}^n = \left(\frac{\mathrm{i}}{2\pi}\right)^n \left(\frac{(\mathrm{d}\, z \mid \mathrm{d}\, z)}{1 + (z \mid z)}\right)^n \\ - n \left(\frac{\mathrm{i}}{2\pi}\right)^n \frac{(\mathrm{d}\, z \mid z) \wedge (z \mid \mathrm{d}\, z)}{(1 + (z \mid z))^2} \left(\frac{(\mathrm{d}\, z \mid \mathrm{d}\, z)}{1 + (z \mid z)}\right)^{n-1} \\ = \left(\frac{\mathrm{i}}{2\pi}\right)^n \frac{(\mathrm{d}\, z \mid \mathrm{d}\, z)^{n-1} \wedge \gamma(z)}{(1 + (z \mid z))^{n+1}},\end{aligned}$$

with

$$\gamma(z) = (1 + (z \mid z))\,(\mathrm{d}\, z \mid \mathrm{d}\, z) - n\,(\mathrm{d}\, z \mid z) \wedge (z \mid \mathrm{d}\, z)\,.$$

Using

$$n\,(\mathrm{d}\, z \mid z) \wedge (z \mid \mathrm{d}\, z) \wedge (\mathrm{d}\, z \mid \mathrm{d}\, z)^{n-1} = (z \mid z)\,(\mathrm{d}\, z \mid \mathrm{d}\, z)^n\,,$$

we get

$$\widetilde{\beta}^n = \frac{\alpha^n}{(1 + (z \mid z))^{n+1}}.$$

From $\mathbb{C}^n$ to B_n (1)

We define the real-analytic map

$$\begin{aligned} \psi : B_n &\longrightarrow \mathbb{C}^n \\ z &\longmapsto \frac{z}{\left(1 - \|z\|^2\right)^{1/2}}. \end{aligned}$$

The map ψ is a diffeomorphism and its inverse is

$$\psi^{-1} : u \longmapsto \frac{u}{\left(1 + \|u\|^2\right)^{1/2}}.$$

Consider the polar coordinates map in $\mathbb{C}^n$

$$\begin{aligned} \Phi :]0, +\infty[\times S^{2n-1} &\longrightarrow \mathbb{C}^n \setminus \{0\} \\ (\lambda, c) &\longmapsto \lambda c. \end{aligned}$$

Transposed by Φ, the map ψ is $\Psi = \Phi^{-1} \circ \psi \circ \Phi$,

$$\begin{aligned} \Psi :]0, 1[\times S^{2n-1} &\longrightarrow]0, +\infty[\times S^{2n-1} \\ (\lambda, c) &\longmapsto \left(\frac{\lambda}{(1 - \lambda^2)^{1/2}}, c \right). \end{aligned}$$

From $\mathbb{C}^n$ to B_n (2)

The pull-back of α^n by the polar coordinates map Φ is

$$\Phi^*\alpha^n = 2n\lambda^{2n-1}\,\mathrm{d}\,\lambda \wedge \Theta_{2n-1},$$

where Θ_{2n-1} is a $U(n)$-invariant form on S^{2n-1}. We have $\int_{B_n} \alpha^n = \int_0^1 2n\lambda^{2n-1}\,\mathrm{d}\,\lambda \int_{S^{2n-1}} \Theta_{2n-1}$, which implies

$$\int_{S^{2n-1}} \Theta_{2n-1} = 1.$$

The pull-back of $\widetilde{\beta}^n$ by Φ is then

$$\Phi^*\widetilde{\beta}^n = \frac{2n\lambda^{2n-1}}{(1+\lambda^2)^{n+1}}\,\mathrm{d}\,\lambda \wedge \Theta_{2n-1}.$$

Proposition.

$$\psi^*\left((1+z\overline{z})^s\,\widetilde{\beta}^n\right) = (1-z\overline{z})^{-s}\,\alpha^n.$$

Proof. Using polar coordinates, let $\mu = \frac{\lambda}{(1-\lambda^2)^{1/2}}$. Then $1+\mu^2 = \frac{1}{1-\lambda^2}$ and

$$\frac{2n\mu^{2n-1}}{(1+\mu^2)^{n+1}}\,\mathrm{d}\,\mu = \frac{n\mu^{2n-2}}{(1+\mu^2)^{n+1}}\,\mathrm{d}\left(\mu^2\right)$$
$$= \frac{n\lambda^{2n-2}}{(1-\lambda^2)^{n-1}(1+\mu^2)^{n+1}}\frac{2\lambda\,\mathrm{d}\,\lambda}{(1-\lambda^2)^2} = 2n\lambda^{2n-1}\,\mathrm{d}\,\lambda.$$

Volume of $\mathbb{P}_n(\mathbb{C})$

Proposition.

$$\int_{\mathbb{P}_n(\mathbb{C})} \beta^n = 1.$$

Proof.

$$\int_{\mathbb{P}_n(\mathbb{C})} \beta^n = \int_{\mathbb{C}^n} \widetilde{\beta}^n = \int_{B_n} \alpha^n = 1.$$

Exercise. Compute $\int_{B_n} \widetilde{\beta}^n$ (the projective volume of the unit Hermitian ball).
Solution. $1/2^n$
Corollary. *1) Let $P_k \subset \mathbb{P}_n(\mathbb{C})$ be a projective plane of complex dimension k. Then*

$$\int_{P_k} \beta^k = 1.$$

2) Let $X \subset \mathbb{P}_n(\mathbb{C})$ be a projective subvariety of pure dimension k. Then

$$\int_X \beta^k = \deg X.$$

The generalized unit ball in $\mathcal{M}_{p,q}(\mathbb{C})$

For $p \geq q \geq 1$, let

$$V = \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \simeq \mathcal{M}_{p,q}(\mathbb{C})$$

(space of complex matrices with p lines and q columns).

Here $\mathbb{C}^q$ and $\mathbb{C}^p$ are Hermitian vector spaces with the standard Hermitian structures; the standard basis of $\mathbb{C}^q$ (resp. $\mathbb{C}^p$) are denoted by $(\eta_1, \ldots, \eta_q)$ (resp. $(\varepsilon_1, \ldots, \varepsilon_p)$). If $u \in V$, $u : \mathbb{C}^q \to \mathbb{C}^p$, then $u^* : \mathbb{C}^p \to \mathbb{C}^q$ denotes the adjoint homomorphism of u w.r. to these structures. We identify u with its matrix in the standard basis; then $u^* = \overline{u}'$, where v' is the transpose of the matrix v.

The *generalized unit ball* of $\mathcal{M}_{p,q}(\mathbb{C})$ is

$$\Omega^I_{p,q} = \{u \in \mathcal{M}_{p,q}(\mathbb{C}) \mid I_q - u^* u >> 0\}\,.$$

Volume forms on $\mathcal{M}_{p,q}(\mathbb{C})$

On $V = \mathcal{M}_{p,q}(\mathbb{C})$, consider the Hermitian scalar product

$$m_1(x, y) = \mathrm{Tr}\left(y^* x\right),$$

where Tr is the trace of matrices, and the associated $(1, 1)$-form

$$\alpha = \frac{\mathrm{i}}{2\pi} \partial\overline{\partial} m_1\left(x, x\right).$$

The *flat volume form* on V is

$$\alpha^n$$

(with $n = \dim V = pq$); the *projective volume form* is

$$\frac{\alpha^n}{\mathrm{Det}\left(I_q + x^* x\right)^{p+q}}$$

(Det denotes the determinant of matrices).

Spectral decomposition in $\mathcal{M}_{p,q}(\mathbb{C})$ (1)

Each matrix $x \in \mathcal{M}_{p,q}(\mathbb{C})$ can be written

$$x = u\Lambda v^*,$$

with $u \in U(p)$, $v \in U(q)$ and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_q \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$
$$= \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_q E_q,$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q \geq 0$. The matrix x is called *regular* if $\lambda_1 > \lambda_2 > \cdots > \lambda_q > 0$. The regular elements form an open dense subset of $\mathcal{M}_{p,q}(\mathbb{C})$.

Spectral decomposition in $\mathcal{M}_{p,q}(\mathbb{C})$ (2)

A sequence $(c_1, \ldots, c_q)$ is called a *frame* for $\mathcal{M}_{p,q}(\mathbb{C})$ if it satisfies

$$c_i c_i^* c_j = \delta_{ij} c_i \qquad (1 \leq i, j \leq q)\,.$$

If $x = u\Lambda v^*$, then $x = \lambda_1 c_1 + \cdots + \lambda_q c_q$ and $\left(c_j = uE_j v^*\right)$ is a frame. Each regular matrix has a unique decomposition

$$x = \lambda_1 c_1 + \cdots + \lambda_q c_q,$$

where $(c_1, \ldots, c_q)$ is a frame and $\lambda_1 > \cdots > \lambda_q > 0$. This decomposition is called *spectral decomposition* of x.
The frames form a real-analytic manifold $\mathcal{F}_{p,q}^I$ (the *Fürstenberg-Satake boundary* of $\Omega_{p,q}^I$). We denote by Φ the map

$$\{\lambda_1 > \lambda_2 > \cdots > \lambda_q > 0\} \times \mathcal{F}_{p,q}^I \to \mathcal{M}_{p,q}(\mathbb{C})$$
$$\Big((\lambda_1, \ldots, \lambda_q)\,, \big(c_1, \ldots, c_j\big)\Big) \mapsto \textstyle\sum_{j=1}^q \lambda_j c_j,$$

which generalizes the polar coordinates map.

Volume forms on $\mathcal{M}_{p,q}(\mathbb{C})$ (2)

Proposition.

$$\Phi^*\alpha^n = \prod_{j=1}^{q} \lambda_j^{2(p-q)+1} \prod_{j<k} \left(\lambda_j^2 - \lambda_k^2\right)^2 \mathrm{d}\,\lambda_1 \wedge \ldots \wedge \mathrm{d}\,\lambda_q \wedge \Theta_{p,q}^I,$$

where $\Theta_{p,q}^I$ *is a* $U(p) \times U(q)$*-invariant volume form on* $\mathcal{F}_{p,q}^I$.

Compactification of $\mathcal{M}_{p,q}(\mathbb{C})$ (1)

Let $(\eta_1, \ldots, \eta_q)$ be the standard basis of $\mathbb{C}^q$. To $u \in \mathcal{M}_{p,q}(\mathbb{C})$, $u = (u_1, \ldots, u_q)$, where u_j is the j-th column of u, we associate $\tilde{u} \in G_{q,q+p}(\mathbb{C})$ (the Grassmannian of q-planes in $\mathbb{C}^q \oplus \mathbb{C}^p$) defined by

$$\tilde{u} = \langle \eta_1 \oplus u_1, \ldots, \eta_q \oplus u_q \rangle .$$

By the Plücker embedding

$$G_{p,q}(\mathbb{C}) \subset \mathbb{P}\left(\bigwedge^q (\mathbb{C}^q \oplus \mathbb{C}^p)\right),$$

$\tilde{u}$ is mapped to

$$\Theta(u) = \hat{u} = [(\eta_1 + u_1) \wedge \ldots \wedge (\eta_q \oplus u_q)] .$$

The map

$$\Theta : \mathcal{M}_{p,q}(\mathbb{C}) \to \mathbb{P}\left(\bigwedge^q (\mathbb{C}^q \oplus \mathbb{C}^p)\right)$$

is injective and the closure of $\Theta\left(\mathcal{M}_{p,q}(\mathbb{C})\right)$ is the Grassmannian $G_{p,q}(\mathbb{C})$. The map Θ is called the *canonical compactification map* of $\mathcal{M}_{p,q}(\mathbb{C})$.

Compactification of $\mathcal{M}_{p,q}(\mathbb{C})$ (2)

Using the isomorphism

$$\mathbb{P}\left(\bigwedge^q \left(\mathbb{C}^q \oplus \mathbb{C}^p\right)\right) \simeq \mathbb{P}\left(\bigoplus_{j=0}^{q} \text{Hom}\left(\bigwedge^j \mathbb{C}^q, \bigwedge^j \mathbb{C}^p\right)\right),$$

the compactification map may also be written

$$\Theta(x) = \left[1 \oplus x \oplus \ldots \oplus \bigwedge^q x\right] = \left[\bigoplus_{j=0}^{q} \bigwedge^j x\right].$$

Let $V_j = \text{Hom}\left(\bigwedge^j \mathbb{C}^q, \bigwedge^j \mathbb{C}^p\right)$ and $W = \bigoplus_{j=0}^{q} V_j$. Let $(x \mid y)_j = \text{Tr}\,(y^* x)$ be the Hermitian scalar product on V_j arising from the standard Hermitian products on $\bigwedge^j \mathbb{C}^q$, $\bigwedge^j \mathbb{C}^p$ and let $(\ \mid\ )$ be the direct sum of these products in W. Denote by β the corresponding Fubini-Study form on $\mathbb{P}(W)$:

$$\beta = \frac{\mathrm{i}}{2\pi} \partial\overline{\partial} \ln (w \mid w).$$

From $\mathcal{M}_{p,q}(\mathbb{C})$ to its unit ball (1)

Define the real-analytic map

$$\psi : \Omega^{I}_{p,q} \to \mathcal{M}_{p,q}(\mathbb{C})$$

by

$$\psi(x) = (I_p - xx^*)^{-1/2}\, x = x\, (I_q - x^*x)^{-1/2}\,.$$

Then ψ is a diffeomorphism and

$$\psi^{-1}(y) = (I_p + yy^*)^{-1/2}\, y = y\, (I_q + y^*y)^{-1/2}\,.$$

Proposition.

$$\psi^* \left(\mathrm{Det}\,(I_q + x^*x)^{s-p-q}\,\alpha^n\right) = \mathrm{Det}\,(I_q - x^*x)^{-s}\,\alpha^n.$$

From $\mathcal{M}_{p,q}(\mathbb{C})$ to its unit ball (2)

Proof. Transposed by Φ, the map ψ is $\Psi = \Phi^{-1} \circ \psi \circ \Phi$,

$$\begin{aligned} \Psi : \{1 > \lambda_1 > \lambda_2 > \cdots > \lambda_q > 0\} \times \mathcal{F}^I_{p,q} \\ \to \{\lambda_1 > \lambda_2 > \cdots > \lambda_q > 0\} \times \mathcal{F}^I_{p,q}, \\ \left(\left(\lambda_j\right), \left(c_j\right)\right) \mapsto \left(\left(\frac{\lambda_j}{\left(1-\lambda_j^2\right)^{1/2}}\right), \left(c_j\right)\right). \end{aligned}$$

As $\Phi^* x = \sum_{j=1}^q \lambda_j c_j$, we have

$$\Phi^* \det\left(I_q + x^* x\right) = \prod_{j=1}^q \left(1 + \lambda_j^2\right)$$

and

$$\begin{aligned} \Phi^* \widetilde{\beta}^n &= \Phi^* \frac{\alpha^n}{\operatorname{Det}\left(I_q + x^* x\right)^{p+q}} \\ &= \frac{\prod_{j=1}^q \lambda_j^{2(p-q)+1} \prod_{j<k}\left(\lambda_j^2 - \lambda_k^2\right)^2}{\prod_{j=1}^q \left(1+\lambda_j^2\right)^{p+q}} \omega\left(\lambda\right) \wedge \Theta^I_{p,q}, \end{aligned}$$

with $\omega\left(\lambda\right) = \mathrm{d}\,\lambda_1 \wedge \ldots \wedge \mathrm{d}\,\lambda_q$.

From $\mathcal{M}_{p,q}(\mathbb{C})$ to its unit ball (3)

Let

$$\mu_j = \frac{\lambda_j}{\left(1-\lambda_j^2\right)^{1/2}}.$$

Then

$$1+\mu_j^2 = \frac{1}{1-\lambda_j^2},$$

$$\frac{\mu_j}{\left(1+\mu_j^2\right)^2}\,\mathrm{d}\,\mu_j = \lambda_j\,\mathrm{d}\,\lambda_j,$$

$$\mu_j^2-\mu_k^2 = \frac{\lambda_j^2-\lambda_k^2}{\left(1-\lambda_j^2\right)\left(1-\lambda_k^2\right)}.$$

Finally

$$\frac{\prod_{j=1}^q \mu_j^{2(p-q)+1}\prod_{j<k}\left(\mu_j^2-\mu_k^2\right)^2}{\prod_{j=1}^q\left(1+\mu_j^2\right)^{p+q}}\omega\left(\mu\right)$$
$$= \frac{\prod_{j=1}^q \lambda_j^{2(p-q)+1}}{\prod_{j=1}^q\left(1-\lambda_j^2\right)^{p-q}}\prod_{j<k}\frac{\left(\lambda_j^2-\lambda_k^2\right)^2}{\left(1-\lambda_j^2\right)^2\left(1-\lambda_k^2\right)^2}\prod_{j=1}^q\left(1-\lambda_j^2\right)^{p+q-2}\omega(\lambda)$$
$$= \prod_{j=1}^q \lambda_j^{2(p-q)+1}\prod_{j<k}\left(\lambda_j^2-\lambda_k^2\right)^2\omega(\lambda),$$

which ends the proof.

The degree of Grassmannians in the Plücker embedding

Proposition.

$$\Theta^*(\beta^n) = \frac{\alpha^n}{\mathrm{Det}\,(I_q + x^* x)^{p+q}}.$$

Theorem. *Let* $G_{p,q}(\mathbb{C}) \subset \mathbb{P}(W)$ *be the Plücker embedding of the Grassmannian. Then*

$$\deg G_{p,q}(\mathbb{C}) = \int_{\Omega^I_{p,q}} \alpha^n$$

($n = \dim V = pq$).

Proof.

$$\deg G_{p,q}(\mathbb{C}) = \int_{G_{p,q}(\mathbb{C})} \beta^n = \int_V \frac{\alpha^n}{\mathrm{Det}\,(I_q + x^* x)^{p+q}} = \int_{\Omega^I_{p,q}} \alpha^n.$$

Bounded symmetric domains

A bounded domain $\Omega \subset \mathbb{C}^n$ is called *symmetric* if for each $x \in \Omega$ there is an involutive holomorphic automorphism s_x ($s_x^2 = \mathrm{id}_\Omega$) such that x is an isolated fixed point of s_x.

Bounded symmetric domains are *homogeneous* (under the group $\mathrm{Aut}\,\Omega$ of holomorphic automorphisms).

Any bounded symmetric domain Ω is biholomorphic to a bounded *circled* homogeneous domains, which is unique up to linear isomorphisms and is called the *circled realization* of Ω.

We will always consider bounded symmetric domains in their circled realization.

A bounded symmetric domain is called *irreducible* if it is not equivalent to the direct product of two bounded symmetric domains.

Jordan triple associated to a bounded symmetric domain

Let Ω be an irreducible bounded circled homogeneous domain in a complex vector space V. Let K be the identity component of the (compact) Lie group of (linear) automorphisms of Ω leaving 0 fixed. Let ω be a volume form on V, invariant by K and by translations. Let $\mathcal{K}$ be the Bergman kernel of Ω with respect to ω. The *Bergman metric* at $z \in \Omega$ is defined by

$$h_z(u,v) = \partial_u \overline{\partial}_v \log \mathcal{K}(z).$$

The *Jordan triple product* on V is defined by

$$h_0(\{uvw\}, t) = \partial_u \overline{\partial}_v \partial_w \overline{\partial}_t \log \mathcal{K}(z) \,|_{z=0} \,.$$

The triple product $(x,y,z) \mapsto \{xyz\}$ is complex bilinear and symmetric with respect to (x,z), complex antilinear with respect to y. It satisfies the *Jordan identity* (J).

Hermitian Jordan triples

Let V be a (finite dimensional) complex vector space, endowed with a triple product

$$(x, y, z) \mapsto \{xyz\}$$

complex bilinear and symmetric with respect to (x, z), complex antilinear with respect to y, satisfying the *Jordan identity*

$$\{xy\{uvw\}\} - \{uv\{xyw\}\} = \{\{xyu\}vw\} - \{u\{vxy\}w\}. \tag{J}$$

Then $(V, \{xyz\})$ is called a *(Hermitian) Jordan triple system*.

For $x, y, z \in V$, denote by $D(x, y)$ and $Q(x, z)$ the operators defined by

$$\{xyz\} = D(x, y)z = Q(x, z)y.$$

Positive Jordan triples

A Jordan triple system is called *(Hermitian) positive* if

$$(u|v) = \operatorname{tr} D(u,v)$$

is positive definite.

An Hermitian positive Jordan triple system is always *semi-simple*, that is, the direct sum of a finite family of simple subsystems with component-wise triple product. It is called *simple* if it not the product of two non-trivial subsystems.

Quadratic operator and Bergman operator

Let $(V, \{\ ,\ ,\ \})$ be a Jordan triple. The *quadratic representation* $Q : V \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$ is defined by

$$2Q(x)y = \{xyx\}.$$

The following fundamental identity is a consequence of the Jordan identity:

$$Q(Q(x)y) = Q(x)Q(y)Q(x).$$

The *Bergman operator* B is defined by

$$B(x,y) = I - D(x,y) + Q(x)Q(y),$$

where I denotes the identity operator in V. It is also a consequence of the Jordan identity that the following fundamental identity holds for the Bergman operator:

$$Q(B(x,y)z) = B(x,y)Q(z)B(y,x).$$

Tripotent elements in Jordan triples

Let $(V, \{\ ,\ ,\ \})$ be a Jordan triple. An element $c \in V$ is called *tripotent* if

$$\{ccc\} = 2c.$$

If c is a tripotent, the operator $D(c,c)$ annihilates the polynomial $T(T-1)(T-2)$.The decomposition

$$V = V_0(c) \oplus V_1(c) \oplus V_2(c),$$

where $V_j(c)$ is the eigenspace

$$V_j(c) = \{x \in V\ ;\ D(c,c)x = jx\},$$

is called the *Peirce decomposition* of V (with respect to the tripotent c).

Orthogonality of tripotents

Two tripotents c_1 and c_2 are called *orthogonal* if $D(c_1, c_2) = 0$. If c_1 and c_2 are orthogonal tripotents, then $D(c_1, c_{1)}$ and $D(c_2, c_2)$ commute and $c_1 + c_2$ is also a tripotent.

A non zero tripotent c is called *primitive* if it is not the sum of non zero orthogonal tripotents. A tripotent c is called *maximal* if there is no non zero tripotent orthogonal to c.

Frames

A *frame* of V is a maximal sequence $(c_1, \ldots, c_r)$ of pairwise orthogonal primitive tripotents.

Let V be a *simple* positive Jordan triple. Then there exist frames for V. All frames have the same number of elements, which is the *rank* r of V.

Let $\mathbf{c} = (c_1, \ldots, c_r)$ be a frame. For $0 \le i \le j \le r$, let

$$V_{ij}(\mathbf{c}) \\ = \left\{ x \in V \mid D(c_k, c_k)x = (\delta_i^k + \delta_j^k)x, \ 1 \le k \le r \right\}.$$

The decomposition $V = \bigoplus_{0 \le i \le j \le r} V_{ij}(\mathbf{c})$ is called the *simultaneous Peirce decomposition* with respect to the frame $\mathbf{c}$.

Numerical invariants

Let V be a *simple* positive Jordan triple.

For any frame $\mathbf{c}$ of V, the subspaces $V_{ij} = V_{ij}(\mathbf{c})$ of the simultaneous Peirce decomposition have the following properties: $V_{00} = 0$; $V_{ii} = \mathbb{C}e_i$ $(0 < i)$; all V_{ij}'s $(0 < i < j)$ have the same dimension a; all V_{0i}'s $(0 < i)$ have the same dimension b.

The *numerical invariants* of V are the rank r and the two integers

$$a = \dim V_{ij} \;\; (0 < i < j),$$
$$b = \dim V_{0i} \;\; (0 < i).$$

The *genus* of V is the number g defined by

$$g = 2 + a(r-1) + b.$$

The positive Jordan triple V is said to be of *tube type* if $b = 0$.

Spectral theory

Let V be a simple positive Jordan triple. Then any $x \in V$ can be written in a unique way

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p,$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $c_1, c_2 \ldots, c_p$ are pairwise orthogonal tripotents. The element x is *regular* iff $p = r$ (the rank of V); then $(c_1, c_2, \ldots, c_r)$ is a frame of V. The decomposition $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is called the *spectral decomposition* of x.

The map $x \mapsto \lambda_1$, where $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is the spectral decomposition of x $(\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0)$ is a norm on V, called the *spectral norm*.

The generic minimal polynomial

Let V be a simple positive Jordan triple of rank r. There exist polynomials $m_1, \dots, m_r$ on $V \times \overline{V}$, homogeneous of respective bidegrees $(1,1), \dots, (r,r)$, such that for each regular $x \in V$, the polynomial

$$m(T, x, y)$$
$$= T^r - m_1(x,y)T^{r-1} + \cdots + (-1)^r m_r(x,y)$$

satisfies

$$m(T, x, x) = \prod_{i=1}^{r} (T - \lambda_i^2),$$

where $x = \sum \lambda_j c_j$ is the spectral decomposition of x. The polynomial $m(T, x, y)$ is called the *generic minimal polynomial* of V (at (x, y)). The (inhomogeneous) polynomial $N : V \times \overline{V} \to \mathbb{C}$ defined by

$$N(x, y) = m(1, x, y)$$

is called the *generic norm*.

The spectral unit ball

If $(V, \{xyz\})$ is the triple system associated to a bounded symmetric domain Ω, the Bergman metric at 0 is related to D by

$$h_0(u,v) = \operatorname{tr} D(u,v).$$

Hence $(V, \{xyz\})$ is Hermitian positive. The *Bergman operator* gets its name from the following property:

$$h_z\left(B(z,z)u, v\right) = h_0(u,v) \quad (z \in \Omega;\ u, v \in V).$$

The bounded symmetric domain Ω is the unit ball of V for the spectral norm.

It is also characterized by the set of polynomial inequalities

$$\left.\frac{\partial^j}{\partial T^j} m(T,x,x)\right|_{T=1} > 0, \qquad 0 \leq j \leq r-1.$$

Volume forms on Jordan triples

Let V be a simple positive Hermitian Jordan triple, with generic norm

$$N(x,y) = 1 - m_1(x,y) + \cdots + (-1)^r m_r(x,y).$$

Consider on V the Hermitian scalar product

$$(x \mid y) = m_1(x,y)$$

and the associated $(1,1)$-form

$$\alpha = \frac{\mathrm{i}}{2\pi} \partial\bar{\partial} m_1(x,x).$$

The *flat volume form* on V is

$$\alpha^n$$

(with $n = \dim V$); the *projective volume form* is

$$\frac{\alpha^n}{N(x,x)^g},$$

where g is the genus of V.

Polar coordinates in Jordan triples

Let V be a simple positive Hermitian Jordan triple and Ω the associated bounded symmetric domain.

The frames of V form a real-analytic manifold $\mathcal{F}$ (the *Fürstenberg-Satake boundary* of Ω). The map Φ

$$\begin{aligned}\{\lambda_1 > \lambda_2 > \cdots > \lambda_q > 0\} \times \mathcal{F} \to V \\ \Big((\lambda_1, \ldots, \lambda_q), \big(c_1, \ldots, c_j\big)\Big) \mapsto \textstyle\sum_{j=1}^{q} \lambda_j c_j\end{aligned}$$

is a diffeomorphism onto the set V_{reg} of regular elements of V.

Let K be the identity component of the (linear Lie) group of automorphisms of V. Then K acts transitively on $\mathcal{F}$ and the map Φ is K-equivariant.

Volume forms in polar coordinates

Proposition. *Let V be a simple positive Jordan triple, with dimension n, rank r, numerical invariants a, b and genus $g = 2 + a(r-1) + b$. Then the pull-back of the flat volume form in generalized polar coordinates is*

$$\Phi^*\alpha^n = \prod_{j=1}^{r} \lambda_j^{2b+1} \prod_{j<k} \left(\lambda_j^2 - \lambda_k^2\right)^a \omega_r(\lambda) \wedge \Theta,$$

where $\omega_r(\lambda) = \mathrm{d}\,\lambda_1 \wedge \ldots \wedge \mathrm{d}\,\lambda_q$ and Θ is a K-invariant volume form on $\mathcal{F}$.

Schmid decomposition

Let V be a simple positive Hermitian Jordan triple of rank r. Let $\mathcal{P}(V)$ be the space of polynomials on V. For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, write $\mathbf{n} \geq 0$ iff $n_1 \geq \cdots \geq n_r \geq 0$.

Theorem. *(Schmid decomposition) The space $\mathcal{P}(V)$ decomposes into irreducible, pairwise unequivalent K-modules:* $\mathcal{P}(V) = \bigoplus_{\mathbf{n} \geq 0} \mathcal{P}_{\mathbf{n}}(V)$.

For $0 \leq j \leq r$, let $\langle j \rangle = (j, 0, \ldots, 0)$. Then $m_j(x, y)$ is a *reproducing kernel* for $\mathcal{P}_{\langle j \rangle}(V)$, endowed with the Hermitian structure induced on $\mathcal{P}(V)$ by the Hermitian scalar product m_1: for each $f \in \mathcal{P}_{\langle j \rangle}(V)$,

$$f(y) = \left(f \mid \left(m_j \right)_y \right),$$

where

$$\left(m_j \right)_y (x) = m_j (x, y).$$

Compactification of Jordan triples

Let V be a simple positive Hermitian Jordan triple of rank r. Let

$$\sigma_j : V \to \mathcal{P}_{\langle j\rangle}(V^*) \subset \bigodot_j V$$

be defined by

$$\sigma_j\,(x)\,(y^*) = m_j\,(x,y)\,,$$

where $y \mapsto y^*$ is the anti-isomorphism of V onto V^* induced by the Hermitian product m_1. Then

$$m_j\,(x,y) = (\sigma_j(x) \mid \sigma_j(y))\,.$$

In particular, $\sigma_0(x) = 1$ and $\sigma_1(x) = x$. Let

$$W = \bigoplus_{j=0}^{r} \mathcal{P}_{\langle j\rangle}(V^*).$$

The *canonical compactification map* of the Jordan triple V is

$$\begin{aligned} \sigma &: V \to P(W) \\ x &\mapsto [1 \oplus \sigma_1(x) \oplus \cdots \oplus \sigma_r(x)]\,. \end{aligned}$$

The closure $X = \overline{\sigma(V)}$ is an algebraic projective variety, called *canonical compactification* of V, or *compact dual* of the bounded symmetric domain Ω associated to V.

Projective volume form of Jordan triples

Let V be a simple positive Hermitian Jordan triple of rank r and genus g. Let

$$W = \bigoplus_{j=0}^{r} \mathcal{P}_{\langle j \rangle}(V^*)$$

be endowed with the Hermitian product induced by m_1. Denote by β the corresponding Fubini-Study form on $P(W)$. Then

$$\sigma^* \beta = \frac{\mathrm{i}}{2\pi} \partial \overline{\partial} \ln N(\mathrm{i}\, x, \mathrm{i}\, x).$$

Proposition.

$$\begin{aligned} \sigma^* \beta^n &= \left(\frac{\mathrm{i}}{2\pi} \partial \overline{\partial} \ln N(\mathrm{i}\, x, \mathrm{i}\, x) \right)^n \\ &= N(\mathrm{i}\, x, \mathrm{i}\, x)^{-g} \alpha^n. \end{aligned}$$

From a Jordan triple to its unit ball (1)

Let V be a simple positive Hermitian Jordan triple and Ω the associated bounded symmetric domain. Denote by B the Bergman operator

$$B(x,y) = \mathrm{id}_V - D(x,y) + Q(x)Q(y).$$

Define the real-analytic map

$$\psi : \Omega \to V$$

by

$$\psi(x) = B(x,x)^{-1/4}x.$$

Then ψ is a diffeomorphism and

$$\psi^{-1}(y) = B(\mathrm{i}\,y, \mathrm{i}\,y)^{-1/4}y.$$

Proposition. *Let N denote the generic norm of V, g the genus of V. Then*

$$\psi^*\left(N(\mathrm{i}\,x, \mathrm{i}\,x)^{s-g}\,\alpha^n\right) = N(x,x)^{-s}\alpha^n.$$

From a Jordan triple to its unit ball (2)

Proof. Transposed by Φ, the map ψ is $\Psi = \Phi^{-1} \circ \psi \circ \Phi$,

$$\begin{aligned} \Psi : \{1 > \lambda_1 > \lambda_2 > \cdots > \lambda_r > 0\} \times \mathcal{F} \\ \to \{\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0\} \times \mathcal{F}, \\ \left(\left(\lambda_j\right), \left(c_j\right)\right) \mapsto \left(\left(\frac{\lambda_j}{\left(1 - \lambda_j^2\right)^{1/2}}\right), \left(c_j\right)\right). \end{aligned}$$

If $\Phi^* x = \sum_{j=1}^r \lambda_j c_j$, we have

$$\Phi^* N\left(\mathrm{i}\, x, \mathrm{i}\, x\right) = \prod_{j=1}^r \left(1 + \lambda_j^2\right)$$

and

$$\begin{aligned} \Phi^* \widetilde{\beta}^n &= \Phi^* \frac{\alpha^n}{N\left(\mathrm{i}\, x, \mathrm{i}\, x\right)^g} \\ &= \frac{\prod_{j=1}^r \lambda_j^{2b+1} \prod_{j<k} \left(\lambda_j^2 - \lambda_k^2\right)^a}{\prod_{j=1}^r \left(1 + \lambda_j^2\right)^g} \omega_r\left(\lambda\right) \wedge \Theta. \end{aligned}$$

From a Jordan triple to its unit ball (3)

Let

$$\mu_j = \frac{\lambda_j}{\left(1 - \lambda_j^2\right)^{1/2}}.$$

Then

$$1 + \mu_j^2 = \frac{1}{1 - \lambda_j^2},$$

$$\frac{\mu_j}{\left(1 + \mu_j^2\right)^2} \,\mathrm{d}\,\mu_j = \lambda_j \,\mathrm{d}\,\lambda_j,$$

$$\mu_j^2 - \mu_k^2 = \frac{\lambda_j^2 - \lambda_k^2}{\left(1 - \lambda_j^2\right)\left(1 - \lambda_k^2\right)}.$$

Finally, using $g = 2 + a(r-1) + b$, we have

$$\frac{\prod_{j=1}^r \mu_j^{2b+1} \prod_{j<k} \left(\mu_j^2 - \mu_k^2\right)^a}{\prod_{j=1}^r \left(1 + \mu_j^2\right)^g} \omega_r(\mu)$$
$$= \frac{\prod_{j=1}^r \lambda_j^{2b+1}}{\prod_{j=1}^r \left(1 - \lambda_j^2\right)^b} \prod_{j<k} \frac{\left(\lambda_j^2 - \lambda_k^2\right)^a}{\left(1 - \lambda_j^2\right)^a \left(1 - \lambda_k^2\right)^a} \prod_{j=1}^r \left(1 - \lambda_j^2\right)^{g-2} \omega_r(\lambda)$$
$$= \prod_{j=1}^r \lambda_j^{2b+1} \prod_{j<k} \left(\lambda_j^2 - \lambda_k^2\right)^a \omega_r(\lambda),$$

which ends the proof.

Volume and degree

Let V be a simple positive Jordan triple and Ω the associated bounded symmetric domain.

Theorem. *Let $\sigma : V \to \mathbb{P}(W)$ be the canonical compactification map and $X = \overline{\sigma(V)}$ the compact dual of Ω. Then*

$$\deg X = \int_\Omega \alpha^n$$

($n = \dim V$).

Proof.

$$\deg X = \int_X \beta^n = \int_V \frac{\alpha^n}{N(\mathrm{i}\,x, \mathrm{i}\,x)^g} = \int_\Omega \alpha^n.$$

Exercise. Compute the projective volume of the domain Ω, embedded in X by σ

$$\int_{\sigma(\Omega)} \beta^n.$$

Solution. $\deg X/2^n$