ON THE TOPOLOGY OF CYCLES IN PSEUDOLINEAR PROGRAMS

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Abstract. By demand coming from the quasicrystal community we present here a construction realizing arbitrary oriented link in R^3 by the graph of 3d pseudolinear (or matroid) program. This gives a hint about possible "topological complexity" of rank 4 oriented matroids $\approx$ 3d pseudoplane arrangements $\approx$ 3d quasicrystal tilings .

0.1. **Oriented matroids, arrangements and quasicrystals.** There is a well known but still surprising connection between the rank $n+1$ generic oriented matroids, arrangements of topological hyperplanes and tilings of n-dimensional Euclidean space by parallelepipeds (which are Minkowski sums of n generic closed intervals = affine deformations of cubes). We will briefly outline this connection. For a complete general reference one can use [OM].

One can imagine a rank $n+1$ oriented matroid as an axiomatic ordered partial n-dimensional projective geometry, i.e. a collection of abstract k-dimensional planes $k = 0...n-1$ which are subject to all usual incidence and order axioms of ordered projective space, but we skip a requirement that any abstract k points in general position define a $(k-1)$ subspace. Everything is done so that any arrangement of projective hyperplanes in $P^n(\mathbb{R})$ produce a correct model of oriented matroid axiomatics. Such a voluntary "finite" shortening of the list of axioms of projective geometry gives an interesting result. It is a theorem that we can not restrict projective geometry axiomatics so that in restricted theory the arrangements of projective hyperplanes will be the only models. We will always get some virtual, nonrealizable models. It turns out that for the case of "oriented matroids restriction" such a virtual models has a very nice description, which is subject to rather topology then geometry. This description is following. Let us pass from $P^n(\mathbb{R})$ to its canonical two-fold covering by the standard sphere S^n. Then an arrangement of projective hyperplanes in $P^n(\mathbb{R})$ is covered by the arrangement of $(n-1)$-dimensional "equatorial" spheres embedded into S^n. So an arrangement of equatorial $(n-1)$-spheres in S^n is the same thing as an arrangement of projective hyperplanes in $P^n(\mathbb{R})$. Call by *arrangement of pseudospheres* a collection of arbitrary topologically embedded $(n-1)$-spheres $\mathcal{H} = \{H_0, ..., H_m\}$ such that two axioms holds:

- **1:** The common intersection of spheres from an arbitrary subset of $\mathcal{H}$ is a sphere of some dimension.
- **2:** For any pair $H_i, H_j, i \neq j$ there is a homeomorphism of S^n sending pair (H_i, H_j) into a pair of equatorial spheres.

It is the content of Folkman-Lawrence theorem that all oriented matroids can be modeled by arrangements of pseudospheres and back, any arrangement of pseudospheres is a model of some oriented matroid, particularly the chamber complexes on S^n generated by arrangements forms nice, tame class of regular cell decompositions of sphere with a finite axiomatic description. Starting from that point those who prefer the language of classical combinatorics can play with matroids axiomatically,

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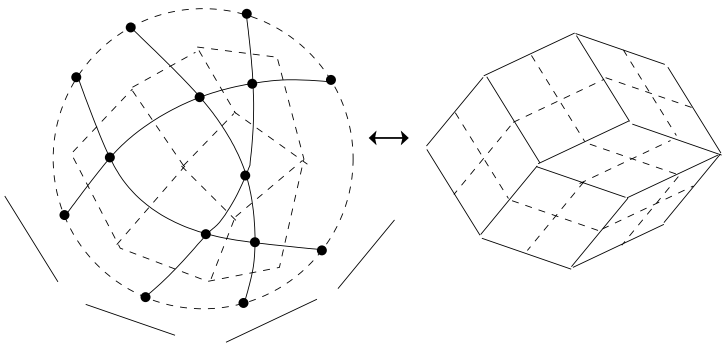
all others can forget axiomatics and draw pictures. The arrangements of pseudospheres have affine and projective models. For the arrangement $\mathcal{H}$ one can find an isotopic model consisting of centrally-symmetric pseudospheres, which can be identified with projective pseudohyperplanes – embeddings of $P^{n-1}(\mathbb{R})$ into $P^n(\mathbb{R})$, isotopic to linear embeddings. For "affine" model one can select a pseudosphere H_0 of $\mathcal{H}$ as "infinity" , choose H_0^+ to be one of the two closed semispheres in S^n with boundary H_0 and form the arrangement of closed $(n-1)$-dimensional topological balls $\{D_1 = H_0^+ \cap H_1, ..., D_m = H_0^+ \cap H_m\}$. We can send H_0^+ by homeomorphism to the standard unit ball D^n and got an object which one can call an "arrangement of affine pseudohyperplanes": a collection $\mathcal{A} = \{(D_1), ...(D_m)\}$ of embeddings of standard disk (D^{n-1}) into standard disk (D^n) such that any intersection of any subcollection is a disk of some dimension and any pair of embeddings is isotopic to the standard coordinate pair.

Suppose we have an arrangement $\mathcal{L} = \{L_0, ..., L_m\}$ of linear n-dimensional subspaces in $\mathbb{R}^{n+1}$. Let $\bar{\mathcal{L}} = \{L_0 \cap S^d, ..., L_k \cap S^d\}$ be the induced arrangement of equatorial spheres on the unit sphere S^d. Let $C(\bar{\mathcal{L}})$ be the set of all chambers in the decomposition of S^d defined by $\bar{\mathcal{L}}$. The set $C(\bar{\mathcal{L}})$ has a natural poset structure by the adjacency of chambers. The nice fact is that the dual poset $C(\bar{\mathcal{L}})^{op}$ has a canonical presentation as a face poset of convex polytope. One can build this polytope as follows. For every $L_i, i = 0...k$ pick an orthogonal closed interval e_i. Than convex polytope (called *zonotope*)

$$P(\mathcal{L}) = \text{Minkovski} \sum_{i=0}^{k} \mathrm{e_i}$$

has $C(\bar{\mathcal{L}})^{op}$ as a face poset. If the arrangement $\mathcal{L}$ is generic (i.e. all its hyperplanes are in general position), then all faces of $P(\mathcal{L})$ are parallelepipeds.

Now we return to the arrangement of affine pseudohyperplanes $\mathcal{A}$. Suppose that the boundary $(n-1)$ arrangement of pseudospheres $\partial\mathcal{A} = \{D_1 \cap S^{n-1}, ..., D_k \cap S^{n-1}\}$ is linearly representable, i.e. it is isotopic to the arrangement of equatorial spheres in S^{n-1} with normal intervals $f_1...f_k$. Considerthe poset of all chambers $C(\mathcal{A})$ of the chamber decomposition of open disk $D^n \setminus S^{n-1}$ defined by $\mathcal{A}$. Poset $C(\mathcal{A})$ contains poset $C(\partial\mathcal{A})$ as a subposet. We know that $C(\partial\mathcal{A})^{op}$ is representable by the face poset of zonotope $P(\partial\mathcal{A}) = \sum_{i=1}^{k} f_\mathrm{i}$. The simple, but surprising theorem is that $C(\mathcal{A})^{op} \supset C(\partial\mathcal{A})^{op}$ is representable by the face poset of a tilling of $P(\partial\mathcal{A})$ using set of tiles $\{\sum_{i\in I} f_i | I \subset 1...k\}$ and this correspondence (extensions of $\partial\mathcal{A}$ up to an arrangement of affine pseudohyperplanes) <–> (tilings of $P(\partial\mathcal{A})$) is one-to-one.

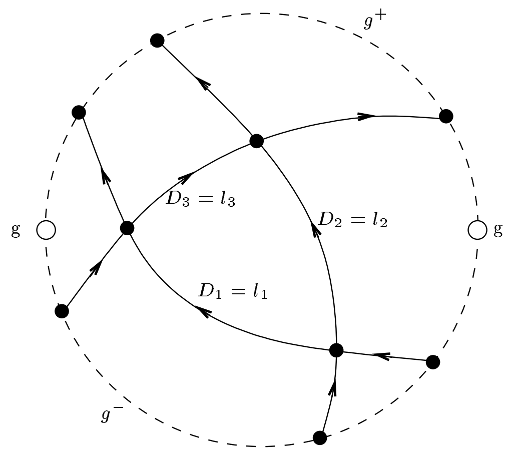


When arrangement $\mathcal{A}$ is generic and $\partial\mathcal{A}$ is linear representable then we obtain that $C(\mathcal{A})^{op}$ is representable by a tilling of generic zonotope by parallelepipeds.

The small-dimensional tilings of zonotopes by parallelepipeds have got a special value as local models for physical quasicrystals as in [W]. (Penrose tilings are the most famous examples of 2-dimensional quasicrystals). Here we are studying a special nonlinear effect which shows up in arrangements (and hence in quasicrystals) starting from dimension 3 – appearance of embedded nontrivial links, which generally can affect the dynamics of quasicrystals.

0.2. **Links in pseudolinear programs.** We will follow the terminology of Oriented Matroids book [OM], where chapter 10 is devoted to (pseudo) linear programming.

The constructions are of topological nature and so, our main object is a generic arrangement of affine pseudohyperplanes. The pseudolinear $n-$program for us is a data $\mathcal{P} = \langle M, g\rangle$, where M is an arrangement of m $(n-1)$-dimensional affine pseudohyperplanes $(H_1...H_m)$ on the unit n-dimensional disk D^n. The object g is supposed to be a $n-2$ dimensional pseudosphere, $g \subset S^{n-1} = D^{n-1}$, which is a generic *extension* of ∂M, i.e. $(g, \partial M)$ is a correct generic $(n-1)$-dimensional arrangement of $m+1$ $(n-2)$-pseudospheres. We suppose also that on the spheres S^{n-1}, g some orientations are fixed. So, two closed (pseudo)semispheres g^+, g^-in the sphere S^{n-1} are defined.



Now we define an oriented topological graph $\Gamma(\mathcal{P})$, together with a canonical embedding $|\Gamma(\mathcal{P})| \hookrightarrow D^n$. For two natural numbers $a \geq b$ denote by $\left[\begin{array}{c} a \\ b \end{array}\right]$ the set of multi-indices $\{(i_1, .., i_b)|1 \leq i_1 < i_2 < .. < i_b \leq a\}$. Consider the set of 1-dimensional discs

$$L = \{l_I | I \in \left[\begin{array}{c} m \\ n-1 \end{array}\right], l_J = \bigcap_{i \in I} D_i\}$$

Every $l \in L$ intersects S^{n-1} in two opposite points $p_l^+ = g^+ \cap l$ and $p_l^- = g^- \cap l$. We can orient interval l as an arc with source p_l^- and target p_l^+.

Now consider the set of points in D^n:

$$P = \{p_J | J \in \left[\begin{array}{c} m \\ n \end{array}\right], p_J = \bigcap_{i \in J} D_i\} \cup \{p_l^-, p_l^+\}_{l \in L}$$

The points from P are siting on the oriented topological intervals from L and break intervals into the set A of smaller arcs with the induced orientation. So we obtain an oriented topological graph $\Gamma(\mathcal{P})$ in H^+ associated to the topological program $\mathcal{P}$.

Now we start to estimate the topological complexity of $\Gamma(\mathcal{P})$ when dimension of program $n = 3$. First define an oriented link in 3-dimensional disk D^3. Take a free sum $\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup ... \sqcup \overrightarrow{S}^1_q$ of oriented one-dimensional topological circles. Then a topological link in D^3 is a topological embedding

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup ... \sqcup \overrightarrow{S}^1_q \xrightarrow{F} D^3$$

Next, we note that the graph $\Gamma(\mathcal{P}) \hookrightarrow D^3$ of 3-dimensional program can have cycles as a oriented graph. The cycles exist together with embedding of correspondent topological circles into D^3. So if we put by $\Omega(\mathcal{P})$ the set of cycles in $\Gamma(\mathcal{P})$ then a map is defined

$$\bigsqcup_{i\in\Omega(\mathcal{P})} \overrightarrow{S}^1_i \xrightarrow{\omega} D^3$$

Usually the map ω is very far from being an embedding, and, probably, always is very singular. Our claim is:

Theorem 0.1. *For any link*

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup ... \sqcup \overrightarrow{S}^1_q \xrightarrow{F} D^3$$

there exists a 3-dimensional program $\mathcal{Q}_F$ and an embedding e of the set of circles $\{1..q\}$ into the set of cycles $\Omega(\mathcal{Q}_F)$ such that the composite map

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup ... \sqcup \overrightarrow{S}^1_q \xrightarrow{id_{e(1)}\sqcup ... \sqcup id_{e(k)}} \bigsqcup_{i\in\Omega(\mathcal{Q})} \overrightarrow{S}^1_i \xrightarrow{\omega(\mathcal{Q}_F)} D^3$$

is isotopic to F. The number of pseudospheres in P is a linear function from the complexity of presentation of link F by piecewise-linear 2-dimensional diagram.

We will describe a construction of $\mathcal{Q}_{\mathcal{F}}$ in the following three paragraphs.

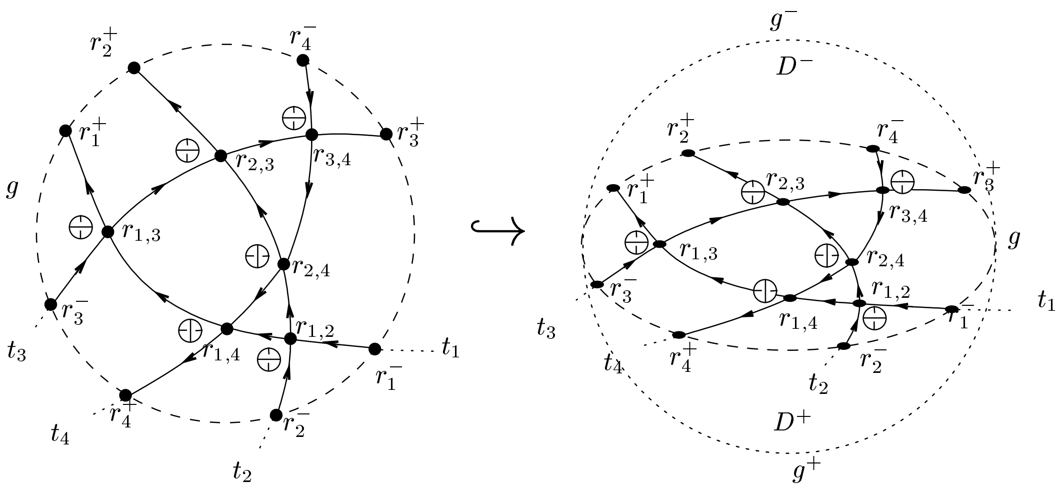
0.3. **PsL-diagrams.** Here we start our constructions. The main point of the construction is a realization of *pseudolines linking diagram* (*PsL-diagram*) by a 3-program.

Consider some generic affine arrangement of *oriented* pseudolines $\mathcal{A}$, as in [OM, Chapter 6]. So $\mathcal{A}$ is a collection of *oriented* topological intervals $(t_1, ..., t_k)$ embedded into the unit 2-dimensional disk D^2. Every interval t_i transversally intersects the boundary unit sphere g in its source and target points r^-_j and r^+_j. Every two intervals $t_i, t_j, i < j$ a intersects transversally at a single point $r_{i,j}$. Let us introduce two symbols: "bridge"$=$ ⦵ and "pass"$=$ ⦶ .

A PsL-diagram is pair $\mathcal{D} = \langle \mathcal{A}, \xi\rangle$, where $\mathcal{A}$ is an arrangement of k numbered oriented pseudolines and ξ is a map $\begin{bmatrix} k \\ 2 \end{bmatrix} \xrightarrow{\xi} \{\text{brige}, \text{pass}\}$. The map ξ we imagine as an assignment of the symbol $\xi(i, j)$ to the point $r_{i,j}$. (See [M] for the developed science on the links of (projective) lines and the diagrams of such links.) The points $\{r_{1,j}, .., r_{j-1,j}, r_{j,j+1}, .., r_{j,k}, r^+_j, r^-_j\}$ split the oriented interval t_j into a set of k oriented arcs B_j. Thus we get a topological oriented graph $\Phi(\mathcal{D}) = (B, R)$ in D^2, where the set of arcs is $B = B_1 \cup ... \cup B_k$ and the set of vertices is

$$R = \{r_{i,j}\}_{(i,j)\in \begin{bmatrix} k \\ 2 \end{bmatrix}} \cup \{r^+_i\}_{i\in 1...k} \cup \{r^-_i\}_{i\in 1...k}$$

Here is a picture:

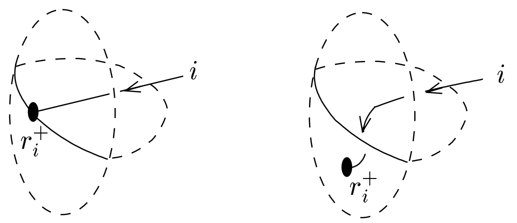


Now we will use PsL-diagram $\mathcal{D}$ to build an oriented topological graph $\Xi(\mathcal{D}) \hookrightarrow D^3$.

First we will embed by horizontal coordinate way a unit disk D^2, carrying the diagram $\mathcal{D}$, into a 3-dimensional unit ball D^3. This induces an embedding of graph $\Phi(\mathcal{P}) \hookrightarrow D^3$ as shown. Now we orient the boundary circle g in the embedding and got a positive and negative closed semispheres g^+ and g^-, $g^+ \cap g^- = g$ in the boundary 2-sphere of ball D^3. Than to the constructed embedding of $\Phi(\mathcal{P})$ we are applying following rules of local surgery:

1a: Local transform around a point of type r_i^+.

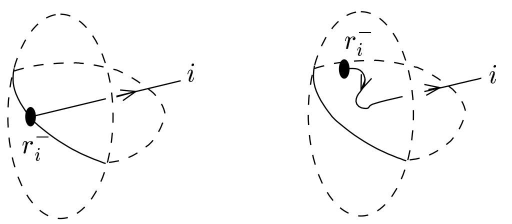
We pick a small neighborhood of point r_i^+ in D^3 and change in it slightly the embedding, mowing r_i^+ to interior of positive semisphere g^+, as shown:



the remaining part of graph leaves unchanged.

2a: Local transform around a point of type r_i^-

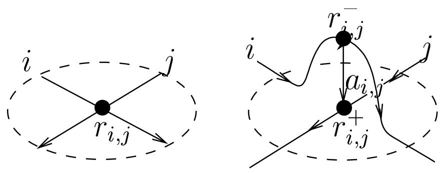
All the same, but move sends r_j^- to interiour of negative semisphere g^-



3a: Local transform around a point of type $r_{i,j}$, when $\xi(i,j) =$"bridge"

We pick a small neighborhood of point $r_{i,j}$ in D^3. We replace the single point r_{ij} by two points $r_{i,j}^+$ and $r_{i,j}^-$, moving slightly $r_{i,j}^-$ up into open negative semiball. Than we introduce a new small oriented straight ark, $a_{i,j}$ with source $r_{i,j}^-$ and target $r_{i,j}^-$. Line t_i we deform to make a "bridge" over t_j with "base" $a_{i,j}$, as shown:

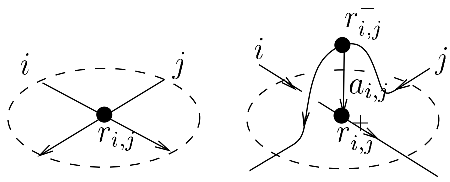
$\xi(i,j) = \oplus$



4a: Local transform around a point of type $r_{i,j}$, when $\xi(i,j) =$"pass".

All the same with i, j mutated. This will create a "pass" of t_i under t_j

$\xi(i,j) = \oplus$



We define a new graph $\Xi(\mathcal{D}) \hookrightarrow D^3$ as a result of applying rules ***1a, 2a, 3a, 4a*** *to all vertices of $\Phi(\mathcal{D}) \hookrightarrow D^3$.*

What remains is to do two things: to embed arbitrary link in graph of type $\Xi()$ and to embed the graph $\Xi()$ into the graph $\Gamma()$ of some 3-program .

0.4. **Diagram of link $\longmapsto$ Ξ(PsL-diagram).** Let we have a link

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \ldots \sqcup \overrightarrow{S}^1_q \xrightarrow{F} D^3$$

For a PsL-diagram $\mathcal{D}$ denote by $\Omega(\mathcal{D})$ the set of cycles in the graph $\Xi(\mathcal{D})$ and by $\omega_{\mathcal{D}}$ we denote naturally defined by embedding $\Xi(\mathcal{D}) \hookrightarrow D^3$ *map*

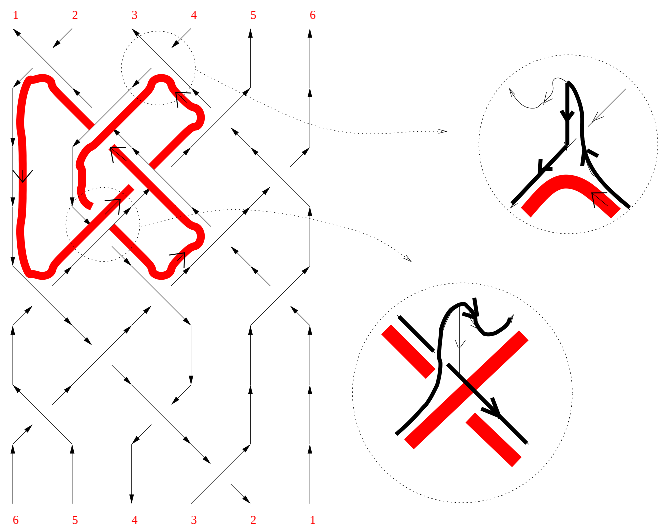
$$\bigsqcup_{i\in\Omega(\mathcal{D})} \overrightarrow{S}^1_i \xrightarrow{\omega_{\mathcal{D}}} D^3$$

Proposition 0.2. *For any piecewise-linear diagram U of the link F on the plane, such that U has q linear intervals, there exist a PsL-diagram $\mathcal{D}(U)$ having q pseudo-lines* and an embedding e of the set of circles $\{1..q\}$ into the set of cycles $\Omega(\mathcal{D}(U))$ such that composite map

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \ldots \sqcup \overrightarrow{S}^1_q \xrightarrow{id_{e(1)}\sqcup\ldots\sqcup id_{e(k)}} \bigsqcup_{i\in\Omega(\mathcal{D}(U))} \overrightarrow{S}^1_i \xrightarrow{\omega_{\mathcal{D}(U))}} \mathcal{D}^3$$

is isotopic to F.

We will not describe the construction here in detail – it is quite obvious: pick up a pl-link-diagram, generate lines by the linear intervals of diagram, orient them by orientation of the intervals induced from the orientations of components of link and than distribute path-bridge information in a way, illustrated here on example of trefoil knot:



We are presenting here the background arrangement of pseudolines using "wiring diagram" as in [OM, page 260].

0.5. **PsL-diagram $\longmapsto$ 3-program.** Now we will embed $\Xi(\mathcal{D})$ into the graph $\Gamma(\mathcal{P}(\mathcal{D}))$ of some affine 3-program $\mathcal{P}(\mathcal{D})$.

Proposition 0.3. *For any PsL-diagram $\mathcal{D}$ having k pseudolines there exist a 3-program $\mathcal{P}(\mathcal{D})$ with $2k$ pseudoplanes such that there is canonical embedding $\Xi(\mathcal{D}) \hookrightarrow \Gamma(\mathcal{P}(\mathcal{D}))$.*

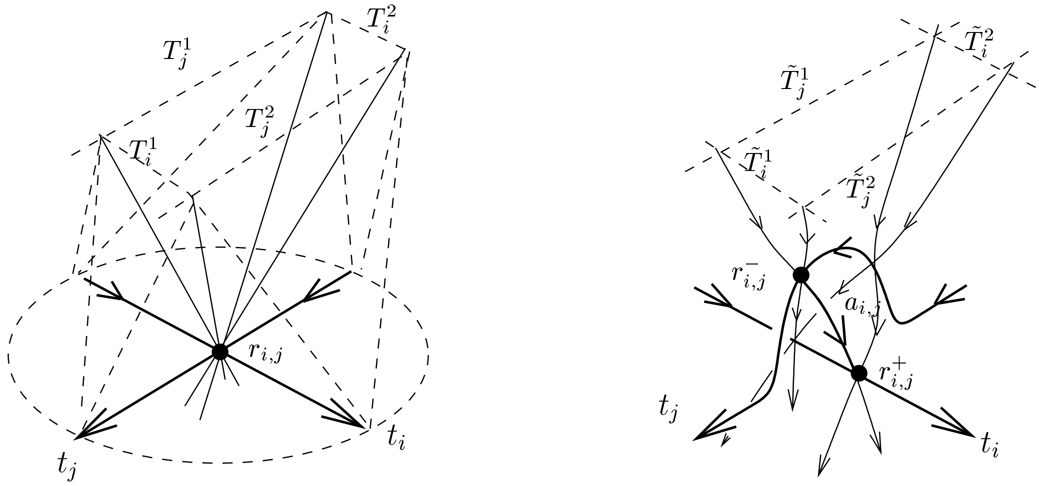
Let we have a PsL-diagram $\mathcal{D}$. We suppose that a unit 2-disk D^2 carrying $\mathcal{D}$ is already embedded in D^3 by horizontal coordinate way. Suppose that all notations of §0.2 are valid. Suppose that the boundary circle $g \hookrightarrow S^2$ of $\mathcal{D}$ is oriented and this will be an "object function" of our future 3-program. Now for any pseudoline t_i of $\mathcal{D}$ we choose two 2-dimensional pseudoplanes $T_i^1$and T_i^2 in D^3 such that (1): $T_i^1 \cap T_i^2 = t_i$ and (2): $T_i^\alpha \cap D^2 = t_i$ for $\alpha = 1, 2$ and $(T_1^1, T_2^2, ..., T_k^1, T_k^2)$ is a correct arrangement. It is always possible to do so. Our program $\mathcal{P}(\mathcal{D})$ will consist from "object function " g and an arrangement of $2k$ pseudoplanes $(\tilde{T}_1^1, \tilde{T}_2^2, ..., \tilde{T}_k^1, \tilde{T}_k^2)$, where $\tilde{T}_i^\alpha$ is a result of small deformation of T_i^α by following local rules:

3b,4b: Working around the interior vertices of type $r_{i,j}$, when $\xi(i, j) =$ "bridge" or "pass". Consider the case of "pass" .

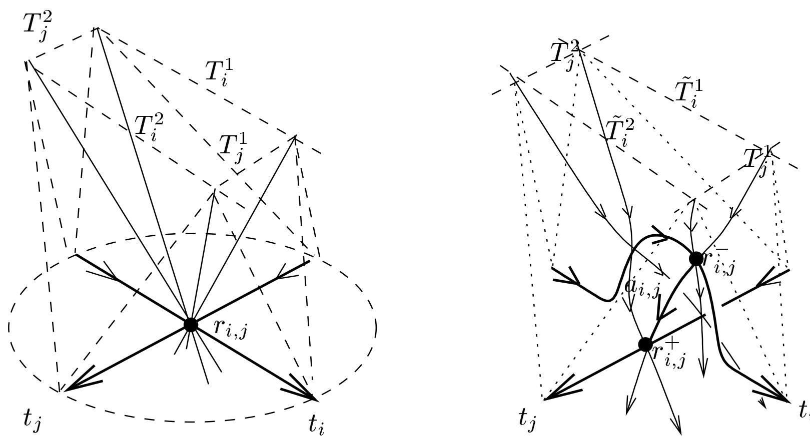
Pick up a small neighborhood V of $r_{i,j}$. We have:

$$r_{i,j} = t_i \cap t_j = (T_i^1 \cap T_i^2) \cap (T_i^1 \cap T_j^2)$$

By continuous isotopy $f : [0, 1] \times D^3 \to D^3$ such that $f|_{[0,1]\times(D^3\setminus V)} \equiv id$ we can slightly shift up a pair of pseudoplanes $\{T_j^1, T_j^2\}$ into $\{\tilde{T}_j^1, \tilde{T}_j^2\}$ which is in general position with $\{T_i^1, T_i^1\}$ and such that $(\tilde{T}_j^1, \tilde{T}_j^2, T_i^1, T_i^2)$ is a correct arrangement. The condition "not to leave the class of correct arrangements" pose a constrains on a continues deformation of pseudoplanes of the arrangement. The important thing that these constrains are topologically local. Hence, together with all other pseudoplanes the deformed ones forms a correct arrangement. The point r_{ij} after such a surgery blows up to simplex. We have some freedom in choice of points $r_{i,j}^+$ $r_{i,j}^-$ and ark $a_{i,j}$ from the elements of this simplex. Hope that all is clear from the picture:



We have a symmetric picture for transform in the case $\xi(i,j)$=“brige”:



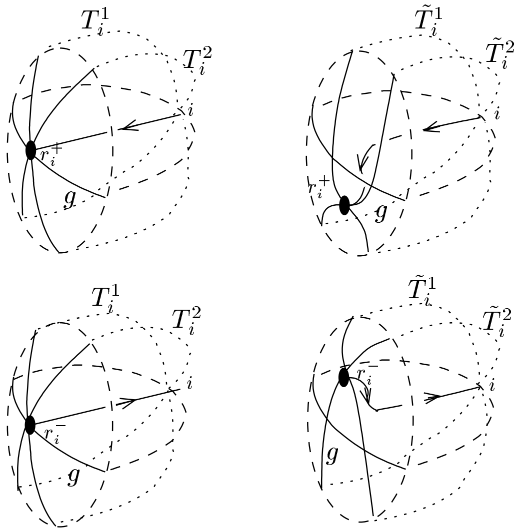
1b,2b: Working around the endpoints r_i^+,r_i^-.

Oriented pseudoline t_i of PsL-diagram $\mathcal{D}$ has two endpoints marked r_i^--“source” and r_i^+-“target”. We has presented t_i as an intersection of the two pseudoplanes T_i^1 and T_i^2. Also the point r_i^-,r_i^+ belongs to the boundary circle g. Let us pick a small neighborhood V of r_i^- in D^3. The arrangement of pseudocircles $T_1^1 \cap S^2, T_1^2 \cap S^2, ..., T_k^1 \cap S^2, T_K^2 \cap S^2$ is a correct centrally-symmetric arrangement. So, $-V$ is a neighborhood of point r_i^+. By continuous centrally-symmetric isotopy

$$f : [0,1] \times D^3 \to D^3$$

such that $f|_{[0,1]\times(D^3\setminus\{V\cup -V)} \equiv id$ we can slightly deform the pair of pseudoplanes $\{T_i^1, T_i^2\}$ into $\{\tilde{T}_i^1, \tilde{T}_i^2\}$ such that r_i^- moves to interior of g^-, r_i^+ moves to interior of g^+ and $(\tilde{T}_i^1, \tilde{T}_i^2)$ together with the remaining pseudoplanes and g forms a correct

3-program. Here are the drawings.



For i=**1**,**2**,**3**,**4** each rule i**b** induces the rule i**a** of cooking the graph $\Xi(\mathcal{D})$ from the PsL-diagram $\mathcal{D}$ together with the embedding of this graph into the graph Γ of the resulting 3-program $\mathcal{P}(\mathcal{D})$. So, we are done.

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