# Compliments to Bad Spaces

Oleg Viro

March 13, 2007

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not differentiable, but differential,

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   In other words, $f \in \mathcal{C}^r(X)$ if for each $a \in X$ there exist $g, h \in \mathcal{C}^r(X)$ such that $h(a) > 0$ and $f(x) = g(x)$ for each $x$ with $h(x) > 0$.

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4. *Topological space.* A topological space $X$ with all continuous functions $X \to \mathbb{R}$.

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Isomorphisms of the category are called *$C^r$-diffeomorphims*.

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do not necessarily constitute a differential structure on $A$.

For example, if $X = \mathbb{R}$ and $A = \mathbb{R}_{>0} = \{x \mid x > 0\}$,
the function $A \to \mathbb{R} : x \mapsto \frac{1}{x}$ is not a restriction of any
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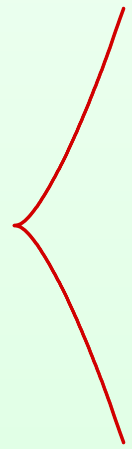

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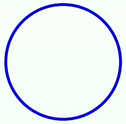
If so, how does the the image look like? Like this:  ?

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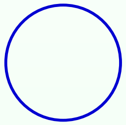
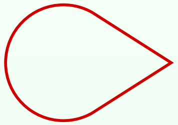
If so, how does the the image look like? Like this:  ?

No, like this:  !

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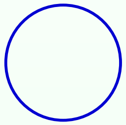
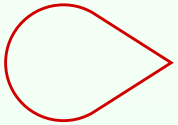
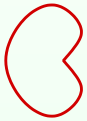
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  !

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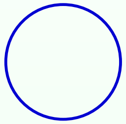
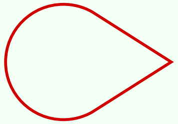
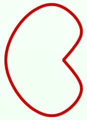
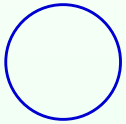
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

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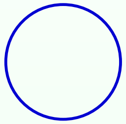
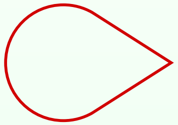
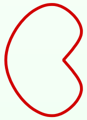
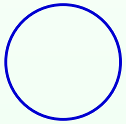
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

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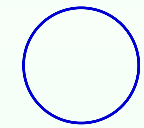
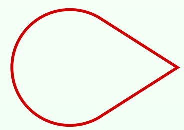
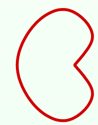
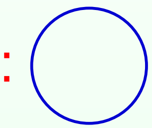
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

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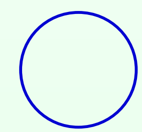
Then we get really a space diffeomorphic to  .

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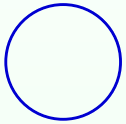
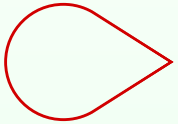
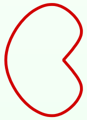
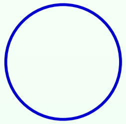
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

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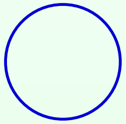
Then we get really a space diffeomorphic to  .

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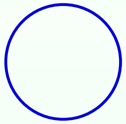
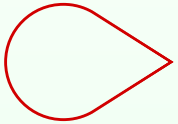
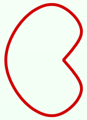
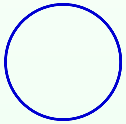
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! **But not this:**  !

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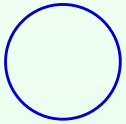
Then we get really a space diffeomorphic to  .

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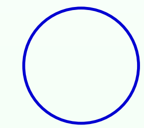
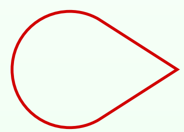
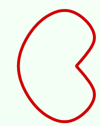
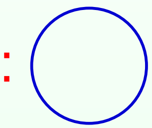
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

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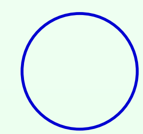
Then we get really a space diffeomorphic to  .

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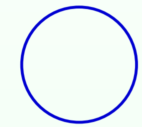
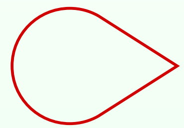
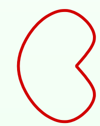
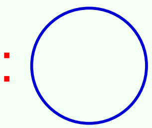
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

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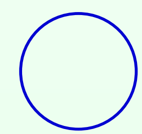
Then we get really a space diffeomorphic to  .

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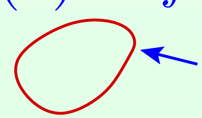
The resulting space:  .

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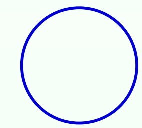
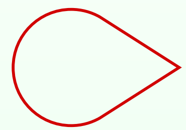
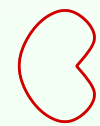
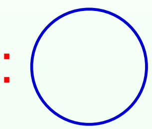
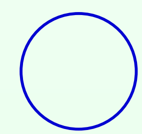
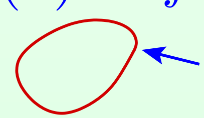
If so, how does the the image look like? Like this:  ?

No, like this:  ! Or that:  ! But not this:  !

2. What if we take $[0, 1.5]$ and identify each $x \in [0, 0.5]$ with $x + 1$ ?

Then we get really a space diffeomorphic to  .

New factorization:
Identifying end points of $[0, 1]$, identify also tangent vectors!
That is consider functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1)$ and $f'(0) = f'(1)$.
The resulting space:  . Smooth, but with jump of curvature.

# Tangent vectors and dimensions



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The quotient space $D^2/\partial D^2$ of disk $D^2$ is homeomorphic to sphere $S^2$. What is $\dim_{\partial D^2/\partial D^2}(D^2/\partial D^2)$? Infinity!

# Metric spaces

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$$\frac{|f(x) - f(p) - \sum a_i(\mathrm{dist}(q_i, x) - \mathrm{dist}(q_i, p))|}{\mathrm{dist}(x, p)} \to 0$$

as $x \to p$ .

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as $x \to p$. Is this definition good? At least, it recovers the smooth structure of a Riemannian manifold.

# Finite Topological Spaces

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At early days of topology, they were the main objects of the *Combinatorial Topology*.

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Especially if the partition was a triangulation.

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The star $St(\sigma)$ of a face $\sigma$ is the union of all faces $\Sigma$ such that $\partial\Sigma \supset \sigma$.

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$P$ can be recovered from $Q$.

# Homotopy

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**Corollary.** *Any compact polyhedron is weak homotopy equivalent to a finite topological space.*

# Digital plane and Jordan Theorem

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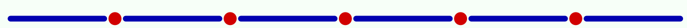

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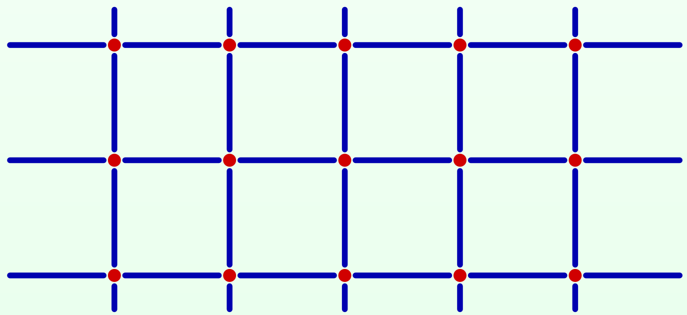

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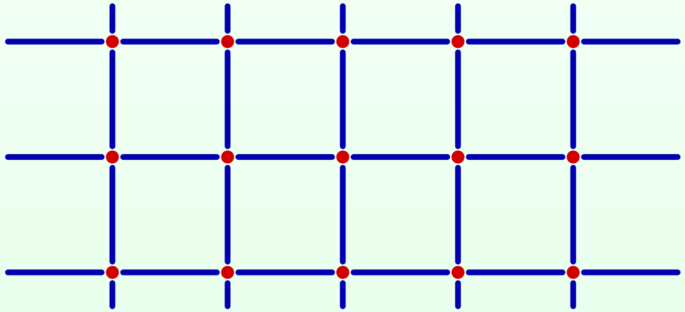

*Digital circle* of length $d$ is the quotient space of the circle $S^1 \subset \mathbb{C}$ by the partition formed by complex roots of unity of degree $d$ and open arcs connecting the roots next to each other.

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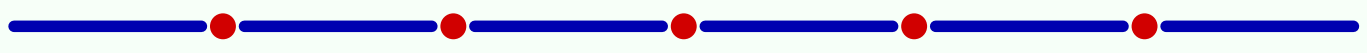

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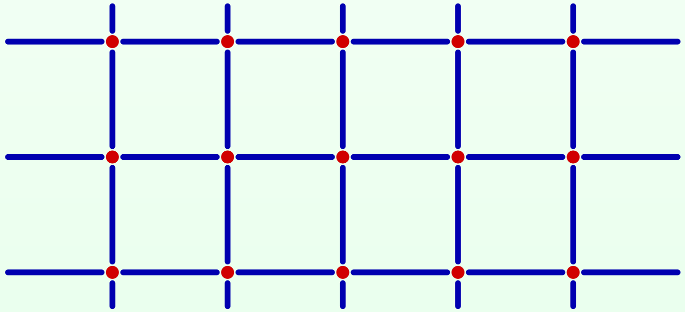
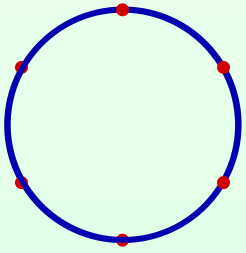


*Digital circle*



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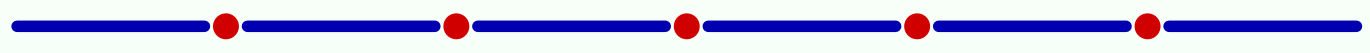

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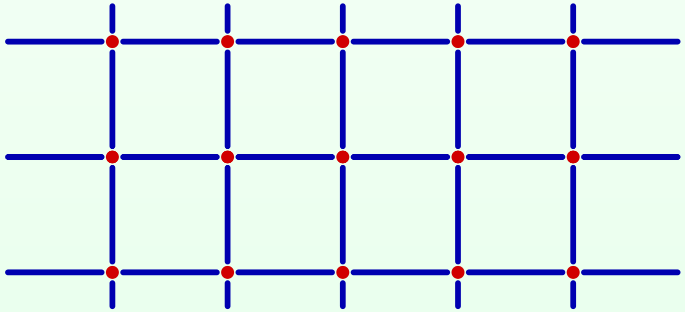

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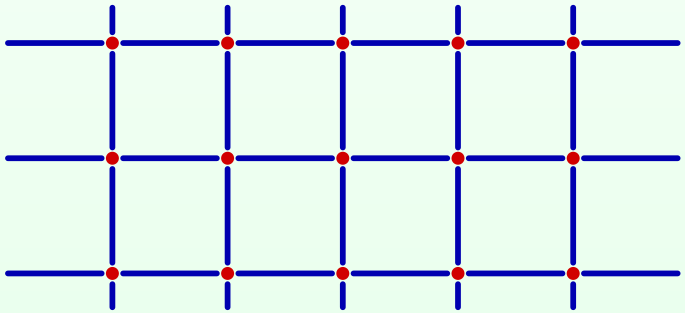

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Not finite, but *locally finite*.

# Arbitrary finite space



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# Baricentric subdivision

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How far is a poset topology from the face space of a polyhedron? Not really far, just one step construction.

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**Theorem.** *Any finite topological space is weak homotopy equivalent to a compact polyhedron.*