On basic notions of the tropical geometry

Oleg Viro

January 14, 2010

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and have amoebas which are tropical varieties.

- The goal

Tropical Geometry
- Tropical algebra
- Tropical polynomials
- Bridges

Multi-valued algebra

Dequantizataion

Equations and varieties

Tropical Geometry

Tropical algebra

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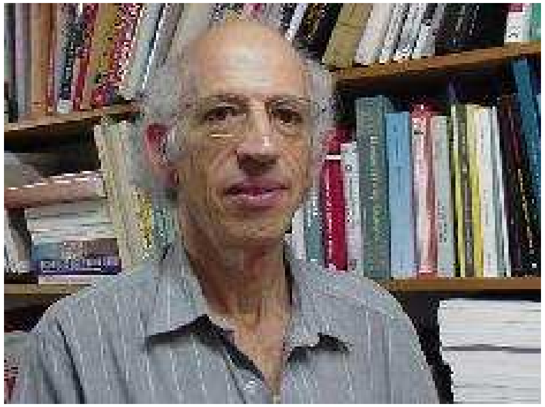

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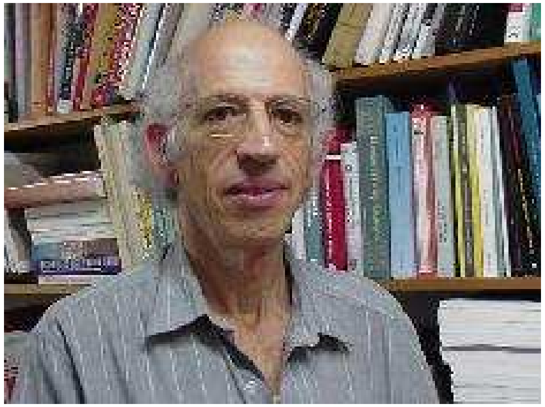

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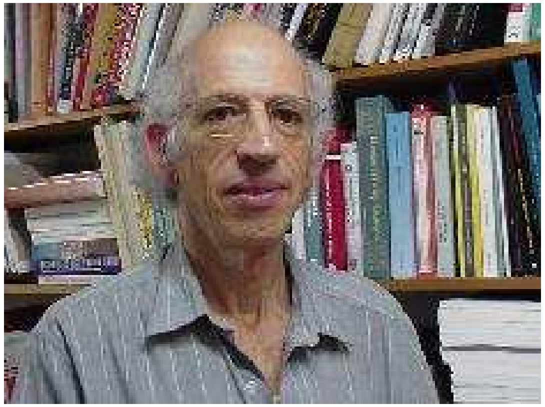

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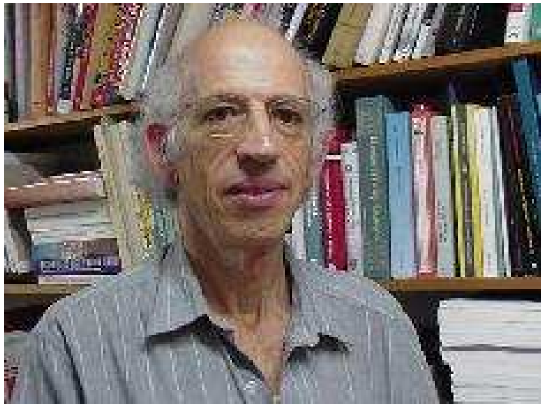

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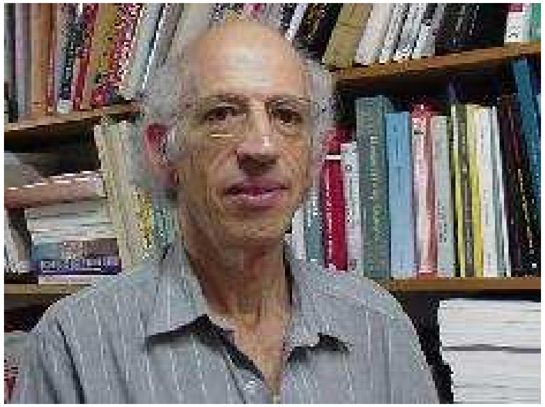

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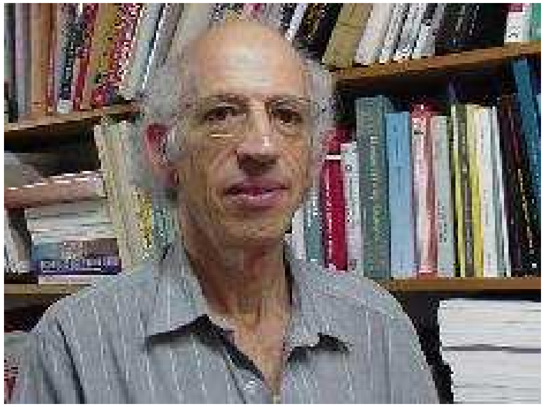

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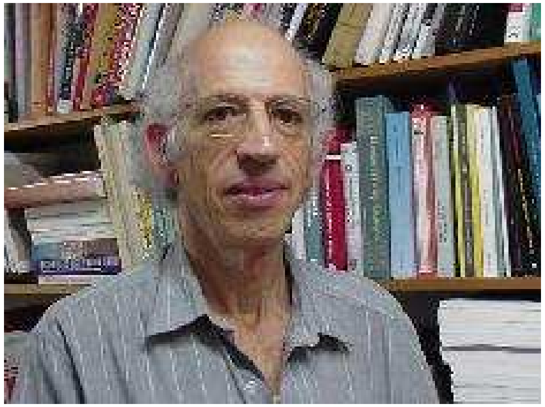

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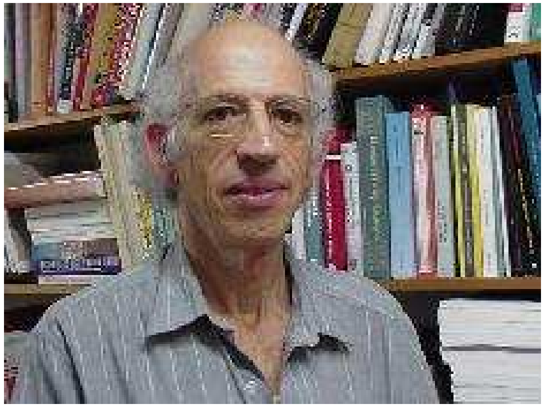

$\mathbb{R}_{\max,+}$ is a semi-ring. Everything is as in a ring, but no subtraction, no 0. Adjoin $-\infty$ as 0, denote by $\mathbb{T}$. This is a semi-field.

Still, no subtraction.

Tropical polynomials

A polynomial over $\mathbb{T}$ is a convex PL-function with integral slopes.

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Indeed, a monomial $ax_1^{k_1}\dots x_n^{k_n}$ is $a + k_1x_1 + \cdots + k_nx_n$.

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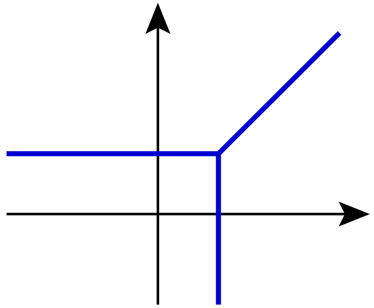

Bridges

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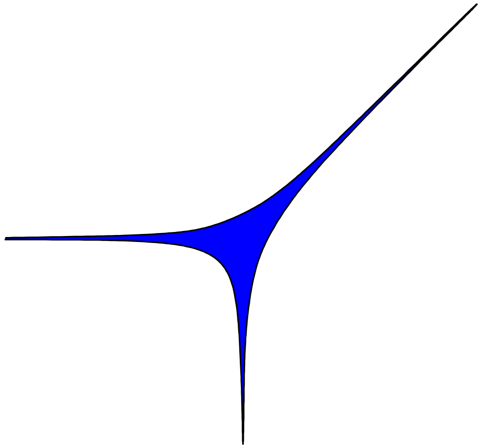

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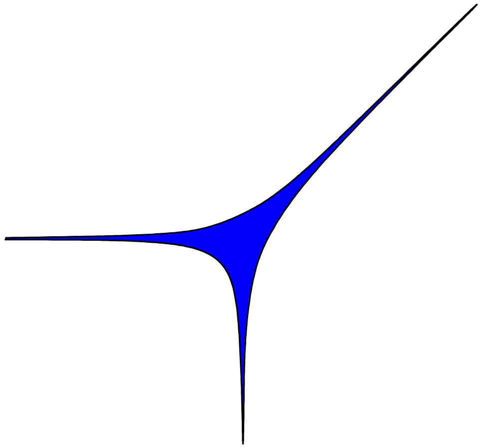

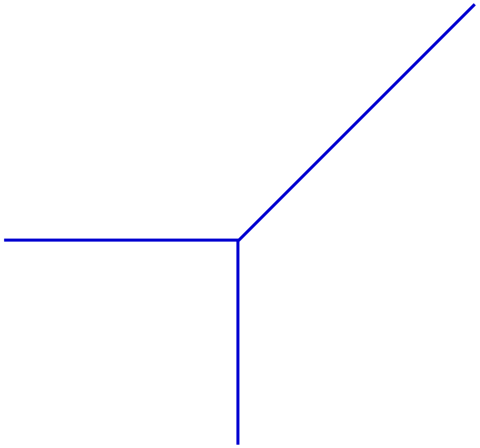

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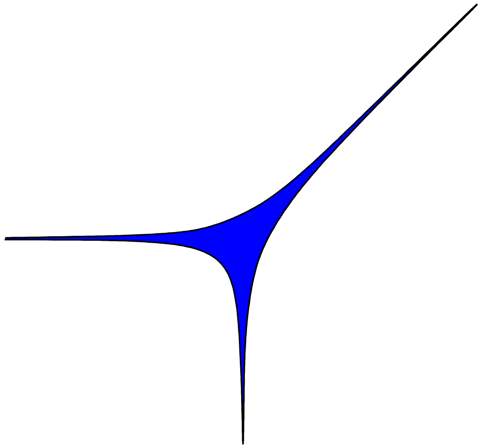

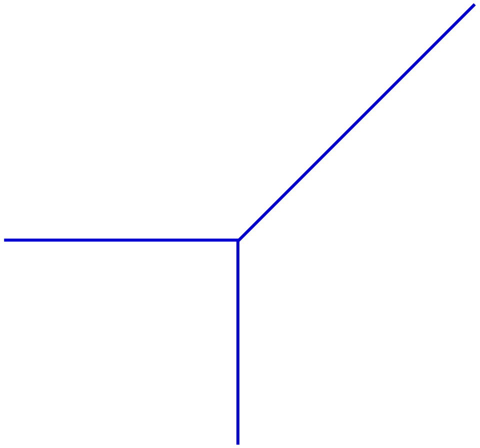

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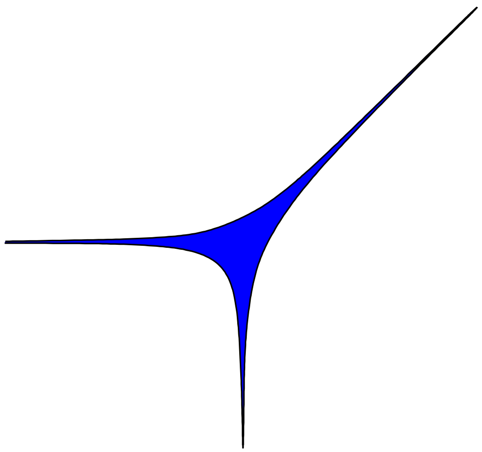

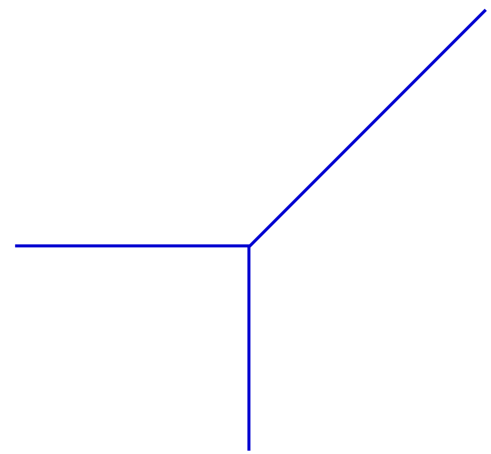

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Another bridge: Algebraic Geometry over the field of Piuseux series, and non-Archimedean amoebas. Instead of $\log|z|$, take $e^{-val(z)}$.

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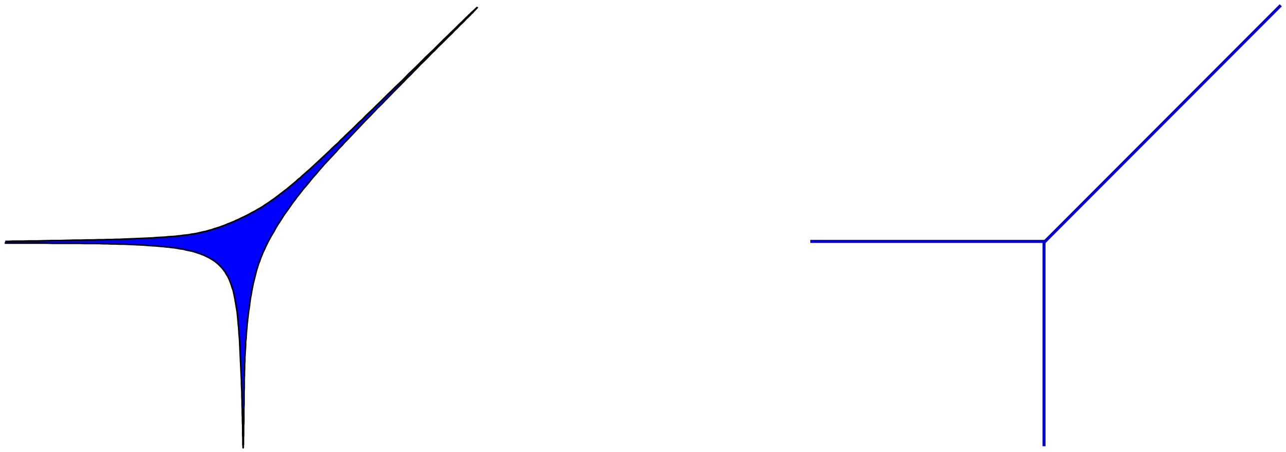

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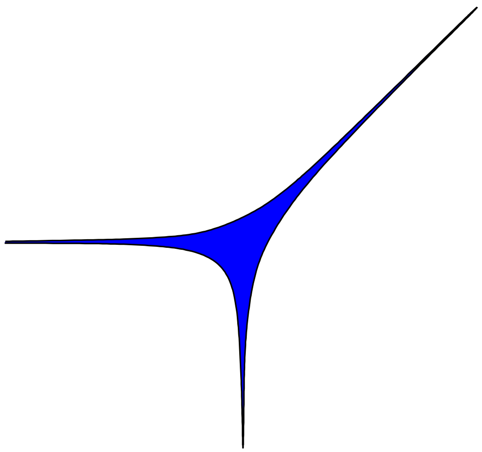

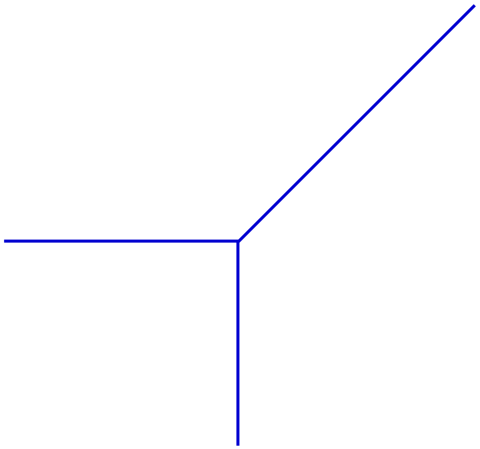

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What are the bridges good for?

- The goal

Tropical Geometry

Multi-valued algebra
- Tropical addition of complex numbers
- Tropical groups
- Operation induced on a subset
- Tropical addition of real numbers
- Homomorphisms
- Tropical rings and fields
- Leading term

Dequantizataion

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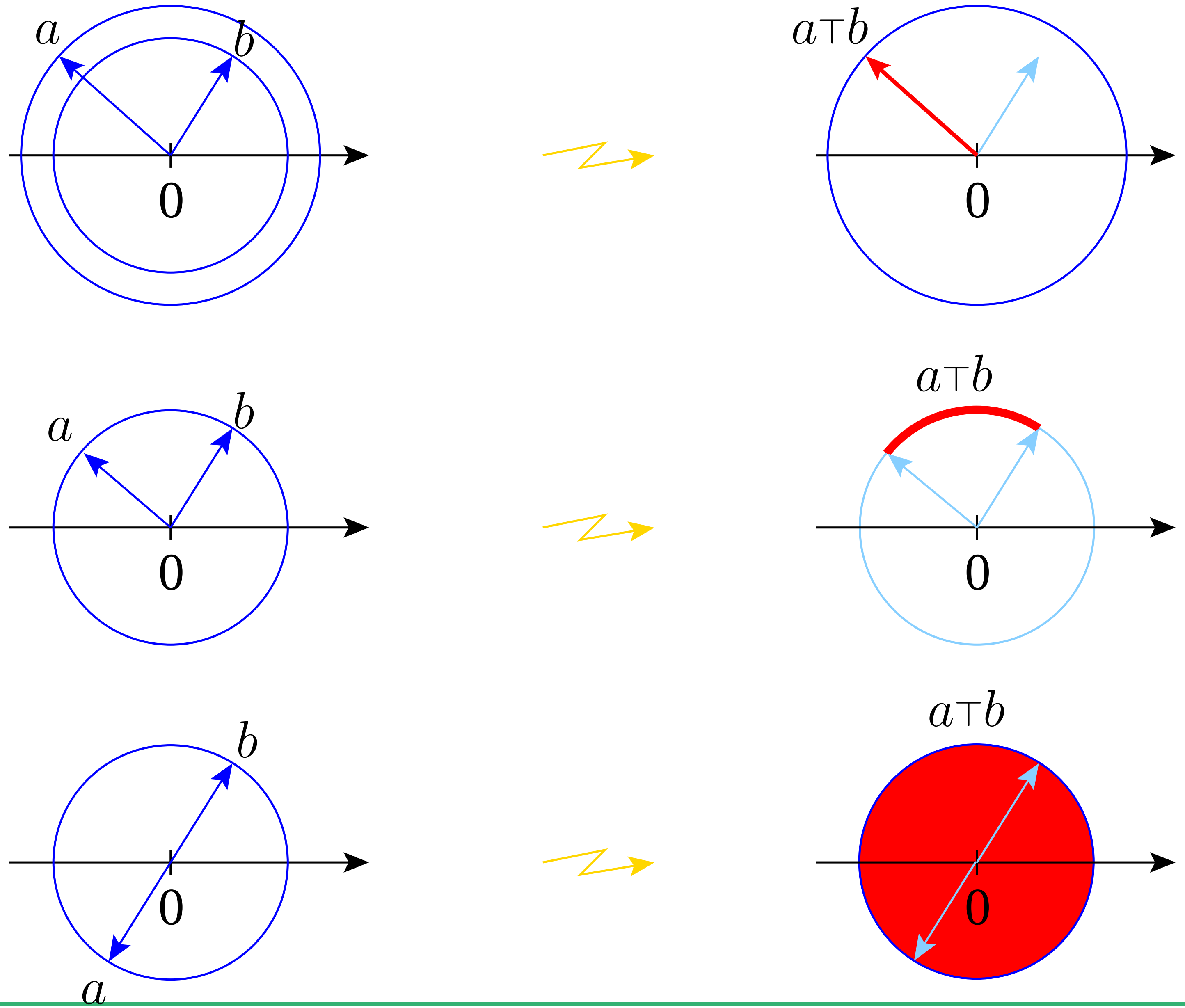

Tropical groups

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A binary multi-valued operation $f : X \times X \to 2^X$ naturally extends to

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$f : X \times X \to 2^X$ is associative

if $f(f(a, b), c) = f(a, f(b, c))$ for any $a, b, c \in X$.

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Any abelian group is a tropical group.

Tropical groups

A binary multi-valued operation in X : a map $X \times X \to 2^X \smallsetminus \{\varnothing\}$.

A set X with a multi-valued operation $(a, b) \mapsto a \top b$ is a commutative tropical group if

1. $\top$ is commutative;
2. $\top$ is associative;
3. X contains 0 such that $0 \top a = a$ for any $a \in X$;
4. for each $a \in X$ there exists a unique $-a \in X$ such that $0 \in a \top (-a)$.

Any abelian group is a tropical group.

Theorem. $(\mathbb{C}, \top)$ is a tropical group.

Operation induced on a subset

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Recall that the definition of multivalued binary operation prohibits $g(a, b)$ to be empty.

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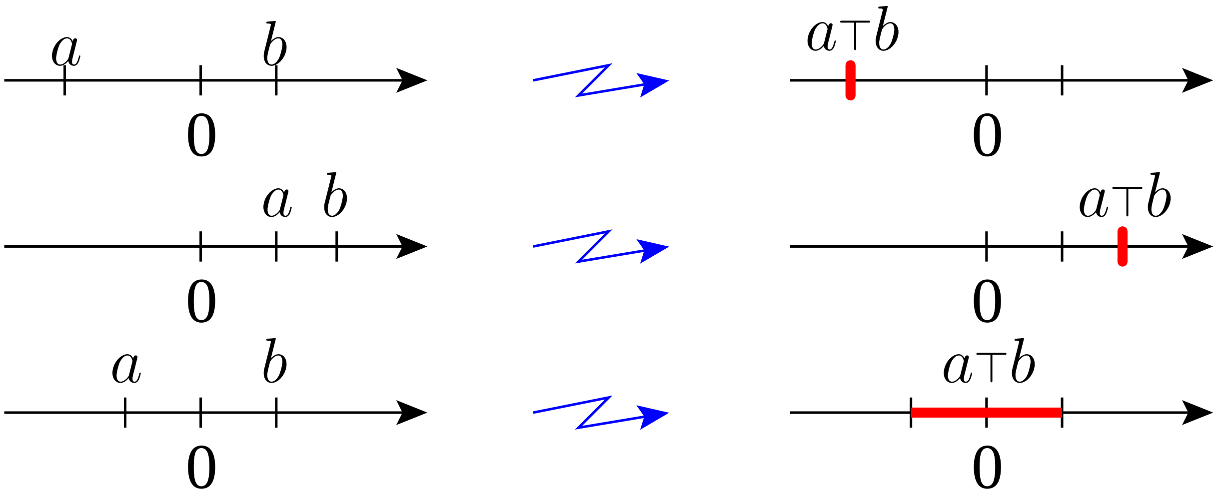

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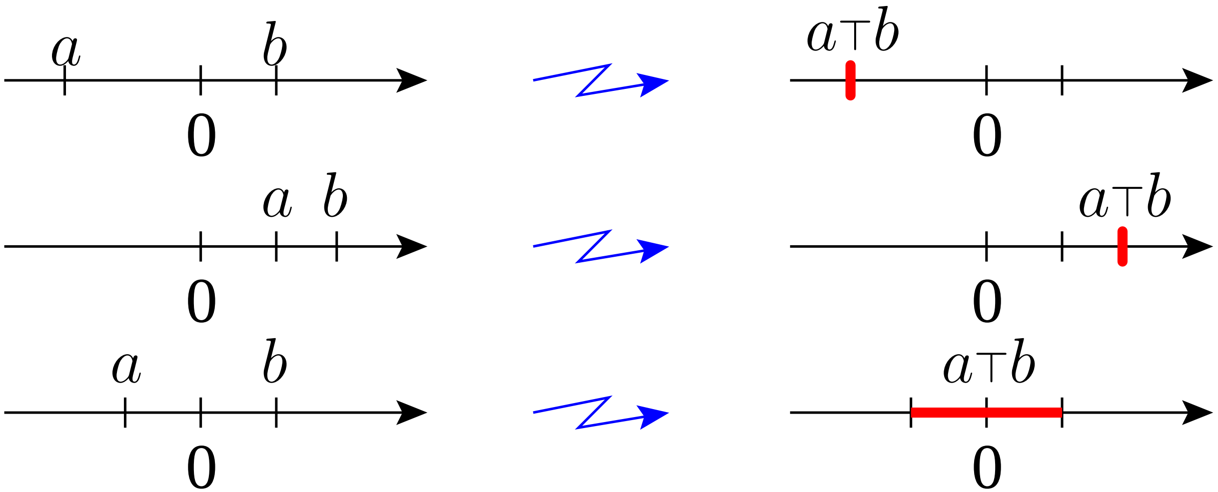

For $a, b \in \mathbb{R}$

$$a \top b = \begin{cases} \{a\}, & \text{if} \quad |a| > |b|, \\ \{b\}, & \text{if} \quad |a| < |b|, \\ \{a\}, & \text{if} \quad a = b, \\ [-|a|, |a|], & \text{if} \quad a = -b. \end{cases}$$

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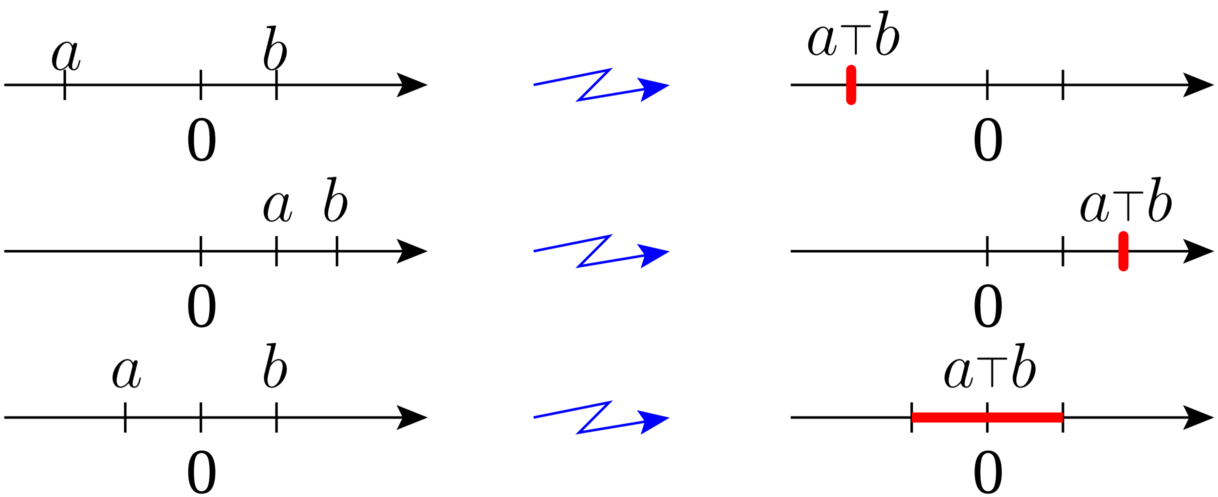

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then $(Y, \top_Y)$ is a tropical group (tropical subgroup of X) and $Y \hookrightarrow X$ is a homomorphism.

Tropical rings and fields

A set X with a binary multi-valued addition $\top$ and a (uni-valent) multiplication is called a tropical ring if

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Warning: a natural map in the opposite direction,

$\mathbb{R}_{\top} \to \mathbb{R}_{\geq 0, \max} : x \mapsto |x|$, is not a homomorphism.

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Tropical semifield $\mathbb{T}$ is a subsemifield of the tropical fields $\mathbb{C}$ and $\mathbb{R}$.

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$f(a+b) \in f(a) \top f(b)$ and $f(ab) = f(a)f(b)$.

- The goal

Tropical Geometry

Multi-valued algebra

Dequantizataion
- Deformation of $\mathbb{C}$
- A look of the limit
- Properties of $+_0$
- Upper Vietoris topology
- Continuity of tropical addition

Equations and varieties

Dequantizataion

Deformation of $\mathbb{C}$

For $h > 0$ consider a map $S_h\colon \mathbb{C} \longrightarrow \mathbb{C}$

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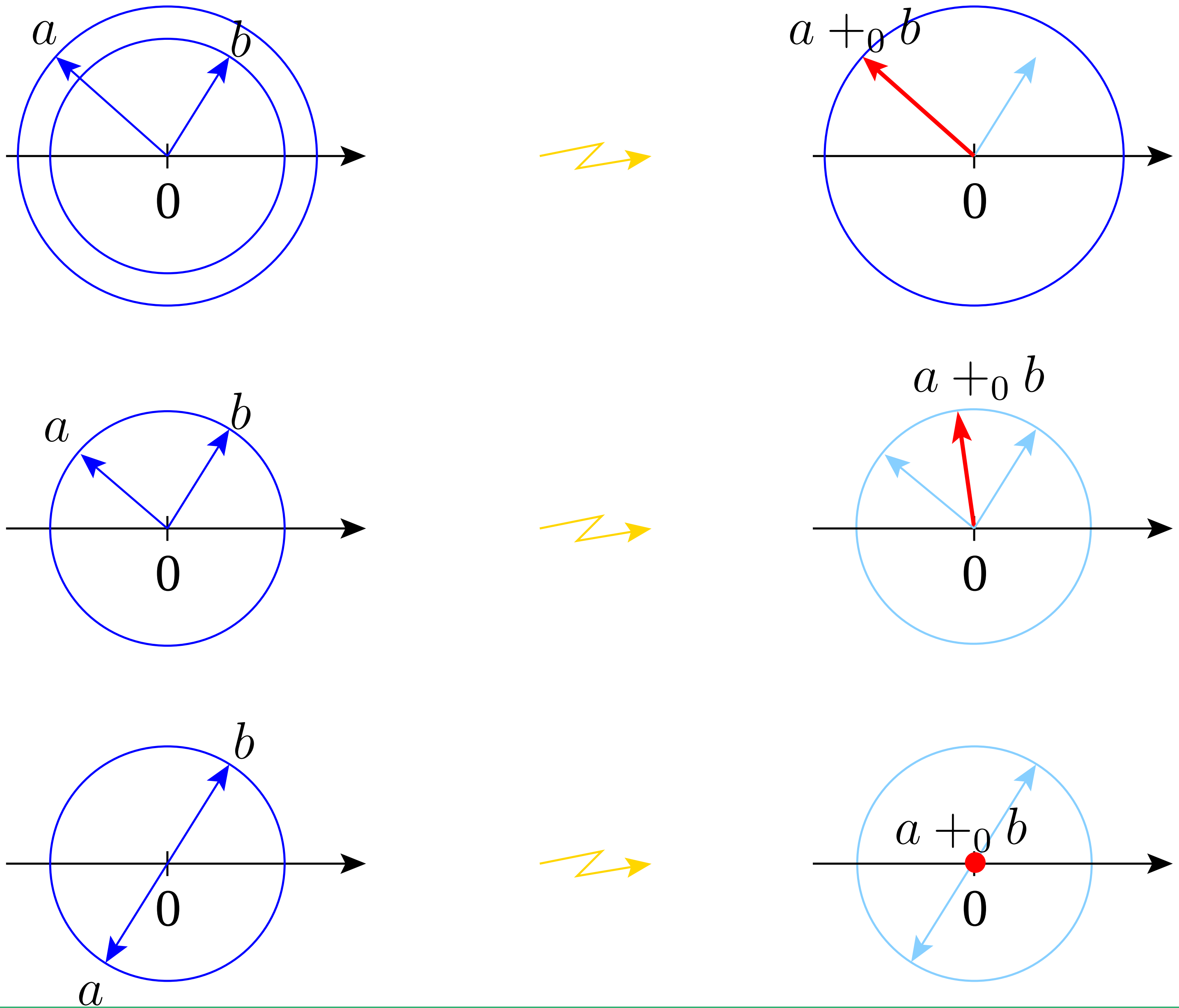

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There is one that
fixes all the defects,
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If the images of points are connected and the map is upper semi-continuous, then the image of a connected set is connected.

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Upper Vietoris topology in the set 2^X of all subsets of X is generated by sets $2^U \subset 2^X$ with U open in X.

A multivalued map $X \to Y$ is called upper semi-continuous if the corresponding univalued map $X \to 2^Y$ is continuous with respect to the upper Vietoris topology of 2^Y and the topology of X.

If the images of points are connected and the map is upper semi-continuous, then the image of a connected set is connected.

If the images of points are compact and the map is upper semi-continuous, then the image of a compact set is compact.

Continuity of tropical addition

Let $\Gamma_h \subset \mathbb{C}^3$ be a graph of $+_h$ for $h > 0$:

$$\Gamma_h = \{(a, b, c) \in \mathbb{C}^3 \mid a +_h b = c\}.$$

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Corollary. The multivalued map defined by a complex tropical polynomial is upper semi-continuous. It preserves connectedness and compactness.

● The goal

Tropical Geometry

Multi-valued algebra

Dequantizataion

Equations and varieties

● Good and bad polynomials

● Exercise in tropical addition

● Amoebas: relation to tropics

● Patchworking of hypersurfaces

● Complex tropical geometry

Equations and varieties

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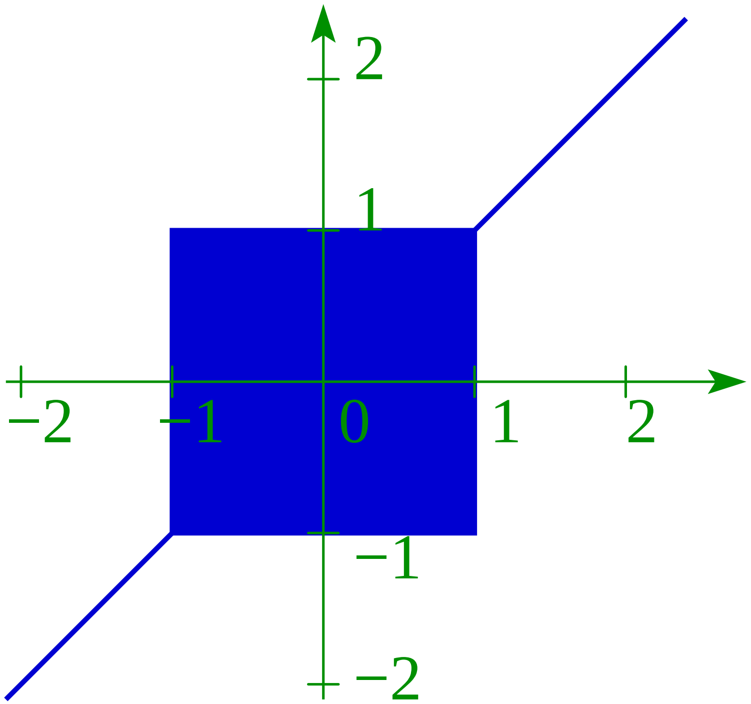

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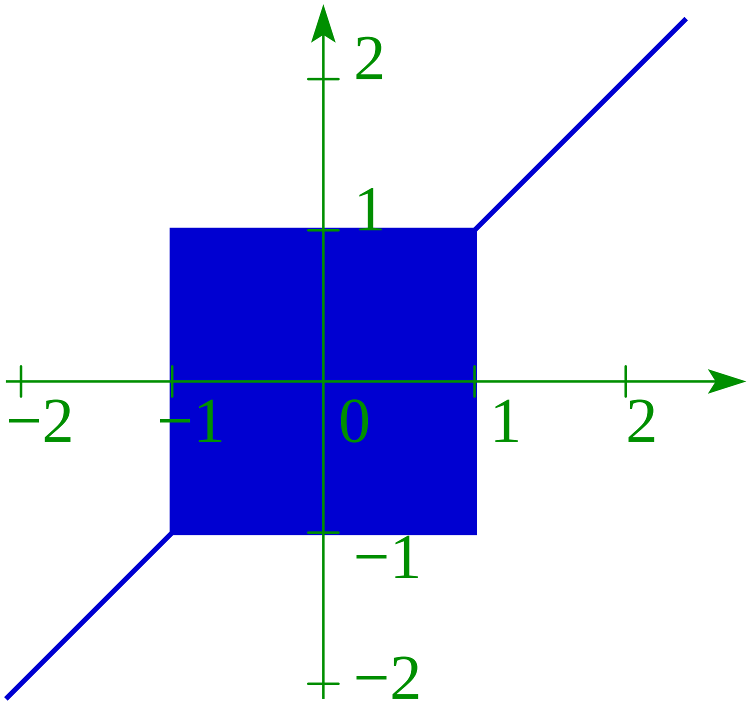

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A polynomial is said to be pure if it has no two monomials with the same exponents.

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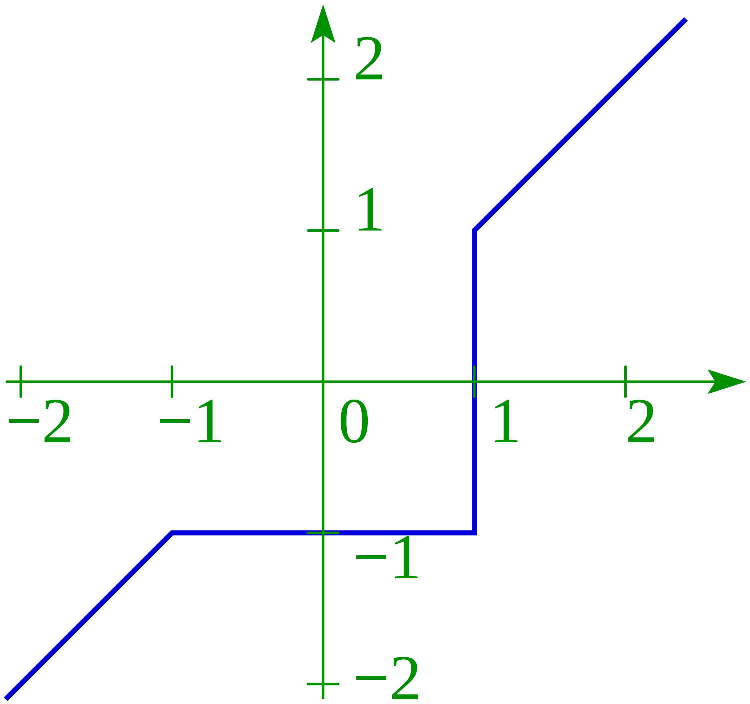

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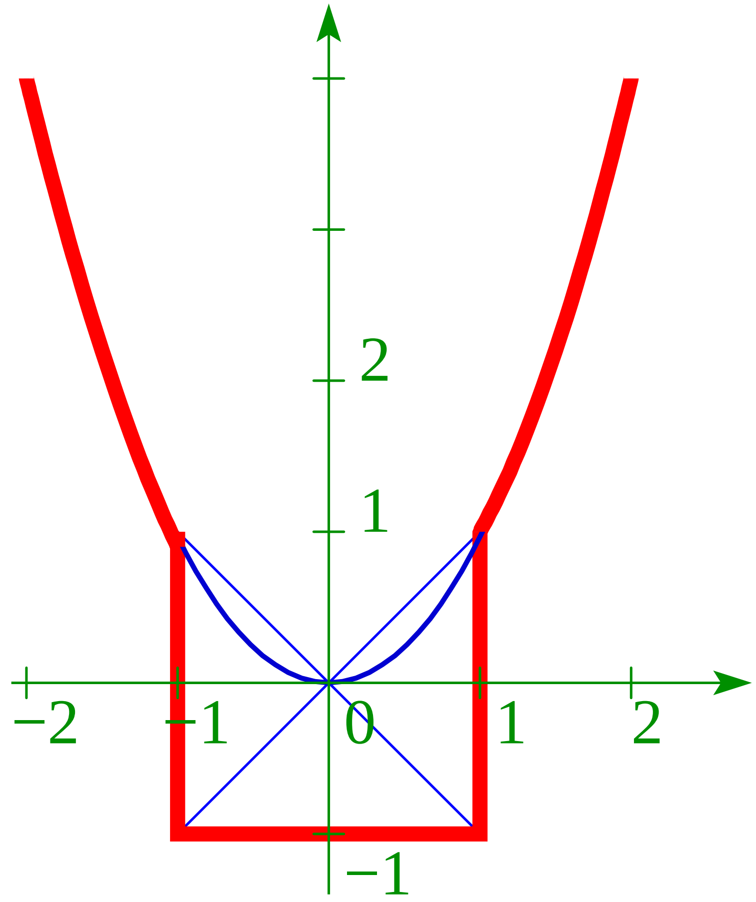

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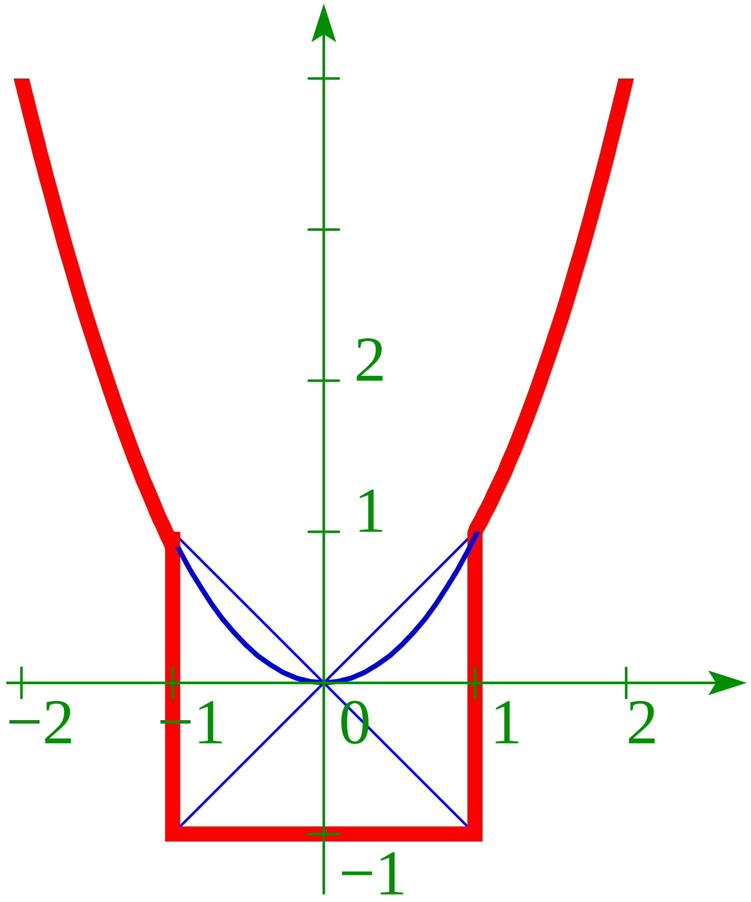

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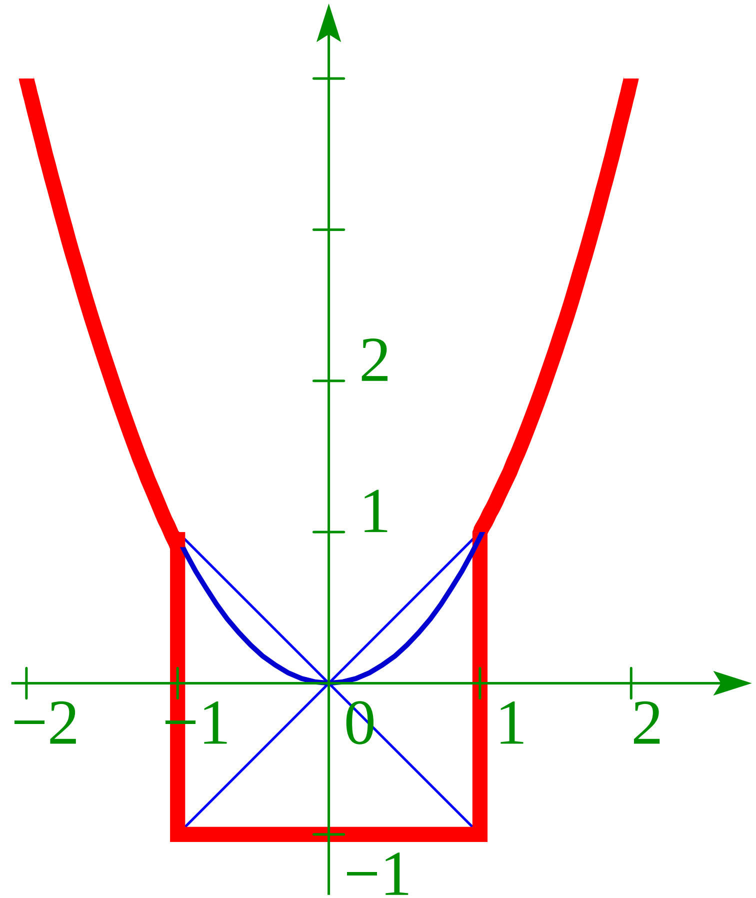

The graph of a polynomial is connected.

Because a polynomial is upper semi-continuous and has connected values.

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What if they have different absolute values?

Then only those with the greatest one matter!

Amoebas: relation to tropics

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$(\mathbb{C} \setminus 0)^n$ is convenient to consider fibred over $\mathbb{R}^n$ via the map $\mathrm{Log} : (\mathbb{C} \setminus \{0\})^n \to \mathbb{R}^n \ : \ (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$.

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The amoeba of a complex tropical hypersurface is the tropical hypersurface (defined by the same polynomial).

Patchworking of hypersurfaces

Let p be a polynomial in n variables.

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There is a real version of this statement.

Complex tropical geometry

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Conjecture. (Itenberg, Mikhalkin, Zharkov) Let X be a complex tropical variety, $X_q = \mathrm{Log}^{-1}(q\text{-skeleton}(\mathrm{Log}(X)))$, $H_n^q(X) = \mathrm{Im}(\mathrm{in}_* : H_n(X_q) \to H_n(X))$, $H_{p,q}(X) = H_{p+q}^q(X)/H_{p+q}^{q-1}(X)$. Then $H_{p,q}(X) \otimes \mathbb{C}$ is isomorphic to $H^{p,q}(X_h)$.

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This is a work in progress started 2 months ago.

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