## 2. A Real Algebraic Curve from the Complex Point of View

2.1. **Complex Topological Characteristics of a Real Curve.** According to a tradition going back to Hilbert, for a long time the main question concerning the topology of real algebraic curves was considered to be the determination of which isotopy types are realized by nonsingular real projective algebraic plane curves of a given degree (i.e., Problem *1.1.B* above). However, as early as in 1876 F. Klein [Kle-22] posed the question more broadly. He was also interested in how the isotopy type of a curve is connected to the way the set $\mathbb{R}A$ of its real points is positioned in the set $\mathbb{C}A$ of its complex points (i.e., the set of points of the complex projective plane whose homogeneous coordinates satisfy the equation defining the curve).

The set $\mathbb{C}A$ is an oriented smooth two-dimensional submanifold of the complex projective plane $\mathbb{C}P^2$. Its topology depends only on the degree of $A$ (in the case of nonsingular $A$). If the degree is $m$, then $\mathbb{C}A$ is a sphere with $\frac{1}{2}(m-1)(m-2)$ handles. (It will be shown in Section 2.3.) Thus the literal complex analogue of Topological Classification Problem *1.1.A* is trivial.

The complex analogue of Isotopy Classification Problem *1.1.B* leads also to a trivial classification: the topology of the pair $(\mathbb{C}P^2, \mathbb{C}A)$ depends only on the degree of $A$, too. The reason for this is that the complex analogue of a more refined Rigid Isotopy Classification problem *1.7.A* has a trivial solution: nonsingular complex projective curves of degree $m$ form a space $\mathbb{C}NC_m$ similar to $\mathbb{R}NC_m$ (see Section 1.7) and this space is connected, since it is the complement of the space $\mathbb{C}SC_m$ of singular curves in the space $\mathbb{C}C_m (= \mathbb{C}P^{\frac{1}{2}m(m+3)})$ of all curves of degree $m$, and $\mathbb{C}SC_m$ has *real* codimension 2 in $\mathbb{C}C_m$ (its *complex* codimension is 1).

The set $\mathbb{C}A$ of complex points of a *real* curve $A$ is invariant under the complex conjugation involution $conj : \mathbb{C}P^2 \to \mathbb{C}P^2 : (z_0 : z_1 : z_2) \mapsto (\overline{z}_0 : \overline{z}_1 : \overline{z}_2)$. The curve $\mathbb{R}A$ is the fixed point set of the restriction of this involution to $\mathbb{C}A$.

The real curve $\mathbb{R}A$ may divide or not divide $\mathbb{C}A$. In the first case we say that $A$ is a *dividing* curve or a curve of *type* I, in the second case we say that it is a *nondividing* curve or a curve of *type* II. In the first case $\mathbb{R}A$ divides $\mathbb{C}A$ into two connected pieces.[3] The natural orientations of these two halves determine two opposite orientations on $\mathbb{R}A$ (which

[3]Proof: the closure of tne union of a connected component of $\mathbb{C}A \smallsetminus \mathbb{R}A$ with its image under *conj* is open and close in $\mathbb{C}A$, but $\mathbb{C}A$ is connected.

is their common boundary); these orientations of $\mathbb{R}A$ are called the *complex orientations* of the curve.

A pair of orientations opposite to each other is called a *semiorientation.* Thus the complex orientations of a curve of type I comprise a semiorientation. Naturally, the latter is called a *complex semiorientation.*

The scheme of relative location of the ovals of a curve is called the *real scheme* of the curve. The real scheme enhanced by the type of the curve, and, in the case of type I, also by the complex orientations, is called the *complex scheme* of the curve.

We say that the real scheme of a curve of degree $m$ is *of type* I (*type* II) if any curve of degree $m$ having this real scheme is a curve of type I (type II). Otherwise (i.e., if there exist curves of both types with the given real scheme), we say that the real scheme is of *indeterminate type*.

The division of curves into types is due to Klein [Kle-22]. It was Rokhlin [Rok-74] who introduced the complex orientations. He introduced also the notion of complex scheme and its type [Rok-78]. In the eighties the point of view on the problems in the topology of real algebraic varieties was broadened so that the role of the main object passed from the set of real points, to this set together with its position in the complexification. This viewpoint was also promoted by Rokhlin.

As we will see, the notion of complex scheme is useful even from the point of view of purely real problems. In particular, the complex scheme of a curve is preserved under a rigid isotopy. Therefore if two curves have the same real scheme, but distinct complex schemes, the curves are not rigidly isotopic. The simplest example of this sort is provided by the curves of degree 5 shown in Figure 8, which are isotopic but not rigidly isotopic.

2.2. **The First Examples.** A complex projective line is homeomorphic to the two-dimensional sphere.[4] The set of real points of a real projective line is homeomorphic to a circle; by the Jordan theorem it divides the complexification. Therefore a real projective line is of type I. It has a pair of complex orientations, but they do not add anything, since the real line is connected and admits only one pair of orientations opposite to each other.

[4] I believe that this may be assumed well-known. A short explanation is that a projective line is a one-point compactification of an affine line, which, in the complex case, is homeomorphic to $\mathbb{R}^2$. A one-point compactification of $\mathbb{R}^2$ is unique up to homeomorphism and homeomorphic to $S^2$.

The action of *conj* on the set of complex points of a real projective line is determined from this picture by rough topological arguments. Indeed, it is not difficult to prove that any smooth involution of a two-dimensional sphere with one-dimensional (and non-empty) fixed point set is conjugate in the group of autohomeomorphisms of the sphere to the symmetry in a plane. ([1])

The set of complex points of a nonsingular plane projective conic is homeomorphic to $S^2$, because the stereographic projection from any point of a conic to a projective line is a homeomorphism. Certainly, an empty conic, as any real algebraic curve with empty set of real points, is of type II. The empty set cannot divide the set of complex points. For the same reasons as a line (i.e. by Jordan theorem), a real nonsingular curve of degree 2 with non-empty set of real points is of type I. Thus the real scheme $\langle 1\rangle$ of degree 2 is of type I, while the scheme $\langle 0\rangle$ is of type II for any degree.

2.3. **Classical Small Perturbations from the Complex Point of View.** To consider further examples, it would be useful to understand what is going on in the complex domain, when one makes a classical small perturbation (see Section 1.5).

First, consider the simplest special case: a small perturbation of the union of two real lines. Denote the lines by $L_1$ and $L_2$ and the result by $C$. As we saw above, $\mathbb{C}L_i$ and $\mathbb{C}C$ are homeomorphic to $S^2$. The spheres $\mathbb{C}L_1$ and $\mathbb{C}L_2$ intersect each other at a single point. By the complex version of the implicit function theorem, $\mathbb{C}C$ approximates $\mathbb{C}L_1 \cup \mathbb{C}L_2$ outside a neighborhood $U_0$ of this point in the sense that $\mathbb{C}C \smallsetminus U_0$ is a section of a tulubular neighborhood $U_1$ of $(\mathbb{C}L_1 \cup \mathbb{C}L_1) \smallsetminus U_0$, cf. *1.5.A*. Thus $\mathbb{C}C$ may be presented as the union of two discs and a part contained in a small neighborhood of $\mathbb{C}L_1 \cap \mathbb{C}L_2$. Since the whole $\mathbb{C}C$ is homeomorphic to $S^2$ and the complement of two disjoint discs embedded into $S^2$ is homeomorphic to the annulus, the third part of $\mathbb{C}C$ is an annulus. The discs are the complements of a neighborhood of $\mathbb{C}L_1 \cap \mathbb{C}L_2$ in $\mathbb{C}L_1$ and $\mathbb{C}L_2$, respectively, slightly perturbed in $\mathbb{C}P^2$, and the annulus connects the discs through the neighborhood $U_0$ of $\mathbb{C}L_1 \cap \mathbb{C}L_2$.

This is the complex view of the picture. Up to this point it does not matter whether the curves are defined by real equations or not.

To relate this to the real view presented in Section 1.5, one needs to describe the position of the real parts of the curves in their complexifications and the action of *conj*. It can be recovered by rough topological agruments. The whole complex picture above is invariant under *conj*. This means that the intersection point of $\mathbb{C}L_1$ and $\mathbb{C}L_2$

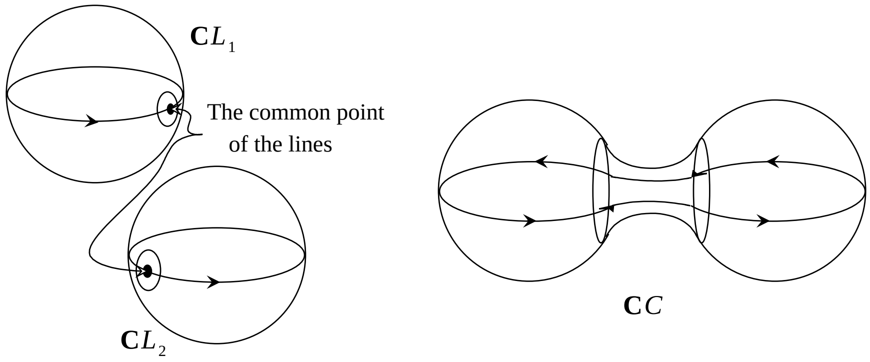


FIGURE 15

is real, its neighborhood $U_0$ can be chosen to be invariant under $conj$. Thus each half of $\mathbb{C}C$ is presented as the union of two half-discs and a half of the annulus: the half-discs approximate the halves of $\mathbb{C}L_1$ and $\mathbb{C}L_2$ and a half of annulus is contained in $U_0$. See Figure 15.

This is almost complete description. It misses only one point: one has to specify which half-discs are connected with each other by a half-annulus.

First, observe, that the halves of the complex point set of any curve of type I can be distinguished by the orientations of the real part. Each of the halves has the canonical orientation defined by the complex structure, and this orientation induces an orientation on the boundary of the half. This is one of the complex orientations. The other complex orientation comes from the other half. Hence the halves of the complexification are in one-to-one correspondence to the complex orientations.

Now we have an easy answer to the question above. The halves of $\mathbb{C}L_i$ which are connected with each other after the perturbation correspond to the complex orientations of $\mathbb{R}L_i$ which agree with some orientation of $\mathbb{R}C$. Indeed, the perturbed union $C$ of the lines $L_i$ is a curve of type I (since this is a nonempty conic, see Section 2.2). Each orientation of its real part $\mathbb{R}C$ is a complex orientation. Choose one of the orientations. It is induced by the canonical orientation of a half of the complex point set $\mathbb{C}C$. Its restriction to the part of the $\mathbb{R}C$ obtained from $\mathbb{R}L_i$ is induced by the orientation of the corresponding part of this half.

The union of two lines can be perturbed in two different ways. On the other hand, there are two ways to connect the halves of their complexifications. It is easy to see that different connections correspond to different perturbations. See Figure 16.

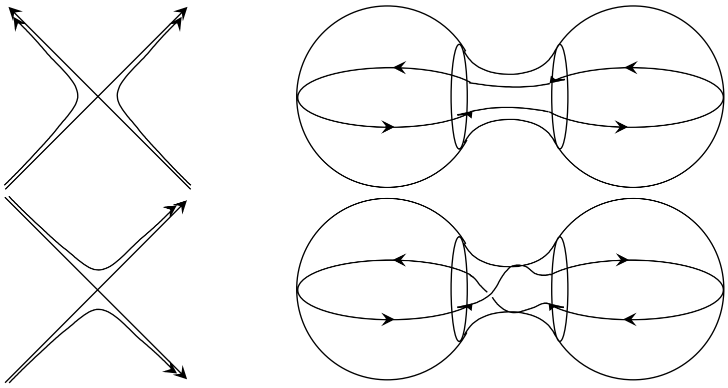

FIGURE 16

The special classical small perturbation considered above is a key for understanding what happens in the complex domain at an arbitrary classical small perturbation. First, look at the complex picture, forgetting about the real part. Take a plane projective curve, which has only nondegenerate double points. Near such a point it is organized as a union of two lines intersecting at the point. This means that there are a neighborhood $U$ of the point in $\mathbb{C}P^2$ and a diffeomorphism of $U$ onto $\mathbb{C}^2$ mapping the intersection of $U$ and the curve onto a union of two complex lines, which meet each other in 0. This follows from the complex version of the Morse lemma. By the same Morse lemma, near each double point the classical small perturbation is organized as a small perturbation of the union of two lines: the union of two transversal disks is replaced by an annulus.

For example, take the union of $m$ projective lines, no three of which have a common point. Its complex point set is the union of $m$ copies of $S^2$ such that any two of them have exactly one common point. A perturbation can be thought of as removal from each sphere $m-1$ disjoint discs and insertion $\frac{m(m-1)}{2}$ tubes connecting the boundary circles of the disks removed. The result is orientable (since it is a complex manifold). It is easy to realize that this is a sphere with $\frac{(m-1)(m-2)}{2}$ handles. One may prove this counting the Euler characteristic, but it may be seen directly: first, by inserting the tubes which join one of the lines with all other lines we get a sphere, then each additional tube gives rise to a handle. The number of these handles is

$$\binom{m-1}{2} = \frac{(m-1)(m-2)}{2}.$$

By the way, this description shows that the complex point set of a nonsingular plane projective curve of degree $m$ realizes the same homology class as the union of $m$ complex projective lines: the $m$-fold generator of $H_2(\mathbb{C}P^2)(=\mathbb{Z})$.

Now let us try to figure out what happens with the complex schemes in an arbitrary classical small perturbation of real algebraic curves. The general case requirs some technique. Therefore we restrict ourselves to the following intermediate assertion.

*2.3.A.* (Fiedler [Rok-78, Section 3.7] and Marin [Mar-80].) *Let $A_1, \dots, A_s$ be nonsingular curves of degrees $m_1, \dots, m_s$ such that no three of them pass through the same point and $A_i$ intersects transversally $A_j$ in $m_i m_j$ real points for any $i$, $j$. Let $A$ be a nonsingular curve obtained by a classical small perturbation of the union $A_1 \cup \dots A_s$. Then $A$ is of type I if and only if all $A_i$ are of type I and there exists an orientation of $\mathbb{R}A$ which agrees with some complex orientations of $A_1, \dots, A_s$ (it means that the deformation turning $A_1 \cup \dots A_s$ into $A$ brings the complex orientations of $A_i$ to the orientations of the corresponding pieces of $\mathbb{R}A$ induced by a single orientation of the whole $\mathbb{R}A$).*

*If it takes place, then the orientation of $\mathbb{R}A$ is one of the complex orientations of $A$.*

*Proof.* If some of $A_i$ is of type II, then it has a pair of complex conjugate imaginary points which can be connected by a path in $\mathbb{C}A_i \smallsetminus \mathbb{R}A_i$. Under the perturbation this pair of points and the path survive (being only slightly shifted), since they are far from the intersection where the real changes happen. Therefore $A$ in this case is also of type II.

Assume now that all $A_i$ are of type I. If $A$ is also of type I then a half of $\mathbb{C}A$ is obtained from halves of $\mathbb{C}A_i$ as in the case considered above. The orientation induced on $\mathbb{R}A$ by the orientation of the half agrees with orientations induced from the halves of the corresponding pieces. Thus a complex orietation of $A$ agrees with complex orientations of $A_i$'s.

Again assume that all $A_i$ are of type I. Let some complex orientations of $A_i$ agree with a single orientation of $\mathbb{R}A$. As it follows from the Morse Lemma, at each intersection point the perturbation is organized as the model perturbation considered above. Thus the halves of $\mathbb{C}A_i$'s defining the complex orientations are connected. It cannot happen that some of the halves will be connected by a chain of halves to its image under $conj$. But that would be the only chance to get a curve of type II, since in a curve of type II each imaginary point can be connected with its image under $conj$ by a path disjoint from the real part. □

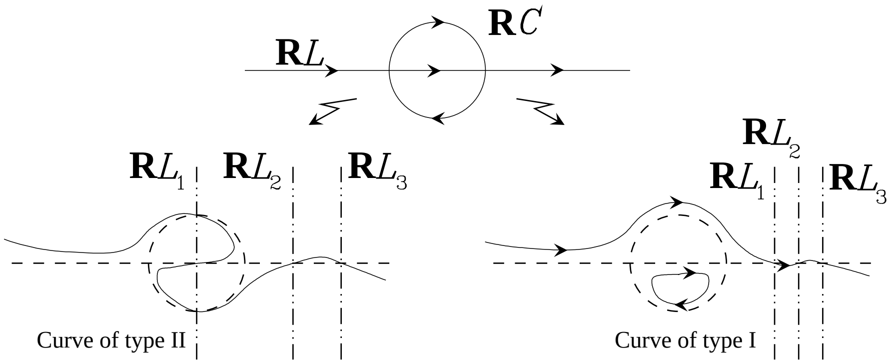


FIGURE 17. Construction of nonsingular cubic curves. Cf. Figure 2.

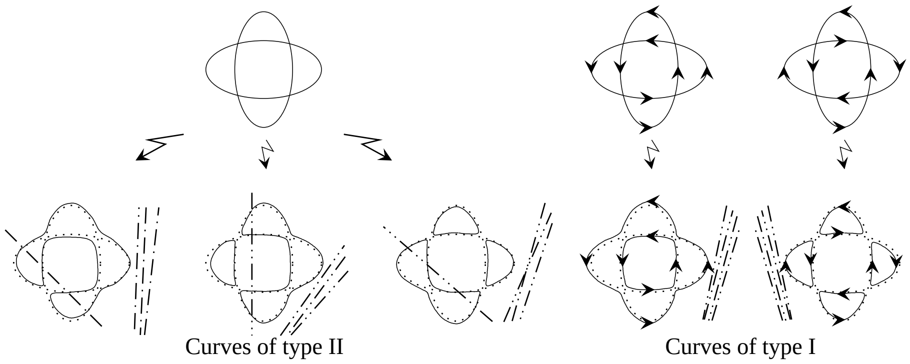


FIGURE 18. Construction of nonsingular quartic curves. Cf. Figure 3.

2.4. **Further Examples.** Although Theorem *2.3.A* describes only a very special class of classical small perturbations (namely perturbations of unions of nonsingular curves intersecting only in real points), it is enough for all constructions considered in Section 1. In Figures 17, 18, 19, 20, 21, 22 and 23 I reproduce the constructions of Figures 2, 3, 4, 6, 7, 10 and 11, enhancing them with complex orientations if the curve is of type I.

2.5. **Digression: Oriented Topological Plane Curves.** Consider an oriented topological plane curve, i. e. an oriented closed one-dimensional submanifold of the projective plane, cf. 1.2.

A pair of its ovals is said to be *injective* if one of the ovals is enveloped by the other.

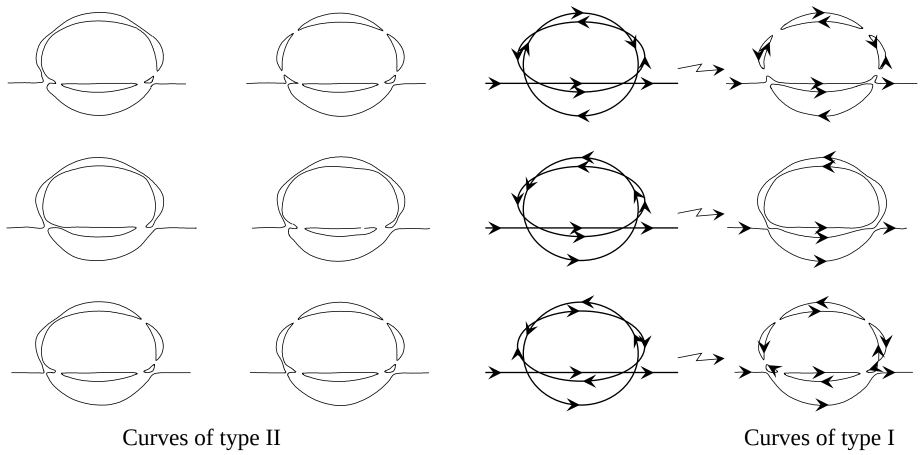


FIGURE 19. Construction of nonsingular quintic curves. Cf. Figure 4.

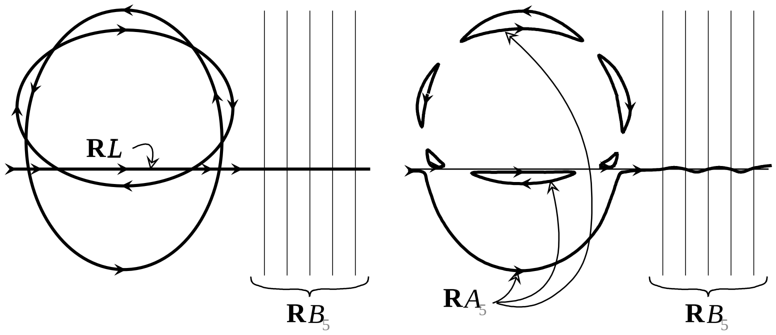


FIGURE 20. Construction of a quintic M-curve with its complex orientation. Cf. Figure 6.

An injective pair of ovals is said to be *positive* if the orientations of the ovals determined by the orientation of the entire curve are induced by an orientation of the annulus bounded by the ovals. Otherwise, the injective pair of ovals is said to be *negative*. See Figure 24. It is clear that the division of pairs of ovals into positive and negative pairs does not change if the orientation of the entire curve is reversed; thus, the injective pairs of ovals of a semioriented curve (and, in particular, a curve of type I) are divided into positive and negative. We let $\Pi^+$ denote the number of positive pairs, and $\Pi^-$ denote the number of negative pairs.

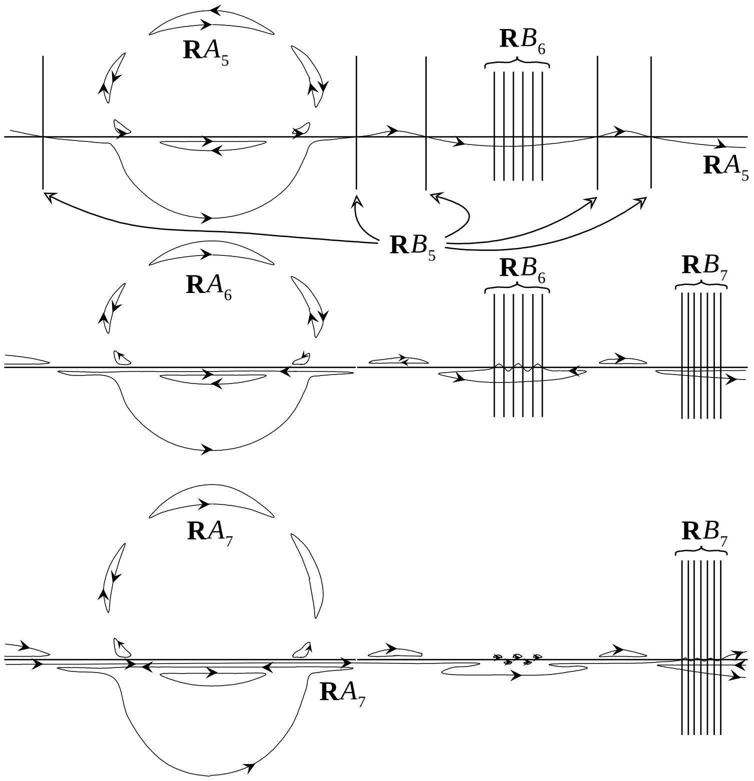


FIGURE 21. Harnack's construction with complex orientations. Cf. Figure 7.

The ovals of an oriented curve one-sidedly embedded into $\mathbb{R}P^2$ can be divided into positive and negative. Namely, consider the Möbius strip which is obtained when the disk bounded by an oval is removed from $\mathbb{R}P^2$. If the integral homology classes which are realized in this strip by the oval and by the doubled one-sided component with the orientations determined by the orientation of the entire curve coincide, we say that the oval is *negative,* otherwise we say that the oval is *positive.* See Figure 25. In the case of a two-sided oriented curve, only the non-outer ovals can be divided into positive and negative. Namely, a non-outer oval is said to be *positive* if it forms a positive pair with the outer oval which envelops it; otherwise, it is said to be *negative*. As in the case of pairs, if the orientation of the curve is reversed, the division of ovals into positive and negative ones does not change. Let $\Lambda^+$ denote the number of positive ovals on a curve, and let $\Lambda^-$ denote the number of negative ones.

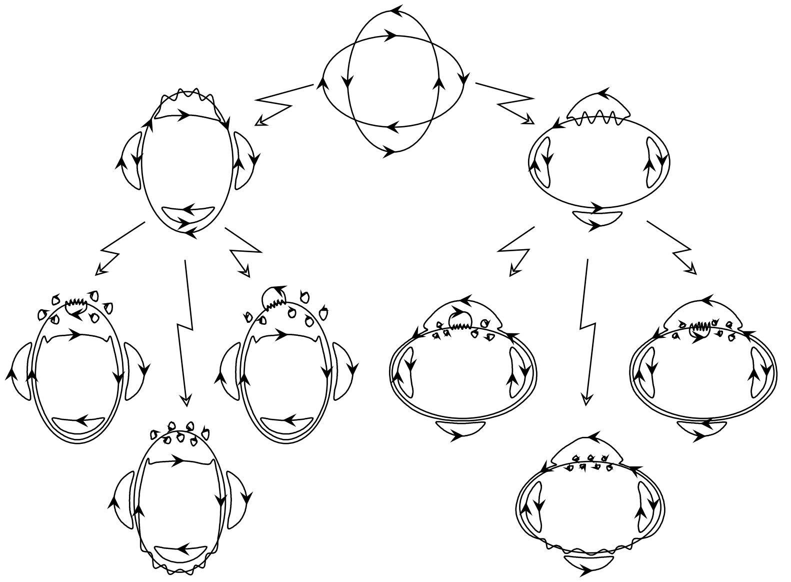

Figure 22. Construction of even degree curves by Hilbert's method. Degrees 4 and 6. Cf. Figure 10.

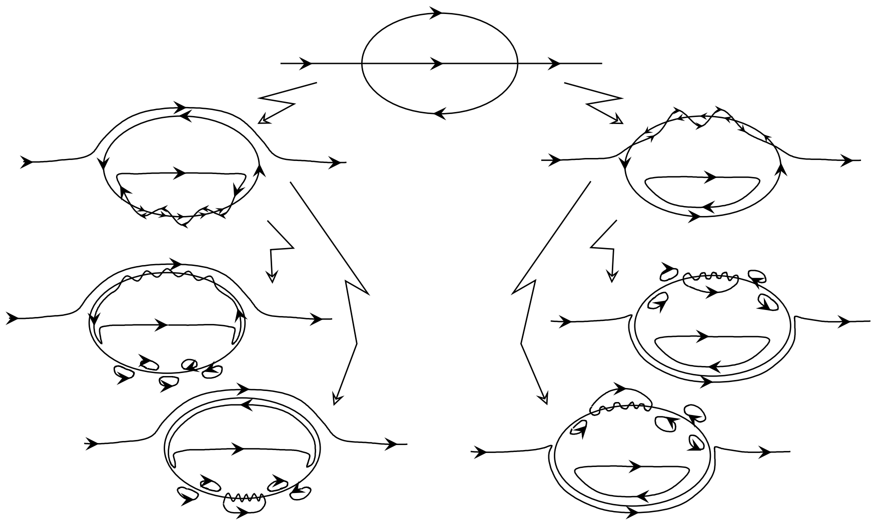

Figure 23. Construction of odd degree curves by Hilbert's method. Degrees 3 and 5. Cf. Figure 11.

To describe a semioriented topological plane curve (up to homeomorphism of the projective plane) we need to enhance the coding system introduced in 1.2. The symbols representing positive ovals will be equipped with a superscript $+$, the symbols representing negative ovals, with a superscript $-$. This kind of code of a semioriented curve


Figure 24

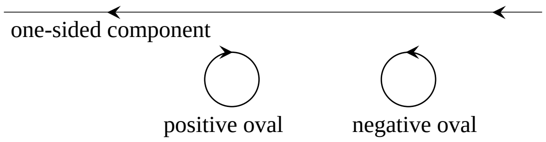


Figure 25

is complete in the following sense: for any two semioriented curves with the same code there exists a homeomorphism of $\mathbb{R}P^2$, which maps one of them to the other preserving semiorientations.

To describe the complex scheme of a curve of degree $m$ we will use, in the case of type I, the scheme of the kind described here, for its complex semiorientation, equipped with subscript I and superscript $m$ and, in the case of type II, the notation used for the real scheme, but equipped with subscript II and superscript $m$.

It is easy to check, that the coding of this kind of the complex scheme of a plane projective real algebraic curve describes the union of $\mathbb{R}P^2$ and the complex point set of the curve up to a homeomorphism mapping $\mathbb{R}P^2$ to itself.

In these notations, the complex schemes of cubic curves shown in Figure 17 are $\langle J\rangle^3_{II}$ and $\langle J \amalg 1^-\rangle^3_I$.

The complex schemes of quartic curves realized in Figure 18 are $\langle 0\rangle^4_{II}$, $\langle 1\rangle^4_{II}$, $\langle 2\rangle^4_{II}$, $\langle 1\langle 1^-\rangle\rangle^4_I$, $\langle 3\rangle^4_{II}$, $\langle 4\rangle^4_I$.

The complex schemes of quintic curves realized in Figure 19 are $\langle J\rangle^5_{II}$, $\langle J \amalg 1\rangle^5_{II}$, $\langle J \amalg 2\rangle^5_{II}$, $\langle J \amalg 1^-\langle 1^-\rangle\rangle^5_I$, $\langle J \amalg 3\rangle^5_{II}$, $\langle J \amalg 4\rangle^5_{II}$, $\langle J \amalg 1^+ \amalg 3^-\rangle^5_I$ $\langle J \amalg 5\rangle^5_{II}$, $\langle J \amalg 3^+ \amalg^-\rangle^5_I$.

In fact, these lists of complex schemes contain all schemes of nonsingular algebraic curves for degrees 3 and 5 and all nonempty schemes for degree 4. To prove this, we need not only constructions, but also restrictions on complex schemes. In the next two sections restrictions sufficient for this will be provided.

2.6. **The Simplest Restrictions on a Complex Scheme.** To begin with, recall the following obvious restriction, which was used in Section 2.2.

*2.6.A*. *A curve with empty real point set is of type* II. □

The next theorem is in a sense dual to *2.6.A*.

*2.6.B*. *An M-curve is of type* I.

*Proof.* Let $A$ be an M-curve of degree $m$. Then $\mathbb{R}A$ is the union of

$$\frac{(m-1)(m-2)}{2}+1$$

disjoint circles lying on $\mathbb{C}A$, which is a sphere with $\frac{(m-1)(m-2)}{2}$ handles. That many disjoint circles necessarily divide a sphere with $\frac{(m-1)(m-2)}{2}$ handles. Indeed, cut $\mathbb{C}A$ along $\mathbb{R}A$. The Euler characteristic of a surface has not changed. It equals

$$2-2\left(\frac{(m-1)(m-2)}{2}\right)=2-(m-1)(m-2).$$

Then cap each boundary circle with a disk. Each component of $\mathbb{R}A$ gives rise to 2 boundary circles. Therefore the number of the boundary circles is $(m-1)(m-2)+2$. The surface which is obtained has Euler characteristic $2-(m-1)(m-2)+(m-1)(m-2)+2=4$. However, there is no connected closed surface with Euler characteristic 4. (A connected closed oriented surface is a sphere with $g$ handles for some $g \geq 0$; it has Euler characteristic $2-2g \leq 2$.) □

*2.6.C* (Klein's Congruence (see [Kle-22, page 172])). *If $A$ is a curve of type* I *of degree $m$ with $l$ ovals, then $l \equiv [\frac{m}{2}] \mod 2$.*

*Proof.* Consider a half of $\mathbb{C}A$ bounded by $\mathbb{R}A$. Its Euler characteristic equals the half of the Euler characteristic of $\mathbb{C}A$, i.e. $1-\frac{(m-1)(m-2)}{2}$. Cap the boundary components of the half with disjoint disks. This increases the Euler characteristics by the number of components of $\mathbb{R}A$. In the case of even degree $m=2k$, the Euler characteristic of the result is $1-(2k-1)(k-1)+l \equiv k+l \mod 2$. In the case of odd degree $m=2k+1$, it is $1-k(2k-1)+l \equiv k+l \mod 2$. In both cases the Euler characteristic should be even, since the surface is closed orientable and connected (i.e. sphere with handles). Thus in both cases $k \equiv l \mod 2$, where $k=[m/2]$. □

*2.6.D* (A Nest of the Maximal Depth (see [Rok-78, 3.6])). *A real scheme of degree $m$ containing a nest of depth $k=[m/2]$ is of type* I.

Such a scheme exists and is unique for any $m$ (for even $m$ it is just the nest, for odd $m$ it consists of the nest and the one-sided component). To realize the scheme, perturb the union of $k$ concentric circles and, in the case of odd $m$, a line disjoint from the circles. The uniqueness was proved in 1.3, see *1.3.C*.

I preface the proof of *2.6.D* with a construction interesting for its own. It provides a kind of window through which one can take a look at the imaginary part of $\mathbb{C}P^2$.

As we know (see Section 2.2), the complex point set of a real line is divided by its real point set into two halves, which are in a natural one-to-one correspondence with the orientations of the real line. The set of all real lines on the projective plane is the real point set of the dual projective plane. The halves of lines comprise a two-dimensional sphere covering this projective plane. An especially clear picture of these identifications appears, if one identifies real lines on the projective plane with real planes in $\mathbb{R}^3$ containing 0. A half of a line is interpreted as the corresponding plane with orientation. An oriented plane corresponds to its positive unit normal vector, which is nothing but a point of $S^2$. The complex conjugation *conj* maps a half of a real line to the other half of the same line. It corresponds to the reversing of the orientation, which, in turn, corresponds to the antipodal involution $S^2 \to S^2 : x \mapsto -x$.

There is a unique real line passing through any imaginary point of $\mathbb{C}P^2$. To construct such a line, connect the point with the conjugate one. The connecting line is unique since a pair of distinct points determines a line, and this line is real, since it coincides with its image under *conj*.

Consequently, there is a unique half of a real line containing an imaginary point of $\mathbb{C}P^2$. This construction determines a fibration $p : \mathbb{C}P^2 \smallsetminus \mathbb{R}P^2 \to S^2$. The fibres of $p$ are the halves of real lines. Note that conjugate points of $\mathbb{C}P^2 \smallsetminus \mathbb{R}P^2$ are mapped to antipodal points of $S^2$.

*Proof of 2.6.D.* Let $A$ be a real projective curve of degree $m$ with a nest of depth $[m/2]$. Choose a point $P \in \mathbb{R}P^2$ from the domain encircled by the interior oval of the nest. Consider the great circle of $S^2$ consisting of halves of real lines which pass through $P$. Since each line passing through $P$ intersects $\mathbb{R}A$ in $m$ points, it cannot intersect $\mathbb{C}A \smallsetminus \mathbb{R}A$. Therefore the great circle has no common point with the image of $\mathbb{C}A \smallsetminus \mathbb{R}A$ under $p : \mathbb{C}P^2 \smallsetminus \mathbb{R}P^2 \to S^2$. But the image contains, together with any of its points, the antipodal point. Therefore it cannot be connected, and $\mathbb{C}A \smallsetminus \mathbb{R}A$ cannot be connected, too. $\square$

2.7. **Rokhlin's Complex Orientation Formula.** Now we shall consider a powerful restriction on a complex orientation of a curve of type I. It is powerful enough to imply restrictions even on *real* schemes of type I. The first version of this restriction was published in 1974, see

[Rok-74]. There Rokhlin considered only the case of an algebraic M-curve of even degree. In [Mis-75] Mishachev considered the case of an algebraic M-curve of odd degree. For an arbitrary nonsingular algebraic curve of type I, it was formulated by Rokhlin [Rok-78] in 1978. The proofs from [Rok-74] and [Mis-75] work in this general case. The only reason to restrict the main formulations in these early papers to M-curves was the traditional viewpoint on the subject of the topology of real plane algebraic curves.

Here are Rokhlin's formulations from [Rok-78].

*2.7.A* (Rokhlin Formula)**.** *If the degree $m$ is even and the curve is of type* I*, then*

$$2(\Pi^+ - \Pi^-) = l - \frac{m^2}{4}.$$

*2.7.B* (Rokhlin-Mishachev Formula)**.** *If $m$ is odd and the curve is of type* I*, then*

$$\Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = l - \frac{m^2 - 1}{4}.$$

Theorems *2.7.A* and *2.7.B* can be united into a single formulation. This requires, however, two preliminary definitions.

First, given an oriented topological curve $C$ on $\mathbb{R}P^2$, for any point $x$ of its complement, there is the index $i_C(x)$ of the point with respect to the curve. It is a nonnegative integer defined as follows. Draw a line $L$ on $\mathbb{R}P^2$ through $x$ transversal to $C$. Equip it with a normal vector field vanishing only at $x$. For such a vector field, one may take the velocity field of a rotation of the line around $x$. At each intersection point of $L$ and $C$ there are two directions transversal to $L$: the direction of the vector belonging to the normal vector field and the direction defined by the local orientation of $C$ at the point. Denote the number of intersection points where the directions are faced to the same side of $L$ by $i_+$ and the number of intersection points where the directions are faced to the opposite sides of $L$ by $i_-$. Then put $i_C(x) = |i_+ - i_-|/2$.[5] It is easy to check that $i_C(x)$ is well defined: it depends neither on the choice of $L$, nor on the choice of the normal vector field. It does not change under reversing of the orientation of

[5]Division by 2 appears here to make this notion closer to the well-known notion for an affine plane curve. In the definition for affine situation one uses a ray instead of entire line. In the projective situation there is no natural way to divide a line into two rays, but we still have an opportunity to divide the result by 2. Another distinction from the affine situation is that there the index may be negative. It is related to the fact that the affine plane is orientable, while the projective plane is not.

$C$. Thus for any nonsingular curve $A$ of type I on the complement $\mathbb{R}P^2 \smallsetminus \mathbb{R}A$, one has well defined function $i_{\mathbb{R}A}$.

The second prerequisite notion is a sort of unusual integration: an integration with respect to the Euler characteristic, in which the Euler characteristic plays the role of a measure. It is well known that the Euler characteristic shares an important property of measures: it is additive in the sense that for any sets $A$, $B$ such that the Euler characteristics $\chi A$, $\chi B$, $\chi(A \cap B)$ and $\chi(A \cup B)$ are defined,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

However, the Euler characteristic is neither $\sigma$-additive, nor positive. Thus the usual theory of integral cannot be applied to it. This can be done though if one restricts to a very narrow class of functions. Namely, to functions which are finite linear combinations of characteristic functions of sets belonging to some algebra of subsets of a topological space such that each element of the algebra has a well defined Euler characteristic. For a function $f = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{S_i}$ set

$$\int f(x)\, d\chi(x) = \sum_{i=1}^{r} \lambda_i \chi(S_i).$$

For details and applications of that notion, see [Vir-88].

Now we can unite *2.7.A* and *2.7.B*:

*2.7.C* (Rokhlin Complex Orientation Formula). *If $A$ is a nonsingular real plane projective curve of type* I *and degree $m$ then*

$$\int (i_{\mathbb{R}A}(x))^2\, d\chi(x) = \frac{m^2}{4}.$$

Here I give a proof of *2.7.C*, skipping the most complicated details. Take a curve $A$ of degree $m$ and type I. Let $\mathbb{C}A_+$ be its half bounded by $\mathbb{R}A$. It may be considered as a chain with integral coefficients. The boundary of this chain (which is $\mathbb{R}A$ equipped with the complex orientation) bounds in $\mathbb{R}P^2$ a chain $c$ with rational coefficients, since $H_1(\mathbb{R}P^2; \mathbb{Q}) = 0$. In fact, in the case of even degree the chain can be taken with integral coefficients, but in the case of odd degree the coefficients are necessarily half-integers. The explicit form of $c$ may be given in terms of function $i_{\mathbb{R}A}$: it is a linear combination of the fundamental cycles of the components of $\mathbb{R}P^2 \smallsetminus \mathbb{R}A$ with coefficients equal to the values of $i_{\mathbb{R}A}$ on the components (taken with appropriate orientations).

Now take the cycle $[\mathbb{C}A_+] - c$ and its image under $conj$, and calculate their intersection number in two ways.

First, it is easy to see that the homology class $\xi$ of $[\mathbb{C}A_+]-c$ is equal to $\frac{1}{2}[\mathbb{C}A] = \frac{m}{2}[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2;\mathbb{Q})$. Indeed, $[\mathbb{C}A_+]-c-conj([\mathbb{C}A_+]-c) = [\mathbb{C}A] + c - conj(c) = [\mathbb{C}A]$, and therefore $\xi - conj_*(\xi) = [\mathbb{C}A] = m[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$. On the other hand, $conj$ acts in $H_2(\mathbb{C}P^2)$ as multiplication by $-1$, and hence $\xi - conj_*(\xi) = 2\xi = m[\mathbb{C}P^1]$. Therefore $\xi \circ conj_*(\xi) = -(\frac{m}{2})^2$.

Second, one may calculate the same intersection number geometrically: moving the cycles into a general position and counting the local intersection numbers. I will perturb the cycle $[\mathbb{C}A_+]-c$. First, choose a smooth tangent vector field $V$ on $\mathbb{R}P^2$ such that it has only nondegenerate singular points, the singular points are outside $\mathbb{R}A$, and on $\mathbb{R}A$ the field is tangent to $\mathbb{R}A$ and directed according to the complex orientation of $A$ which comes from $\mathbb{C}A_+$. The latter means that at any point $x \in \mathbb{R}A$ the vector $\sqrt{-1}V(x)$ is directed inside $\mathbb{C}A_+$ (the multiplication by $\sqrt{-1}$ makes a real vector normal to the real plane and lieves any vector tangent to $\mathbb{R}A$ tangent to $\mathbb{C}A$). Now shift $\mathbb{R}A$ inside $\mathbb{C}A_+$ along $\sqrt{-1}V$ and extend this shift to a shift of the whole chain $c$ along $\sqrt{-1}V$. Let $c'$ denote the result of the shift of $c$ and $h$ denote the part of $\mathbb{C}A_+$ which was not swept during the shift. The cycle $[h]-c'$ represents the same homology class $\xi$ as $[\mathbb{C}A_+]-c$, and we can use it to calculate the intersection number $\xi \circ conj_*(\xi)$. The cycles $[h]-c'$ and $conj([\mathbb{C}A_+]-c)$ intersect only at singular points of $V$. At a singular point $x$ they are smooth transversal two-dimensional submanifolds, each taken with multiplicity $-i_{\mathbb{R}A}(x)$. The local intersection number at $x$ is equal to $(i_{\mathbb{R}A}(x))^2$ multiplied by the local intersection number of the submanifolds supporting the cycles. The latter is equal to the index of the vector field $V$ at $x$ multiplied by $-1$.

I omit the proof of the latter statement. It is nothing but a straightforward checking that multiplication by $\sqrt{-1}$ induces isomorphism between tangent and normal fibrations of $\mathbb{R}A$ in $\mathbb{C}A$ reversing orientation.

Now recall that the sum of indices of a vector field tangent to the boundary of a compact manifold is equal to the Euler characteristic of the manifold. Therefore the input of singular points lying in a connected component of $\mathbb{R}P^2 \smallsetminus \mathbb{R}A$ is equal to the Euler characteristic of the component multiplied by $-(i_{\mathbb{R}A}(x))^2$ for any point $x$ of the component. Summation over all connected components of $\mathbb{R}P^2 \smallsetminus \mathbb{R}A$ gives $-\int (i_{\mathbb{R}A}(x))^2\, d\chi(x)$. Its equality to the result of the first calculation is the statement of *2.7.C*. $\square$

*2.7.D* (Corollary 1. Arnold Congruence). *For a curve of an even degree $m = 2k$ and type* I

$$p - n \equiv k^2 \mod 4.$$

*Proof.* Observe that in the case of an even degree $i_{\mathbb{R}A}(x)$ is even, iff $x \in \mathbb{R}P^2_+$. Therefore

$$(i_{\mathbb{R}A}(x))^2 \equiv \begin{cases} 0 \mod 4, & \text{if } x \in \mathbb{R}P^2_+ \\ 1 \mod 4, & \text{if } x \in \mathbb{R}P^2_-. \end{cases}$$

Thus

$$\int_{\mathbb{R}P^2} (i_{\mathbb{R}A}(x))^2 \, d\chi(x) \equiv \chi(\mathbb{R}P^2_+) \mod 4.$$

Recall that $\chi(\mathbb{R}P^2_+) = p - n$, see 1.11. Hence the left hand side of Rokhlin's formula is $p - n$ modulo 4. The right hand side is $k^2$. □

Denote the number of all injective pairs of ovals for a curve under consideration by $\Pi$.

*2.7.E* (Corollary 2)**.** *For any curve of an even degree $m = 2k$ and type* I *with $l$ ovals*

$$\Pi \geq \frac{1}{2}|l - k^2|.$$

*Proof.* By *2.7.A* $\Pi^+ - \Pi_- = \frac{1}{2}(l - k^2)$. On the other hand, $\Pi = \Pi_+ + \Pi_- \geq |\Pi_+ - \Pi_-|$. □

*2.7.F* (Corollary 3)**.** *For any curve of an odd degree $m = 2k + 1$ and type* I *with $l$ ovals*

$$\Pi + l \geq \frac{1}{2}k(k+1).$$

*Proof.* Since $l = \Lambda_+ + \Lambda_-$, the Rokhlin - Mishachev formula *2.7.B* can be rewritten as follows:

$$\Lambda_- + \Pi_- - \Pi_+ = \frac{1}{2}k(k+1).$$

On the other hand, $\Pi \geq \Pi_- - \Pi_+$ and $l \geq \Lambda_-$. □

2.8. **Complex Schemes of Degree $\leq 5$.** As it was promised in Section 2.5, we can prove now that only schemes realized in Figures 17, 18 and 19 are realizable by curves of degree 3, 4 and 5, respectively. For reader's convinience, I present here a list of all these complex schemes in Table 5.

**Degree 3.** By Harnack's inequality, the number of components is at most 2. By *1.3.B* a curve of degree 3 is one-sided, thereby the number of components is at least 1. In the case of 1 component the real scheme is $\langle J \rangle$, and the type is II by Klein's congruence *2.6.C*. In the case of 2 components the type is I by *2.6.B*. The real scheme is $\langle J \amalg 1 \rangle$. Thus we have 2 possible complex schemes: $\langle J \amalg 1^- \rangle^3_I$ (realized above) and $\langle J \amalg 1^+ \rangle^3_I$. For the first one $\int (i_{\mathbb{R}A}(x))^2 \, d\chi(x) = 9/4$ and

Table 5

| $m$ | Complex schemes of nonsingular plane curves of degree $m$ |
|---|---|
| 1 | $\langle J\rangle^1_I$ |
| 2 | $\langle 1\rangle^2_I$<br>$\langle 0\rangle^2_{II}$ |
| 3 | $\langle J\amalg 1^-\rangle^3_I$<br>$\langle J\rangle^3_{II}$ |
| 4 | $\langle 4\rangle^4_I$<br>$\langle 3\rangle^4_{II}$<br>$\langle 1\langle 1^-\rangle\rangle^4_I$ $\langle 2\rangle^4_{II}$<br>$\langle 1\rangle^4_{II}$<br>$\langle 0\rangle^4_{II}$ |
| 5 | $J\amalg 3^+\amalg 3^-\rangle^5_I$<br>$\langle J\amalg 5\rangle^5_{II}$<br>$\langle J\amalg 1^+\amalg 3^-\rangle^5_I$ $\langle J\amalg 4\rangle^5_{II}$<br>$\langle J\amalg 3\rangle^5_{II}$<br>$\langle J\amalg 1^-\langle 1^-\rangle\rangle^5_I$ $\langle J\amalg 2\rangle^5_{II}$<br>$\langle J\amalg 1\rangle^5_{II}$<br>$\langle J\rangle^5_{II}$ |

for the second $\int(i_{\mathbb{R}A}(x))^2\,d\chi(x) = 1/4$. Since the right hand side of the complex orientation formula is $m^2/4$ and $m = 3$, only the first possibility is realizable. $\square$

**Degree 4.** By Harnack's inequality the number of components is at most 4. We know (see 1.4) that only real schemes $\langle 0\rangle$, $\langle 1\rangle$, $\langle 2\rangle$, $\langle 1\langle 1\rangle\rangle$, $\langle 3\rangle$ and $\langle 4\rangle$ are realized by nonsingular algebraic curves of degree 4. From Klein's congruence *2.6.C* it follows that the schemes $\langle 1\rangle$ and $\langle 3\rangle$ are of type II. The scheme $\langle 0\rangle$ is of type II by *2.6.A*. By *2.6.B* $\langle 4\rangle$ is of type I.

The scheme $\langle 2\rangle$ is of type II, since it admits no orientation satisfying the complex orientation formula. In fact, for any orientation $\int(i_{\mathbb{R}A}(x))^2\,d\chi(x) = 2$ while the right hand side is $m^2/4 = 4$.

By *2.6.D* the scheme $\langle 1\langle 1\rangle\rangle$ is of type I. A calculation similar to the calculation above on the scheme $\langle 2\rangle$, shows that only one of the two semiorientations of the scheme $\langle 1\langle 1\rangle\rangle$ satisfies the complex orientation formula. Namely, $\langle 1\langle 1^-\rangle\rangle$. It was realized in Figure 18.

**Degree 5.** By Harnack's inequality the number of components is at most 7. We know (see 1.4) that only real schemes $\langle J\rangle$, $\langle J\amalg 1\rangle$, $\langle J\amalg 2\rangle$, $\langle J\amalg 1\langle 1\rangle\rangle$, $\langle J\amalg 3\rangle$, $\langle J\amalg 4\rangle$, $\langle J\amalg 5\rangle$, $\langle J\amalg 6\rangle$ are realized by nonsingular algebraic curves of degree 5. From Klein's congruence *2.6.C* it follows that the schemes $\langle J \amalg 1\rangle$, $\langle J \amalg 3\rangle$, $\langle J \amalg 5\rangle$ are of type II. By *2.7.F* $\langle J\rangle$ and $\langle J \amalg 2\rangle$ are of type II.

By *2.6.B* $\langle J \amalg 6\rangle$ is of type I. The complex orientation formula gives the value of $\Lambda_-$ (cf. Proof of *2.7.F*): $\Lambda_- = \frac{1}{2}k(k+1) = 3$. This determines the complex scheme. It is $\langle J \amalg 3^- \amalg 3^+\rangle^5_I$.

By *2.6.D* $\langle J \amalg 1\langle 1\rangle\rangle$ is of type I. The complex orientation formula allows only the semiorientation with $\Lambda_- = 2$. Cf. Figure 19.

The real scheme $\langle J \amalg 4\rangle$ is of indefinite type, as follows from the construction shown in Figure 19. In the case of type I only one semiorientation is allowed by the the complex orientation formula. It is $\langle J \amalg 3^- \amalg 1^+\rangle^5_I$.

**Exercises. 2.1** Prove that for any two semioriented curves with the same code (of the kind introduced in 3.7) there exists a homeomorphism of $\mathbb{R}P^2$ which maps one of them to another preserving semiorientations.

**2.2** Prove that for any two curves $A_1$, $A_2$ with the same code of their complex schemes (see Subsection 2.5) there exists a homeomorphism $\mathbb{C}A_1 \cup \mathbb{R}P^2 \to \mathbb{C}A_2 \cup \mathbb{R}P^2$ commuting with *conj*.

**2.3** Deduce *2.7.A* and *2.7.B* from *2.7.C* and, vise versa, *2.7.C* from *2.7.A* and *2.7.B*.