3. The Topological Point of View on Prohibitions

3.1. **Flexible Curves.** In Section 1 all prohibitions were deduced from the Bézout Theorem. In Section 2 many proofs were purely topological. A straightforward analysis shows that the proofs of all prohibitions are based on a small number of basic properties of the complexification of a nonsingular plane projective algebraic curve. It is not difficult to list all these properties of such a curve A:

(1) Bézout's theorem;
(2) $\mathbb{C}A$ realizes the class $m[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$;
(3) $\mathbb{C}A$ is homeomorphic to a sphere with $(m-1)(m-2)/2$ handles;
(4) $conj(\mathbb{C}A) = conj$;
(5) the tangent plane to $\mathbb{C}A$ at a point $x \in \mathbb{R}A$ is the complexification of the tangent line of $\mathbb{R}A$ at x.

The last four are rough topological properties. Bézout's theorem occupies a special position. If we assume that some surface smoothly embedded into $\mathbb{C}P^2$ intersects the complex point set of any algebraic curve as, according to Bézout's theorem, the complex point set of an algebraic curve, then this surface is the complex point set of an algebraic curve. Thus the Bézout theorem is completely responsible for the whole set of properties of algebraic curves. On the other hand, its usage in obtaining prohibitions involves a construction of auxiliary curves, which may be very subtle.

Therefore, along with algebraic curves, it is useful to consider objects which imitate them topologically.

An oriented smooth closed connected two-dimensional submanifold S of the complex projective plane $\mathbb{C}P^2$ is called a *flexible curve of degree* m if:

(i) S realizes $m[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$;
(ii) the genus of S is equal to $(m-1)(m-2)/2$;
(iii) $conj(S) = S$;
(iv) the field of planes tangent to S on $S \cap \mathbb{R}P^2$ can be deformed in the class of planes invariant under $conj$ into the field of (complex) lines in $\mathbb{C}P^2$ which are tangent to $S \cap \mathbb{R}P^2$.

A flexible curve S intersects $\mathbb{R}P^2$ in a smooth one-dimensional submanifold, which is called the *real part* of S and denoted by $\mathbb{R}S$. Obviously, the set of complex points of a nonsingular algebraic curve of degree m is a flexible curve of degree m. Everything said in Section 2.1 about algebraic curves and their (real and complex) schemes carries over without any changes to the case of flexible curves. We say that a prohibition on the schemes of curves of degree m *comes from* *topology* if it can be proved for the schemes of flexible curves of degree m. The known classification of schemes of degree ≤ 6 can be obtained using only the prohibitions that come from topology. In other words, for $m \leq 6$ all prohibitions come from topology.

3.2. **The Most Elementary Prohibitions on Real Topology of a Flexible Curve.** The simplest prohibitions are not related to the position of $\mathbb{R}S$ in $\mathbb{R}P^2$, but deal with the following purely topological situation: a surface S, which is homeomorphic to a sphere with g ($= (m-1)(m-2)/2$) handles, and an involution c ($=$ *conj*) of S reversing orientation with fixed point set F ($= \mathbb{R}S$).

The most important of these prohibitions is Harnack's inequality. Recall that it is

$$L \leq \frac{(m-1)(m-2)}{2} + 1,$$

where L is the number of connected components of the real part a curve and m is its degree. Certainly, this formulation given in Section 1.3 can be better adapted to the context of flexible curves. The number $\frac{(m-1)(m-2)}{2}$ is nothing but the genus. Therefore the Harnack inequality follows from the following theorem.

3.2.A. *For a reversing orientation involution $c : S \to S$ of a sphere S with g handles, the number L of connected components of the fixed point set F is at most $g+1$.*

In turn, *3.2.A* can be deduced from the following purely topological theorem on involutions:

3.2.B (Smith-Floyd Theorem). *For any involution i of a topological space X,*

$$\dim_{\mathbb{Z}_2} H_*(fix(i); \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_*(X; \mathbb{Z}_2).$$

This theorem is one of the most famous results of the Smith theory. It is deduced from the basic facts on equivariant homology of involution, see, e. g., [Bre-72, Chapter 3].

Theorem *3.2.A* follows from *3.2.B*, since

$$\dim_{\mathbb{Z}_2} H_*(S; \mathbb{Z}_2) = 2 + 2g,$$

and

$$\dim_{\mathbb{Z}_2} H_*(F; \mathbb{Z}_2) = 2L.$$

Smith - Floyd Theorem can be applied to high-dimensional situation, too. See Sections 5.3 and **??**. In the one-dimensional case, which we deal with here, Theorem *3.2.B* is easy to prove without any homology tool, like the Smith theory. Namely, consider the orbit space S/c of the involution. It is a connected surface with boundary. The boundary is the image of the fixed point set. The Euler characteristic of the orbit space is equal to the half of the Euler characteristic of S, i.e. it is $\frac{2-2g}{2} = 1-g$. Cap each boundary circle with a disk. The result is a closed connected surface with Euler characteristic $1-g+L$. On the other hand, as it is well known, the Euler characteristic of a connected closed surface is at most 2. (Remind that such a surface is homeomorphic either to the sphere, which has Euler characteristic 2, or the sphere with h handles, whose Euler characteristic is $2-2h$, or sphere with h Möbius strips having Euler characteristic $2-h$.) Therefore $1-g+L \le 2$, and $L \le g+1$. □

These arguments contain more than just a proof of *2.3.A*. In particular, they imply that

3.2.C. *In the case of an M-curve (i.e., if* $L = g+1$*) and only in this case, the orbit space is a sphere with holes.*

Similarly, in the case of an $(M-1)$-curve, the orbit space is homeomorphic to the projective plane with holes.

If F separates S (i.e., $S \setminus F$ is not connected), the involution c is said to be of *type* I, otherwise it is said to be of *type* II. The types correspond to the types of real algebraic curves (see Section 2.1).

Note that F separates S at most into two pieces. To prove this, we can use the same arguments as in a footnote in Section 2.1: the closure of tne union of a connected component of $S \setminus F$ with its image under c is open and close in S, but S is connected.

3.2.D. *The orbit space* S/c *is orientable if and only if* F *separates* S.

Proof. Assume that F separates S. Then the halves are homeomorphic, since the involution maps each of them homeomorphically onto the other one. Therefore, each of the halves is homeomorphic to the orbit space. The halves are orientable since the whole surface is.

On the other hand, if F does not separate S, then one can connect a point of $S \setminus F$ to its image under the involution by a path in the complement $S \setminus F$. Such a path covers a loop in the orbit space. This is an orientation reversing loop, since the involution reverses orientation. □

3.2.E ((Cf. *2.6.C*)). *If the curve is of type* I, *then* $L \equiv \left[\frac{m+1}{2}\right] \mod 2$.

Proof. This theorem follows from *3.2.C* and the calculation of the Euler characteristic of S/c made in the proof of the Harnack inequality above. Namely, $\chi(S/c) = 1-g$, but for any orientable connected surface with Euler characteristic χ and L boundary components $\chi + L \equiv 0 \mod 2$. Therefore $1-g+L \equiv 0 \mod 2$. Since $g = (m-1)(m-2)/2 \equiv \left[\frac{m-1}{2}\right]$

mod 2, we obtain $1 - \left[\frac{m-1}{2}\right] + L \equiv 0 \mod 2$ which is equivalent to the desired congruence. □

3.2.F ((Cf. *2.6.B*)). *Any M-curve is of type* I.

Proof. By *3.2.C*, in the case of M-curve the orbit space S/c is homeomorphic to a sphere with holes. In particular, it is orientable. By *3.2.D*, this implies that F separates S. □

Now consider the simplest prohibition involving the placement of the real part of the flexible curve in the projective plane.

3.2.G. *The real part of a flexible curve is one-sided if and only if the degree is odd.*

Proof. The proof of *3.2.G* coincides basically with the proof of the same statement for algebraic curves. One has to consider a real projective line transversal to the flexible curve and calculate the intersection number of the complexification of this line and the lfexible curve. On one hand, it is equal to the degree of the flexible curve. On the other hand, the intersection points in $\mathbb{C}P^2 \setminus \mathbb{R}P^2$ give rise to an even contribution to the intersection number. □

Rokhlin's complex orientation formula also comes from topology. The proof presented in Section 2.7 works for a flexible curve.

At this point I want to break a textbook style exposition. Escaping a detailed exposition of prohibitions, I switch to a survey.

In the next two sections, the current state of prohibitions on the topology of a flexible curve of a given degree is outlined. (Recall that all formulations of this sort are automatically valid for real projective algebraic plane curves of the same degree.) After the survey a light outline of some proofs is proposed. It is included just to convey a general impression, rather than for more serious purposes. For complete proofs, see the surveys [Wil-78], [Rok-78], [Arn-79], [Kha-78], [Kha-86], [Vir-86] and the papers cited there.

3.3. **A Survey of Prohibitions on the Real Schemes Which Come from Topology.** In this section I list all prohibitions on the real scheme of a flexible curve of degree m that I am aware of, including the ones already referred to above, but excluding prohibitions which follow from the other prohibitions given here or from the prohibitions on the complex schemes which are given in the next section.

3.3.A. *A curve is one-sided if and only if it has odd degree.*

This fact was given before as a corollary of Bézout's theorem (see Section 1.3) and proved for flexible curves in Section 3.2 (Theorem *3.2.G*).

3.3.B (Harnack's Inequality). *The number of components of the set of real points of a curve of degree m is at most $\frac{(m-1)(m-2)}{2}+1$.*

Harnack's inequality is undoubtedly the best known and most important prohibition. It can also be deduced from Bézout's theorem (cf. Section 1.3) and was proved for flexible curves in Section 3.2 (Theorem *3.2.A*).

In prohibitions *3.3.C*–*3.3.P* the degree m of the curve is even: $m = 2k$.

Extremal Properties of Harnack's Inequality

3.3.C (Gudkov-Rokhlin Congruence). *In the case of an M-curve (i.e., if $p+n=(m-1)(m-2)/2+1$),*

$$p-n \equiv k^2 \mod 8.$$

3.3.D (Gudkov-Krakhnov-Kharlamov Congruence). *In the case of an $(M-1)$-curve (i.e., if $p+n=\frac{(m-1)(m-2)}{2}$),*

$$p-n \equiv k^2 \pm 1 \mod 8.$$

The Euler characteristic of a component of the complement of a curve in $\mathbb{R}P^2$ is called the *characteristic* of the oval which bounds the component from outside. An oval with a positive characteristic is said to be *elliptic,* an oval with the zero characteristic is said to be *parabolic* and an oval with a negative characteristic is said to be *hyperbolic.*

3.3.E (Fiedler's Congruence). *If the curve is an M-curve, $m \equiv 4 \mod 8$, and every even oval has an even characteristic, then*

$$p-n \equiv -4 \mod 16.$$

3.3.F (Nikulin's Congruence). *If the curve is an M-curve, $m \equiv 0 \mod 8$, and the characteristic of every even oval is divisible by 2^r, then*

$$\text{either} \quad p-n \equiv 0 \mod 2^{r+3}, \tag{3}$$

$$\text{or else} \quad p-n = 4^q\chi, \tag{4}$$

where $q \geq 2$ and $\chi \equiv 1 \mod 2$.

3.3.G (Nikulin's Congruence). *If the curve is an M-curve, $m \equiv 2 \mod 4$ and the characteristic of every odd oval is divisible by 2^r, then*

$$p-n \equiv 1 \mod 2^{r+3}.$$

Denote the number of even ovals with positive characteristic by p^+, the number of even ovals with zero characteristic by p^0, and the number of even ovals with negative characteristic by p^-. Similarly define n^+, n^0 and n^- for the odd ovals; and let l^+, l^0 and l^- be the corresponding numbers for both even and odd ovals together.

Refined Petrovsky Inequalities

3.3.H. $p - n^- \leq \frac{3k(k-1)}{2} + 1$.

3.3.I. $n - p^- \leq \frac{3k(k-1)}{2}$.

Refined Arnold Inequalities

3.3.J. $p^- + p^0 \leq \frac{(k-1)(k-2)}{2} + \frac{1+(-1)^k}{2}$.

3.3.K. $n^- + n^0 \leq \frac{(k-1)(k-2)}{2}$.

Extremal Properties of the Refined Arnold Inequalities

3.3.L. *If k is even and $p^- + p^0 = \frac{(k-1)(k-2)}{2} + 1$, then $p^- = p^+ = 0$.*

3.3.M. *If k is odd and $n^- + n^0 = \frac{(k-1)(k-2)}{2}$, then $n^- = n^+ = 0$ and there is only one outer oval at all.*

Viro-Zvonilov Inequalities

Besides Harnack's inequality, we know only one family of prohibition coming from topology which extends to real schemes of both even and odd degree. For proofs see [VZ-92].

3.3.N (Bound of the Number of Hyperbolic Ovals). *The number of components of the complement of a curve of odd degree m that have a negative Euler characteristic does not exceed $\frac{(m-3)^2}{4}$. In particular, for any odd m*

$$l^- \leq \frac{(m-3)^2}{4}.$$

The latter inequality also holds true for even $m \neq 4$, but it follows from Arnold inequalities *3.3.J* and *3.3.K*.

3.3.O (Bound of the Number of Nonempty Ovals). *If h is a divisor of m and a power of an odd prime, and if $m \neq 4$, then*

$$l^- + l^0 \leq \frac{(m-3)^2}{4} + \frac{m^2 - h^2}{4h^2}.$$

If m is even, this inequality follows from *3.3.J–3.3.L*.

3.3.P (Extremal Property of the Viro-Zvonilov Inequality). *If*

$$l^- + l^0 = \frac{(m-3)^2}{4} + \frac{m^2 - h^2}{4h^2},$$

where h is a divisor of m and a power of an odd prime p, then there exist $\alpha_1, \dots, \alpha_r \in \mathbb{Z}_p$ and components $B_1, \dots, B_r$ of the complement $\mathbb{R}P^2 \setminus \mathbb{R}A$ with $\chi(B_1) = \dots = \chi(B_r) = 0$, such that the boundary of the chain $\sum_{i=1}^r \alpha_i[B_i] \in C_2(\mathbb{R}P^2; \mathbb{Z}_p)$ is $[\mathbb{R}A] \in C_1(\mathbb{R}P^2; \mathbb{Z}_p)$.

3.4. **Survey of Prohibitions on the Complex Schemes Which Come From Topology.** Recall that l denotes the total number of ovals on the curve. The following theorem is a reformulation of *3.2.E*.

3.4.A (See *2.6.A*). *A curve with empty real point set is of type* II.

3.4.B ((See *2.6.C*)). *If the curve is of type* I, *then*

$$l \equiv \left[\frac{m}{2}\right] \mod 2.$$

3.4.C (Rokhlin Complex Orientation Formula (see *2.7.C*)). *Let A be a nonsingular curve of type* I *and degree m. Then*

$$\int (i_{\mathbb{R}A}(x))^2 \, d\chi(x) = \frac{m^2}{4}$$

Extremal Properties of Harnack's Inequality

3.4.D ((Cf. *2.6.B*)). *Any M-curve is of type* I.

3.4.E (Kharlamov-Marin Congruence). *Any $(M-2)$-curve of even degree $m = 2k$ with*

$$p - n \equiv k^2 + 4 \mod 8$$

is of type I.

Extremal Properties of the Refined Arnold Inequalities

3.4.F. *If $m \equiv 0 \mod 4$ and $p^- + p^0 = \frac{(m-2)(m-4)}{8} + 1$, then the curve is of type* I.

3.4.G. *If $m \equiv 0 \mod 4$ and $n^- + n^0 = \frac{(m-2)(m-4)}{8}$, then the curve is of type* I.

Extremal Properties of the Viro-Zvonilov Inequality

3.4.H. *Under the hypothesis of 3.3.P, the curve is of type* I.

Congruences

3.4.I (Nikulin-Fiedler Congruence). *If* $m \equiv 0 \mod 4$*, the curve is of type* I*, and every even oval has even characteristic, then* $p - n \equiv 0 \mod 8$.

The next two congruences are included violating a general promise given at the beginning of the previous section. There I promised exclude prohibitions which follow from other prohibitions given here. The following two congruences are consequences of Rokhlin's formula *3.4.C*. The first of them was discovered long before *3.4.C*. The second was overlooked by Rokhlin in [Rok-74], where he even mistakenly proved that such a result cannot exist. Namely, Rokhlin proved that the complex orientation formula does not imply any result which would not follow from the prohibitions known by that time and could be formulated solely in terms of the real scheme. Slepian congruence *3.4.K* for M-curves is the only counter-example to this Rokhlin's statement. Slepian was Rokhlin's student, he discovered a gap in Rokhlin's arguments and deduced *3.4.K*.

3.4.J (Arnold Congruence (see *2.7.D*)). *If* m *is even and the curve is of type I, then*

$$p - n \equiv \frac{m^2}{4} \mod 4.$$

3.4.K (Slepian Congruence). *If* m *is even, the curve is of type* I*, and every odd oval has even characteristic, then*

$$p - n \equiv \frac{m^2}{4} \mod 8.$$

Rokhlin Inequalities

Denote by π and ν the number of even and odd nonempty ovals, respectively, bounding from the outside those components of the complement of the curve which have the property that each of the ovals bounding them from the inside envelops an odd number of other ovals.

3.4.L. *If the curve is of type* I *and* $m \equiv 0 \mod 4$*, then*

$$4\nu + p - n \le \frac{(m-2)(m-4)}{2} + 4.$$

3.4.M. *If the curve is of type* I *and* $m \equiv 2 \mod 4$*, then*

$$4\pi + n - p \le \frac{(m-2)(m-4)}{2} + 3.$$

3.5. **Ideas of Some Proofs.** Theorems formulated in 3.3 and 3.4 are very different in their profundity. The simplest of them were considered in Subsection 3.2.

Congruences

There are two different approaches to proving congruences. The first is due basically to Arnold [Arn-71] and Rokhlin [Rok-72]. It is based on consideration of the intersection form of two-fold covering Y of $\mathbb{C}P^2$ branched over the complex point set of the curve. The complex conjugation involution $conj : \mathbb{C}P^2 \to \mathbb{C}P^2$ is lifted to Y in two different ways, and the liftings induce involutions in $H_2(Y)$, which are isometries of the intersection form. One has to take an appropriate eigenspace of one of the liftings and consider the restriction of the intersection form to the eigenspace. The signature of this restriction can be calculated in terms of $p - n$. On the other hand, it is involved into some congruences of purely arithmetic nature relating it with the discriminant of the form and the value of the form on some of characteristic vectors. The latters can be calculated sometimes in terms of degree and the difference between the number of ovals and the genus of curve. Realizations of this scheme can be found in [Arn-71] for *3.4.J*, [Rok-72] for *3.3.C*, [Kha-73] and [GK-73] for *3.3.D*, [Nik-83] for *3.3.F*, *3.3.G*, *3.4.I* and a weakened form of *3.3.E*. In survey [Wil-78] this method was used for proving *3.3.C*, *3.3.D* and *3.4.J*.

The second approach is due to Marin [Mar-80]. It is based on application of the Rokhlin-Guillou-Marin congruence modulo 16 on characteristic surface in a 4-manifold, see [GM-77]. It is applied either to the surface in the quotient space $\mathbb{C}P^2/conj$ (diffeomorphic to S^4) made of the image of the flexible curve S and a half of $\mathbb{R}P^2$ bounded by $\mathbb{R}S$ (as it is the case for proofs of *3.3.C*, *3.3.D* and *3.4.E* in [Mar-80]), or to the surface in $\mathbb{C}P^2$ made of a half of S and a half of $\mathbb{R}P^2$ (as it is the case for proofs of *3.3.E*, *3.4.I* and special cases of *3.3.F* and *3.3.G* in [Fie-83]).

The first approach was applied also in high-dimensional situations. The second approach worked better than the second one for curves on surfaces distinct from projective plane, see [Mik-94]. Both were used for singular curves [KV-88].

Inequalities

Inequalities *3.3.H*, *3.3.I*, *3.3.J*, *3.3.K*, *3.4.J* and *3.4.K* are proved along the same scheme, originated by Arnold [Arn-71]. One constructs an auxiliary manifold, which is the two-fold covering of $\mathbb{C}P^2$ branched over S in the case of *3.3.H*, *3.3.I*, *3.3.J* and *3.3.K* and the two-fold covering of $\mathbb{C}P^2/conj$ branched over the union of $S/conj$ and a half of $\mathbb{R}P^2$ in the case of *3.4.J* and *3.4.K*. Then preimages of some of the components of $\mathbb{R}P^2 \smallsetminus \mathbb{R}S$ gives rise to cycles in this manifold. Those cycles define homology classes with special properties formulated in terms of their behavior with respect to the intersection form and the complex conjugation involutions. On the other hand, the numbers of homology classes with these properties are estimated. See [Arn-71], [Gud-74], [Wil-78] and [Rok-80].

3.6. **Flexible Curves of Degrees ≤ 5.** In this subsection, I show that for degrees ≤ 5 the prohibitions coming from topology allow the same set of complex schemes as all prohibitions. The set of complex schemes of algebraic curves of degrees ≤ 5 was described in 2.8. In fact the same is true for degree 6 too. For degree greater than 6, it is not known, but there is no reason to believe that it is the case.

Degrees ≤ 3. Theorems *3.3.A* and the Harnack inequality *3.3.B* prohibit all non realizable real schemes for degree ≤ 3. To obtain the complete set of prohibitions for complex schemes of degrees ≤ 3 one has to add the Klein congruence *3.4.B*, *3.4.D* and the complex orientation formula *3.4.C*; cf. Section 2.8.

Degree 4. By the Arnold inequlity *3.3.K*, a flexible curve of degree 4 cannot have a nest of depth 3. By the Arnold inequality *3.3.J*, it has at most one nonempty positive oval, and if it has a nonempty oval then, by the extremal property *3.3.L* of this inequality, the real scheme is $\langle 1\langle 1\rangle\rangle$. Together with *3.3.A* and the Harnack inequality *3.3.B*, this forms the complete set of prohibitions for real schemes of degree 4.

From the Klein congruence *3.4.B*, it follows that the real schemes $\langle 1\rangle$ and $\langle 3\rangle$ are of type II. The empty real scheme $\langle 0\rangle$ is of type II by *3.4.A*. By the extremal property *3.4.D* of the Harnack inequality, $\langle 4\rangle$ is of type I. The real scheme $\langle 2\rangle$ is of type II by the complex orientation formula *3.4.C*, cf. Section 2.8. By *3.4.F*, the scheme $\langle 1\langle 1\rangle\rangle$ is of type I. By the complex orientation formula, it admits only the complex orientation $\langle 1\langle 1^-\rangle\rangle$.

Degree 5. By the Viro-Zvonilov inequality *3.3.O*, a flexible curve of degree 5 can have at most one nonempty oval. By the extremal property of this inequality *3.3.P*, if a flexible curve of degree 5 has a nonempty oval, then its real scheme is $\langle J \amalg 1\langle 1\rangle\rangle$. Together with *3.3.A* and the Harnack inequality *3.3.B*, this forms the complete set of prohibitions for real schemes of degree 5.

From the Klein congruence *3.4.B*, it follows that the real schemes $\langle J\amalg 1\rangle$, $\langle J\amalg 3\rangle$, and $\langle J\amalg 5\rangle$ are of type II. From the complex orientation formula, one can deduce that the real schemes $\langle J\rangle$ and $\langle J \amalg 2\rangle$ are of type II, cf. 2.8. By the extremal property *3.4.D* of the Harnack inequality, $\langle J \amalg 6\rangle$ is of type I. The complex orientation formula allows only one complex semiorientation for this scheme, namely $\langle J \amalg 3^- \amalg 3^+\rangle$. By the *3.4.H*, the real scheme $\langle J \amalg 1\langle 1\rangle\rangle$ is of type I. The complex orientation formula allows only one complex semiorientation for this scheme, namely $\langle J \amalg 1^-\langle 1^-\rangle\rangle$, cf. 2.8. The real scheme $\langle J \amalg 4\rangle$ is of indefinite type (even for algebraic curves, see 2.8). In the case of type I, only one semiorientation is allowed by the the complex orientation formula. It is $\langle J \amalg 3^- \amalg 1^+\rangle$.

3.7. **Sharpness of the Inequalities.** The arsenal of constructions in Section 1 and the supply of curves constructed there, which are very modest from the point of view of classification problems, turn out to be quite rich if we are interested in the problem of sharpness of the inequalities in Section 3.3.

The Harnack curves of even degree m with scheme

$$\langle (3m^2 - 6m)/8 \amalg 1\langle m^2 - 6m + 8)/8\rangle\rangle$$

which were constructed in Section 1.6 (see also Section 1.9) not only show that Harnack's inequality *3.3.B* is the best possible, but also show the same for the refined Petrovsky inequality *3.3.H*.

One of the simplest variants of Hilbert's construction (see Section 1.10) leads to the construction of a series of M-curves of degree $m \equiv 2 \mod 4$ with scheme $\left\langle \frac{(m-2)(m-4)}{8} \amalg 1 \left\langle \frac{3m(m-2)}{8} \right\rangle\right\rangle$. This proves that the refined Petrovsky inequality *3.3.I* for $m \equiv 2 \mod 4$ is sharp. If $m \equiv 0 \mod 4$, the methods of Section 1 do not show that this inequality is the best possible. This fact will be proved below in **??**.

The refined Arnold inequality *3.3.J* is best possible for any even m. If $m \equiv 2 \mod 4$, this can be proved using the Wiman M-curves (see the end of Section 1.12). If $m \equiv 0 \mod 4$, it follows using curves obtained from a modification of Wiman's construction: the construction proceeds in exactly the same way, except that the opposite perturbation is taken, as a result of which one obtains a curve that can serve as the boundary of a tubular neighborhood of an M-curve of degree $m/2$.

The last construction (doubling), if applied to an M-curve of odd degree, shows that the refined Arnold inequality *3.3.K* is the best possible for $m \equiv 2 \mod 4$. If $m \equiv 0 \mod 4$, almost nothing is known about sharpness of the inequality *3.3.K*, except that for $m = 8$ the right side can be lowered by 2.

3.8. **Prohibitions not Proven for Flexible Curves.** In conclusion of this section, let us come back to algebraic curves. We see that to a great extent the topology of their real point sets is determined by the properties which were included into the definition of flexible curves. In fact, it has not been proved that it is not determined by these properties completely. However some known prohibitions on topology of real algebraic curves have not been deduced from them.

As a rule, these prohibitions are hard to summarize, in the sense that it is difficult to state in full generality the results obtained by some particular method. To one extent or another, all of them are consequences of Bézout's theorem.

Consider first the restrictions which follow directly from the Bézout theorem. To state them, we introduce the following notations. Denote by h_r the maximum number of ovals occurring in a union of $\leq r$ nestings. Denote by h'_r the maximum number of ovals in a set of ovals contained in a union of $\leq r$ nests but not containing an oval which envelops all of the other ovals in the set. Under this notations Theorems *1.3.C* and *1.3.D* can be stated as follows:

3.8.A. *$h_2 \leq m/2$; in particular, if $h_1 = [m/2]$, then $l = [m/2]$.*

3.8.B. *$h'_5 \leq m$; in particular, if $h'_4 = m$, then $l = m$.*

These statements suggest a whole series of similar assertions. Denote by $c(q)$ the greatest number c such that there is a connected curve of degree q passing through any c points of $\mathbb{R}P^2$ in general position. It is known that $c(1) = 2$, $c(2) = 5$, $c(3) = 8$, $c(4) = 13$

3.8.C ((Generalization of Theorem *3.8.A*)). *If $r \leq c(q)$ with q odd, then*

$$h_r + \left[c(q) - \frac{r}{2}\right] \leq \frac{qm}{2}.$$

In particular, if $h_{c(q)-1} = \left[\frac{qm}{2}\right]$, then $l = \left[\frac{qm}{2}\right]$.

3.8.D ((Generalization of Theorem *3.8.B*)). *If $r \leq c(q)$ with q even, then*

$$h'_r + [(c(q) - r)/2] \leq qm/2.$$

In particular, if $h'_{c(q)-1} = qm/2$, then $l = qm/2$.

The following two restrictions on complex schemes are similar to Theorems *3.8.A* and *3.8.B*. However, I do not know the corresponding analogues of *3.8.C* and *3.8.D*.

3.8.E. *If $h_1 = \left[\frac{m}{2}\right]$, then the curve is of type* I.

3.8.F. *If $h'_4 = m$, then the curve is of type* I.

Here I will not even try to discuss the most general prohibitions which do not come from topology. I will only give some statements of results which have been obtained for curves of small degree.

3.8.G. *There is no curve of degree* 7 *with the real scheme* $\langle J \amalg 1\langle 14\rangle\rangle$.

3.8.H. *If an M-curve of degree* 8 *has real scheme* $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ *with nonzero* β, γ *and* δ*, then* β, γ *and* δ *are odd.*

3.8.I. *If an* $(M-2)$*-curve of degree* 8 *with* $p - n \equiv 4 \mod 8$ *has real scheme* $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ *with nonzero* β, γ *and* δ*, then two of the numbers* β, γ, δ *are odd and one is even.*

Proofs of *3.8.G* and *3.8.H* are based on technique initiated by Fiedler [Fie-82]. It will be developed in the next Section.