4. The Comlexification of a Curve from a Real Viewpoint

In the previous two sections we discovered that a knowledge on topology of the complexification gives restriction on topology of real part of the curve under consideration. More detailed topological information on complexification can be obtained using geometric constructions involving auxiliary curves. They use Bézout theorem. Therefore they cannot be applied to flexible curves. Here we consider first the simplest of arguments of that sort, and then obtain some special results on curves of low degrees (up to 8) which, together with forthcoming constructions will be useful in solution of some classification problems.

We will use the simplest auxiliary curves: lines. Consideration of a pencil of lines (the set of all lines passing through a point) and intersection of a curve with lines of this pencil can be thought of as a study of the curve by looking at it from the common point of the lines. However, since imaginary lines of the pencil can be included into this study and even real lines may intersect the curve in imaginary points, we have a chance to find out something on the complex part of the curve.

4.1. **Curves with Maximal Nest Revised.** To begin with, I present another proof of Theorem *2.6.D*. It gives slightly more: not only that a curve with maximal nest has type I, but that its complex orientation is unique. This is not difficult to obtain from the complex orientation formula. The real cause for including this proof is that it is the simplest application of the technique, which will work in this section in more complicated situations. Another reason: I like it.

4.1.A. *If a nonsingular real plane projective curve A of degree m has a nest of ovals of depth* $[m/2]$ *then A is of type* I *and all ovals (except for the exterior one, which is not provided with a sign in the case of even m) are negative.*

Recall that by Corollary *1.3.C* of the Bézout theorem a nest of a curve of degree m has depth at most $m/2$, and if a curve of degree m has a nest of depth $[m/2]$, then it does not have any ovals not in the nest. Thus the real scheme of a curve of *4.1.A* is $\langle 1\langle 1\dots\langle 1\rangle\dots\rangle\rangle$, if m is even, and $\langle J\langle 1\langle\dots 1\langle 1\rangle\dots\rangle\rangle$ if m is odd. Theorem *4.1.A* says that the complex scheme in this case is defined by the real one and it is

$$\langle 1\langle 1^-\dots\langle 1^-\rangle\dots\rangle\rangle_I^m$$

for even m and

$$\langle J\langle 1^-\langle\dots 1^-\langle 1^-\rangle\dots\rangle\rangle_I^m$$

if m is odd.

Proof of 4.1.A. Take a point P inside the smallest oval in the nest. Project the complexification $\mathbb{C}A$ of the curve A from P to a real projective line $\mathbb{C}L$ not containing P. The preimage of $\mathbb{R}L$ under the projection is $\mathbb{R}A$. Indeed, the preimage of a point $x \in \mathbb{R}L$ is the intersection of $\mathbb{C}A$ with the line connecting P with x. But since P is inside all ovals of the nest, any real line passing through it intersects $\mathbb{C}A$ only in real points.

The real part $\mathbb{R}L$ of L divides $\mathbb{C}L$ into two halves. The preimage of $\mathbb{R}L$ divides $\mathbb{C}A$ into the preimages of the halves of $\mathbb{R}L$. Thus $\mathbb{R}A$ divides $\mathbb{C}A$.

The projection $\mathbb{C}A \to \mathbb{C}L$ is a holomorphic map. In particular, it is a branched covering of positive degree. Its restriction to a half of $\mathbb{C}A$ is a branched covering of a half of $\mathbb{C}L$. Therefore the restriction of the projection to $\mathbb{R}A$ preserves local orientations defined by the complex orientations which come from the halves of $\mathbb{C}A$ and $\mathbb{C}L$. $\square$

4.2. **Fiedler's Alternation of Orientations.** Consider the pencil of real lines passing through the intersection point of real lines L_0, L_1. It is divided by L_0 and L_1 into two segments. Each of the segments can be described as $\{L_t\}_{t\in[0,1]}$, where L_t is defined by equation $(1-t)\lambda_0(x) + t\lambda_1(x) = 0\}$ under an appropriate choice of equations $\lambda_0(x) = 0$ and $\lambda_1(x) = 0$ defining L_0 and L_1, respectively. Such a family $\{L_t\}_{t\in[0,1]}$ is called *a segment of the line pencil connecting L_0 with L_1.*

A point of tangency of two oriented curves is said to be *positive* if the orientations of the curves define the same orientation of the common tangent line at the point, and *negative* otherwise.

The following theorem is a special case of the main theorem of Fiedler's paper [Fie-82].

4.2.A (Fiedler's Theorem)**.** *Let A be a nonsingular curve of type I. Let L_0, L_1 be real lines tangent to $\mathbb{R}A$ at real points x_0, x_1, which are not points of inflection of A. Let $\{L_t\}_{t\in I}$ is a segment of the line pencil, connecting L_0 with L_1. Orient the lines $\mathbb{R}L_0$, $\mathbb{R}L_1$ in such a way that the orientations turn to one another under the isotopy $\mathbb{R}L_t$. If there exists a path $s : I \to \mathbb{C}A$ connecting the points x_0, x_1 such that for $t \in (0,1)$ the point $s(t)$ belongs to $\mathbb{C}A \smallsetminus \mathbb{R}A$ and is a point of transversal intersection of $\mathbb{C}A$ with $\mathbb{C}L_t$, then the points x_0, x_1 are either both positive or both negative points of tangency of $\mathbb{R}A$ with $\mathbb{R}L_0$ and $\mathbb{R}L_1$ respectively.*

I give here a proof, which is less general than Fiedler's original one. I hope though that it is more visualizable.

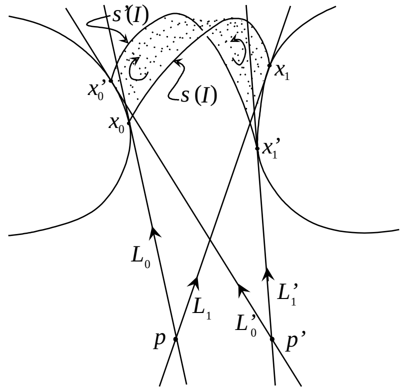

FIGURE 26

Roughly speaking, the main idea of this proof is that, looking at a curve, it is useful to move slightly the viewpoint. When one looks at the intersection of the complexification of a real curve with complexification of real lines of some pencil, besides the real part of the curve only some arcs are observable. These arcs connect ovals of the curve, but they do not allow to realize behavior of the complexification around. However, when the veiwpoint (= the center of the pencil) is moving, the arcs are moving too sweeping ribbons in the complexification. The ribbons bear orientation inherited from the complexification and thereby they allow to trace relation between the induced orientation of the ovals connected by the arcs. See Figure 26

Proof of 4.2.A. The whole situation described in the *4.2.A* is stable under small moves of the point $P = L_0 \cap L_1$. It means that there exists a neighbourhood U of P such that for each point $P' \in U$ there are real lines L'_0, L'_1 passing through P' which are close to L_0, L_1, and tangent to A at points x'_0, x'_1; the latter are close to x_0, x_1; there exists a segment $\{L'_t\}_{t\in I}$ of the line pencil connecting L'_0 with L'_1 which consists of lines close to L_t, and, finally, there exists a path $s' : I \to \mathbb{C}A$ connecting the points x'_0 and x'_1, which is close to s, such that $s'(t) \in \mathbb{C}A \cap \mathbb{C}L'_t$.

Choose a point $P' \in U \smallsetminus \bigcup_{t\in I} \mathbb{R}L_t$. Since, obviously, $\mathbb{R}A$ is tangent to the boundary of the angle $\bigcup_{t\in I} L_t$ from outside at x_0, x_1, the new points x'_0, x'_1 of tangency are obtained from the old ones by moves, one of which is in the direction of the orientation of $\mathbb{R}L_t$, the other – in the opposite direction (see Figure 26). Since $P' \notin \bigcup_{t\in I} L_t$, it follows that no line of the family $\{L_t\}_{t\in I}$ belongs to the family $\{L'_t\}_{t\in I}$ and thus

$$s(\operatorname{Int} I) \cap s'(\operatorname{Int} I) \subset (\bigcup_{t\in I}(\mathbb{C}L_t - \mathbb{R}L_t) \cap (\bigcup_{t\in I}(\mathbb{C}L']_t - \mathbb{R}L'_t)) = \varnothing.$$

Thus the arcs $s(I)$ and $s'(I)$ are disjoint, and bound in $\mathbb{C}A$, together with the arcs $[x_0, x_0']$ and $[x_1', x_1]$ of $\mathbb{R}A$, a ribbon connecting arcs $[x_0, x_0']$, $[x_1, , x_1']$. This ribbon lies in one of the halves, into which $\mathbb{R}A$ divides $\mathbb{C}A$ (see Figure 26). Orientation, induced on the arcs $[x_0, x_0']$, $[x_1, x_1']$ by an orientation of this ribbon, coincides with a complex orientation. It proves, obviously, *4.2.A* $\square$

The next thing to do is to obtain prohibitions on complex schemes using Fiedler's theorem. It takes some efforts because we want to deduce topological results from a geometric theorem. In the theorem it is crucial how the curve is positioned with respect to lines, while in any theorem on topology of a real algebraic curve, the hypothesis can imply some particular position with respect to lines only implicitely.

Let A be a nonsingular curve of type I and $P \in \mathbb{R}P^2 \smallsetminus \mathbb{R}A$. Let $Z = \{L_t\}_{t\in I}$ be a segment of the pencil of lines passing through P, which contains neither a line tangent to $\mathbb{R}A$ at a point of inflection of $\mathbb{R}A$ nor a line, whose complexifications is tangent to $\mathbb{C}A$ at an imaginary point. Denote $\bigcup_{t\in I} \mathbb{R}L_t$ by C.

Fix a complex orientations of A and orientations of the lines $\mathbb{R}L_t$, $t \in I$, which turn to one another under the natural isotopy. Orient the part C of the projective plane in such a way that this orientation induces on $\mathbb{R}L_0$, as on a part of its boundary, the orientation selected above. An oval of A, lying in C is said to be *positive with respect to* Z if its complex orientation and orientation of C induce the same orientation of its interior; otherwise the oval is said to be *negative with respect to* Z.

A point of tangency of A and a line from Z is a nondegenerate critical point of the function $A \cap C \to I$ which assigns to x the real number $t \in I$ such that $x \in L_t$. By *index* of the point of tangency we shall call the Morse index of this function at that point (zero, if it is minimum, one, if it is maximum). A pair of points of tangency of $\mathbb{R}A$ with lines from Z is said to be *switching*, if the points of the pair has distinct indices and one of the points is positive while the other one is negative; otherwise the pair is said to be *inessential*. See Figure 27.

If A is a nonsingular conic with $\mathbb{R}A \neq \varnothing$ and $\mathbb{R}A \subset C$ then the tangency points make a switching pair. The same is true for any convex oval. When an oval is deforming and loses its convexity, new points of tangency may appear. If the deformation is generic, the points of tangency appear and disappear pairwise. Each time appearing pair is an inessential pair of points with distinct indices. Any oval can be deformed (topologically) into a convex one. Tracing the births and

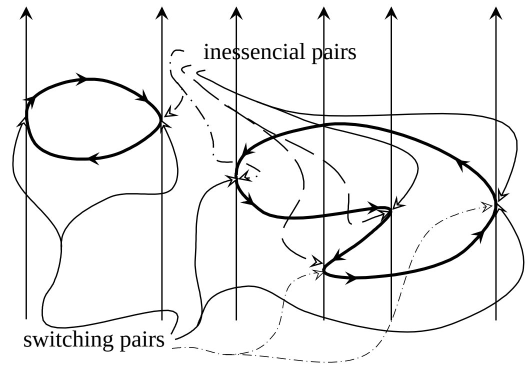

FIGURE 27

deaths of points of tangency it is not difficult to prove the following lemma.

4.2.B ([Fie-82, Lemma 2]). *Let Γ be a component of $\mathbb{R}A \cap C$ and M be the set of its points of tangency with lines from Z. If $\Gamma \cap \partial C = \varnothing$, then M can be divided into pairs, one of which is switching, and all others are inessential. If $\Gamma \cap \partial C \neq \varnothing$, and Γ connects distinct boundary lines of C then M can be decomposed into inessential pairs. If the end points of Γ are on the same boundary lines of C then M with one point deleted can be decomposed into inessential pairs.* $\square$

Denote the closure of $(\mathbb{C}A \smallsetminus \mathbb{R}A) \cap (\bigcup_{t \in I} \mathbb{C}L_t)$ by S. Fix one of the decomposions into pairs of the set of points of tangency of lines from Z with each component of $\mathbb{R}A \cap C$ existing by *4.2.B*. By *a chain of points of tangency* call a sequence of points of tangency, in which any two consecutive points either belong to one of selected pairs or lie in the same component of S. A sequence consisting of ovals, on which the selected switching pairs of points of tangency from the chain lie, is called a *chain of ovals*. Thus the set of ovals of A lying in C appeared to be decomposed to chains of ovals. The next theorem follows obviously from *4.2.A*.

4.2.C. *The signs of ovals with respect to Z in a chain alternate (i.e. an oval positive with respect to Z follows by an oval negative with respect to Z, the latter oval follows by an oval positive with respect to Z).* $\square$

The next theorem follows in an obvious way from *4.2.C*. Contrary to the previous one, it deals with the signs of ovals with respect to the one-sided component in the case of odd degree and outer ovals in the case of even degree.

4.2.D ([Fie-82, Theorem 3]). *If the degree of a curve A is odd and ovals of a chain are placed in the same component of the set*

$$C \smallsetminus (\textit{one-sided component of}\, \mathbb{R}A)$$

then the signs of these ovals alternate. If degree of A is even and ovals of a chain are placed in the same component of intersection of $\operatorname{Int} C$ *with the interior of the outer oval enveloping these ovals, then the signs of ovals of this chain alternate.* □

4.3. **Complex Orientations and Pencils of Lines. Alternative Approach.** In proofs of *3.8.G*, *3.8.H* and *3.8.I*, the theory developed in the previous section can be replaced by the following Theorem *4.3.A*. Although this theorem can be obtained as a corollary of Theorem *4.2.C*, it is derived here from Theorem *2.3.A* and the complex orientation formula, and in the proof no chain of ovals is used. The idea of this approach to Fiedler's alternation of orientations is due to V. A. Rokhlin.

4.3.A. *Let A be a non-singular dividing curve of degree m. Let L_0, L_1 be real lines, C be one of two components of $\mathbb{R}P^2 \smallsetminus (\mathbb{R}L_0 \cup \mathbb{R}L_1)$. Let $\mathbb{R}L_0$ and $\mathbb{R}L_1$ be oriented so that the projection $\mathbb{R}L_0 \to \mathbb{R}L_1$ from a point lying in $\mathbb{R}P^2 \smallsetminus (C \cup \mathbb{R}L_0 \cup \mathbb{R}L_1)$ preserves the orientations. Let ovals u_0, u_1 of A lie in $\mathbb{R}P^2 - C$ and u_i is tangent to L_i at one point $(i = 1, 2)$. If the intersection $\mathbb{R}A \cap C$ consists of $m-2$ components, each of which is an arc connecting $\mathbb{R}L_0$ with $\mathbb{R}L_1$, then points of tangency of u_0 with L_0 and u_1 with L_1 are positive with respect to one of the complex orientations of A.*

Proof. Assume the contrary: suppose that with respect to a complex orientation of A the tangency of u_0 with L_0 is positive and the tangency of u_1 with L_1 is negative. Rotate L_0 and L_1 around the point $L_0 \cap L_1$ in the directions out of C by small angles in such a way that each of the lines L_0' and L_1' obtained intersects transversally $\mathbb{R}A$ in m points. Perturb the union $A \cup L_0'$ and $A \cup L_1'$ obeying the orientations. By *2.3.A*, the nonsingular curves B_0 and B_1 obtained are of type I. It is easy to see that their complex schemes can be obtained one from another by relocating the oval, appeared from u_1 (see Figure 28). This operation changes one of the numbers $\Pi^+ - \Pi^-$ and $\Lambda^+ - \Lambda^-$ by 1. Therefore the left hand side of the complex orientation formula is changed. It means that the complex schemes both of B_0 and B_1 can not satisfy the complex orientation formula. This proves that the assumption is not true. □

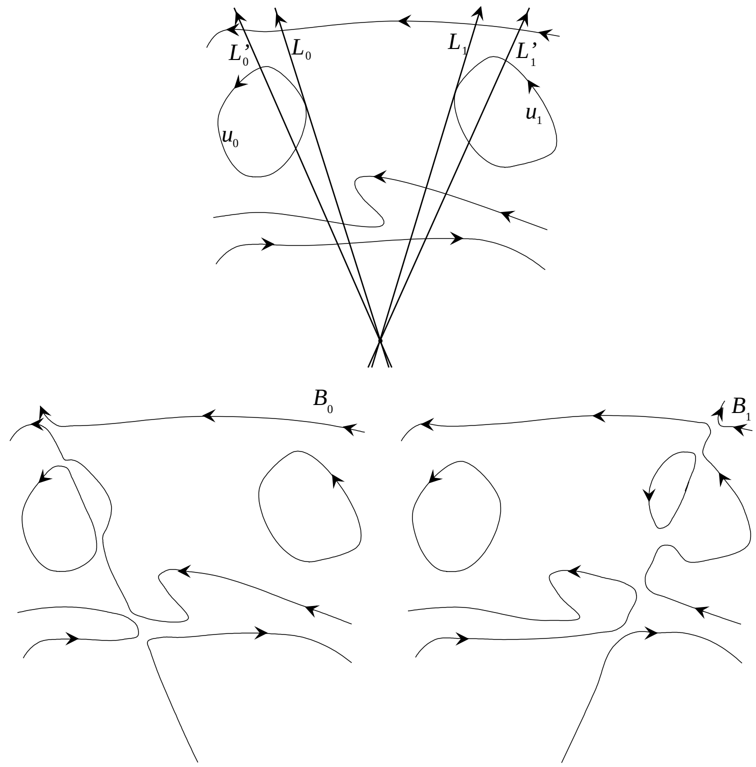

FIGURE 28

4.4. **Curves of Degree 7.** In this section Theorem *3.8.G* is proved, i.e. it is proved that there is no nonsingular curve of degree 7 with real scheme $\langle J \amalg 1\langle 14\rangle\rangle$.

Assume the contrary: suppose that there exists a nonsingular curve X of degree 7 with real scheme $\langle J \amalg 1\langle 14\rangle\rangle$.

Being an M-curve, X is of type I (see *2.6.B*) and, hence, has a complex orientation.

4.4.A. **Lemma.** *X cannot have a complex scheme distinct from* $\langle J \amalg 1^+\langle 6^+ \amalg 8^-\rangle\rangle^7_I$.

Proof. Let ε be the sign of the outer oval, i.e.

$$\varepsilon = \begin{cases} +1, & \text{if the outer oval is positive} \\ -1, & \text{otherwise.} \end{cases}$$

It is clear that

$$\Lambda^+ = \begin{cases} \Pi^- + 1, & \text{if } \varepsilon = +1 \\ \Pi^+, & \text{if } \varepsilon = -1 \end{cases}, \qquad \Lambda^- = \begin{cases} \Pi^+, & \text{if } \varepsilon = +1 \\ \Pi^- + 1, & \text{if } \varepsilon = -1. \end{cases}$$

Therefore, $\Lambda^+ - \Lambda^- = \varepsilon(\Pi^- + 1 - \Pi^+)$. On the other hand, by *2.7.B*, $\Lambda^+ - \Lambda^- = 2(\Pi^- - \Pi^+) + 3$. From these two equalities we have

$$\varepsilon = 2 + \frac{1}{\Pi^- + 1 - \Pi^+}$$

and, since $|\varepsilon| = 1$, it follows that $\varepsilon = +1$ and $\Pi^- + 1 - \Pi^+ = -1$, i.e. $\Pi^+ - \Pi^- = 2$. Finally, since $\Pi^+ + \Pi^- = 14$, it follows that $\Pi^+ = 8$ and $\Pi^- = 6$. This gives the desired result. $\square$

The next ingredient in the proof of Theorem *3.8.G* is a kind of convexity in disposition of interior ovals. Although we study a projective problem, it is possible to speak about convexity, if it is applied to interior ovals. The exact sense of this convexity is provided in the following statement.

4.4.B. **Lemma.** *Let A be any nonsingular curve of degree 7 with real scheme $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ and the number of ovals ≥ 6. Then for each of β interior ovals there exists a pair of real lines L_1, L_2 intersecting inside this oval such that the rest $\beta - 1$ interior ovals lie in one of three domains into which $\mathbb{R}L_1 \cup \mathbb{R}L_2$ cut the disk bounded by the exterior oval.*

Proof. A line intersecting two interior ovals cannot intersect any other interior oval. Furthermore, it intersects each of these two interior ovals in two points, meets the nonempty oval in two points and the one-sided component in one point. (This follows from the following elementary arguments: the line intersects the one-sided component with odd multiplicity, it has to intersect the nonempty oval, since it intersects ovals inside of it, it can intersect any oval with even multiplicty and by Bézout theorem the total number of ontersection points is at most 7.) The real point set of the line is divided by the intersection points with the nonempty oval into two segments. One of these segments contains the intersection point with the one-sided component, the other one is inside the nonempty oval and contains the intersections with the interior ovals. A smaller segment connects the interior ovals inside the nonempty ovals. Thus any points inside two interior ovals can be connected by a segment of a line inside the exterior nonemty oval. See Figure 29.

Choose a point inside each interior oval and connect these points by segments inside the exterior oval. If the lines guaranteed by *4.4.B* exist, then the segments comprise a convex polygon. Otherwise, there exist interior ovals u_0, u_1, u_2 and u_3 such that u_0 is contained inside the triangle made of the segments connecting inside the exterior oval the points q_1, q_2, q_3 chosen inside u_1, u_2 and u_3. See Figure 30.

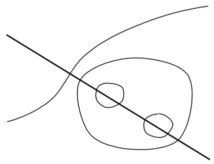

Figure 29

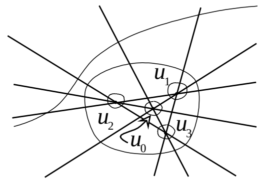

Figure 30

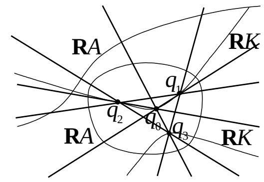

Figure 31

To prove that this is impossible, assume that this is the case and construct a conic K through q_1, q_2, q_3, the point q_0 chosen inside u_0 and a point q_4 chosen inside some empty oval u_4 distinct from u_0, u_1, u_2 and u_3 (recall that the total number of ovals is at least 6, thereby u_4 exists). Since the space of conics is a 5-dimensional real projective space and the conics containing a real point form a real hyperplane, there exists a real conic passing through any 5 real points. If the conic happened to be singular, we could make it nonsingular moving the points. However it cannot happen, since then the conic would be decomposed into two lines and at least one of the lines would intersect with 3 empty ovals and with the nonempty oval, which would contradict the Bézout theorem.

Now let us estimate the number of intersection points of the conic and the original curve A of degree 7. The conic $\mathbb{R}K$ passes through the vertices of the triangle $q_1q_2q_3$ and through the point q_0 inside it. The component of the intersection of $\mathbb{R}K$ with the interior of the triangle has to be an arc connecting two points of q_1, q_2, q_3. Let they be q_1 and q_2. Then the segment $[q_0, q_3]$ lies outside the disk bounded by $\mathbb{R}K$. This segment together with an arc q_0, q_1, q_3 of $\mathbb{R}K$ is a one-sided circle in $\mathbb{R}P^2$, which has to intersect the one-sided component of $\mathbb{R}A$. Since neither the segment nor the arc q_0, q_1 intersect $\mathbb{R}A$, the arc q_1, q_3 does intersect. The intersection point is outside the nonempty oval, while both q_1 and q_3 are inside. Therefore the same arc has at least 2 common points with the nonempty oval. Similar arguments show that the arc q_2, q_3 intersects the one-sided component of $\mathbb{R}A$ and has at least 2 common points with the nonempty oval. Thus $\mathbb{R}K$ intersects the one-sided component of A at least in 2 points and the nonempty oval at least in 4 points. See Figure 31. Together with 10 intersection points with ovals u_i, $i = 0, 1, \ldots, 4$ (2 points with each) it gives 16 points, which contradicts the Bézout theorem. (^{2})

□

End of Proof of Theorem 3.8.G. Assume that a curve X prohibited by Theorem *3.8.G* does exist. According to Lemma *4.4.A*, its complex scheme is $\langle J \amalg 1^+\langle 6^+ \amalg 8^-\rangle\rangle^7_I$. Take a point inside a positive interior oval. Consider the segment of the pencil of line passing through this point. The other interior ovals compose a chain. By Lemma *4.4.B* they lie in one connected component of the intersection of the domain swept by the lines of the segment of the pencil with the interior domain of the nonempty oval. By Theorem *4.2.C* signs of ovals in this chain alternate. Therefore the difference between the numbers of positive and negative ovals is 1, while it has to be 3 by Lemma *4.4.A*. $\square$