5. Introduction to Topological Study of Real Algebraic Spatial Surfaces

5.1. **Basic Definitions and Problems.** Our consideration of real algebraic surfaces will be based on definitions similar to the definitions that we used in the case of curves. In particular, by a *real algebraic surface of degree* m *in the 3-dimensional projective space* we shall mean a real homogeneous polynomial of degree m in four variables considered up to a constant factor.

Obvious changes adapt definitions of sets of real and complex points, singular points, singular and nonsingular curves and rigid isotopy to the case of surfaces in $\mathbb{R}P^3$. Exactly as in the case of curves one formulates the topological classification problem (cf. *1.1.A* above):

5.1.A (Topological Classification Problem)**.** *Up to homeomorphism, what are the possible sets of real points of a nonsingular real projective algebraic surface of degree* m *in* $\mathbb{R}P^3$*?*

However, the isotopy classification problem *1.1.B* splits into two problems:

5.1.B (Ambient Topological Classification Problem)**.** *Classify up to homeomorphism the pairs* $(\mathbb{R}P^3, \mathbb{R}A)$ *where* A *is a nonsingular real projective algebraic surface of degree* m *in* $\mathbb{R}P^3$*?*

5.1.C (Isotopy Classification Problem)**.** *Up to ambient isotopy, what are the possible sets of real points of a nonsingular a nonsingular real projective algebraic surface of degree* m *in* $\mathbb{R}P^3$*?*

The reason for this splitting is that, contrary to the case of projective plane, there exists a homeomorphism of $\mathbb{R}P^3$ non-isotopic to the identity. Indeed, 3-dimensional projective space is orientable, and the mirror reflection of this space in a plane reverses orientation. Thus the reflection is not isotopic to the identity. However, there are only two isotopy classes of homeomorphisms of $\mathbb{R}P^3$. It means that the difference between *5.1.B* and *5.1.C* is not really big. Although the isotopy classification problem is finer, to resolve it, one should add to a solution of the ambient topological classification problem an answer to the following question:

5.1.D (Amphichirality Problem)**.** *Which nonsingular real algebraic surfaces of degree* m *in* $\mathbb{R}P^3$ *are isotopic to its own mirror image?*

Each of these problems has been solved only for $m \leq 4$. The difference between *5.1.B* and *5.1.C* does not appear: the solutions of *5.1.B* and *5.1.C* coincide with each other for $m \leq 4$. (Thus Problem *5.1.D* has a simple answer for $m \le 4$: any nonsingular real algebraic surface of degree ≤ 4 is isotopic to its mirror image.) For $m \le 3$ solutions of *5.1.A* and *5.1.B* also coincide, but for $m = 4$ they are different: there exist nonsingular surfaces of degree 4 in $\mathbb{R}P^3$ which are homeomorphic, but embedded in $\mathbb{R}P^3$ in a such a way that there is no homeomorphism of $\mathbb{R}P^3$ mapping one of them to another. The simplest example is provided by torus defined by equation

$$(x_1^2 + x_2^2 + x_3^2 + 3x_0^2)^2 - 16(x_1^2 + x_2^2)x_0^2 = 0$$

and the union of one-sheeted hyperboloid and an imaginary quadric (perturbed, if you wish to have a surface without singular points even in the complex domain)

Similar splitting happens with the rigid isotopy classification problem. Certainly, it may be transferred literally:

5.1.E (Rigid Isotopy Classification Problem). *Classify up to rigid isotopy the nonsingular surfaces of degree m.*

However, since there exists a projective transformation of $\mathbb{R}P^3$, which is not isotopic to the identity (e.g., the mirror reflection in a plane) and a real algebraic surface can be nonisotopic rigidly to its mirror image, one may consider the following rougher problem:

5.1.F (Rough Projective Classification Problem). *Classify up to rigid isotopy and projective transformation the nonsingular surfaces of degree m.*

Again, as in the case of topological isotopy and homeomorphism problem, the difference between these two problems is an amphichirality problem:

5.1.G (Rigid Amphichirality Problem). *Which nonsingular real algebraic*
surfaces of degree m in $\mathbb{R}P^3$ are rigidly isotopic to its mirror image?

Problems *5.1.E*, *5.1.F* and *5.1.G* have been solved also for $m \le 4$. For $m \le 3$ the solutions of *5.1.E* and *5.1.F* coincide with each other and with the solutions of *5.1.A*, *5.1.B* and *5.1.C*. For $m \le 2$ all these problems belong to the traditional analytic geometry. The solutions are well-known and can be found in traditional textbooks on analytic geometry. The case $m = 3$ is also elementary. It was studied in the nineteenth century. The solution is associated with names of Schläfli and Klein. The case $m = 4$ is really difficult. Although the first attempts of a serious attack were undertaken in the nineteenth century, too, and among the attackers we see D. Hilbert and K. Rohn, the complete solutions of all classification problems listed above were obtained only in the seventies and eighties. Below, in Subsection **??**, I will discuss the results and methods. In higher degrees even the most rough problems, like the Harnack problem on the maximal number of components of a surface of degree m are still open.

5.2. **Digression: Topology of Closed Two-Dimensional Submanifolds of $\mathbb{R}P^3$.** For brevity, we shall refer to closed two-dimensional submanifolds of $\mathbb{R}P^3$ as *topological spatial surfaces,* or simply *surfaces* when there is no danger of confusion.

Since the homology group $H_2(\mathbb{R}P^3; \mathbb{Z}_2)$ is $\mathbb{Z}_2$, a connected surface can be situated in $\mathbb{R}P^3$ in two ways: zero-homologous, and realizing the nontrivial homology class.

In the first case it divides the projective space into two domains being the boundary for both domains. Hence, the surface divides its tubular neighborhood, i. e. it is *two-sided.*

In the second case the complement of the surface in the projective space is connected. (If it was not connected, the surface would bound and thereby realize the zero homology class.) Moreover, it is *one-sided.*

The latter can be proved in many ways. For example, if the surface was two-sided and its complement was connected, there would exist a nontrivial infinite cyclic covering of $\mathbb{R}P^3$, which would contradict the fact that $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$. The infinite cyclic covering could be constructed by gluing an infinite sequence of copies of $\mathbb{R}P^3$ cut along the surface: each copy has to be glued along one of the sides of the cut to the other side of the cut in the next copy.

Another proof: take a projective plane, make it transversal to the surface, and consider the curve which is their intersection. Its homology class in $\mathbb{R}P^2$ is the image of the nontrivial element of $H_2(\mathbb{R}P^3; \mathbb{Z}_2)$ under the inverse Hopf homomorphism $in^! : H_2(\mathbb{R}P^3; \mathbb{Z}_2) \to H_1(\mathbb{R}P^2; \mathbb{Z}_2)$. This is an isomorphism, as one can see taking the same construction in the case when the surface is another projective plane. Thus the intersection is a one-sided curve in $\mathbb{R}P^2$. Hence the normal fibration of the original surface in $\mathbb{R}P^3$ is not trivial. This means that the surface is one-sided.

A connected surface two-sidedly embedded in $\mathbb{R}P^3$ is orientable, since it bounds a part of the ambient space which is orientable. Therefore, such a surface is homeomorphic to sphere or to sphere with handles. There is no restriction to the number of handles: one can take an embedded sphere bounding a small ball, and adjoin to it any number of handles.

A one-sidedly embedded surface is nonorientable. Indeed, its normal bundle is nonorientable, while the restriction of the tangent bundle of $\mathbb{R}P^3$ to the surface is orientable (since $\mathbb{R}P^3$ is). The restriction of the tangent bundle of $\mathbb{R}P^3$ to the surface is the Whitney sum of the normal and tangent bundles of the surface. Therefore it cannot happen that only one of these three bundles is not orientable.

Contrary to the case of two-sided surfaces, in the case of one-sided surfaces there is an additional restriction on their topological types.

*5.2.A***.** *The Euler characteristic of a connected surface one-sidedly embedded to $\mathbb{R}P^3$ is odd.*

In particular, it is impossible to embed a Klein bottle to $\mathbb{R}P^3$. (The Euler characteristic of a connected surface two-sidedly embedded into $\mathbb{R}P^3$ is even, but it follows from orientability: the Euler characteristic of any closed oriented surface is even.) By topological classification of closed surfaces, a nonorientable connected surface with odd Euler characteristic is homeomorphic to the projective plane or to the projective plane with handles. Any surface of this sort can be embedded into $\mathbb{R}P^3$: for the projective plane $\mathbb{R}P^3$ is the native ambient space, and one can adjoin to it in $\mathbb{R}P^3$ any number of handles. We denote a sphere with g handles by S_g and a projective plane with g handles by P_g.

Proof of 5.2.A. Let S be a connected surface one-sidedly embedded into $\mathbb{R}P^3$. By a small shift it can be made transversal to the projective plane $\mathbb{R}P^2$ standardly embedded into $\mathbb{R}P^3$. Since both surfaces are embedded one-sidedly, they realize the same homology class in $\mathbb{R}P^3$. Therefore their union bounds in $\mathbb{R}P^3$: one can color the complement $\mathbb{R}P^3 \smallsetminus (S \cup \mathbb{R}P^2)$ into two colors in such a way that the components adjacent from the different sides to the same (two-dimensional) piece of $S \cup \mathbb{R}P^2$ would be of different colors. It is a kind of checkerboard coloring.

Consider the disjoint sum Q of the closures of those components of $\mathbb{R}P^3 \smallsetminus (S \cup \mathbb{R}P^2)$ which are colored with the same color. It is a compact 3-manifold. It is oriented since each of the components inherits orientation from $\mathbb{R}P^3$. The boundary of this 3-manifold is composed of pieces of S and $\mathbb{R}P^2$. It can be thought of as the result of cutting both surfaces along their intersection curve and regluing. The intersection curve is replaced by its two copies, while the rest part of S and $\mathbb{R}P^2$ does not change. Since the intersection curve consists of circles, its Euler characteristic is zero. Therefore $\chi(\partial Q) = \chi(S) + \chi(\mathbb{R}P^2) = \chi(S) + 1$.

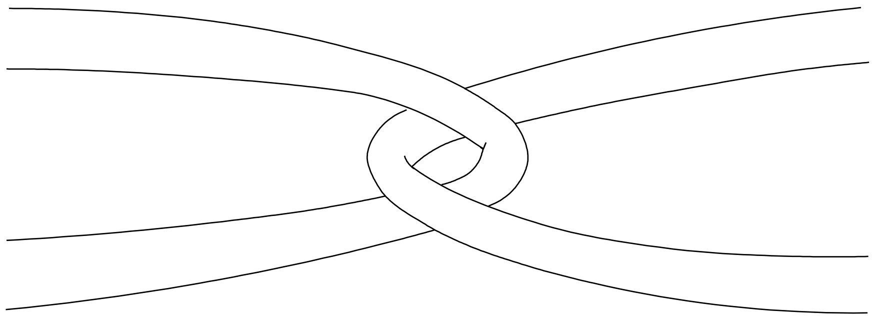

FIGURE 32

On the other hand, $\chi(\partial Q)$ is even since ∂Q is a closed oriented surface (∂Q inherits orientation from Q). Thus $\chi(S)$ is odd. $\square$

A one-sided connected surface in $\mathbb{R}P^3$ contains a loop which is not contractible in $\mathbb{R}P^3$. Such a loop can be detected in the following way: Consider the intersection of the surface with any one-sided transversal surface (e. g., $\mathbb{R}P^2$ or a surface obtained from the original one by a small shift). The homology class of the intersection curve is the self-intersection of the nonzero element of $H_2(\mathbb{R}P^3\,;\mathbb{Z}_2)$. Since the self-intersection is the nonzero element of $H_1(\mathbb{R}P^3\,;\mathbb{Z}_2)$, the intersection curve contains a component noncontractible in $\mathbb{R}P^3$.

A two-sided connected surface in $\mathbb{R}P^3$ can contain no loops noncontractible in $\mathbb{R}P^3$ (this happens, for instance, if the surface lies in an affine part of $\mathbb{R}P^3$). Of course, if a surface contains a loop noncontractible in $\mathbb{R}P^3$, it is not contractible in $\mathbb{R}P^3$ itself. Moreover, then it meets any one-sided surface, since the noncontractible loop realizes the nonzero element of $H_1(\mathbb{R}P^3\,;\mathbb{Z}_2)$ and this element has nonzero intersection number with the homology class realized by a one-sided surface.

If any loop on a connected surface S embedded in $\mathbb{R}P^3$ is contractible in $\mathbb{R}P^3$ (which means that the embedding homomorphism $\pi_1(S) \to \pi_1(\mathbb{R}P^3)$ is trivial), then there is no obstruction to contract the embedding, i. e., to construct a homotopy between the embedding $S \to \mathbb{R}P^3$ and a constant map. One can take a cell decomposition of S, contract the 1-skeleton (extending the homotopy to the whole S), and then contract the map of the 2-cell, which is possible, since $\pi_2(\mathbb{R}P^3) = 0$. A surface of this sort is called *contractible (in $\mathbb{R}P^3$).*

It may happen, however, that there is no isotopy relating the embedding of a contractible surface with a map to an affine part of $\mathbb{R}P^3$. The simplest example of a contractible torus which cannot be moved by an isotopy to an affine part of $\mathbb{R}P^3$ is shown in Figure 32.

As it was stated above, the complement $\mathbb{R}P^3 \smallsetminus S$ of a connected surface S two-sidedly embedded in $\mathbb{R}P^3$ consists of two connected components. If S is not contractible in $\mathbb{R}P^3$ then both of them are not contractible, since a loop on S noncontractible in $\mathbb{R}P^3$ can be pushed to each of the components. They may be positioned in $\mathbb{R}P^3$ in the same way.

The simplest example of this situation is provided by a one-sheeted hyperboloid. It is homeomorphic to torus and its complement consists of two solid tori. So, this is a Heegaard decomposition of $\mathbb{R}P^3$. There exists an isotopy of $\mathbb{R}P^3$ made of projective transformation exchanging the components. ([3])

A connected surface decomposing $\mathbb{R}P^3$ into two handlebodies is called a *Heegaard surface.* Heegaard surfaces are the most unknotted surfaces among two-sided noncontractible connected surfaces. They may be thought of as unknotted noncontractible surfaces.

If a connected surface S is contractible in $\mathbb{R}P^3$, then the components C_1 and C_2 can be distinguished in the following way: for one of them, say C_1, the inclusion homomorphism $\pi_1(C_1) \to \pi_1(\mathbb{R}P^3)$ is trivial, while for the other one the inclusion homomorphism $\pi_1(C_2) \to \pi_1(\mathbb{R}P^3)$ is surjective. This follows from the van Kampen theorem. The component with trivial homomorphism is called the *interior* of the surface. It is contractible in $\mathbb{R}P^3$ (in the same sense as the surface is).

A contractible connected surface S in $\mathbb{R}P^3$ is said to be *unknotted,* if it is contained in some ball B embedded into $\mathbb{R}P^3$ and divides this ball into a ball with handles (which is the interior of S) and a ball with handles with an open ball deleted. Any two unknotted contractible surfaces of the same genus are ambiently isotopic in $\mathbb{R}P^3$. Indeed, first the balls containing them can be identified by an ambient isotopy (see, e. g., Hirsch [Hir-76], Section 8.3), then it follows from uniqueness of Heegaard decomposition of sphere that there is an orientation preserving homeomorphism of the ball mapping one of the surfaces to the other. Any orientation preserving homeomorphism of a 3-ball is isotopic to the identity.

At most one component of a (closed) surface embedded in $\mathbb{R}P^3$ may be one-sided. Indeed, a one-sided closed surface cannot be zero-homologous in $\mathbb{R}P^3$ and the self-intersection of its homology class (which is the only nontrivial element of $H_2(\mathbb{R}P^3\,;\,\mathbb{Z}_2)$) is the nonzero element of $H_1(\mathbb{R}P^3\,;\,\mathbb{Z}_2)$. Therefore any two one-sided surfaces in $\mathbb{R}P^3$ intersect.

Moreover, if an embedded surface has a one-sided component, then all other components are contractible. The contractible components are naturally ordered: a contractible component of a surface can contain other contractible component in its interior and this gives rise to a partial order in the set of contractible components. If the interior of contractible surface A contains a surface B, then one says that A *envelopes* B.

The connected components of a surface embedded in $\mathbb{R}P^3$ divide $\mathbb{R}P^3$ into connected regions. Let us construct a graph of adjacency of these regions: assign a vertex to each of the regions and connect two regions with an edge if the corresponding regions are adjacent to the same connected two-sided component of the surface. Since the projective space is connected and its fundamental group is finite, the graph is contractible, i. e., it is a tree. It is called *region tree* of the surface.

Consider now a (closed) surface without one-sided components. It may contain several noncontractible components. They decompose the projective space into connected domains, each of which is not contractible in $\mathbb{R}P^3$. Let us construct a graph of adjacency of these domains: assign a vertex to each of the domains and connect two vertices with an edge if the corresponding domains are adjacent. Edges of the graph correspond to noncontractible components of the surface. For the same reasons as above, this graph is contractible, i. e. it is a tree. This tree is called the *domain tree* of the surface.

Contractible components of the surface are distributed in the domains. Contractible components which are contained in different domains cannot envelope one another. Contractible components of the surface which lie in the same domain are partially ordered by enveloping. They divide the domain into regions. Each domain contains only one region which is not contractible in $\mathbb{R}P^3$. If the domain does not coincide with the whole $\mathbb{R}P^3$ (i.e., the surface does contain noncontractible components), then this region can be characterized also as the only region which is adjacent to all the noncontractible components of the surface comprising the boundary of the domain. Indeed, contractible components of the surface cannot separate noncontractible ones.

The region tree of a surface contains a subtree isomorphic to the domain tree, since one can assign to each domain the unique noncontractible region contained in the domain and two domains are adjacent iff the noncontractible regions contained in them are adjacent. The complement of the noncontractible domains tree is a union of adjacency trees for contractible subdomains contained in each of the domains.

Let us summarize what can be said about topology of a spatial surface in the terms described above.

If a surface is one-sided (i. e., contains a one-sided component), then it is a disjoint sum of a projective plane with handles and several (maybe none) spheres with handles. Thus, it is homeomorphic to

$$P_g \amalg S_{g_1} \amalg \dots S_{g_k},$$

where $\amalg$ denotes disjoint sum.

All two-sided components are contractible and ordered by enveloping. The order is easy to incorporate into the notation of the topological type above. Namely, place notations for components enveloped by a component A immediately after A inside brackets $\langle\ \rangle$. For example,

$$P_0 \amalg S_1 \amalg S_1 \amalg S_0\langle S_1 \amalg S_0 \amalg S_2\langle S_1 \amalg S_0\rangle \amalg S_2\langle S_1 \amalg S_0\rangle\rangle$$

denotes a surface consisting of a projective plane, two tori, which do not envelope any other component, a sphere, which envelopes a torus and a sphere without components inside them and a two spheres with two handles each of which envelopes empty sphere and torus. To make the notations shorter, let us agree to skip index 0, i. e. denote projective plane P_0 by P and sphere S_0 by S, and denote the disjoint sum of k fragments identical to each other by k followed by the notation of the fragment. These agreements shorten the notation above to

$$P \amalg 2S_1 \amalg S\langle S_1 \amalg S \amalg 2S_2\langle S_1 \amalg S\rangle\rangle.$$

If a surface is two-sided (i. e. does not contain a one-sided component), then it is a disjoint sum $S_{g_1} \amalg \dots S_{g_k}$, of spheres with handles. To distinguish in notations the components noncontractible in $\mathbb{R}P^3$, we equip the corresponding symbols with upper index 1. Although we do not make any difference between two components of the complement of noncontractible connected surface (and there are cases when they cannot be distinguished), in notations we proceed as if one of the components is interior: the symbols denoting components of the surface which lie in one of the components of the complement of the noncontractible component A are placed immediately after the notation of A inside braces $\{\ \}$. Our choice is the matter of convenience. It correspond to the well-known fact that usually, to describe a tree, one introduces a partial order on the set of its vertices.

In these notations,

$$S_1 \amalg S\langle 3S\rangle \amalg S_1^1\{S_3 \amalg 2S_2^1\{3S \amalg S_1\}\}$$

denotes a two-sided surface containing three noncontractible components. One of them is a torus, two others are spheres with two handles. The torus bounds a domain containing a contractible empty torus and a sphere enveloping three empty spheres. There is a domain bounded by all three noncontractible components. It contains a contractible empty sphere with three handles. Each of the noncontractible spheres with two handles bounds a domain containing empty contractible torus and three empty spheres.

This notation system is similar to notations used above to described isotopy types of curves in the projective plane. However, there is a fundamental difference: the notations for curves describe the isotopy type of a curve completely, while the notations for surfaces are far from being complete in this sense. Although topological type of the surface is described, knotting and linking of handles are completely ignored. In the case when there is no handle, the notation above does provide a complete description of isotopy type.

5.3. **Restrictions on Topology of Real Algebraic Surfaces.** As in the case of real plane projective curves, the set of real points of a nonsingular spatial surface of degree m is one-sided, if m is odd, and two-sided, if m is even. Indeed, by the Bézout theorem a generic line meets a surface of degree m in a number of points congruent to m modulo 2. On the other hand, whether a topological surface embedded in $\mathbb{R}P^3$ is one-sided or two-sided, can be detected by its intersection number modulo 2 with a generic line: a surface is one-sided, iff its intersection number with a generic line is odd.

There are some other restrictions on topology of a nonsingular surface of degree m which can be deduced from the Bézout theorem.

5.3.A (On Number of Cubic's Components)**.** *The set of real points of a nonsingular surface of degree three consists of at most two components.*

Proof. Assume that there are at least three components. Only one of them is one-sided, the other two are contractible. Connect with a line two contractible components. Since they are zero-homologous, the line should intersect each of them with even intersection number. Therefore the total number of intersection points (counted with multiplicities) of the line and the surface is at least four. This contradicts to the Bézout theorem, according to which it should be at most three. $\square$

5.3.B (On Two-Component Cubics)**.** *If the set of real points of a nonsingular surface of degree 3 consists of two components, then the components are homeomorphic to the sphere and projective plane (i. e., this is $P \amalg S$).*

Proof. Choose a point inside the contractible component. Any line passing through this point intersects the contractible component at least in two points. These points are geometrically distinct, since the line should intersect also the one-sided component. On the other hand, the total number of intersection points is at most three according to the Bézout theorem. Therefore any line passing through the selected point intersects one-sided component exactly in one point and two-sided component exactly in two points. The set of all real lines passing through the point is $\mathbb{R}P^2$. Drawing a line through the selected point and a real point of the surface defines a one-to-one map of the one-sided component onto $\mathbb{R}P^2$ and two-to-one map of the two-sided component onto $\mathbb{R}P^2$. Therefore the Euler characteristic of the one-sided component is equal to $\chi(\mathbb{R}P^2) = 1$, and the Euler characteristic of the two-sided component is $2\chi(\mathbb{R}P^2) = 2$. This determines the topological types of the components. □

5.3.C (Estimate for Diameter of Region Tree)**.** *The diameter of the region tree*[6] *of a nonsingular surface of degree m is at most* $[m/2]$.

Proof. Choose two vertices of the region tree the most distant from each other. Choose a point in each of the coresponding regions and connect the points by a line. □

*5.3.D***.** *The set of real points of a nonsingular surface of degree 4 has at most two noncontractible components. If the number of noncontractible components is 2, then there is no other component.*

Proof. First, assume that there are at least three noncontractible components. Consider the complement of the union of three noncontractible components. It consists of three domains, and at least two of them are not adjacent (cf. the previous subsection: the graph of adjacency of the domains should be a tree). Connect points of nonadjacent domains with a line. It has to intersect each of the three noncontractible components. Since they are zero-homologous, it intersects each of them at least in two points. Thus, the total number intersection points is at least 6, which contradicts to the Bézout theorem.

Now assume that there are two noncontractible components and some contractible component. Choose a point p inside the contractible component. The noncontractible components divide $\mathbb{R}P^3$ into 3 domains. One of the domains is adjacent to the both noncontractible components, while each of the other two domains is bounded by a single noncontractible component. If the contractible component lies in a domain bounded by a single noncontractible component, then take a point q in the other domain of the same sort, and connect p and q with

[6]Here by the diameter of a tree it is understood the maximal number of edges in a simple chain of edges of the tree, i. e., the diameter of the tree in the internal metric, with respect to which each edge has length 1.

a line. This line meets each of the three components at least twice, which contradicts to the Bézout theorem.

Otherwise (i. e. if the contractible component lies in the domain adjacent to both noncontractible components), choose inside each of the two other domains an embedded circle, which does not bound in $\mathbb{R}P^3$. Denote these circles by L_1 and L_2. Consider a surface C_i swept by lines connecting p with points of L_i. It realizes the nontrivial homology class. Indeed, take any line L transversal to it. Each point of $L \cap C_i$ corresponds to a point of the intersection of L_i and the plane consisting of lines joining p with L. Since L_i realizes the nonzero homology class, the intersection number of L_i with a plane is odd. Therefore the intersection number of L and C_i is odd. Since both C_1 and C_2 realizes the nontrivial homology class, their intersection realizes the nontrivial one-dimensional homology class. This may happen only if there is a line passing through p and meeting L_1 and L_2. Such a line has to intersect all three components of the quadric surface. Each of the components has to be met at least twice. This contradicts to the Bézout theorem. $\square$

5.3.E. Remark. In fact, if a nonsingular quartic surface has two noncontractible components then each of them is homeomorphic to torus. It follows from an extremal property of the refined Arnold inequality *5.3.L*. I do not know, if it can be deduced from the Bézout theorem. However, if to assume that one can draw lines in the domains of the complement which are not adjacent to both components, then it is not difficult to find homeomorphisms between the components of the surface and the torus, which is the product of these two lines. Cf. the proof of *5.3.B*.

5.3.F (Generalization of *5.3.D*)**.** *Let A be a nonsingular real algebraic surface of degree m in the 3-dimensional projective space. Then the diameter of the adjacency tree of domains of the complement of $\mathbb{R}A$ is at most $[m/2]$. If the degree is even and the diameter of the adjacency tree of the connected components of the complement of the union of the noncontractible components is exactly $m/2$, then there is no contractible components.*

The proof is a straightforward generalization of the proof of *5.3.D*. $\square$

Surprisingly, Bézout theorem gave much less restrictions in the case of surfaces than in the case of plane curves. In particular, it does not give anything like Harnack Inequality. Most of restrictions on topology of surfaces are analogous to the restrictions on flexible curves and were obtained using the same topological tools. Here is a list of the restrictions, though it is non-complete in any sense.

The restrictions are formulated below for a nonsingular real algebraic surface A of degree m in the 3-dimensional projective space. In these formulations and in what follows we shall denote the i-th Betti number of X over field $\mathbb{Z}_2$ (which is nothing but $\dim_{\mathbb{Z}_2} H_i(X\,;\mathbb{Z}_2)$) by $b_i(X)$. In particular, $b_0(X)$ is the number of components of X. By $b_*(X)$ we denote the total Betti number, i. e. $\sum_{i=0}^{infty} b_i(X) = \dim_{\mathbb{Z}_2} H_*(X\,;\mathbb{Z}_2)$.

5.3.G (Generalized Harnack Inequality).

$$b_*(\mathbb{R}A) \le m^3 - 4m^2 + 6m.$$

5.3.H. Remark. This is a special case of Smith-Floyd Theorem *3.2.B*, which in the case of curves implies Harnack Inequality, see Subsections 3.2. It says that for any involution i of a topological space X

$$b_*(fix(i)) \le b_*(X).$$

Applying this to the complex conjugation involution of the complexification $\mathbb{C}A$ of A and taking into account that $\dim_{\mathbb{Z}_2} H_*(\mathbb{C}A\,;\mathbb{Z}_2) = m^3 - 4m^2 + 6m$ one gets *5.3.G*. Applications to high-dimensional situation is discussed in Subsection **??** below.

5.3.I (Extremal Congruences of Generalized Harnack Inequality). *If*

$$b_*(\mathbb{R}A) = m^3 - 4m^2 + 6m,$$

then

$$\chi(\mathbb{R}A) \equiv (4m - 3m^2)/3 \mod 16.$$

If $b_*(\mathbb{R}A) = m^3 - 4m^2 + 6m - 2$, *then*

$$\chi(\mathbb{R}A) \equiv (4m - m^3 \pm 6)/3 \mod 16.$$

5.3.J (Petrovsky - Oleinik Inequalities).

$$-(2m^3 - 6m^2 + 7m - 6)/3 \le \chi(\mathbb{R}A) \le (2m^3 - 6m^2 + 7m)/3.$$

Denote the numbers of orientable components of $\mathbb{R}A$ with positive, zero and negative Euler characteristic by k^+, k^0 and k^- respectively.

5.3.K (Refined Petrovsky - Oleinik Inequality). *If* $m \ne 2$ *then*

$$-(2m^3 - 6m^2 + 7m - 6)/3 \le \chi(\mathbb{R}A) - 2k^+ - 2k^0.$$

5.3.L (Refined Arnold Inequality). *Either* m *is even,* $k^+ = k^- = 0$ *and*

$$k^0 = (m^3 - 6m^2 + 11m)/6,$$

or

$$k^0 + k^- \le (m^3 - 6m^2 + 11m - 6)/6.$$

5.4. **Surfaces of Low Degree.** Surfaces of degree 1 and 2 are well-known. Any surface of degree 1 is a projective plane. All of them are transformed to each other by a rigid isotopy consisting of projective transformations of the whole ambient space $\mathbb{R}P^3$.

Nonsingular surfaces of degree 2 (nonsingular *quadrics*) are of three types. It follows from the well-known classification of real nondegenerate quadratic forms in 4 variables up to linear transformation. Indeed, by this classification any such a form can be turned to one of the following:

(1) $+x_0^2 + x_1^2 + x_2^2 + x_3^2$,
(2) $+x_0^2 + x_1^2 + x_2^2 - x_3^2$,
(3) $+x_0^2 + x_1^2 - x_2^2 - x_3^2$,
(4) $+x_0^2 - x_1^2 - x_2^2 - x_3^2$,
(5) $-x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Multiplication by -1 identifies the first of them with the last and the second with the fourth reducing the number of classes to three. Since the reduction of a quadratic form to a canonical one can be done in a continuous way, all quadrics belonging to the same type also can be transformed to each other by a rigid isotopy made of projective transformations.

The first of the types consists of quadrics with empty set of real points. In traditional analytic geometry these quadrics are called imaginary ellipsoids. A canonical representative of this class is defined by equation $x_0^2 + x_1^2 + x_3^2 + x_4^2 = 0$.

The second type consists of quadrics with the set of real points homeomorphic to sphere. In the notations of the previous section this is S. The canonical equation is $x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0$.

The third type consists of quadrics with the set of real points homeomorphic to torus. They are known as one-sheeted hyperboloids. The set of real points is not contractible (it contains a line), so in the notations above it should be presented as S_1^1. The canonical equation is $x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0$.

Quadrics of the last two types (i. e., quadrics with nonempty real part) can be obtained by small perturbations of a union of two real planes. To obtain a quadric with real part homeomorphic to sphere, one may perturb the union of two real planes in the following way. Let the plane be defined by equations $L_1(x_0, x_1, x_2, x_3) = 0$ and $L_2(x_0, x_1, x_2, x_3) = 0$. Then the union is defined by equation $L_1(x_0, x_1, x_2, x_3)L_2(x_0, x_1, x_2, x_3) = 0$. Perturb this equation adding a small positive definite quadratic form. Say, take

$$L_1(x_0, x_1, x_2, x_3)L_2(x_0, x_1, x_2, x_3) + \varepsilon(x_0^2 + x_1^2 + x_2^2 + x_3^2) = 0$$

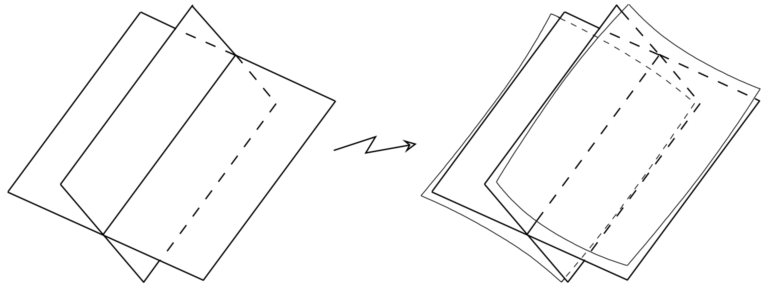

Figure 33. Two-sheeted hyperboloid as a result of small perturbation of a pair of planes.

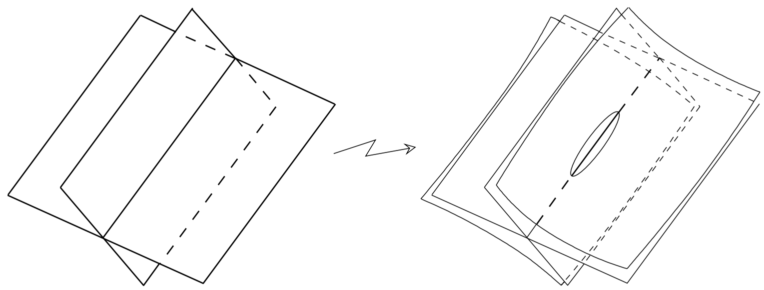

Figure 34. One-sheeted hyperboloid as a result of small perturbation of a pair of planes.

with a small $\varepsilon > 0$. This equation defines a quadric. Its real part does not meet plane $L_1(x_0, x_1, x_2, x_3) = L_2(x_0, x_1, x_2, x_3)$, since on the real part of the quadric the product $L_1(x_0, x_1, x_2, x_3)L_2(x_0, x_1, x_2, x_3)$ is negative. Therefore the real part of the quadric is contractible in $\mathbb{R}P^3$. Since it is obtained by a perturbation of the union of two planes, it is not empty, provided $\varepsilon > 0$ is small enough. As easy to see, it is not singular for small $\varepsilon > 0$. Cf. Subsection **??**. Of course, this can be proved explicitely, as an exercise in analytic geometry. See Figure 33

To obtain a noncontractible nonsingular quadric (one-sheeted hyperboloid), one can perturb the same equation $L_1(x_0, x_1, x_2, x_3)L_2(x_0, x_1, x_2, x_3) = 0$, but by a small form which takes both positive and negative values on the intersection line of the planes. See Figure 34.

Nonsingular surfaces of degree 3 (nonsingular *cubics*) are of five types. Here is the complete list of there topological types:

$$P, \quad P \amalg S, \quad P_1, \quad P_2, \quad P_3.$$

Let us prove, first, that only topological types from this list can be realized. Since the degree is odd, a nonsingular surface has to be one-sided. By *5.3.D* if it is not connected, then it is homeomorphic to

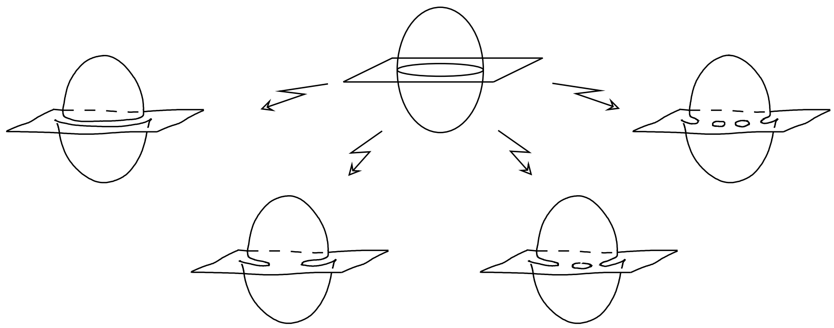

FIGURE 35. Constructing cubic surfaces of types $P \amalg S$, P, P_1 and P_2.

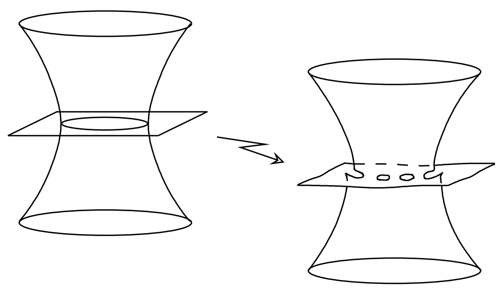

FIGURE 36. Constructing a cubic surface of type P_3.

$P \amalg S$. By the Generalized Harnack Inequality *5.3.G*, the total Betti number of the real part is at most $3^3 - 4 \times 3^2 + 6 \times 3 = 9$. On the other hand, the first Betti number of a projective plane with g handles is $1 + 2g$ and the total Betti number $b_*(P_g)$ is $3 + 2g$. Therefore in the case of a nonsingular cubic with connected real part, it is of the type P_g with $g \leq 3$.

All the five topological types are realized by small perturbations of unions of a nonsingular quadric and a plane transversal to one another. This is similar to the perturbations considered above, in the case of spatial quadrics. See Figures 35 and 36.

An alternative way to construct nonsingular surfaces of degree 3 of all the topological types is provided by a connection between nonsingular spatial cubics and plane nonsingular quartics. More precisely, there is a correspondence assigning a plane nonsingular quartic with a selected real double tangent line to a nonsingular spatial cubic with a selected real point on it. It goes as follows. Consider the projection of the cubic from a point selected on it to a plane. The projection is similar to the well-known stereographic projection of a sphere to plane.