Linking Lines

Oleg Viro

November 2, 2006

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Can a set of disjoint lines be *rearranged*
by a continuous movement during which they stay disjoint?
Any two sets of parallel lines with the same number of lines in each set can be transformed to each other in this sense.

Combing lines

Can an arbitrary set of lines be moved ("combed") into a set of parallel lines?

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by fastening the lines at those points.

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but only as a point on that plane, which we allow to slide along the lines.

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In other words, drill small holes in the second plane where it intersects with the lines.

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Another description: expand the space away from the first plane in a direction perpendicular to it, driving the expansion factor rapidly to infinity in a finite length of time. The lines rotate around their points of intersection with the first plane, and in the limit become perpendicular to the planes.

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Let us decide not to allow parallel lines anymore!

Equivalence of links made of skew lines

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Such movements of lines is called an *isotopy* .
If one set of lines can be obtained from another by an isotopy, then the two sets are *isotopic*.

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1. find the common perpendicular,
2. rotate one of the lines around it to make the lines perpendicular,
3. adjust the length of the common perpendicular.

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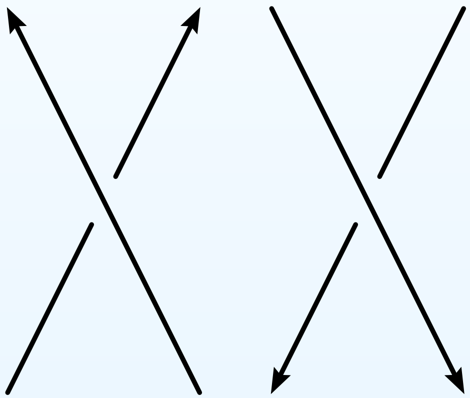

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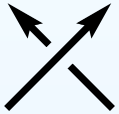
The linking number is $+1$ if the lines look like this: 

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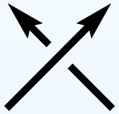
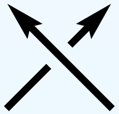
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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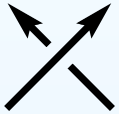
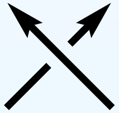
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

Exercise: Prove that the look does not depend on the point of view using only arguments of Elementary Geometry!

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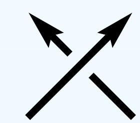
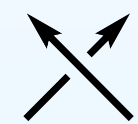
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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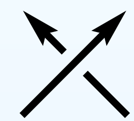
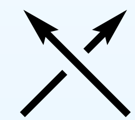
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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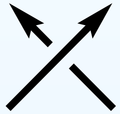
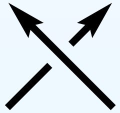
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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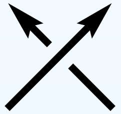
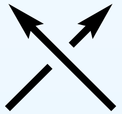
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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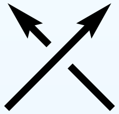
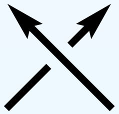
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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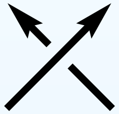
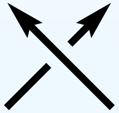
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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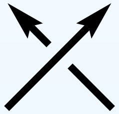
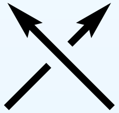
The linking number is $+1$ if the lines look like this:  ,

the linking number is -1 if the lines look like that:  .

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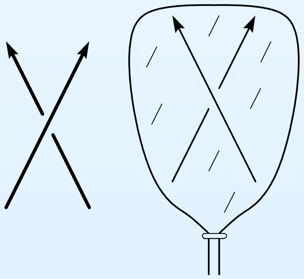

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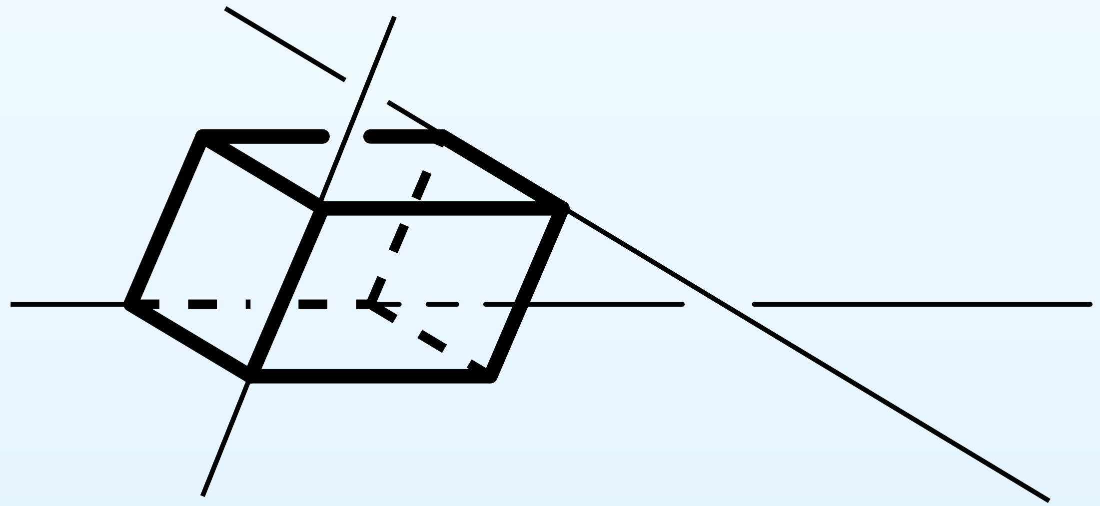
Our lines are the extensions of three of its skew edges:



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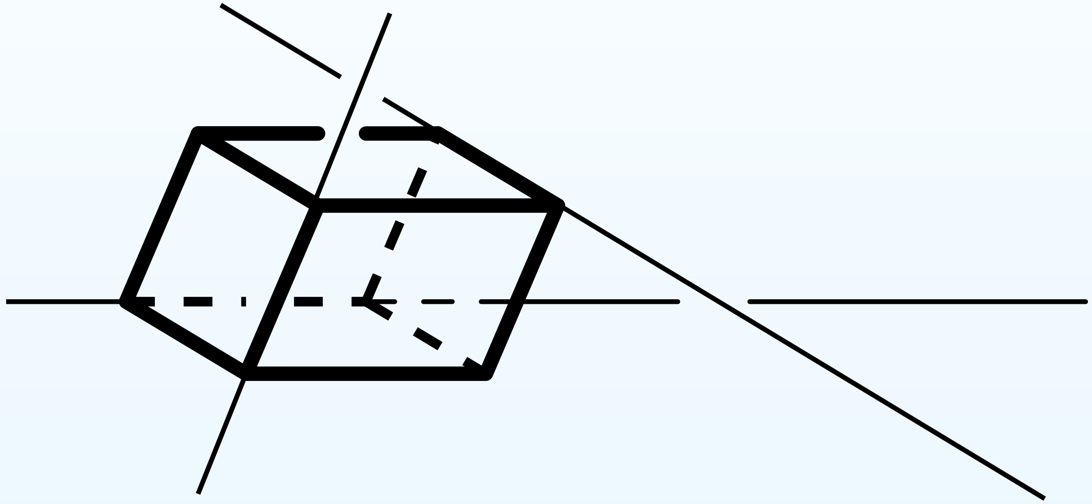

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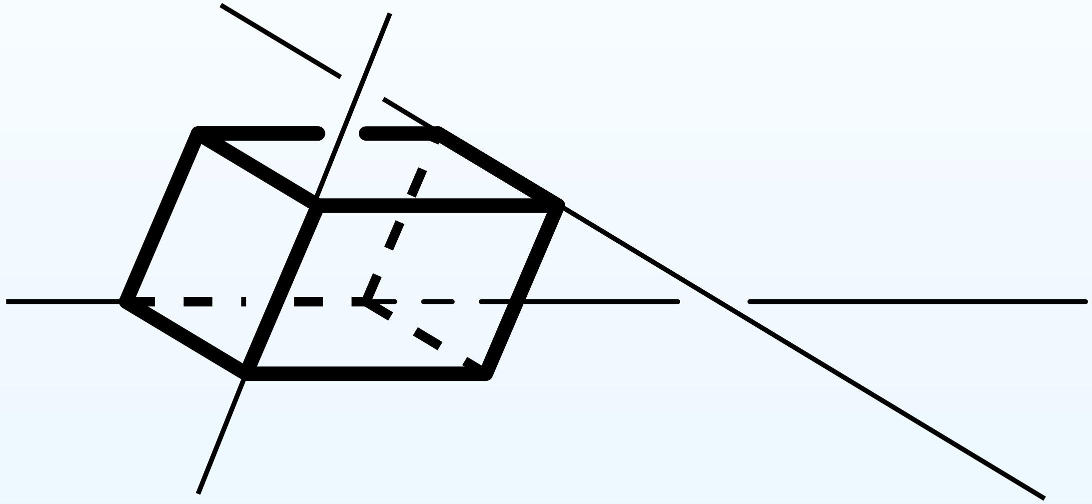

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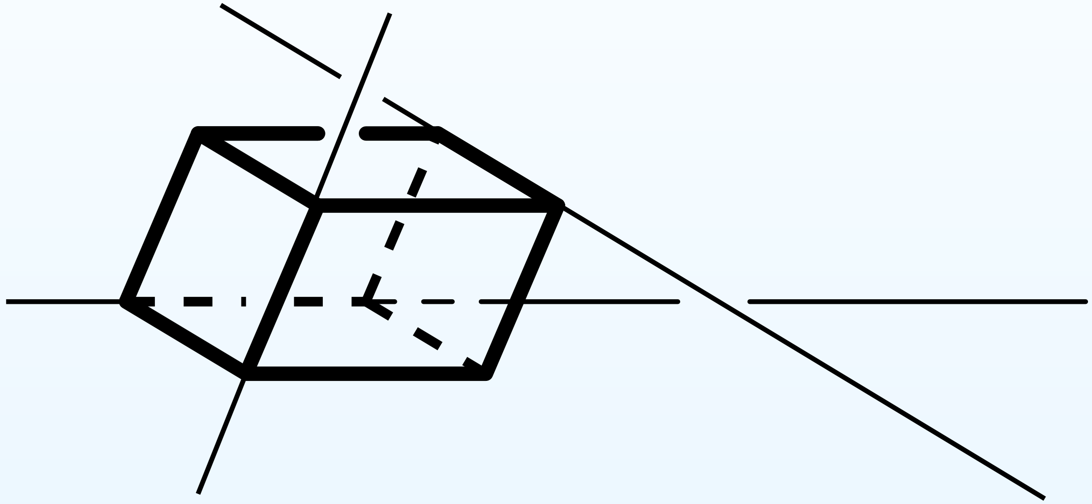

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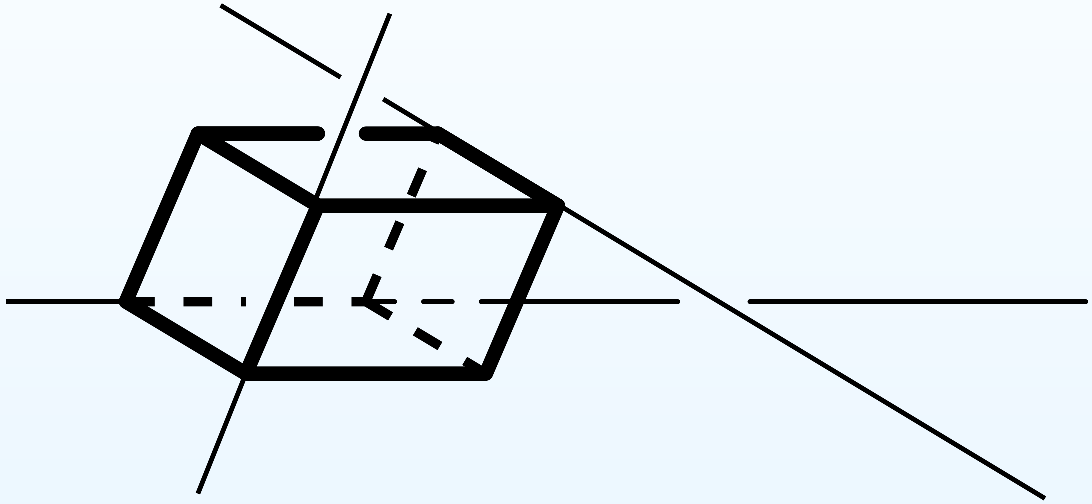

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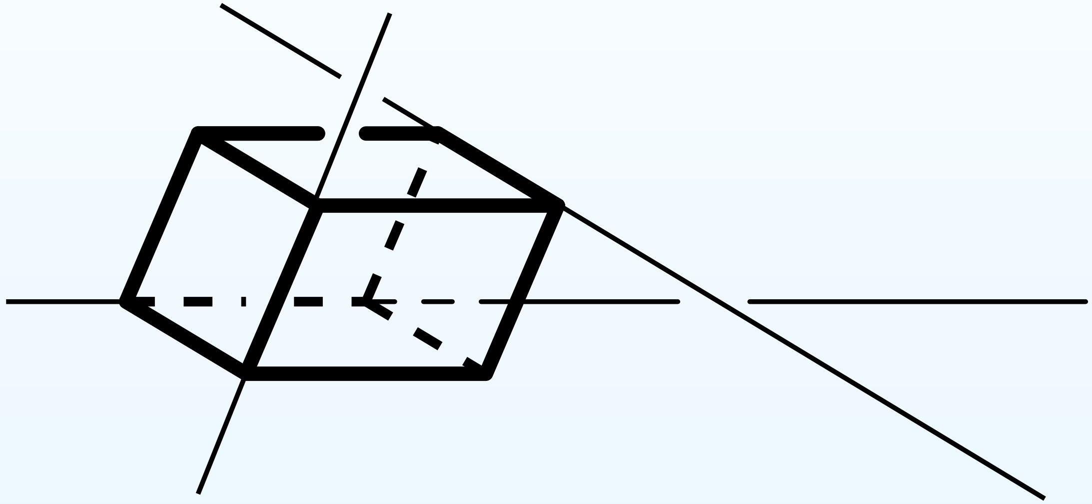

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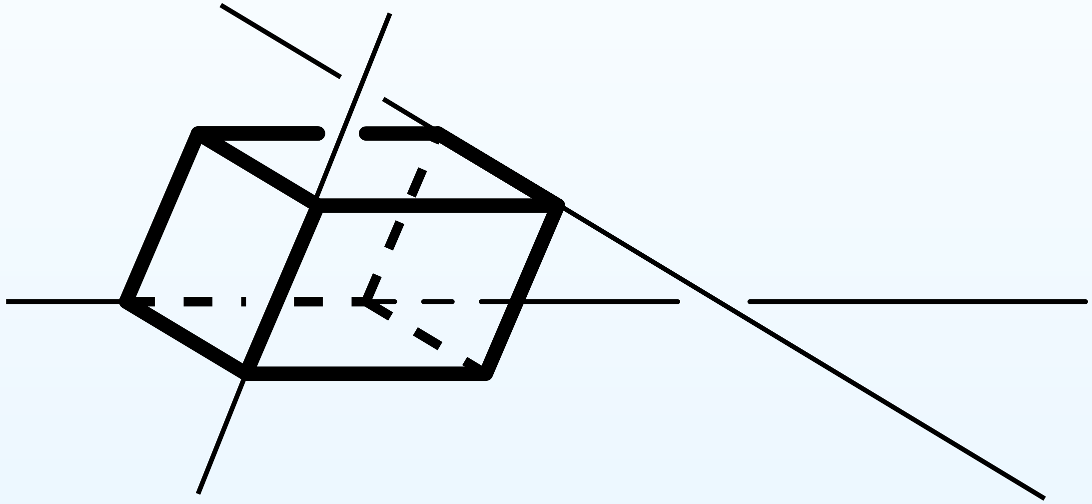

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Hence, the six planes are uniquely determined by the original triple of lines.

Deforming parallelipiped

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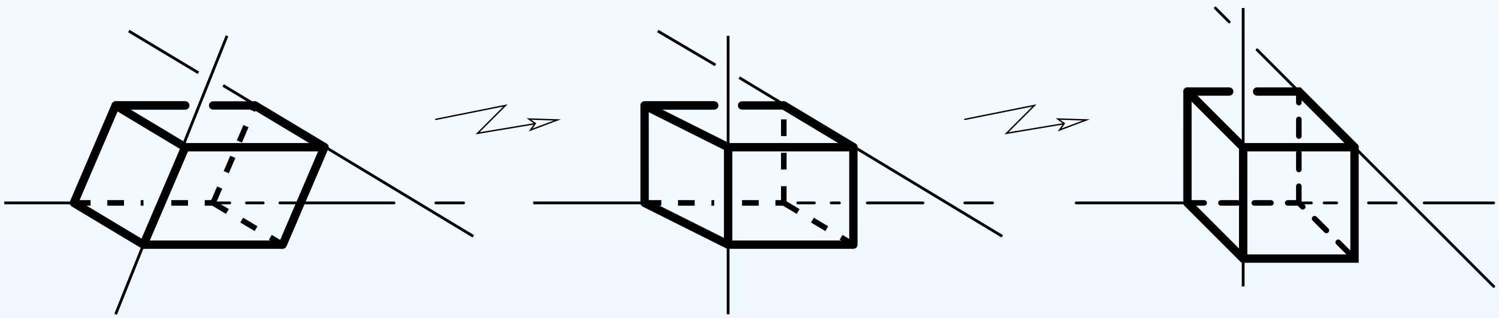

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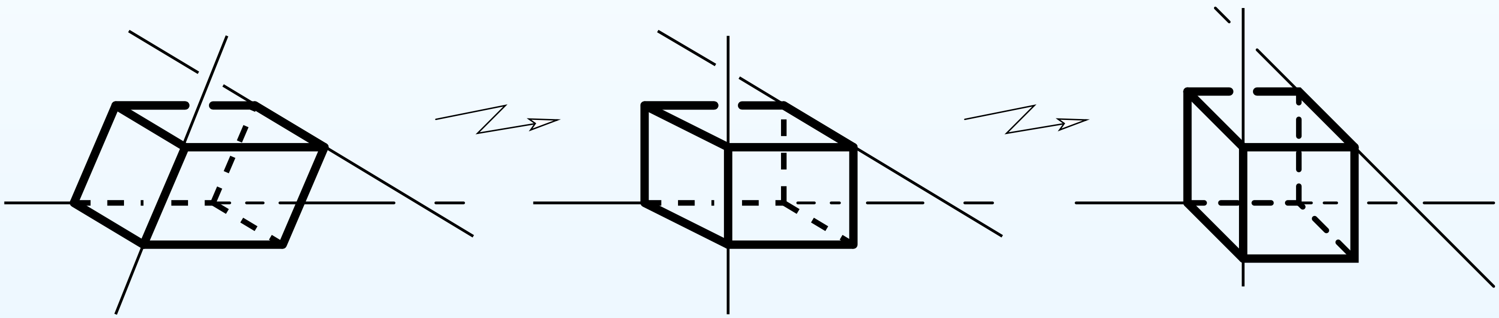

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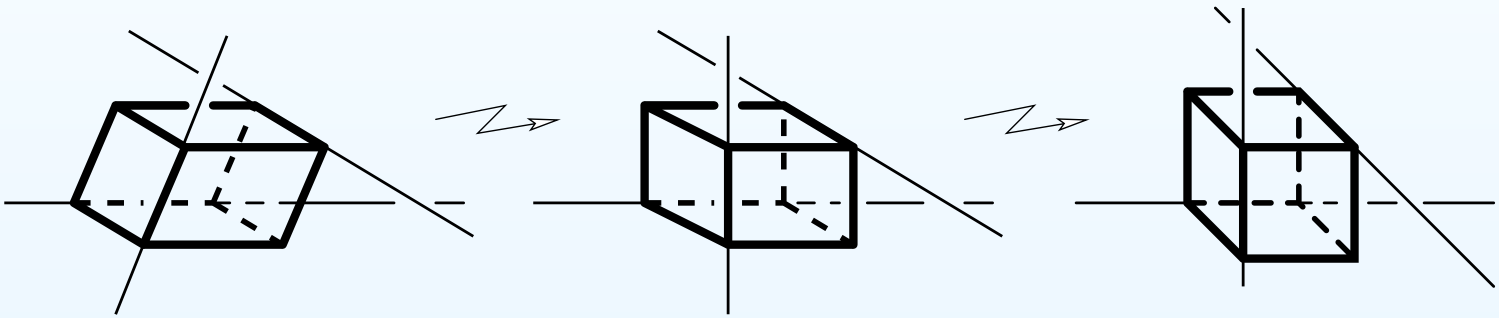

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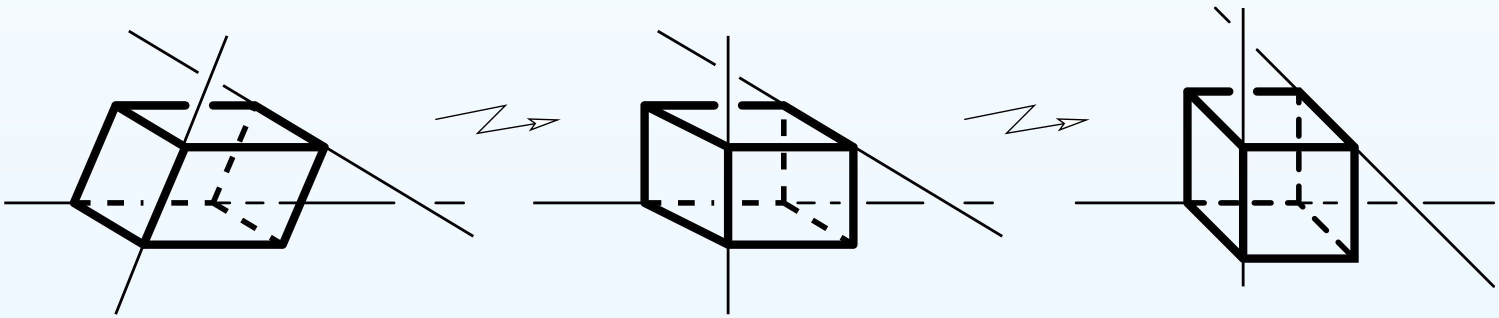

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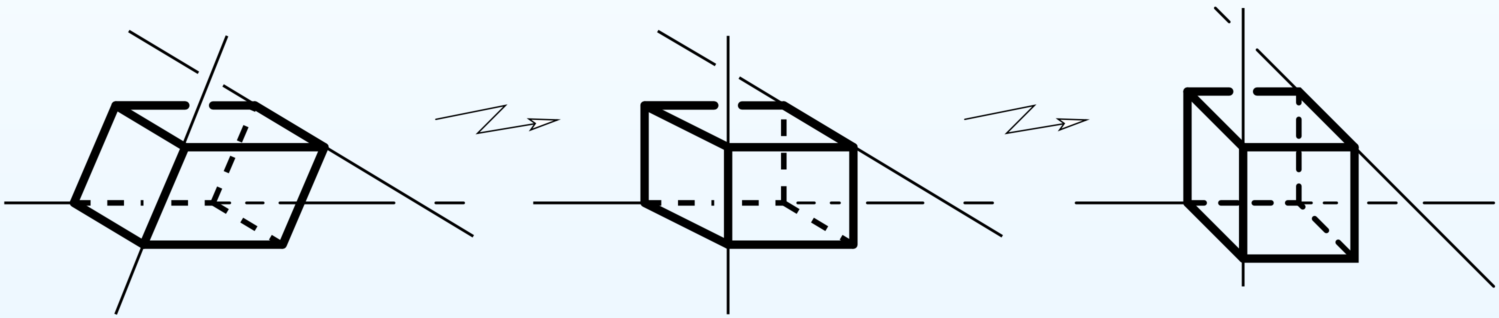

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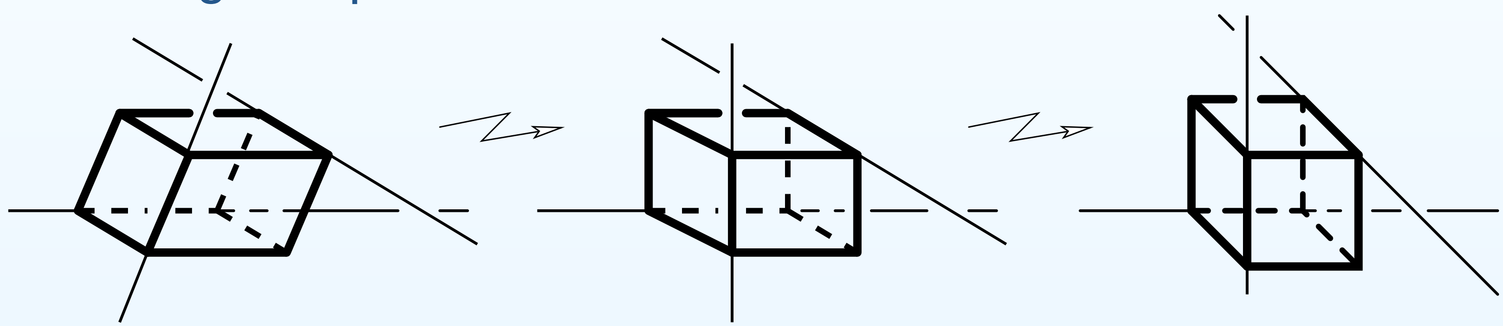

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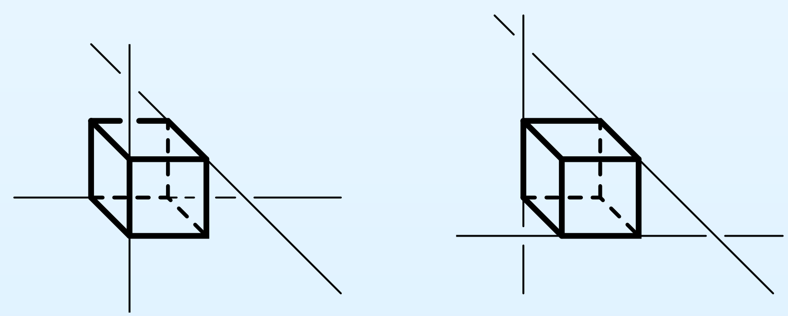
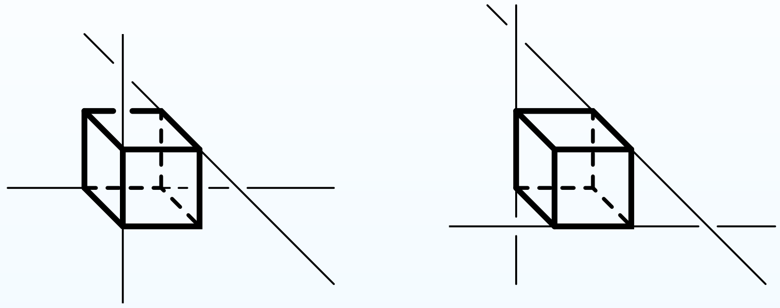


Linking number of a triple



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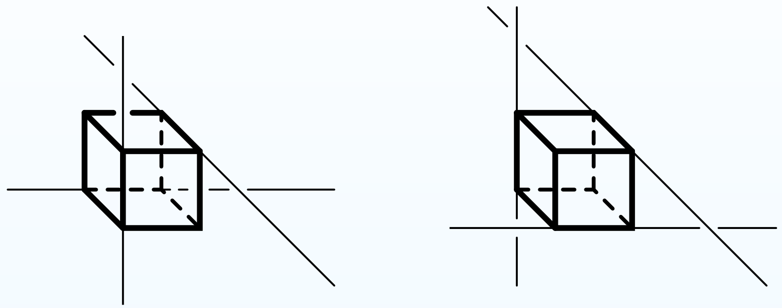

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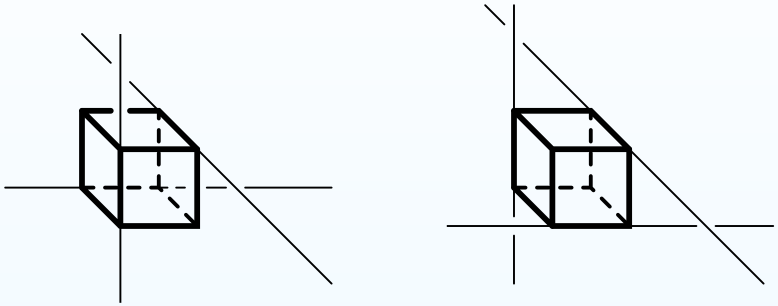

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They are not isotopic.

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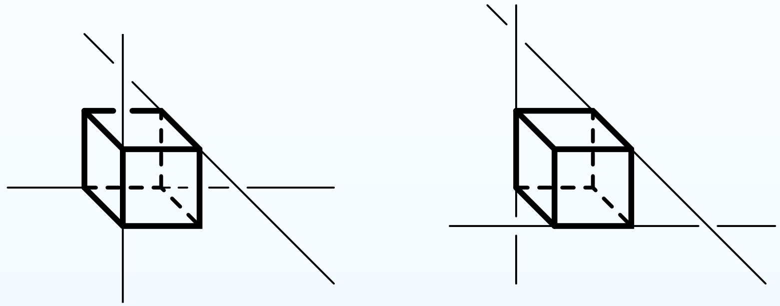

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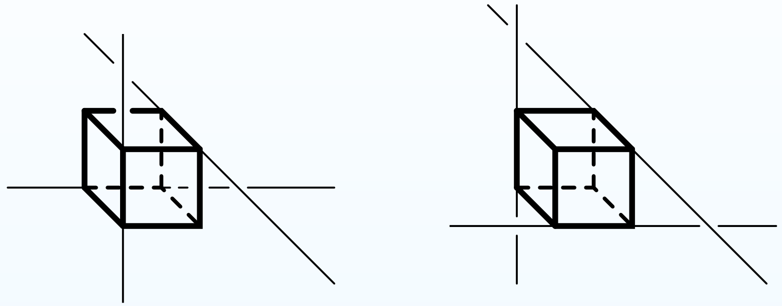

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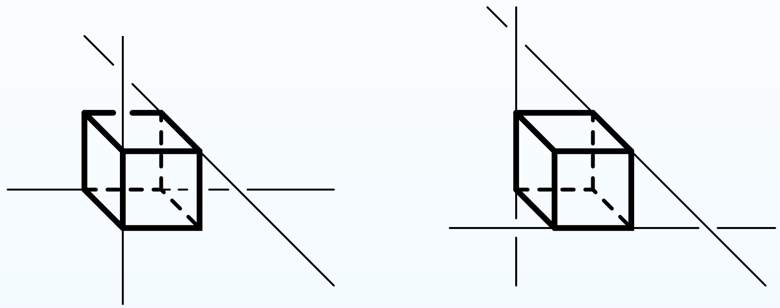

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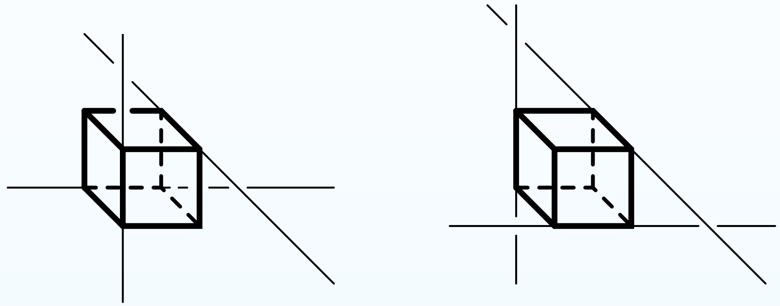

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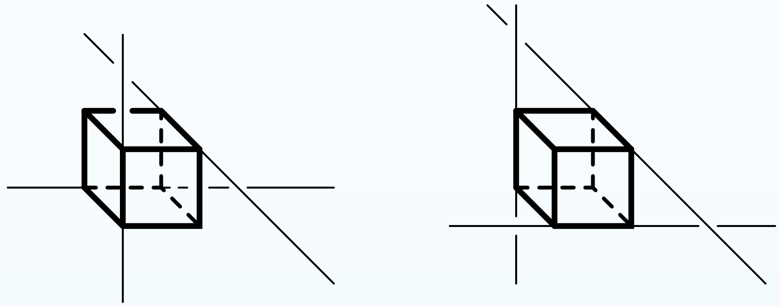

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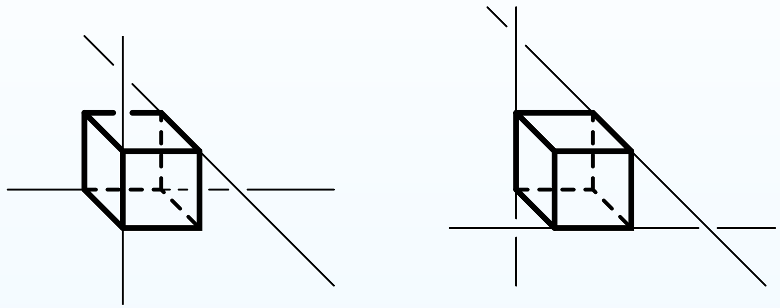

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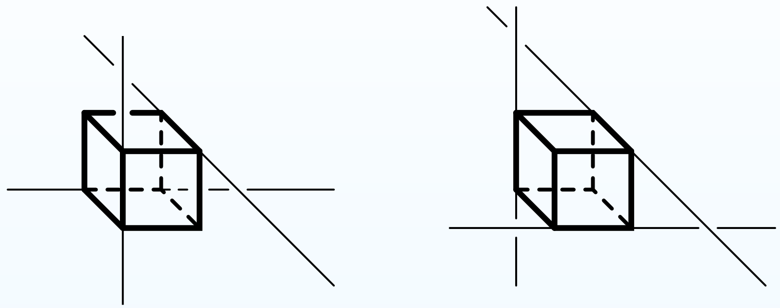

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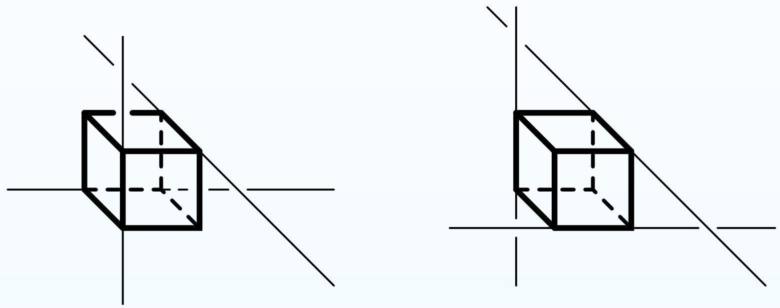

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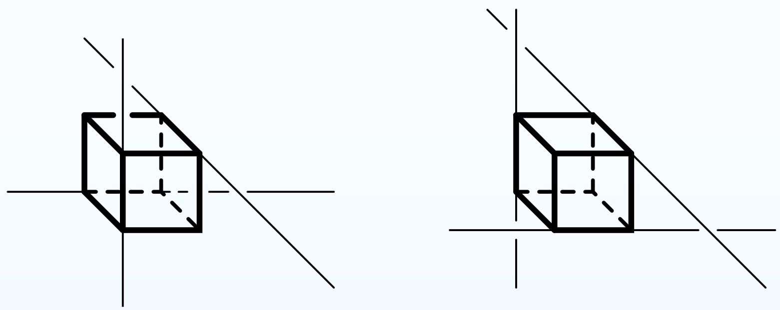

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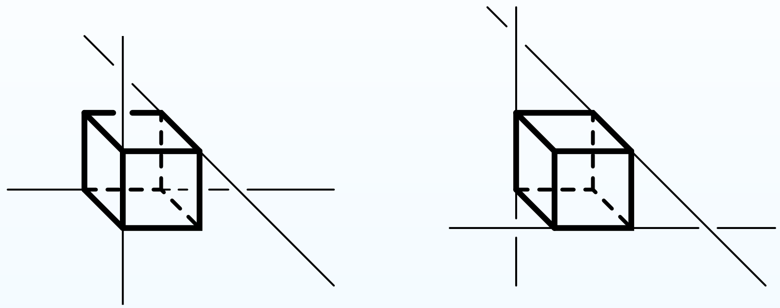

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Amphicheiral and Nonamphicheiral

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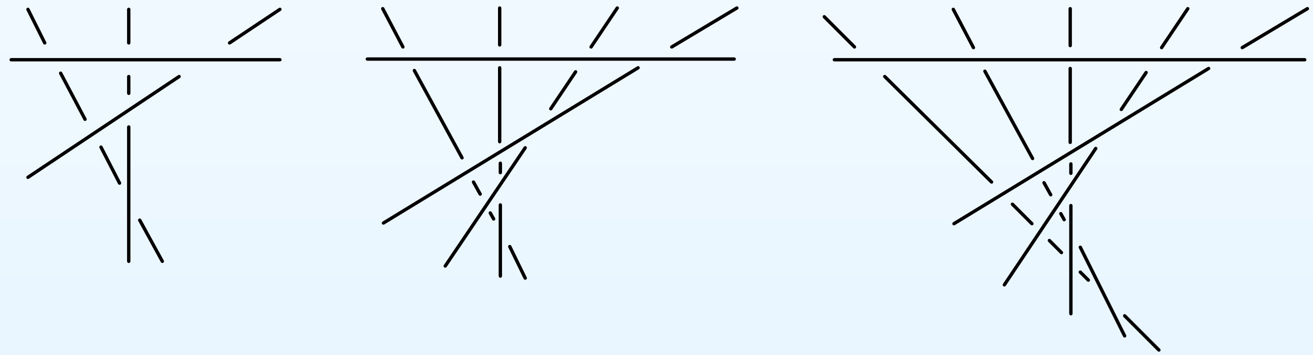

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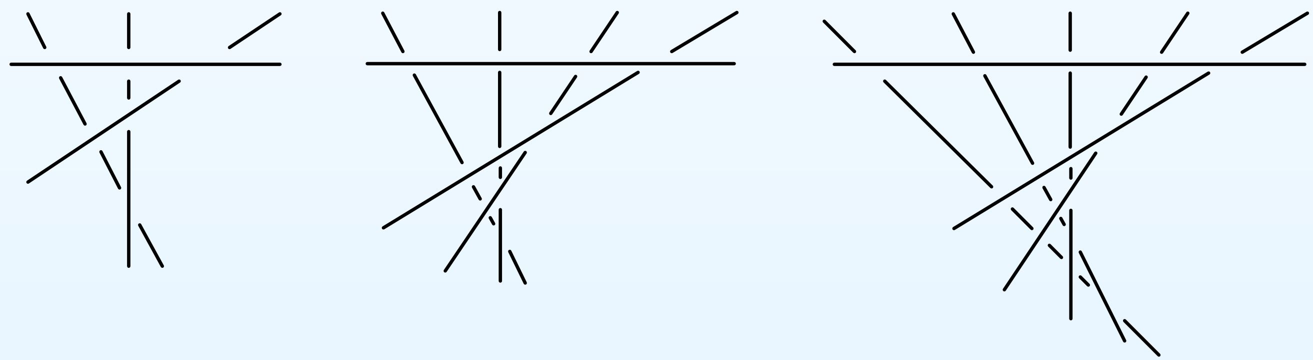

The links are nonamphicheiral since all of the triples of lines in them have the same linking number.

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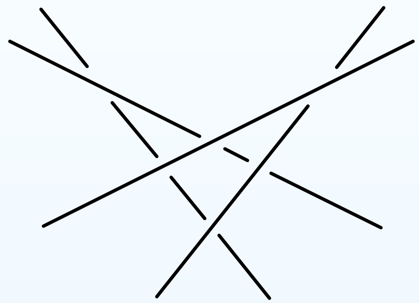

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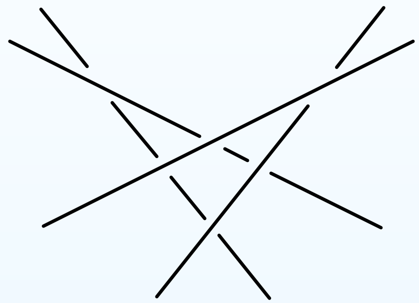
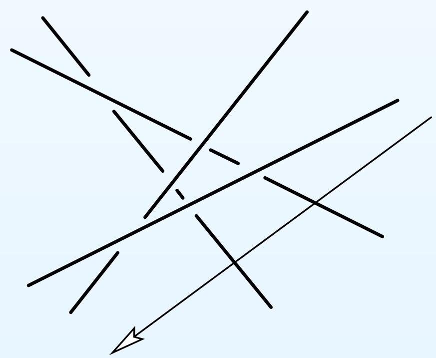


To prove amphicheirality of he link, move the upper two lines in such a way that the part of its projection which contains all of the intersections (in the projection) passes over and above the projection of the other two lines:



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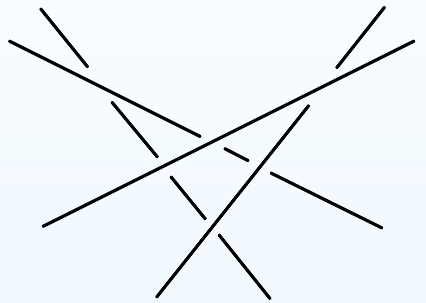

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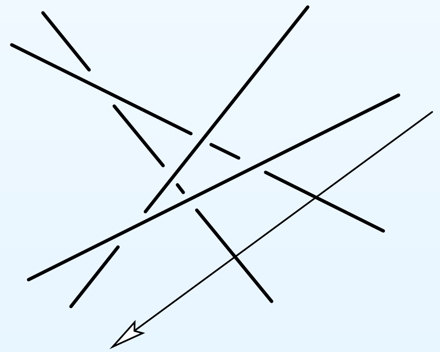

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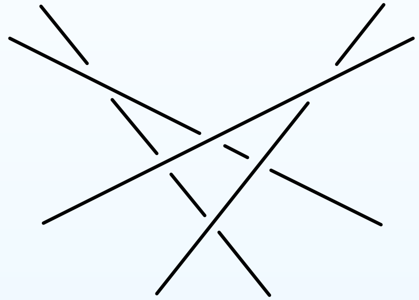

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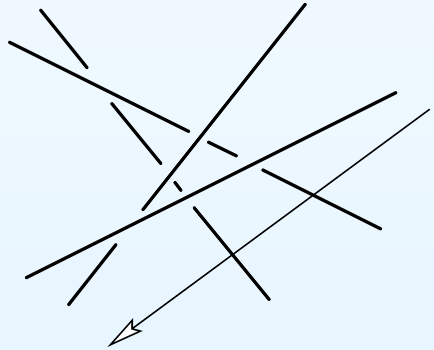

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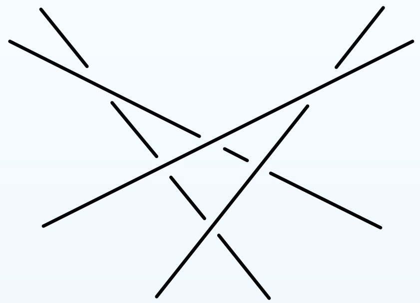

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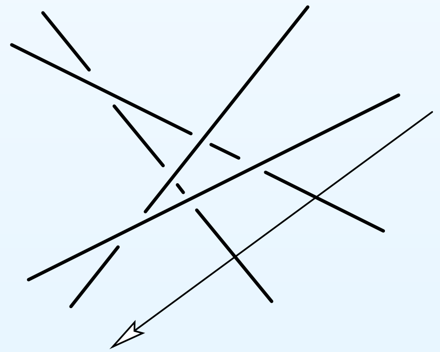

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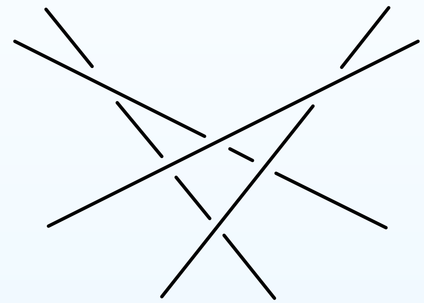

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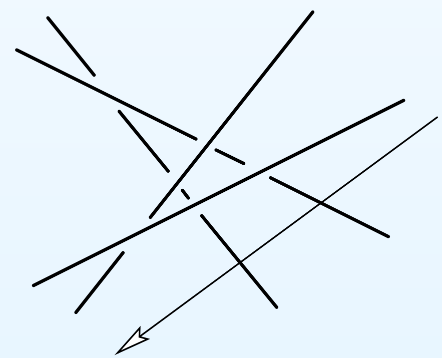

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Take the lines of the upper set from the sequence of nonamphicheiral links constructed earlier. The other $p/2$ lines is obtained from the first ones by rotating and then reflecting in a mirror.

Amphicheiral examples

Exercise. Construct similar examples for $p \equiv 1 \mod 4$, i.e., amphicheiral links of $p = 4k + 1$ lines.

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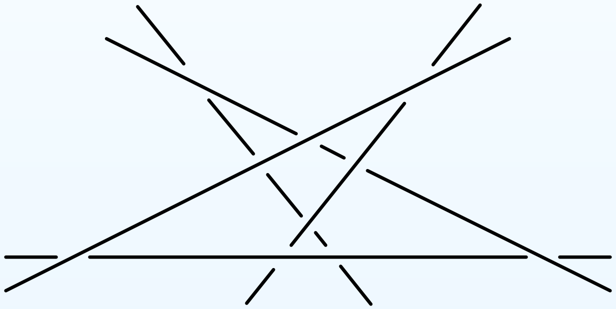
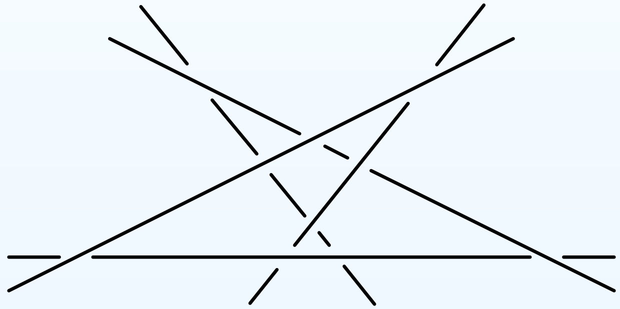
Here is the simplest link of this sort, with $p = 5$:



Amphicheiral examples

Exercise. Construct similar examples for $p \equiv 1 \mod 4$, i.e., amphicheiral links of $p = 4k + 1$ lines.

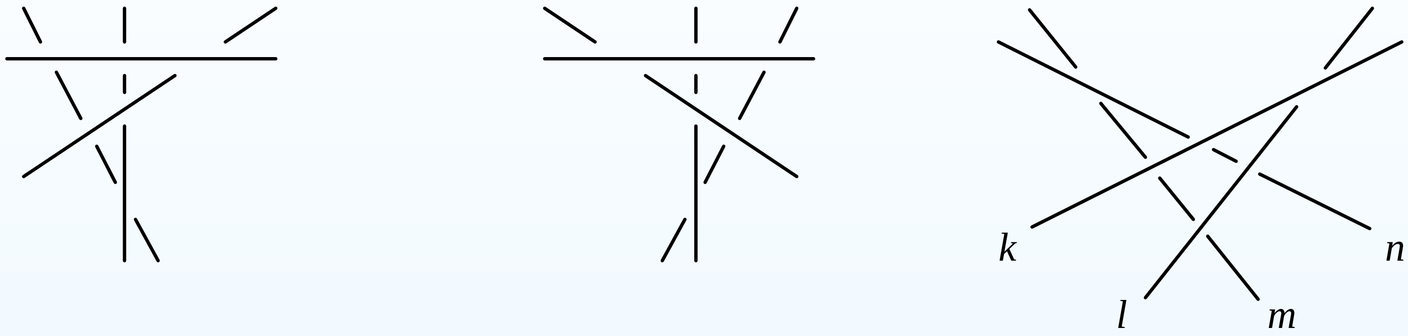
Here is the simplest link of this sort, with $p = 5$:



Exercise. Prove, at least, that this link is amphicheiral.

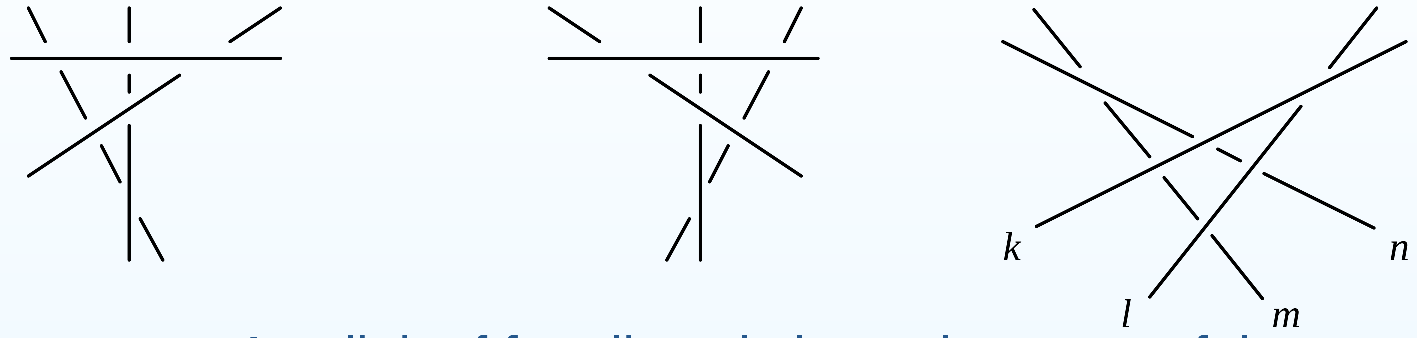
Four lines

Links of four lines:



Four lines

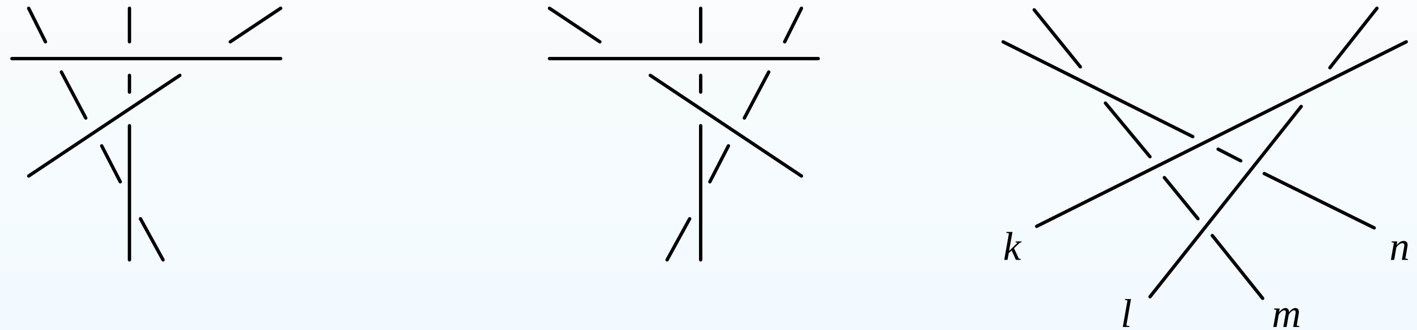
Links of four lines:



Theorem. Any link of four lines is isotopic to one of these links.

Four lines

Links of four lines:

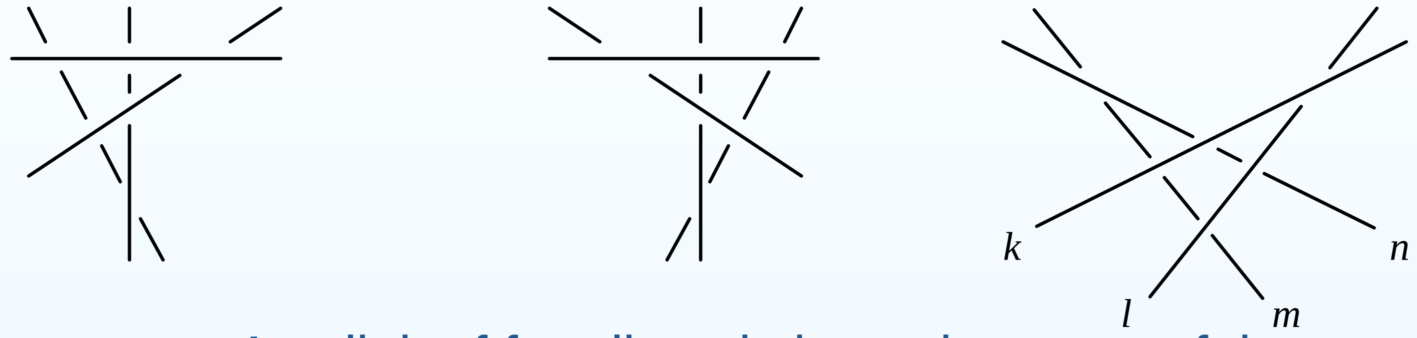


Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines.

Four lines

Links of four lines:

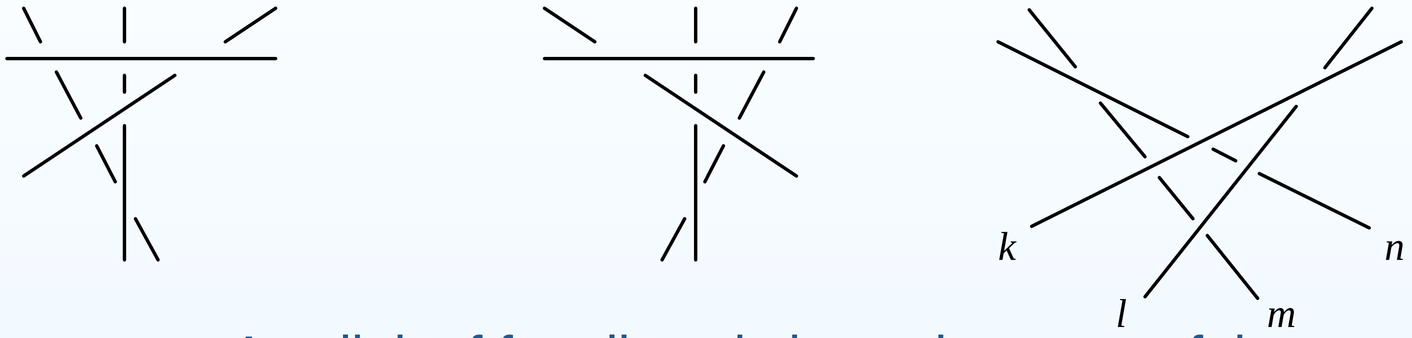


Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. By moving it slightly, if necessary, make three of the four lines (it makes no difference which three) do not lie in parallel planes.

Four lines

Links of four lines:

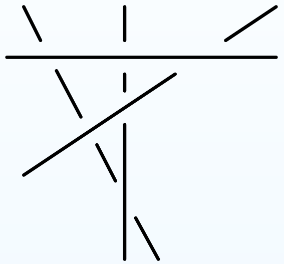


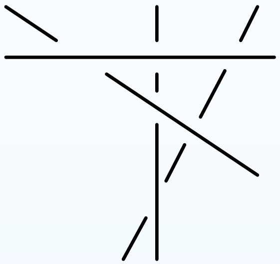
Theorem. Any link of four lines is isotopic to one of these links.

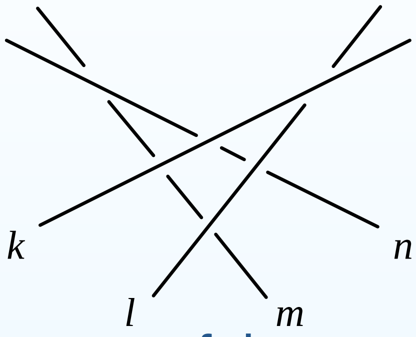
Proof. Take an arbitrary link of four lines. By moving it slightly, if necessary, make three of the four lines (it makes no difference which three) do not lie in parallel planes. Construct a hyperboloid through these three lines.

Four lines

Links of four lines:





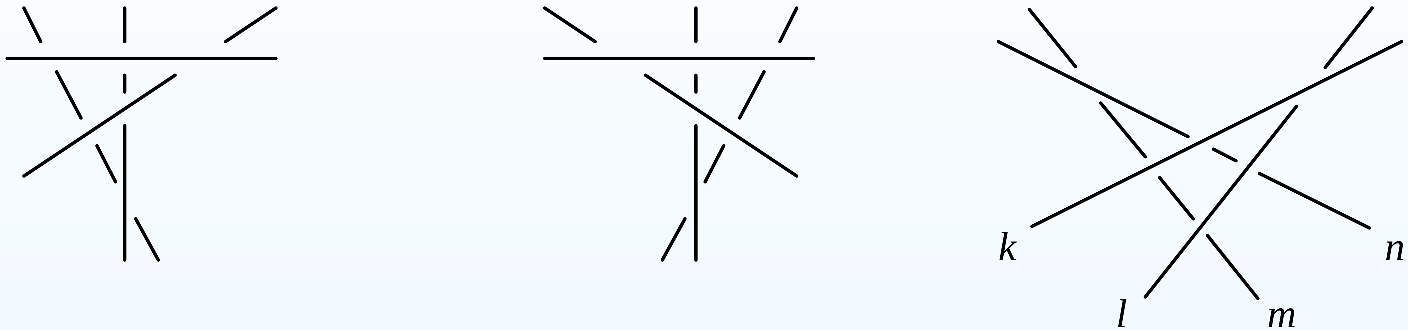


Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

Four lines

Links of four lines:



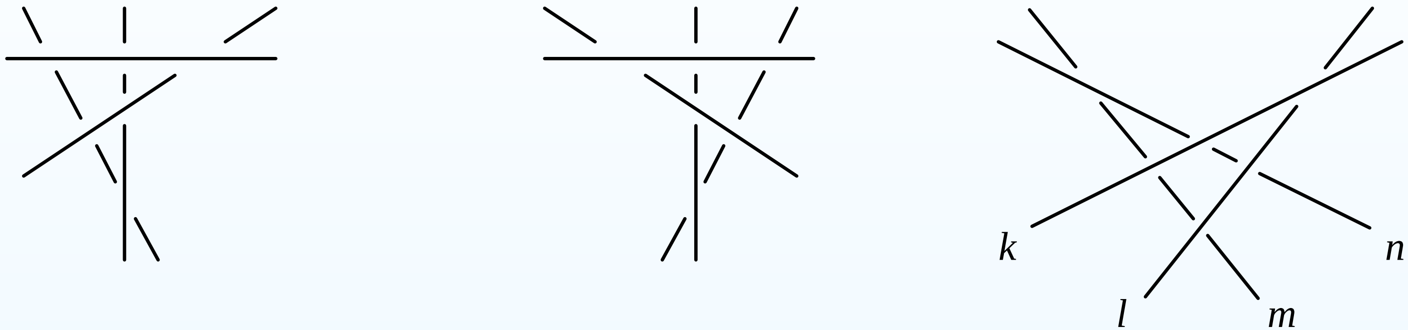
Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

1. either does not intersect the hyperboloid

Four lines

Links of four lines:



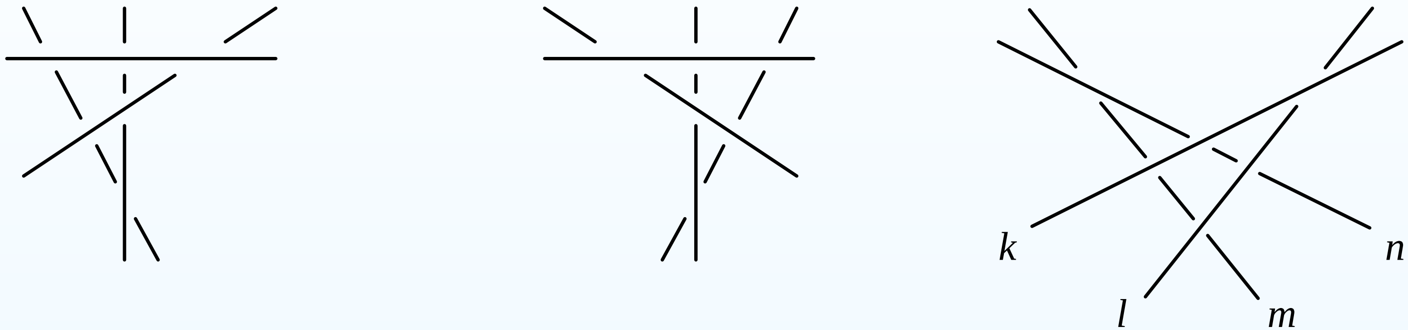
Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

1. either does not intersect the hyperboloid
2. or intersect it in a single point,

Four lines

Links of four lines:



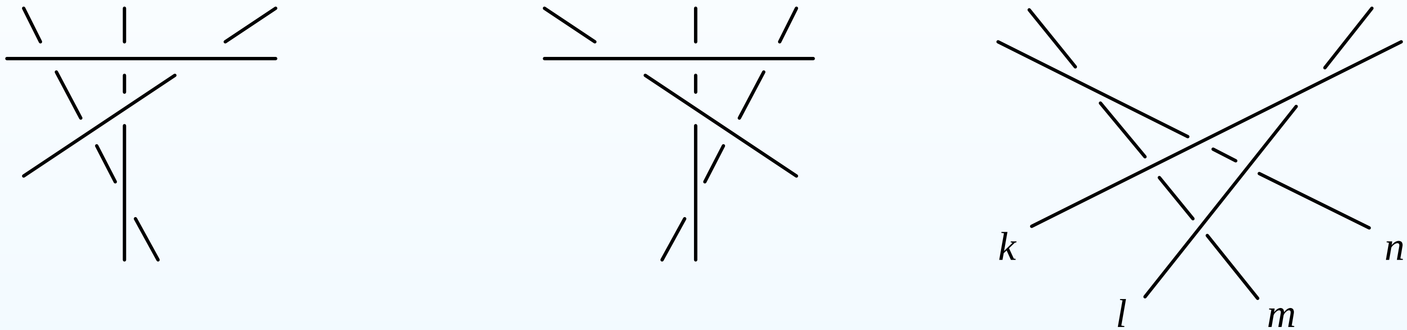
Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

1. either does not intersect the hyperboloid
2. or intersect it in a single point,
3. or intersect it in two points,

Four lines

Links of four lines:



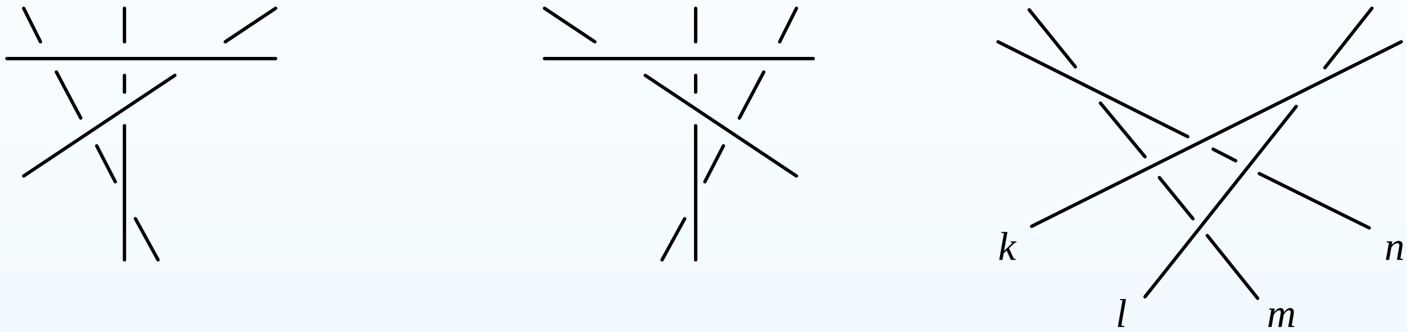
Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

1. either does not intersect the hyperboloid
2. or intersect it in a single point,
3. or intersect it in two points,
4. or lies on the hyperboloid.

Four lines

Links of four lines:



Theorem. Any link of four lines is isotopic to one of these links.

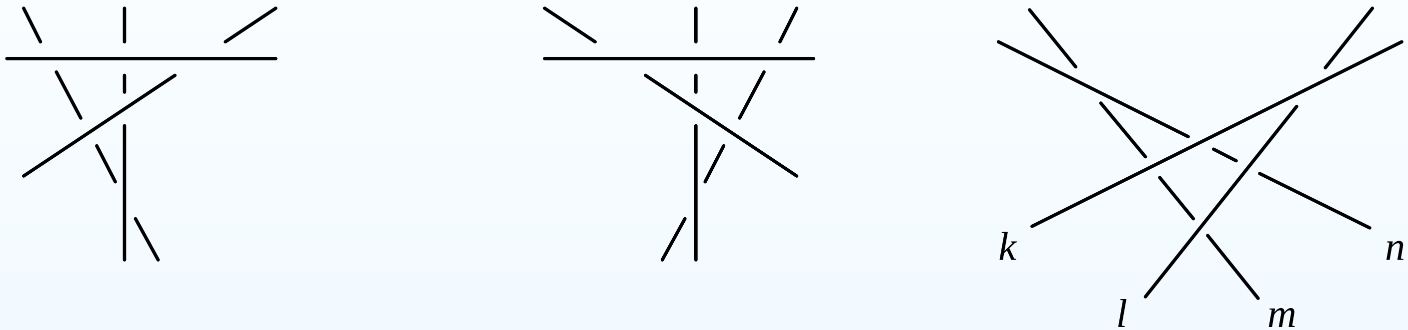
Proof. Take an arbitrary link of four lines. The fourth line:

1. either does not intersect the hyperboloid
2. or intersect it in a single point,
3. or intersect it in two points,
4. or lies on the hyperboloid.

In the last case the link is isotopic either to the left or the center link.

Four lines

Links of four lines:



Theorem. Any link of four lines is isotopic to one of these links.

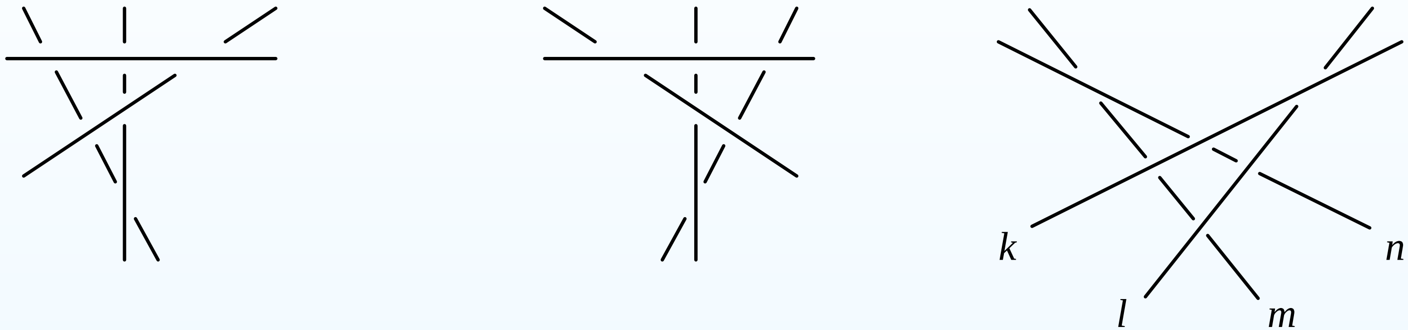
Proof. Take an arbitrary link of four lines. The fourth line:

1. either does not intersect the hyperboloid
2. or intersect it in a single point,
3. or intersect it in two points,

If the fourth line does not intersect the hyperboloid, then it can be brought in toward the hyperboloid until it is tangent to the hyperboloid, i.e., the case 1 is reduced to case 2.

Four lines

Links of four lines:



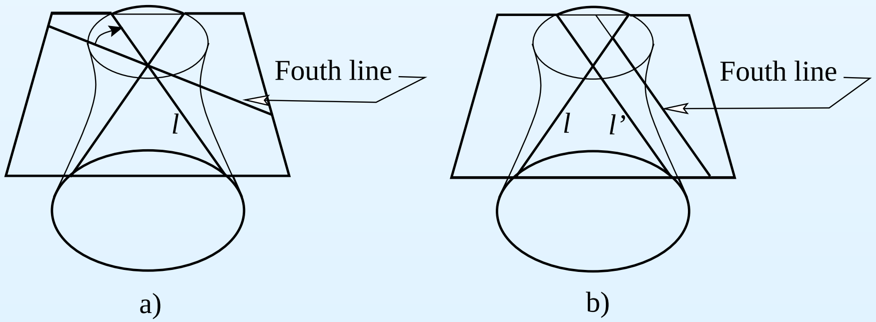
Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

2. or intersect it in a single point,

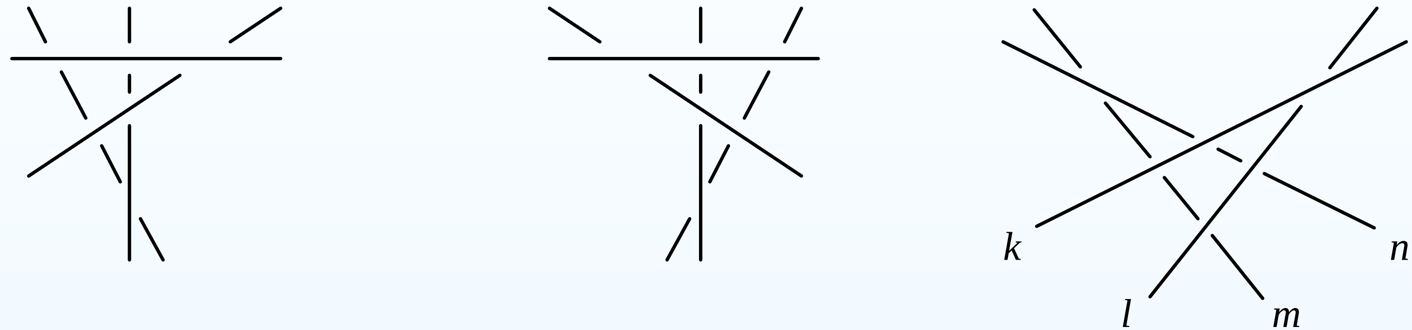
3. or intersect it in two points,

The case 2 reduces to 4 or 3:



Four lines

Links of four lines:



Theorem. Any link of four lines is isotopic to one of these links.

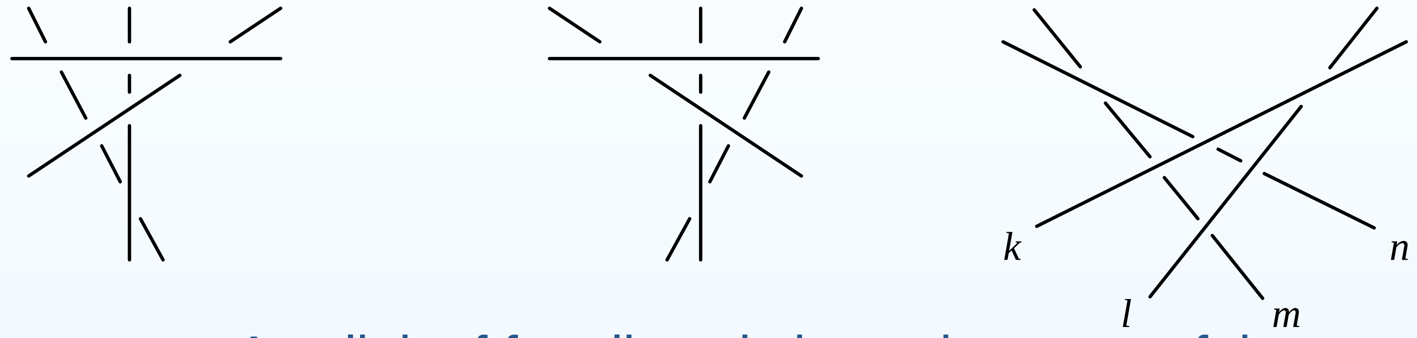
Proof. Take an arbitrary link of four lines. The fourth line:

3. or intersect it in two points,

If the fourth line intersects the hyperboloid in two pints, then everything depends on whether these points are in the same part of the hyperboloid into which the first three lines divide it, or are in different parts (the hyperboloid is divided into three sections).

Four lines

Links of four lines:



Theorem. Any link of four lines is isotopic to one of these links.

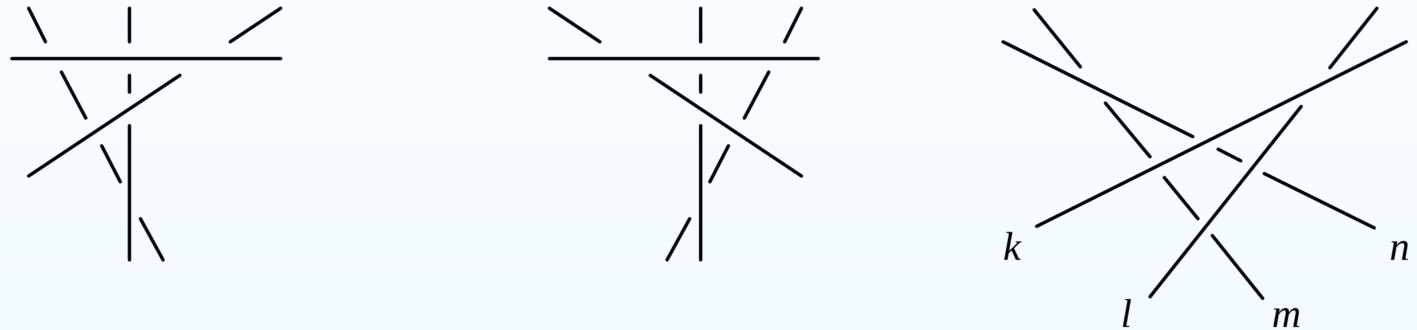
Proof. Take an arbitrary link of four lines. The fourth line:

3. or intersect it in two points,

If the points are in the same part, then the fourth line can be placed on the hyperboloid without the first three lines interfering.

Four lines

Links of four lines:



Theorem. Any link of four lines is isotopic to one of these links.

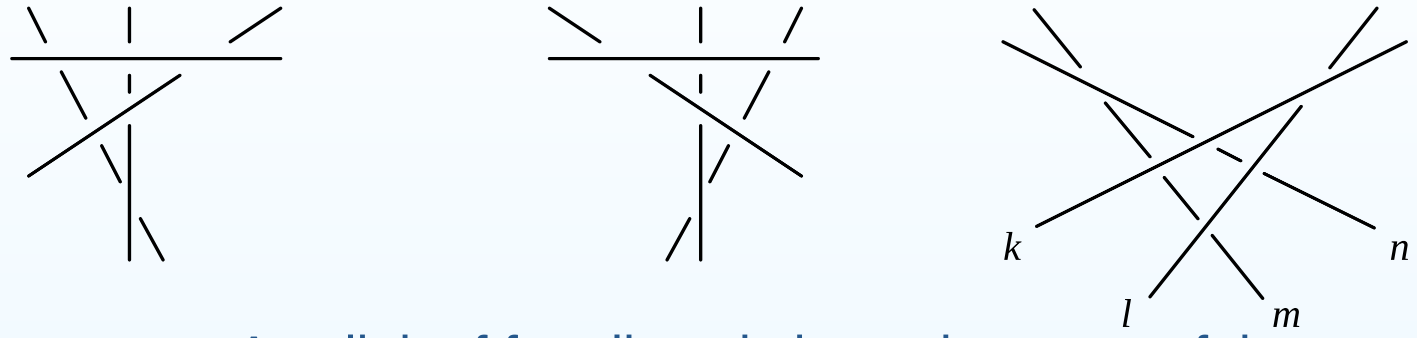
Proof. Take an arbitrary link of four lines. The fourth line:

3. or intersect it in two points,

If the points are in the same part, then the fourth line can be placed on the hyperboloid without the first three lines interfering. Then the fourth line becomes a generatrix, and we are in case 4.

Four lines

Links of four lines:



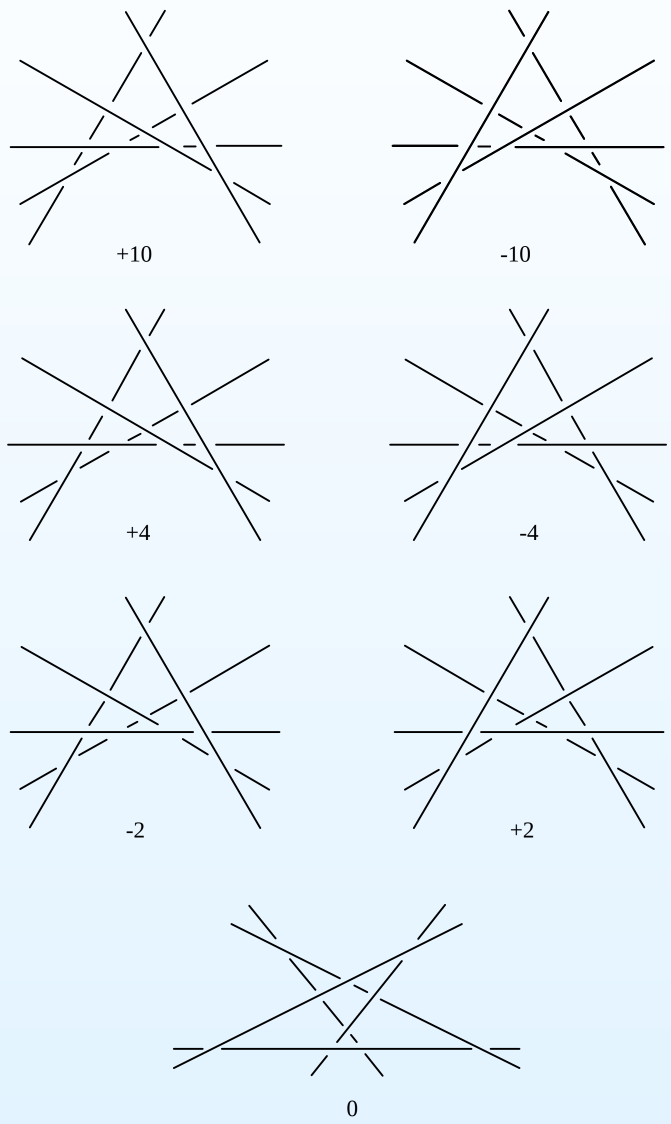
Theorem. Any link of four lines is isotopic to one of these links.

Proof. Take an arbitrary link of four lines. The fourth line:

3. or intersect it in two points,

If the fourth line intersects the hyperboloid in different parts, then the link is isotopic to the rightmost one.

Five lines



- Can lines be linked?
- Combing lines
- No parallel lines
- Equivalence of links made of skew lines
- Pair of lines
- Orientations and Semi-Orientations
- Linking number
- Triples of lines
- Parallelipiped
- Deforming parallelipiped
- Linking number of a triple

Amphicheiral and Nonamphicheiral
- Amphicheirality problems
- Nonamphicheiral examples
- Amphicheiral examples
- Amphicheiral examples
- Four lines
- Five lines

Six lines