SELF-LINKING NUMBER OF REAL ALGEBRAIC LINK

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ABSTRACT. For a nonsingular real algebraic curve in the 3-dimensional projective space or sphere, a new integer valued characteristic is introduced. It is invariant under rigid isotopy and multiplied by -1 under a mirror reflection. In a sense, it is a Vassiliev invariant of degree 1 and a counter-part of a link diagram writhe.

1. INTRODUCTION

This paper is a detailed version of my preprint [9] which was written about 5 years ago. Here I do not discuss results which have appeared since then. I plan to survey them soon in another paper. The subject is developing evolving now to a *real algebraic knot theory.*

This paper is dedicated to the memory of my teacher Vladimir Abramovich Rokhlin. It was V. A. Rokhlin, who suggested to me, a long time ago, in 1977, to develop a theory of real algebraic knots. He suggested this as a topic for my second dissertation (post PhD, like habilitation). Following this suggestion, I moved then from the knot theory and low-dimensional topology to the topology of real algebraic varieties. However in the topology of real algebraic varieties problems on spatial surfaces and plane curves were more pressing than problems on spatial curves, and my second dissertation defended in 1983 was devoted to constructions of real algebraic plane curves and spatial surfaces with prescribed topology.

The change of the topic had happened mainly because I managed to obtain decent results in other direction, on plane curves. There was also a less respectable reason: I failed to relate the traditional techniques of the classical knot theory to real algebraic knots. One of the obstacles was a phenomenon which became the initial point of this paper. A large part of the traditional techniques in the knot theory uses plane knot diagrams, i.e. projections of knots to plane. The projection of an algebraic curve is algebraic, and one could try to apply results on plane real algebraic curves. However, the projection contains extra real points, which do not correspond to real points of the knot. These points are discussed below. In the seventies they ruined my weak attempts to study real algebraic knots. Now they allow us to detect crucial differences between topological and real algebraic knots.

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A lengthy informal introduction, which follows, is intended to explain the matter prior to going into details. I cannot resist the temptation to write in the style of popular mathematics and apologize to the reader whom this style may irritate.

1.1. **Knot Theory and Algebraic Geometry.** In the classical knot theory, by a link one means a smooth closed 1-dimensional submanifold of the 3-dimensional sphere S^3, i.e. a union of several disjoint circles smoothly embedded into S^3. A link may be equipped with various additional structures such as orientation or framing and considered up to various equivalence relations like smooth (or ambient) isotopy, PL-isotopy, cobordism or homotopy. See, e.g., [5] or [1].

In algebraic geometry classical links naturally appear as links of singular points of complex plane algebraic curves. Given a singular point p of a complex plane algebraic curve C, the intersection of C with the boundary of a sufficiently small ball centered at p is called *the link of the singularity.* It provides a base for a fruitful interaction between topology and algebraic geometry with a long history and lots of important results.

Another obvious opportunity for an interaction between algebraic geometry and knot theory is based on the fact that a classical link may emerge as the set of real points of a real algebraic curve. This opportunity was completely ignored, besides that a number of times it was proven that any classical link is approximated by (and hence isotopic to) the set of real points of a real algebraic curve. There are two natural directions in which the algebraic geometry and knot theory may interact in the study of real algebraic links: first, a study of relations between invariants which are provided by the link theory and algebraic geometry, second, developing a theory parallel to the classical link theory, but taking into account the algebraic nature of the objects. From the viewpoint of this second direction, it is more natural to consider real algebraic links up to isotopy consisting of real algebraic links, which belong to the same continuous family of algebraic curves, rather than up to smooth isotopy in the class of classical links. I call an isotopy of the former kind a *rigid isotopy* following the terminology established by Rokhlin [4] in a similar study of real algebraic plane projective curves and the likes (see, e.g., a survey [8]). Of course, there is a forgetting functor: any real algebraic link can be considered as a classical link and a rigid isotopy as a smooth isotopy. It is interesting, how much is lost under that transition.

In this paper I point out a real algebraic link invariant which is lost. It is unexpectedly simple. In an obvious sense it is a nontrivial Vassiliev invariant of degree 1 on the class of real algebraic knots (Recall that a knot is a link consisting of one component). In the classical knot theory the lowest degree of a nontrivial Vassiliev knot invariant is 2. Thus there is an essential difference between the classical knot theory and the theory of real algebraic knots. Of course this difference has a simple topological explanation: a real algebraic link is more complicated topologically, besides its set of real points it contains the set of complex points invariant under the complex conjugation and a rigid isotopy induces an equivariant smooth isotopy of this set.

The invariant of real algebraic links which is defined below is very similar to the self-linking number of a framed knot. I call it also a *self-linking number.* Its definition looks like a refinement of an elementary definition of the writhe of a knot diagram, but taking into consideration the imaginary part of the knot.

1.2. **Self-Linking of Non-Algebraic Knot.** The linking number is a well-known numerical characteristic of a pair of disjoint oriented circles embedded in the three-dimensional Euclidean space. Roughly speaking, it measures how many times one of the circles runs around the other. It is one of the most classical topological invariants, introduced in the nineteenth century by Gauss [3].

In the classical theory, a self-linking number of a knot is defined if the knot is equipped with an additional structure like a framing or just a vector field nowhere tangent to the knot.[1] The self-linking number is the linking number of the knot oriented somehow and its copy obtained by a small shift in the direction specified by the vector field. It does not depend on the orientation, since reversing the orientation of the knot is compensated by reversing the induced orientation of its shifted copy. Of course, the self-linking number depends on the homotopy class of the vector field.

A knot has no natural preferable homotopy class of framings, which would allow us to speak about a self-linking number of the knot without a special care on the choice of a framing.[2] Some framings appear naturally in geometric situations. For example, if one fixes a generic projection of a knot to a plane, the vector field of directions of the projection appears. The corresponding self-linking number is called the *writhe* of the knot. However, it depends on the choice of the projection and changes under isotopy.

The linking number is a Vassiliev invariant of order 1 of two-component oriented links. This means that it changes by a constant (in fact, by 2) when the link experiences a homotopy with a generic appearance of an intersection point of the components. Whether the linking number increases or decreases depends only on the local picture of orientations near the double point: when it passes from through to , the linking number increases by 2. Generalities on Vassiliev invariants see, e.g., in [7].

In a sense the linking number is the only Vassiliev invariant of degree 1 of two-component oriented links: any Vassiliev invariant of degree 1 of two-component oriented links is a linear function of the linking number. Similarly, the self-linking number is a Vassiliev invariant of degree 1 of framed knots (it changes by 2 when the knot experiences a homotopy with a generic appearance of a self-intersection point) and it is the only Vassiliev of degree 1 of framed knots in the same sense. Necessity of a framing for a definition of self-linking number can be formulated now more rigorously: only constants are Vassiliev invariants of degree 1 of (non-framed) knots.

The diagrammatical definition of the writhe, which is imitated below, runs as follows: for each crossing point of the knot projection one defines a *local writhe* equal to $+1$ if near the point the knot diagram looks like and -1 if it looks like . Then one sums up the local writhes over all double points of the projection. The sum is the writhe.

[1] A framing is a pair of orthogonal to each other normal vector fields on a knot. There is an obvious construction which makes a framing from a non-tangent vector field and establishes one to one correspondence between homotopy classes of framings and non-tangent vector fields. The vector fields are more flexible and relevant to the case.

[2] Moreover, the self-linking number is used to define a natural class of framings: namely, the framings with the self-linking number zero.

A continuous change of the projection may cause vanishing of a crossing point. This happens under the first Reidemeister move shown in the left hand half of Figure 1. This move changes the writhe by ± 1.

1.3. **Algebraicity Enhances Self-Linking Number.** If a link is algebraic, then its projection to a plane is algebraic, too. A generic projection has only ordinary double points and the total number of its complex double points is constant.[3] The number of real double points can vary, but only by an even number. A real double point cannot turn alone into an imaginary one, as it seems to happen under the first Reidemeister move. Under an algebraic version of the first Reidemeister move, the double point stays in the real domain, but becomes solitary, like the only real point of the curve $x^2+y^2=0$. The algebraic version of the first Reidemeister move is shown in the right hand half of Figure 1.

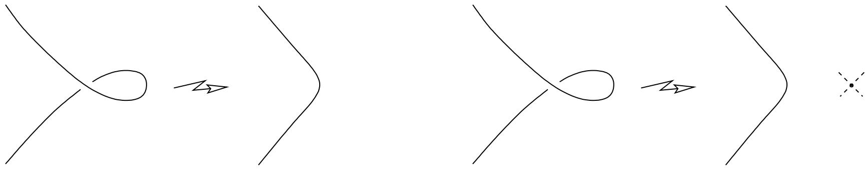

FIGURE 1. Topological (left) and real algebraic (right) versions of the first Reidemeister move. At the solitary crossing point, which is on the right hand side of the picture, the conjugate imaginary branches are indicated by dashed segments, according to an outdated tradition of Analytic Geometry.

It is not difficult to prove that a family of spatial curves, which realizes this move, can be transformed by a local diffeomorphism to the family of the affine curves defined by the following system of equations

$$\begin{cases} xz+y &= 0 \\ x+z^2+\tau &= 0, \end{cases}$$

where τ is the parameter of the deformation. These are rational curves, admitting a rational parametrization

$$\begin{cases} x &= -t^2-\tau \\ y &= -t(t^2+\tau) \\ z &= -t. \end{cases}$$

The projection corresponds to the standard projection $(x,y,z) \mapsto (x,y)$ to the coordinate xy-plane. It maps these curves to the family of affine plane rational cubic curves defined by $y^2+x^2(\tau+x)=0$ with $\tau \in \mathbb{R}$.

A solitary double point of the projection is not the image of any real point of the link. It is the image of two imaginary complex conjugate points of the

[3]Here by a generic projection we mean a projection from a generic point. When one says that a generic projection has some properties, this means that for an open everywhere dense set of points the projection from any point of this set has these properties. For the properties under consideration, there is no need even to improve the ambient space: in the projective space there are lesser types of undesirable points than in the affine space, but anyway the whole set of undesirable points is closed nowhere dense (although it depends on the space and the properties under consideration). A proof is an easy exercise either on Sard Lemma, or Bertini Theorem.

complexification of the link. The preimage of the point in the 3-space under the projection is a real line. It is disjoint from the real part of the link, but intersects its complexification in a couple of complex conjugate imaginary points.

In the model of the first Reidemeister move above, $(0,0)$ is the double point of the projection for each $\tau \neq 0$. If $\tau < 0$, it is a usual crossing point. Its preimage consists of two real points $(0,0,\sqrt{-\tau})$ and $(0,0,-\sqrt{-\tau})$. If $\tau > 0$, it is a solitary double point. Its preimage consists of two imaginary conjugate points $(0,0,i\sqrt{\tau})$ and $(0,0,-i\sqrt{\tau})$, which lie on a real line $x = y = 0$

Below, in Section 2.2, with any solitary double point of the projection, a local writhe equal to ± 1 is associated. This is done in such a way that the local writhe of the crossing point vanishing in the first Reidemeister move is equal to the local writhe of the new-born solitary double point. In the case of an algebraic knot the sum of local writhes of all double points, both solitary and crossings, does not depend on the choice of projection and is invariant under rigid isotopy. This sum is the self-linking number.

1.4. **Self-Linking Numbers for Non-Oriented and Semi-Oriented Links.** A construction similar to the construction of the self-linking number of an algebraic knot can be applied to an algebraic *link.* However in this case there are two versions of the construction.

In the first of them, we define a self-linking number generalizing the self-linking number defined above for knots. We consider a link diagram and the sum of local writhes at solitary double points and crossing points where the branches belong the same connected component of the set of real points. At these crossing point, to define a local writhe we need orientations of the branches. As above, we choose an orientation on each of the components. If we make another choice, at a crossing point, where the branches belong the same component, either both orientations changes or none. Hence the local writhe numbers at crossing points of this kind does not depend on the choice. In Section 2 below we prove that the whole sum of local writhes over crossing points of this kind and solitary double points does not depend on the projection and is invariant under rigid isotopy. We call the sum the *self-linking number* of the link A and denote by $slk(A)$.

In the second version of the construction we consider a real algebraic link which is equipped with an orientation of the set of real points, use these orientations to define local writhe numbers at all crossing points and sum up the local writhe numbers over all crossing points and all solitary double points. The result is called the *self-linking number of an oriented real algebraic link.* This self-linking number does not change when the orientation reverses. An orientation considered up to reversing is called a *semi-orientation.* Thus the self-linking number depends only on the semi-orientation of the link.

The (semi-)orientation may be an artificial extra structure, but it may also appear in a natural way, say, as a complex orientation, if the set of real points divides the set of real points, see [4]. In fact, the complex orientation is defined up to reversing, so this is a semi-orientation. Another important class of semi-oriented algebraic links appears as transversal intersections of two real algebraic surfaces of degrees p and q with $p \equiv q \mod 2$.

The self-linking number of (semi-)oriented real algebraic link differs from the self-linking number of the same link without orientation by the sum of all pairwise linking numbers of the components multiplied by 2: let A be a real algebraic link, let $\bar{A}$ be the same link equipped with an orientation of its set of real points and $\bar{A}_1$, $\dots$ $\bar{A}_n$ the (oriented) connected components of this set, then

$$slk(\bar{A}) = slk(A) + 2 \sum_{1 \le i \le j \le n} \mathrm{lk}(\bar{A}_i, \bar{A}_j).$$

1.5. **Self-Linking and Framings.** In the case of a knot, the self-linking number defines a natural class of framings, since homotopy classes of framings are enumerated by their self-linking numbers and we can choose the framing having the self-linking number equal to the algebraic self-linking number constructed here. I do not know any direct construction of this framing. Moreover, there seems to be a reason for absence of such a construction. In the case of links the construction above gives a single number, while framings are enumerated by sequences of numbers with entries corresponding to components.

2. Real Algebraic Projective Links

Let A be a nonsingular real algebraic curve in the 3-dimensional projective space. Then the set $\mathbb{R}A$ of its real points is a smooth closed 1-dimensional submanifold of $\mathbb{R}P^3$, i. e. a smooth projective link. The set $\mathbb{C}A$ of its complex points is a smooth complex 1-dimensional submanifold of $\mathbb{C}P^3$.

Let c be a point of $\mathbb{R}P^3$. Consider the projection $p_c : \mathbb{C}P^3 \smallsetminus c \to \mathbb{C}P^2$ from c. Assume that c is such that the restriction to $\mathbb{C}A$ of p_c is generic. This means that it is an immersion without triple points and at each double point the images of the branches have distinct tangent lines. As it follows from well-known theorems, those c's for which this is the case form an open dense subset of $\mathbb{R}P^3$ (in fact, it is the complement of a 2-dimensional subvariety).

The real part $p_c(\mathbb{C}A) \cap \mathbb{R}P^2$ of the image consists of the image $p_c(\mathbb{R}A)$ of the real part and, maybe, several solitary points, which are double points of $p_c(\mathbb{C}A)$.

2.1. **The Local Writhe of a Crossing.** There is a purely topological construction which assigns a local writhe equal to ± 1 to a crossing belonging to the image of only one component of $\mathbb{R}A$. This construction is well-known in the case of classical knots. Here is its projective version. I borrow it from Drobotukhina's paper [2] on generalization of Kauffman brackets to links in the projective space.

Let K be a smooth connected one-dimensional submanifold of $\mathbb{R}P^3$, and c be a point of $\mathbb{R}P^3 \smallsetminus K$. Let x be a generic double point of the projection $p_c(K) \subset \mathbb{R}P^2$ and $L \subset \mathbb{R}P^3$ be the line which is the preimage of x under the projection. Denote by a and b the points of $L \cap \mathbb{R}P^3$. The points a and b divide the line L into two

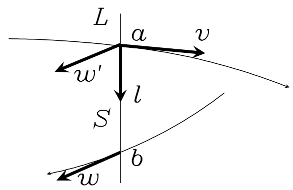

Figure 2. Construction of the frame v, l, w'.

segments. Choose one of them and denote it by S. Choose an orientation of K.

Let v and w be tangent vectors of K at a and b respectively directed along the selected orientation of K.

Let l be a vector tangent to L at a and directed inside S. Let w' be a vector at a such that it is tangent to the plane containing L and w and is directed to the same side of S as w (in an affine part of the plane containing S and w). See Figure 2. The triple v, l, w' is a base of the tangent space $T_a\mathbb{R}P^3$. Define the local writhe of x to be the value taken by the orientation of $\mathbb{R}P^3$ on this frame.

The construction of the local writhe of x contains several choices. Here is a proof that the result does not depend on them.

We have chosen an orientation of K. If the opposite orientation was selected then v and w' would be replaced by the opposite vectors $-v$ and $-w'$. It would not change the result, since $-v, l, -w'$ defines the same orientation as v, l, w'.

We have chosen the segment S. If the other half of L was selected, then l and w' would be replaced by the opposite vectors. But $v, -l, -w'$ defines the same orientation as v, l, w'.

The construction depends on the order of points a and b. The other choice (with the same choice of the orientation of K and segment S) gives a triple of vectors at b. It can be moved continuously without degeneration along S into the triple $w', -l, v$, which defines the same orientation as v, l, w'. □

2.2. **Local Writhe of a Solitary Double Point.** Let A, c and p_c be as in the beginning of Section 2 and let $s \in \mathbb{R}P^2$ be a solitary double point of p_c. Here is a construction assigning ± 1 to s. I will call the result also a *local writhe* of s.

Denote the preimage of s under p_c by L. This is a real line in $\mathbb{R}P^3$ connecting c and s. It intersects $\mathbb{C}A$ in two imaginary complex conjugate points, say, a and b. Since a and b are conjugate they belong to different components of $\mathbb{C}L \smallsetminus \mathbb{R}L$.

Choose one of the common points of $\mathbb{C}A$ and $\mathbb{C}L$, say, a. The natural orientation of the component of $\mathbb{C}L \smallsetminus \mathbb{R}L$ defined by the complex structure of $\mathbb{C}L$ induces an orientation on $\mathbb{R}L$ as on the boundary of its closure. The image under p_c of the local branch of $\mathbb{C}A$ passing through a intersects the plane of the projection $\mathbb{R}P^2$ transversally at s. Take the local orientation of the plane of projection such that the local intersection number of the plane and the image of the branch of $\mathbb{C}A$ is $+1$.

Thus the choice of one of two points of $\mathbb{C}A \cap \mathbb{C}L$ defines an orientation of $\mathbb{R}L$ and a local orientation of the plane of projection $\mathbb{R}P^2$ (we can speak only on a local orientation of $\mathbb{R}P^2$, since the whole $\mathbb{R}P^2$ is not orientable). The plane of projection intersects[4] transversally $\mathbb{R}L$ in s. The local orientation of the plane, orientation of $\mathbb{R}L$ and the orientation of the ambient $\mathbb{R}P^3$ determine the intersection number. This is the local writhe.

It does not depend on the choice of a. Indeed, if one chooses b instead, then both the orientation of $\mathbb{R}L$ and the local orientation of $\mathbb{R}P^2$ would be reversed. The orientation of $\mathbb{R}L$ would be reversed, because $\mathbb{R}L$ receives opposite orientations from the different halves of $\mathbb{C}L \smallsetminus \mathbb{R}L$. The local orientation of $\mathbb{R}P^2$ would be reversed, because the complex conjugation involution $conj : \mathbb{C}P^2 \to \mathbb{C}P^2$ preserves

[4]We may think on the plane of projection as embedded into $\mathbb{R}P^3$. If you would like to think on it as on the set of lines of $\mathbb{R}P^3$ passing through c, please, identify it in a natural way with any real projective plane contained in $\mathbb{R}P^3$ and disjoint from c. All such embeddings $\mathbb{R}P^2 \to \mathbb{R}P^3$ are isotopic.

the complex orientation of $\mathbb{C}P^2$, preserves $\mathbb{R}P^2$ (point-wise) and maps one of the branches of $p_c(\mathbb{C}A)$ at s to the other reversing its complex orientation.

2.3. **Self-Linking Number and Its Invariance.** Now for any real algebraic projective link A choose a point $c \in \mathbb{R}P^3$ such that the projection of A from c is generic and sum up writhes of all crossing points of the projection belonging to image of only one component of $\mathbb{R}A$ and writhes of all solitary double points. The sum is called the *self-linking number of A*.

I have to show that it does not depend on the choice of projection. The prove given below proves more: the sum is invariant under *rigid isotopy* of A. By rigid isotopy we mean an isotopy consisting of nonsingular real algebraic curves. The effect of a movement of c on the projection can be achieved by a rigid isotopy defined by a path in the group of projective transformations of $\mathbb{R}P^3$. Therefore the following theorem implies both independence of the self-linking number on the choice of projection and its invariance under rigid isotopy.

2.A. Theorem. *For any two rigidly isotopic real algebraic projective links A_1 and A_2 such that their projections from the same point $c \in \mathbb{R}P^3$ are generic, the self-linking numbers of A_1 and A_2 defined via c are equal.*

This theorem is proved in Section 2.5.

2.B. Corollary 1. *The self-linking number of a real algebraic projective link does not depend on the choice of projection which is involved in its definition.*

Proof of 2.B. A projection depends only on the center from which it is done. The effect on the projection of a movement of the center can be achieved by a rigid isotopy defined by a path in the group of projective transformations of $\mathbb{R}P^3$. $\square$

Thus the self-linking number is a characteristic of a real algebraic link.

2.C. Corollary 2. *The self-linking number of a real algebraic projective link is invariant under rigid isotopy.* $\square$

2.4. **Algebraic Counter-Parts of Reidemeister Moves.** As in purely topological situation of an isotopy of a classical link, a generic rigid isotopy of a real algebraic link may be decomposed to a composition of rigid isotopies, each of which involves a single local standard move of the projection. There are 5 local standard moves. They are similar to the Reidemeister moves. The first of these 5 moves is shown in the right hand half of Figure 1. The other moves are shown in Figure 3. The next two coincide with the second and third Reidemeister moves. The fourth

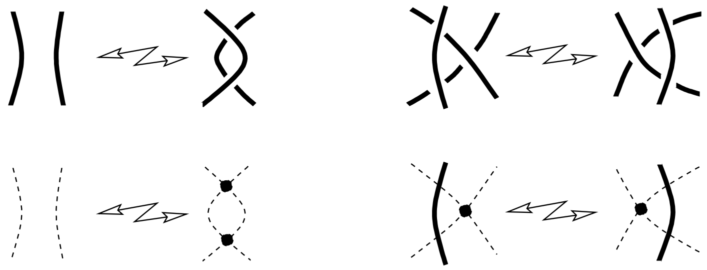

FIGURE 3.

move is similar to the second Reidemeister move: also two double points of projection come to each other and disappear. However the double points are solitary. The fifth move is similar to the third Reidemeister move: also a triple point appears for a moment. But at this triple point only one branch is real, the other two are imaginary conjugate to each other. In this move a solitary double point traverses a real branch.

2.5. **Reduction of Theorem *2.A* to Lemmas.** To prove Theorem *2.A*, first replace the rigid isotopy by a generic one and then decompose the latter into local moves described above, in Section 2.4. Only in the first, fourth and fifth moves solitary double points are involved. The invariance under the second and the third move follows from the well-known fact of knot theory that the topological writhe is invariant under the second and third Reidemeister moves. Cf. [2]. Thus the following three lemmas imply Theorem *2.A*.

***2.D*. Lemma.** *In the fifth move the writhe of the solitary point does not change.*

***2.E*. Lemma.** *In the fourth move the writhes of the vanishing solitary points are opposite*

***2.F*. Lemma.** *In the first move the writhe of vanishing crossing point is equal to the writhe of the new-born solitary point.*

2.6. **Proof of Lemmas *2.D* and *2.E*.** Proof of Lemma *2.F* is postponed to Section 2.7. Note that although Lemma *2.F* is the most difficult to prove, it is the least significant: here its only role is to justify the choice of sign made in the definition of local writhe in solitary double point of the projection. It is clear that the writhes of vanishing double points involved in the first move are related, and if they were opposite to each other, then the definition of the self-linking number should be changed, but would not be destroyed irrecoverably.

Proof of Lemma 2.D. This is obvious. Indeed, the real branch of the projection does not interact with the imaginary branches, it just passes through their intersection point. □

Proof of Lemma 2.E. At the moment of the fourth move take a small ball B in the complex projective plane centered in the solitary self-tangency point of the projection of the curve. Its intersection with the projection of the complex point set of the curve consists of two smoothly embedded disks tangent to each other and to the disk $B \cap \mathbb{R}P^2$. Under the move each of the disks experiences a diffeotopy. Before and after the move the intersection the curve with B is the union of the two disks meeting each other transversally in two points, but before the move the disks do not intersect $\mathbb{R}P^2$, while after the move they intersect $\mathbb{R}P^2$ in their common points.

To calculate writhe in both vanishing solitary double points, let us select the same imaginary branch of the projection of the curve passing through the points. It means that we select one of the disks described above. The sum of the local intersection numbers of this disk (equipped with the complex orientation) and $B \cap \mathbb{R}P^2$ (equipped with some orientation) is zero since under the fourth move the intersection disappears, while in the boundary of B no intersection happens.

Therefore the local orientations of the projective plane in the vanishing solitary double points defined by this branch define opposite orientations of $B \cap \mathbb{R}P^2$. (Recall that the local orientations are distinguished by the condition that the local intersection numbers are positive.)

On the other hand, under the move the preimages of the vanishing solitary double points come to each other up to coincidence at the moment of the move and their orientations defined by the choice of the same imaginary branch are carried to the same orientation of the preimage of the point of solitary self-tangency. Indeed, the preimages are real lines and points of intersection of their complexifications with the selected imaginary branch of the curve also come to the same position. Therefore the halves of the complexifications containing the points come to coincidence, as well as the orientations defined by the halves on the real lines.

It follows that the intersection numbers of B with the preimages of the vanishing solitary double points equipped with these orientations are equal. Since the local orientations of the projective plane in the vanishing solitary double points define distinct orientations of $B \cap \mathbb{R}P^2$, the writhes are opposite to each other. $\square$

2.7. **Proof of Lemma *2.F*.** It is sufficient to consider the model family of curves described in Section 1.3. Recall that the curves of this family are defined by the following system of equations

$$\begin{cases} xz + y &= 0 \\ x + z^2 + \tau &= 0, \end{cases}$$

where τ is the parameter of the deformation. They admit a rational parametrization

$$\begin{cases} x &= -t^2 - \tau \\ y &= -t(t^2 + \tau) \\ z &= -t. \end{cases}$$

The projection corresponds to the standard projection $(x, y, z) \mapsto (x, y)$ to the coordinate xy-plane. It maps these curves to the family of affine plane rational cubic curves defined by $y^2 + x^2(\tau + x) = 0$ with $\tau \in \mathbb{R}$.

We have to prove that the local writhe at $(0, 0)$ for $\tau < 0$ coincides with the local writhe at $(0, 0)$ for $\tau > 0$.

Let us calculate the local writhe for $\tau < 0$. Denote $\sqrt{-\tau}$ by ρ. The preimage of $(0, 0)$ consists of points $a = (0, 0, \rho)$ and $b = (0, 0, -\rho)$ corresponding to the values $-\rho$ and ρ of t, respectively, see Figure 4. The tangent vectors to the curve at these points are $v = (2\rho, -2\rho^2, -1)$ and $w = (-2\rho, -2\rho^2, -1)$. The vector l connecting a and b is $(0, 0, -2\rho)$. By the definition, the writhe is the value taken by the orientation of $\mathbb{R}P^3$ on the frame v, l, w'. This value equals the value of this orientation on the frame $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ multiplied by the sign of

$$\det \begin{pmatrix} 2\rho & -2\rho^2 & -1 \\ 0 & 0 & -2\rho \\ -2\rho & -2\rho^2 & -1 \end{pmatrix} = -16\rho^4 < 0.$$

Let us calculate the local writhe for $\tau > 0$. Denote $\sqrt{\tau}$ by ρ. The preimage of $(0, 0)$ consists of points $a' = (0, 0, i\rho)$ and $b' = (0, 0, -i\rho)$ corresponding to the values $-i\rho$ and $i\rho$ of t. Choose the branch which passes through a'. It belongs to the upper half of the line $x = y = 0$, which induces the positive orientation of the real part directed along $(0, 0, 1)$. At a' the branch of the curve has tangent

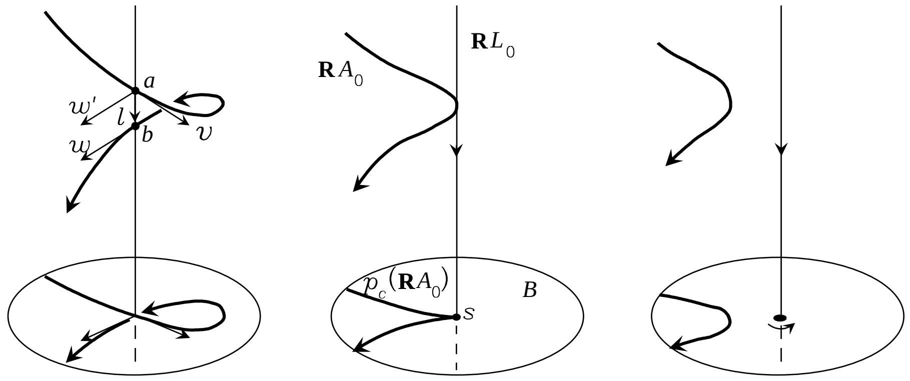

FIGURE 4. Real algebraic version of the first Reidemeister move.

vector $v = (2i\rho, 2\rho^2, -1)$ and the real basis consisting of v and $iv = (-2\rho, 2i\rho^2, -i)$ positively oriented with respect to the complex orientation of this branch. The projection maps this basis to the positively oriented basis $(2i\rho, 2\rho^2)$, $(-2\rho, 2i\rho^2)$ of the projection of the branch. The intersection number of this projection and $\mathbb{R}^2$ in $\mathbb{C}^2$ is the sign of

$$\det \begin{pmatrix} 0 & 2\rho & 2\rho^2 & 0 \\ -2\rho & 0 & 0 & 2\rho^2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -4\rho^3 < 0$$

Hence the orientation of $\mathbb{R}^2$ such that its local intersection number with the selected branch of the projection does not coincide with the orientation defined by the standard basis. The intersection number of the line $x = y = 0$ with the standard orientation and the xy-plane with the standard orientation is the value of the orientation of the ambient space $\mathbb{R}^3$ taken on the standard basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Therefore the local writhe is opposite to this value. $\square$

Remark 1. There is a more conceptual proof of Lemma *2.F*. It is based on a local version of the Rokhlin Complex Orientation Formula, see [4] and [8]. In fact, the original proof was done in that way. However, the Complex Orientation Formula is more complicated than the calculation above.

2.8. **Self-Linking of an Algebraic Link as a Vassiliev Invariant of Degree One.** To speak about Vassiliev invariants, we need to fix a connected family of curves, in which links under consideration comprise the complement to a hypersurface. In the case of classical knots one could include all knots in such a family by adjoining knots with self-intersections and other singularities. A singular knot is a right equivalence class of a smooth map of the circle to the space (recall that two maps from a circle are right equivalent if one of them is a composition of a self-diffeomorphism of the circle with the other one).

In the case of real algebraic knots such a family including all real algebraic knots does not exists. Even the space of complex curves in the three-dimensional projective space consists of infinitely many components. It is impossible to change the homology class realized by the set of complex points of an algebraic curve in $\mathbb{C}P^3$ in a continuous deformation. Recall that the homology class belongs to the group

$H_2(\mathbb{C}P^3) = \mathbb{Z}$ and is a positive multiple $d[\mathbb{C}P^1]$of the natural generator of $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^3)$ realized by a line. The coefficient d is called the *order* of the curve. The genus is another numerical characteristic of a complex curve which takes the same value for all non-singular curves in any irreducible family. As is well known, the non-singular complex curves of given order and genus in the three-dimensional projective space are parametrized by a finite union of quasi-projective varieties. For each of these varieties, one can try to build a separate theory of Vassiliev invariants on a class of non-singular real algebraic curves whose complexifications are parametrized by points of this variety. (A similar phenomenon takes place in topology: links with different numbers of components cannot be included into a reasonable connected family, and therefore for each number of components there is a separate theory of Vassiliev invariants.)

Among the varieties of algebraic curves in the three-dimensional projective space, there are two special families: for each natural number d there is an irreducible variety of rational curves of order d (recall that a an algebraic curve is called rational if it admits an algebraic parametrization by a line), and for each pair of natural number p and q there is an irreducible variety of curves which can be presented as intersection of surfaces of degrees p and q.

In the class of real algebraic rational curves of order d, singular curves comprise a *discriminant hypersurface* in which a generic point is a rational curve such that it has exactly one singular point and this point is an ordinary double point. An ordinary double point may be of one of the following two sorts: either it is an intersection point of two real branches, or two imaginary conjugate branches.

Any two real algebraic rational non-singular curves of order d can be connected by a path in the space of real rational curves of degree d which intersects the discriminant hypersurface only transversally at finite number of generic points. Such a path can be considered as a deformation of a curve to the other one. When it intersects the discriminant hypersurface in a point, which is a curve with singularity on real branches, the set of real points of the curve behaves as in the classical knot theory: two pieces of the set of real points come to each other and pass through each other. As in the classical knot theory, at the moment of intersection a generic projection of the curve experiences an isotopy. Nothing happens besides that one crossing point becomes for a moment the image of a double point and then turns back into a crossing point, but with the opposite writhe. When the path intersects the discriminant hypersurface in a point, which is a curve with singularity on imaginary branches, two complex conjugate imaginary branches pass through each other. At the moment of passing, they intersect in a real isolated double point. At this moment the set of real points of a generic projection experiences an isotopy. No event happens besides that a solitary double point becomes for a moment the image of a solitary real double point of the curve and then turns back into a usual solitary double point of the projection (which is not an image of a real point of the knot), but with the opposite writhe number.

It is clear that the self-linking number of an algebraic curve changes under a modification of each of these kinds by ± 2, with the sign depending only on the local structure of the modification near the double point. This means that *the self-linking number on the family of real rational curves under consideration is a Vassiliev invariant of degree 1.*

This is true also for any space of non-singular real algebraic curves which can be included into a connected family of real algebraic curves by adjoining a hypersurface, penetration through which in a generic point looks as in the family of rational curves described above.

There are many families of this kind besides the families of rational knots. However, in many families of algebraic curves a transversal penetration through the discriminant hypersurface in a generic point looks differently. In particular, for intersections of two surfaces it is a Morse modification of the real part of the curve. At the moment, the old double points of the projection, both solitary and crossing, do not change. An additional double point appears just for a moment. However the division of crossing points to self-crossing points of a single component and crossing points of different components may change. Therefore the self-linking number changes in a complicated way. If the degrees of the surfaces defining the curve are of the same parity, the real part of the curve has a natural semi-orientation. The Morse modification respects this semi-orientation. Therefore the self-linking number of the semi-oriented curve does not change.

2.G. Theorem. *The self-linking number of any non-singular semi-oriented real algebraic link which is a transversal intersection of two real algebraic surfaces whose degrees are of the same parity is zero.*

Proof. Any two non-singular real curves of the type under consideration can be connected by a path as above. Hence their self-linking numbers coincide. On the other hand, it is easy to construct, for any pair of natural numbers p and q of the same parity, a pair of transversal to each other non-singular real algebraic surfaces of degrees p and q in the three-dimensional projective space such that their intersection has zero self-linking number. $\square$

In contrast to this vanishing result, one can prove that the *self-linking number of a real algebraic rational knots of degree d can take any value in the interval between $-\frac{(d-1)(d-2)}{2}$ and $\frac{(d-1)(d-2)}{2}$ including these limits and congruent to them modulo 2.* $\square$

3. Generalizations

3.1. **The Case of an Algebraic Link with Imaginary Singularities.** The same construction may be applied to real algebraic curves in $\mathbb{R}P^3$ having singular imaginary points, but no real singularities. In the construction we can eliminate projections from the points such that some singular point is projected from them to a real point. Indeed, for any imaginary point there exists only one real line passing through it (the line connecting the point with its complex conjugate), thus we have to exclude a finite number of real lines.

This gives a generalization of self-linking numbers with the same properties: it is invariant with respect to rigid isotopies (i.e., isotopies made of curves from this class), and is multiplied by -1 under a mirror reflection.

3.2. **Real Algebraic Links in Sphere.** The construction of this paper can be applied to algebraic links in the sphere S^3. Although from the viewpoint of the knot theory this is the most classical case, from the viewpoint of algebraic geometry the case of curves in the projective space is simpler. The three-dimensional sphere S^3 is a real algebraic variety. It is a quadric in the four-dimensional real affine space. A stereographic projection is a birational isomorphism of S^3 onto $\mathbb{R}P^3$. It defines a diffeomorphism between the complement of the center of projection in S^3 and a real affine space.

Given a real algebraic link in S^3, one may choose a real point of S^3 from the complement of the link and project the link from this point to an affine space. Then include the affine space into the projective space and apply the construction above. The image has no real singular points, therefore we can use the result of the previous section.

This construction blows up the center of projection, making a real projective plane out of it, and maps the complement of the center of the projection in the set of real points of the sphere isomorphically onto the complement of the projective plane. In the imaginary domain, it contracts each generatrix of the cone which is the intersection of the sphere with its tangent plane at the center of projection. The image of the cone is an imaginary quadric curve contained in the projective plane which appeared as the result of blowing up of the central point.

3.3. **Other Generalizations.** It is difficult to survey all possible generalizations. Here I indicate only two directions.

First, consider the most straightforward generalization. Let L be a nonsingular real algebraic $(2k-1)$-dimensional subvariety in the projective space of dimension $4k-1$. Its generic projection to $\mathbb{R}P^{4k-2}$ has only ordinary double points. At each double point either both branches of image are real or they are imaginary complex conjugate. If the set of real points is orientable then one can repeat everything with obvious changes and obtain a definition of a numeric invariant generalizing the self-linking number defined above.

Let M be a nonsingular three-dimensional real algebraic variety with oriented set of real points equipped with a real algebraic fibration over a real algebraic surface F with fiber a projective line. There is a construction which assigns to a real algebraic link (i. e., a nonsingular real algebraic curve in M) with a generic projection to F an integer, which is invariant under rigid isotopy, multiplied by -1 under reversing of the orientation of M and is a Vassiliev invariant of degree 1. This construction is similar to the one presented above, but uses, instead of projection to $\mathbb{R}P^2$, an algebraic version of Turaev's shadow descriptions of links [6].

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