TEACHING STATEMENT

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1. Teaching experience. I started to teach in 1968. Then I was a third year undergraduate student. My first class was a mathematical circle, an evening voluntary class for high school students of the 7th grade. The classes that I have happened to teach since then are quite diverse: from classes at high school and a lecture course on elementary mathematics to prepare for the entrance exams to the Leningrad Gornyj (geological) Institute up to advanced graduate courses at Leningrad State University, UC Berkeley and Paris VII (Jussieu). Seventeen graduate students have defended their dissertations under my supervision. Some of them became well-known mathematicians.

This experience has forced me to think about various problems related to teaching of Mathematics. Below I present my views on few of them, those which seem to be most important.

2. Profound difficulties. Mathematics requires more precise language and logic than those which are in common use. The study of Mathematics involves more or less explicit improvement of the general ability to understand, think and speak, due to a better grasp of language and logic. This is a difficult and important job, which is especially difficult because a student usually is not aware of it, and especially important because success in the study of Mathematics comes along with success in the accomplishing of this job.

The most difficult pedagogical task that a teacher of mathematics meets in a class is to evaluate the differences between the teacher and students in understanding of mathematical language. For a teacher it is crucial to have a clear idea how deep and confusing the misunderstanding can be. Unfortunately most mathematicians have forgotten how they thought before they became mathematicians. Some clues about this can be picked up from students' tests, or conversations with students during office hours. I am very grateful for the extensive practice I had in oral examinations of first and second year students at Leningrad State University. I hope that asking questions and listening to students' presentations of theory (formulation and proofs of theorems) has cured me from the disease at least to some extent: I think I recognize the differences in the way of thinking between a mathematician and a person who has not yet been exposed to Mathematics.

Problems of this kind are especially noticeable in the first courses in abstract mathematics, such as group theory or elementary topology. At the Department of Mathematics of Leningrad State University, where I started my university teaching and gave a course in elementary topology about 15 times, these problems were aggravated by the diversity of the students: from winners of International Mathematical Olympiads to students who had passed the entrance exams by a miracle (if not by a mistake). It was most pleasant, but most misleading, to watch the best students' reaction to the lectures; the reaction of other students was quite different.

3. A problem book. Together with my colleagues, O. A. Ivanov, N. Yu. Netsvetaev and V. M. Kharlamov, I wrote a problem book on elementary topology [4]. One of the main goals was to provide reasonable material for learning to a student of any level and at any pace. We wanted to make sure that a weak student could find there enough exercises and simple concrete examples to understand the matter and to gain confidence in correct understanding. For such a student the subject is especially important as an introduction to abstract mathematics. Advanced students can find there lots of additional material, or have the luxury of proving all standard theorems by solving a chain of problems.

The "collection of problems" format emphasizes the importance of formulations. Careful attention to formulations is one of the main skills needed in a study of Mathematics. It is crucial in any intellectual activity. The problem book format allowed us to force a reader to think through questions, which otherwise would be skipped.

From teachers, who used this book in their classes in various countries, I heard that students of different backgrounds used the book successfully. The book was published twice in Russian; we have prepared the third Russian and the first English editions. In English it will be published soon by the AMS. A preliminary version is available from my web site (http://www.pdmi.ras,ru/˜olegviro).

4. LOMI track. In a class, it is difficult to keep students of different levels happy. At Leningrad State University, even among first year undergraduate students, there were many students who had been well-prepared for studying mathematics (due to participation in mathematical olympiads, mathematical circles, attending mathematical high schools). They demonstrated an excellent understanding of the subject and showed that they could learn more. It is difficult to find a manner of teaching which would be appropriate both for them and for the rest of the students.

In 1989, I organized the "LOMI track" (Lomi potok), a separate program for advanced first and second year students in the Department of Mathematics and Mechanics of Leningrad State University. Administratively, it was a joint project of the Department and LOMI, the Leningrad Branch of the Steklov Institute of the Soviet Academy of Science. I was lucky in setting up this cooperation to have the support of L. D. Faddeev, the director of LOMI, and G. A. Leonov, the chairman of the Mathematics Department. The classes met in the building of the institute. Some teachers were researchers affiliated at the institute, while other teachers were University professors. We managed to build a new coherent system of main mathematical courses. Many topics which had appeared before only in special lecture courses at the graduate level, were included.

Organizing and maintaining LOMI track gave me indispensable administrative experience. Each teacher had his own views on everything. Sometimes these views were quite strong and contradicted each other. I had to understand all of them, work out a reasonable decision and convince all the people involved that this decision was the best one.

At the first stage the teachers worked in LOMI track for free. At a hard time, right after the collapse of the Soviet Union, the AMS supported LOMI track financially. Later it got financing from various Russian funds, and, finally, from St. Petersburg State University. Now this is a small but well-established unit of the Mathematics Department.

In 2001-2002 I initiated a similar program at Uppsala University, Sweden. The most exciting feature of this was a lecture course in the first semester calculus. I will come back to this later.

5. General goals. Certainly, the teaching of each piece of Mathematics has its own specific goals, but there are general goals. Different people formulate them differently. Here is how I do it.

If the students are prospective mathematicians, the main goal is to transfer mathematical culture to them. Here I mean mathematical culture in a wide sense, including skills in manipulations with mathematical objects, knowledge of numerous mathematical stories and relationships between mathematical notions, which at first glance may appear quite distant from each other, understanding and mastering of mathematical language, and the culture of proofs. In other words, the teacher has to welcome these students into our world of mathematics, show each remarkable landmark and give friendly instructions about how to live in this world.

A student who is not going to become a professional mathematician should get some useful skills, a confident understanding of the Mathematics related to these skills, and a general feeling that: "Mathematics is great."

6. A challenge to the mathematical community. Is it that important to make a student feeling that Mathematics is great?

Mathematicians are responsible for the future of both Mathematics and the mathematical community. This responsibility is built on the fact that almost every student studies Mathematics and most mathematicians teach. The impression that mathematicians make on their students will bear fruit when one of the students grows and becomes a national leader responsible for the funding of science.

Mathematicians performed pretty well for centuries, but in the twentieth century their performance was undermined. Due to stupidity and lack of responsibility, the mathematical community disregarded the one subject, which was the most effective from this perspective, Geometry. Generations of educated people had been grateful to mathematicians for teaching them via Geometry how to argue, how to prove, how to think. But in many countries Geometry has been entirely eliminated from high school and its presence in universities has also been dramatically reduced.

What has replaced Geometry in curricula is Calculus. Is Calculus any better than Geometry?

Geometry, as a school subject, appeared as a snapshot of the initial development of Mathematics, as fixed in Euclid's "Elements". It was a naive and fresh stage. Even after hundreds of years spent at school, it was quite charming.

Modern university Calculus stemmed from the foundational stage of Analysis. In comparison with Euclidean Geometry, it was not that fresh stage. It claims for credits as being more useful for applications. It required from a student much higher logical skills.

It was acceptable as a sequel to Euclidean Geometry and Elementary Algebra. However, the decline of educational standards, which was caused by extending the higher education to the whole population and the withdrawal of Geometry, forced the study of Calculus to be reduced to the solving of standard problems. Proofs of its main results (at least, the proofs that can be found in traditional textbooks) are exceedingly difficult for students whose logical skills have not matured in Geometry classes. Hence the proofs are postponed to the course of Analysis. Therefore most students never see them, and cannot appreciate Calculus as a subject which teaches us to think. Usually, Calculus is quite boring. A student who suffers at Calculus becomes a potential enemy of Mathematics. Some students become powerful, and may end up at the command of resources.

Together with many other mathematicians, I see here a great challenge. The very survival of the mathematical community, and Mathematics as a part of human culture, if not the whole human culture itself, are at stake. Either we will manage to make the study of Mathematics one of the most appreciated learning experiences, or we will face the inevitable decline of financing, a further reduced share of Mathematics in curricula, marginalization of the subject, and decline of the role of science in the life of our civilization. These processes are already quite noticeable, although most mathematicians have not yet come to understand that these processes were initiated through the fault of the mathematical community.

7. Ways to please students. Pleasant emotions are necessary for success in learning of any subject. In the modern western tradition positive emotions are often narrowed down to so-called fun. I think this impoverishes the spectrum of positive emotions and hence the toolbox of a teacher. The sources of positive emotions must be diverse. Here is an incomplete list.

1. It is very useful to surprise students. Mathematics should not appear to be boring. It provides huge opportunities for for excitement. Excitement in an unexpected twist of the subject or an unexpected relation between two things which appeared to be absolutely unrelated to each other, can be a great source of pleasant emotions.

For instance, is it not exciting to see a proof that $\sum_{n=0}^{\infty} 2^n = -1$, to understand that this sum is nothing but the infinity, but nonetheless to see a situation in which the equality $\sum_{n=0}^{\infty} 2^n = -1$ makes practical sense? See [3].

2. Students should feel that they acquire new skills almost constantly. Not only knowledge, but skills. Even if the subject seems to be dry and theoretic, a teacher is expected (usually silently) to provide opportunities to do something about this and to make students proud of newly yearned abilities. For a student this is a job, rather than fun; but a rewarding job.

Which skills are useful, is a delicate question. At school I learned how to calculate with a slide rule, and since then have happened to use this at most 10 times overall. Most of so-called practical skills share this fate. Since it is impossible to foresee in detail which skills will be handy, priority should be given to the most profound ones and the ones which are useful for mastering of material further along.

3. From time to time a teacher should organize students to discover a piece of mathematics. Some teachers, especially in the U.S., think that this should be the only style. I do not think so. It slows down study, and hence reduces the total amount of material that can be studied. Emotionally, it is one of the most exciting ways of learning, but it should be used cautiously, to prevent its devaluation. Students need to see perfect samples of Mathematics, and to get enjoyment not only from doing Mathematics, but also from understanding it.

8. What else can we do? As a mathematician, I can contribute to the solution of this problem, firstly, by analysis of the Mathematics that we teach. The results of that analysis can then be used to make study of Mathematics more attractive. The idea is to reshuffle the system of mathematical notions studied in the standard courses to make the courses easier, without loss of valuable mathematical content, and to add new content, enriching the Mathematics presented.

Here are few more specific ideas which I have come across and worked on.

9. University course of Elementary Geometry. First, for the countries where Geometry was practically eliminated from the high school curriculum (like the USA and Sweden), a short nice university course of Elementary Geometry can be developed. Such a course can be given at the same time as Calculus, or even Pre-Calculus. It should not be a course on Foundations of Geometry, but on Geometry itself.

To make it attractive and useful for developing skills in arguing and geometric vision, I suggest a focus on problems of construction. This is a large traditional class of elementary problems. Almost all geometric theorems can be applied in their solutions. The geometric creativity of students can be trained and rewarded. Unfortunately, many mathematicians nowadays have never heard about these problems. I gave such a course several times. It was an exciting experience both for me and the students. I plan to come back to this later and, probably, to write an appropriate textbook.

10. Limitless Calculus. Fifty years ago the logical complexity of the basic definitions and proofs of Calculus did not matter. Now it does. A sequence of three quantifiers makes the definition of limit incomprehensible for most first year students.

The complexity can be reduced without loss of content, as is well known. For this one should replace the dynamic approach, based on the notion of limits, by a static approach, based on inequalities and descriptive definitions in the style of ancient Euclidean Geometry. For details, see, e.g., the textbook "Calculus unlimited", [2] by J. Marsden and A. Weinstein.

Although known for quite a while, this approach is still needs developing to become a reliable replacement for the traditional one. Simplification of the main definitions allows one to incorporate more interesting topics, which have been lost under the evolution of Classical Analysis into Calculus.

11. Shchepin & Co. on Euler footsteps. As I mentioned above, in the Fall of 2001 I invited E. V. Shchepin from the Steklov Institute (Moscow) to give a lecture course in Calculus for a specially selected group of first year students at Uppsala University. He developed his very original course giving lectures for about 10 years at the Kolmogorov High School in Moscow. The lectures are documented in his book "Uppsala Lectures on Calculus. On Euler's footsteps" which can be found at the web site of Topology Atlas, [3]. The course starts with summation of divergent series. Absolutely convergent series, integrals and derivatives appear in it prior to limits, but rigorously. Analytic functions, residues, finite differences, Bernoulli numbers and the Gamma function also are treated in this first semester course.

Students were quite excited, and managed to learn all of this and pass the exam. It was really the success of a team. The students were thoroughly selected. A great contribution to the success of the course was made by Anders Vrettblad, who edited Shchepin's notes and gave complementary lectures.

Shchepin's lectures cannot be considered as the final solution of the Calculus problem, but appear to be an interesting step in a right direction. I would guess that about a half of the mathematical observations needed for a real solution of the

Calculus pedagogical problem have been already made, but even this half has to be polished.

12. Graduate teaching projects. I have also a few projects on teaching at the graduate level.

For example, I plan to rewrite elementary differential topology in the style of algebraic geometry and general topology, as a theory of spaces with differential structure, admitting, from the very beginning, singularities. The basics of this theory have been known from the sixties, but never came into the mainstream mathematics. It provides a simple and flexible terminology, which is especially valuable for work with quotient spaces. It would put differential manifolds (with or without singularities) on a par with objects of other geometries.

Another project is to collect matters related to topology of finite spaces. Yes, topological spaces with finitely many points. Many mathematicians still hold them in contempt, as these spaces usually are not Hausdorff. However, combinatorial topology started as the topology of finite spaces. Any compact polyhedron is weak homotopy equivalent to a finite space. Topology seems to be the only branch of mathematics which pretends to have no decent finite objects.

13. Invitation to the Universe of Mathematics. My most ambitious writing project is to write a book for the general public, addressed mainly to students. A book about Mathematics from an insider point of view. I still postpone it, but collect thoughts to be included.

It is easier to write Mathematics instead of writing about Mathematics. This is what mathematicians tend to do when claiming that they write about Mathematics. All the texts pretending to address the question "What is Mathematics?" are useful and, to some extent, they explain what Mathematics is. Each paper on a specific mathematical topic written for a general public also does this.

I have written two papers of this kind, [5], [1]. They have been published in Kvant, a Russian journal on Mathematics and Physics for high school students. In both of the papers, and in particular, in the second one written together with Julia Viro, the nature of Mathematics was demonstrated by creating an intelligible piece of Mathematics from scratch, with detailed motivations typical for mathematicians.

However, one cannot express everything by examples or allusions. How to explain, for instance, that mathematical objects are objective, although most of them were apparently created by mathematicians? There are difficult topics like: what is considered obvious by mathematicians, and is there a mathematical vision. I want to convey also my feeling of Mathematics as a fantastic universe, in which mathematicians travel and which they study almost solely by power of their thoughts and imagination.

The main difficulty which I face, and which may undermine the whole project, is that I want to eliminate all possible misunderstandings; these might well be unfavorable for Mathematics and mathematicians, and such misunderstandings are difficult to foresee.

14. Democracy in education. Let me finish with a general statement on teaching, with a slightly political flavor, but with practical personal conclusions.

There are basically two points of view of what democracy should do for an education system. According to one of them, it should provide the same education for as many people as possible. According to the other one, it should provide as much as possible diverse opportunities for everyone.

I cannot accept the first viewpoint. If it was accepted in the Soviet Union, as it is accepted, say, now in Sweden, I would have lost a lot in my professional background, and the whole Russian mathematical community would be much weaker than it is now. Ironically, this "democratic" viewpoint is most damaging for children of poor uneducated people.

The second point of view implies that mathematicians should welcome, organize and support advanced mathematical classes on each available level, including math clubs for high school students, advanced classes at high school, honors classes at universities, summer research camps, etc. I have benefited from all of this and feel that I have to do my best to develop, encourage and support these activities.

References

[1] J. V. Drobotukhina, O. Ya. Viro, Interlacing of skew lines, Kvant (1988) No. 3, 12-19 (Russian);
an extended version under rubric Light Reading for a Professional: Configurations of skew lines, *Algebra i analiz* **1:4** (1989) 222-246 (Russian);
English translation in *Leningrad Math. J.* **1:4** (1990) 1027-1050;
Updated revision: http://www.pdmi.ras.ru/~olegviro/skewlines/

[2] Jerrold E. Marsden, Alan Weinstein, Calculus Unlimited, The Benjamin/Cummings Publishing Company, Inc., 1981.

[3] E. V. Shchepin, Uppsala Lectures on Calculus. On Euler's footsteps, Topology Atlas, http://at.yorku.ca/i/a/a/z/20.htm

[4] O. Ya. Viro, O. A. Ivanov, N. Yu. Netsvetaev and V. M. Kharlamov, Zadachi po topologii, Leningrad, LGU, 1988.
Second, extended edition: St. Petersburg, SPbU, 2000.
English translation, preliminary version:
http:/www.pdmi.ras.ru/~olegviro/topoman/

[5] O. Ya. Viro, Colored knots, *Kvant* (1981) No. 3, 8-14 (Russian);
English translation: Tied into Knot Theory: unraveling the basics of mathematical knots. *Quantum* **8** (1998), no. 5, 16–20.