ON ASYMPTOTIC VOLUME OF TORI

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Introduction

0.1. In this paper we improve a method of [BuI] to deal with asymptotic behavior of volume. We combine our approach from [BuI] with the technique for estimates of volume used in numerous papers—see [GLP], [G1], [B] and references there. We prove the estimate conjectured in [B] (also the same inequality was proven in [C] for manifolds without conjugate points). For further information concerning asymptotic volume growth see [G3] and references there.

We consider a universal cover of a Riemannian n-torus (and some more general $\mathbf{Z}^n$-periodic metrics). The main term of the volume of a ball of radius R in such a metric is cR^n for some $c > 0$ (see [GLP]). We show that $c \geq c_E$ where c_E is the constant for a flat metric, and the equality holds iff our metric is flat (Theorem 1).

As another application of the method we prove that passing to the limit of Riemannian metrics on the same manifold decreases the volume, whenever the limit metric is a Finsler one (Theorem 2).

0.2. Certain arguments of the proof of the volume growth theorem are essentially Riemannian, as well as in the proof of the Hopf conjecture ([BuI]). The latter is hardly surprising since the examples of non-flat Finsler tori without conjugate points do exist. However it is unclear whether the volume growth theorem holds in the Finsler case, and this problem seems to be rather intriguing.

To the best of our knowledge, all the known examples of Finsler tori without conjugate points are obtained by some simplectic transformations from a flat one, and they have the same asymptotic volume growth and C^∞-smooth horospherical foliation. On the other hand, if there exists a Finsler contrexample for the volume growth theorem, then rather likely it can be found among metrics without conjugate points. Indeed, if we slightly decrease the metric in a small neighborhood of a vector tangent to a geodesic with conjugate points, we do not influence on the shape of large balls and just decrease their volume. Such examples of Finsler metrics without conjugate points should have completely other nature than the known ones and might give another view on the geodesic flow on manifolds without conjugate points.

0.3. The asymptotic behavior of the distance function of a $\mathbf{Z}^n$-periodic metric is the same as of a Banach norm (see 3.1). Then some estimates may be obtained by comparing the norm with a Euclidean one. In particular, inscribing the norm's unit ball into a proper (affine) cube may be often useful in order to apply arguments of

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the type "in the base directions our metric is greater than Euclidean, so ...". For instance, in [B] the volume growth constant of the metric is estimated from below by inscribing the norm's unit ball into a cube of minimal possible volume and then applying the Besikovitch inequality. Unfortunately, the best cube may be not so close to the norm's ball as the standard cube is to the standard Euclidean ball, so the estimate obtained this way is not exact.

To avoid that difficulties we use another method. Roughly (and wrongly) speaking, we replace the cube to inscribe a body into, by a polyhedron. More precisely, for every symmetric convex body (the unit ball of a norm) we find a number of supporting it linear functions in terms of which an ellipsoid inscribed into the body is expressed in a nice way (similar to the expression of the standard Euclidean ball in terms of coordinate functions). To get the volume growth estimate we apply to these linear functions (considered as "directions" in our metric space) a version of Besikovitch inequality described in §2.

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§1. Representation of Inscribed Ellipsoid

1.1. Let $\|\cdot\|$ be a Banach norm in a linear space V^n. We fix the notation F for its unit sphere, $F=\{x\in V^n : \|x\|=1\}$. For a linear function $L: V^n\to\mathbf{R}$ denote by $\|L\|$ its norm in the space $(V^n,\|\cdot\|)^*$, i.e. $\|L\|=\max\{|L(x)| : \|x\|=1\}$. We say that a linear function L *supports* F at a point $p\in F$ if $\|L\|=1$ and $L(p)=1$. The set of linear functions supporting F is denoted by F^*.

1.2. For a convex surface F we construct an inscribed ellipsoid whose quadratic form admits a nice representation as a sum of a finite number of squares of supporting F linear functions with positive coefficients whose sum is equal to n. This construction has been used in [BuI, Lemma 4.2] to distinguish ellipsoids among all convex surfaces by an extremal property in terms of integrals of squares of linear functions. We extract this construction as a separate lemma.

1.3. Lemma 1. *There exists a finite collection of linear functions $L_i\in F^*$ and real numbers $a_i>0$ ($i=1,\dots,N\le n(n+1)/2+1$) with $\sum a_i=n$ such that*

- **a)** *Quadratic form $Q=\sum a_iL_i^2$ satisfies $Q(x)\ge\|x\|^2$ for all $x\in V^n$. In particular, Q is positive definite and the unit ball of Q lies inside F.*
- **b)** *For each $i\le N$, L_i supports F at some point p_i with $Q(p_i)=1$.*

Proof. Let $A_F=\{nL^2 : L\in F^*\}$ and $\bar A_F$ be the convex hull of A_F in the space of all quadratic forms on V^n. By the Caratheodory theorem every $Q\in\bar A_F$ can be represented as

$$Q=\sum_{i=1}^N a_iL_i^2,\quad N\le n(n+1)/2+1,\ L_i\in F^*,\ a_i>0,\ \sum a_i=n.$$

For $Q\in\bar A_F$ we denote its unit ball $\{x\in V^n : Q(x)\le 1\}$ by Ball_Q and define $v(Q)=\mathrm{Vol}^{-2}(\mathrm{Ball}_Q)$, where Vol is the Lebesgue measure (if $\mathrm{Vol}(\mathrm{Ball}_Q)$ is infinite we put $v(Q)=0$).

For a positive definite $Q\in\bar A_F$ and $L\in F^*$ let $\|L\|_Q=\max\{L(x) : Q(x)=1\}$. Then we have

$$(*)\qquad \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} v\left((1-\varepsilon)Q+n\varepsilon L^2\right)=nv(Q)(\|L\|_Q^2-1).$$

Indeed, in coordinates ($x_1 = L/\|L\|, x_2, \dots, x_n$) orthonormal with respect to Q the quadratic form $(1-\varepsilon)Q + n\varepsilon L^2$ has a diagonal matrix with diagonal elements $1 + (n\|L\|_Q^2 - 1)\varepsilon$, $1-\varepsilon$, …, $1-\varepsilon$. Hence

$$v\left((1-\varepsilon)Q + n\varepsilon L^2\right) = (1-\varepsilon)^{n-1}\left(1 + (n\|L\|_Q^2 - 1)\varepsilon\right) v(Q)$$

and then (*) follows directly.

Let $Q = \sum_1^N a_i L_i^2$ maximize the function v over $\bar{A}_F$. Clearly Q is positive definite. Then (*) implies that $\|L\|_Q \le 1$ for all $L \in F^*$ and moreover $\|L_i\|_Q = 1$ for each i. This means that no hyperplane supporting F crosses Ball_Q and ones of the form $L_i^{-1}(1)$ touch it at some points which we denote by p_i. Thus Ball_Q is inscribed into F and L_i support F at p_i. The lemma follows. □

1.4. If the norm $\|\cdot\|$ is Euclidean then for Q from Lemma 1 we have $Q = \|\cdot\|^2$. Since Q is a volume-minimizer we get the following

Lemma 2. *If $\|\cdot\|$ is a Euclidean norm on some n-dimensional space then any quadratic form represented as $\sum a_i L_i^2$ (where $\|L_i\| \le 1$, $a_i \ge 0$ and $\sum a_i = n$) has the volume of its unit ball greater or equal to the one of $\|\cdot\|$, and the equality holds only if $Q = \|\cdot\|^2$.*

1.5. Lemma 2 implies that the unit ball of Q from Lemma 1 is actually the maximal-volume ellipsoid lying inside F. Therefore such a quadratic form Q (but not the representation as $\sum a_i L_i^2$) is uniquely determined by F.

§2. Generalized Besikovitch Inequality

2.1. Many inequalities were inspired by the Besikovitch inequality (see [G1], [BurZ, pp. 294–296] and references there). In this section we generalize Derrick's proof ([D1], [D2]) for the Besikovitch inequality for the case where the number of functions is greater than the dimension.

2.2. Lemma 3. *Let M^n be a Riemannian manifold and $B_i : M \to \mathbf{R}$ ($i = 1, \dots, N$) be Lipschitz functions with Lipschitz constant 1. Then for any collection of nonnegative numbers a_i with $\sum a_i = n$ the mapping $B : M \to \mathbf{R}^N$ defined by*

$$B(x) = (\sqrt{a_1}B_1(x), \dots, \sqrt{a_N}B_N(x)) \tag{1}$$

is volume non-increasing (with respect to Riemannian volume on M and n-dimensional Hausdorff measure on $\mathbf{R}^N$).

Proof. B is differentiable almost everywhere since B is Lipschitz. It suffices to show that the Jacobian $\mathrm{Jac}(B)$ is not greater than 1 a.e. Let B be differentiable at x and then $d_xB = (\sqrt{a_1}dB_1, \dots, \sqrt{a_N}dB_N) : \mathrm{T}_x M \to \mathbf{R}^N$. The pre-image of the unit ball of $\mathbf{R}^N$ under d_xB is the unit ball of the quadratic form $\sum a_i(d_xB_i)^2$. Applying Lemma 2 we obtain that this ball is not less by the volume than the unit ball of Riemannian scalar product. Thus d_xB is not volume increasing and hence $\mathrm{Jac}_x(B) \le 1$. □

2.3. Let M be a region $\Omega \subset V^n$ with a Riemannian metric and $B_i : \Omega \to \mathbf{R}$ ($i = 1, \dots, N$) be Lipschitz-1 functions. Let $c \in \mathbf{R}$ and $L_i : V^n \to \mathbf{R}$ be linear functions such that $|B_i - L_i| \le c$ for all $i \le N$, $x \in \Omega$.

We consider a Euclidean metric d_Q determined by a quadratic form $Q = \sum a_i L_i^2$ where $a_i \ge 0$, $\sum a_i = n$ (supposing that Q is non-degenerated). Denote by Vol_Q the volume of this Euclidean structure.

Lemma 4. *In the above notations, we have*

$$\mathrm{Vol}(\Omega) \geq \mathrm{Vol}_Q(\mathrm{Int}_{Q,c\sqrt{n}}(\Omega))$$

where $\mathrm{Int}_{Q,c\sqrt{n}}(\Omega) = \{x \in \Omega : d_Q(x, \partial\Omega) > c\sqrt{n}\}$.

Proof. Define a linear map $L : V^n \to \mathbf{R}^N$ by

$$L(x) = (\sqrt{a_1}L_1(x), \dots, \sqrt{a_N}L_N(x))$$

An obvious computation shows that L is an isometric embedding of the Euclidean space (V^n, Q) into $\mathbf{R}^N$. Denote by P the orthogonal projector of $\mathbf{R}^N$ onto its subspace $L(V^n)$. The idea is to compare L and the function B defined by (1). The n-volume of $L(\mathrm{Int}_{Q,c\sqrt{n}}(\Omega))$ is equal to that of $\mathrm{Int}_{Q,c\sqrt{n}}(\Omega)$ and B is volume non-increasing by Lemma 3. Hence $P \circ B$ is also volume non-increasing and it suffices to show that the image $P \circ B(\Omega)$ contains $L(\mathrm{Int}_{Q,c\sqrt{n}}(\Omega))$. Notice that $|B - L| \leq c\sqrt{n}$ since $|B_i - L_i| \leq c$ and $\sum a_i = n$. Hence $|P \circ B - L| \leq c\sqrt{n}$ on Ω.

Let x be a point of Ω with $d_Q(x, \partial\Omega) > c\sqrt{n}$. The estimate above implies that $(P \circ B)$-image of d_Q-sphere of radius $c\sqrt{n}$ centered at x encloses $L(x)$ in the space $L(V^n)$ and hence $L(x) \in P \circ B(\Omega)$. $\square$

§3. Volume Growth Theorem

3.1. Theorem 1. *Let d be a Riemannian $\mathbf{Z}^n$-periodic metric on a vector space V^n, i.e. it is invariant under a co-compact action of $\mathbf{Z}^n$ by translations. Denote by* Ball_r *the ball of radius r in (V^n, d) centered at the origin. Let ε_n be the standard volume of the standard Euclidean ball. Then*

- **a)** $\liminf\limits_{r\to\infty} \dfrac{\mathrm{Vol}(\mathrm{Ball}_r)}{\varepsilon_n r^n} \geq 1.$
- **b)** *If the inequality turns out to be equality then d is flat.*

Proof. **a)** It is known (see [Bu1]) that there exists a norm $\|\cdot\|$ on V^n and a constant c such that

$$\forall x, y \in V^n \qquad |\, d(x,y) - \|x - y\|\, | \leq c \tag{2}$$

Let $L_i \in F^*$, $a_i > 0$ $(i = 1, \dots, N)$ and $Q = \sum a_i L_i^2$ be the quadratic form guaranteed by Lemma 1 for the norm $\|\cdot\|$ (we keep all the notations introduced in the first section).

For each $i \leq N$ we define a function $B_i : V^n \to \mathbf{R}$ by

$$B_i(x) = \limsup_{\|y\|\to\infty}(L_i(y) - d(x,y))$$

B_i is Lipschitz-1 as an upper limit of distance functions. Applying (2) and the equality

$$L_i(x) = \limsup_{\|y\|\to\infty}(L_i(y) - \|x - y\|)$$

(which holds since $\|L_i\| = 1$) we have $|B_i - L_i| \leq c$.

Denote the unit ball of $\|\cdot\|$ by D. Applying Lemma 4 for $\Omega = (r-c)D$ we have

$$\operatorname{Vol}(\mathrm{Ball}_r) \geq \operatorname{Vol}(\Omega) \geq \operatorname{Vol}_Q((r-c-c\sqrt{n})D) = (r-c-c\sqrt{n})^n \operatorname{Vol}_Q(D)$$

since $Ball_r \supset \Omega$ by (2) and D contains the unit ball of Q by Lemma 1. Therefore

$$\liminf_{r\to\infty} \frac{\operatorname{Vol}(\mathrm{Ball}_r)}{r^n} \geq \operatorname{Vol}_Q(D) \geq \varepsilon_n \tag{3}$$

b) Since we have equality in the second part of (3), $\|\cdot\|$ is the Euclidean norm determined by Q. Represent Q as $\sum_1^n L_i^2$ where $L_i \in F^*$ form an orthonormal basis of $(V^n, \|\cdot\|)^*$ and consider the functions B_i $(i = 1, \dots, n)$ defined as in the first part of the proof. The periodicity of the metric implies that

$$B_i(x+k) = B_i(x) + L_i(k) \qquad \forall x \in V^n,\ k \in \mathbf{Z}^n(0)$$

i.e. the functions $(B_i - L_i)$ are $\mathbf{Z}^n$-periodic.

Consider $B = (B_1, \dots, B_n) : (V^n, d) \to (\mathbf{R}^n, \text{standard metric})$. Since B is periodic it does not decrease volume, otherwise we would get a strict inequality in the first part of (3). From the proof of Lemma 3 it follows that $\operatorname{Jac}(B) \leq 1$ almost everywhere. Therefore $\operatorname{Jac}(B) = 1$ a.e. and Lemma 2 implies that for almost all $x \in V^n$ the derivative $d_xB : \mathrm{T}_x(V^n, <,>_d) \to (\mathbf{R}^n, \text{standard } <,>)$ is a linear isometry. The following Sublemma concludes the proof.

Sublemma. *Let (M^n, d) be Riemannian manifold and $B : M^n \to \mathbf{R}^n$ be Lipschitz volume-preserving map whose derivative is a linear isometry almost everywhere. Then B is an isometry and hence d is a flat metric.*

Proof. Since dB is a linear isometry a.e., the Lipschitz constant of B is 1. We prove that B is bi-Lipschitz homeomorphism. Let $x, y \in V^n$ and choose $\rho > |B(x) - B(y)|$. Then

$$\operatorname{Vol}\bigl(\mathrm{Ball}_\rho(x) \cup \mathrm{Ball}_\rho(y)\bigr) \leq \operatorname{Vol}\bigl(\mathrm{Ball}_\rho(B(x)) \cup \mathrm{Ball}_\rho(B(y))\bigr) \leq (2 - 1/2^n)\varepsilon_n \rho^n$$

since the common part of $\mathrm{Ball}_\rho(B(x))$ and $\mathrm{Ball}_\rho(B(y))$ contains a ball of radius $\rho/2$ and hence takes at least $\varepsilon_n(\rho/2)^n$ off their volume. Suppose that ρ is so small that $\operatorname{Vol}(\mathrm{Ball}_\rho(x))/\varepsilon_n\rho^n > 1 - 1/2^{n+1}$ and the same holds for $\mathrm{Ball}_\rho(y)$. Then the sum of volumes of these balls exceeds the left part of the above inequality, so they must intersect. Hence $d(x, y) < 2\rho$.

We have proved that $d(x, y) \leq 2|B(x) - B(y)|$, if $B(x)$ and $B(y)$ are close enough to each other. Thus B is injective and B^{-1} is Lipschitz. Therefore the derivative $d(B^{-1}) = (dB)^{-1}$ does exist almost everywhere. Hence B^{-1} is Lipschitz-1 as well as B, so B preserves the distance. □

3.2. Note that the second part of the theorem is more delicate than the first one. One can prove the volume growth estimate (3) for any Riemannian metric (not necessarily periodic) satisfying a weaker condition: $d(x, y)/\|x - y\| \to 1$ as $\|x - y\| \to \infty$. It will follow from Theorem 2.

3.3. Some of the above arguments on periodic metrics do not require the vector-space topology. We say a metric space (M, d) to be $\mathbf{Z}^n$-periodic if d is invariant under a co-compact totally disconnected action of $\mathbf{Z}^n$ on M. For such a metric, assertion (2) may be re-read in terms of a norm $\|\cdot\|$ on $\mathbf{Z}^n$ as

$$|\,d(x, x+k) - \|k\|\,| \leq c, \qquad x \in M, k \in \mathbf{Z}^n$$

and holds in the most general cases (e.g. for all $\mathbf{Z}^n$-periodic inner metric spaces).

As pointed out by M. Gromov, Theorem 1 also remains true under weaker topology assumptions. Namely, it holds for a $\mathbf{Z}^n$-periodic Riemannian metric on a manifold M^n whose quotient $M^n/\mathbf{Z}^n$ admits a nonzero-degree map onto an n-torus which can be lifted as a $\mathbf{Z}^n$-invariant map from M^n onto R^n. The proof for this version of Theorem 1 is similar to the original one; the nonzero-degree map is needed for arguments from Lemma 4.

For manifolds of arbitrary topology Theorem 1 does not hold. As a counter-example, consider the surface of a small neighborhood of a 2-dimensional grid in $\mathbf{R}^3$

§4. Volume of Limit Finsler Metric

4.1. Let M^n be a smooth manifold and d be a Finsler metric on M. That is, d is determined in a usual way by a family of Banach norms $\{\|\cdot\|_x : x \in M\}$ on the tangent spaces T_xM and these norms form a continuous vector-length function on TM. It is known that d may be represented as a limit of a sequence of Riemannian metrics d_k on M where convergence $d_k \to d$ is uniform on compact subsets of $M \times M$. (A class of such sequences of metrics whose limits coincide with all Finsler metric is investigated in [Bu2]).

In this section we investigate the relations between the volume of our Finsler manifold (M, d) and volumes of its approximations (M, d_k). By volume we mean here the Hausdorff measure (normalized to have Riemannian volume for Riemannian metrics). However, all the sequel is valid for any volume definition, whenever the volume is monotonous with respect to metric and coincides with the standard Riemannian one for Riemannian manifolds.

4.2. Theorem 2. *If a sequence of Riemannian metrics d_k on M converges to d as above, then*

$$\operatorname{Vol}(M, d) \leq \liminf \operatorname{Vol}(M, d_k) \tag{4}$$

Moreover, if the equality holds then d is a Riemannian metric.

Remark. In [BuI] we have formulated this theorem not quite correctly. This theorem holds for uniform convergence of metric function but not for Hausdorff convergence, because for a metric close by Hausdorff it still may be no almost isometry of nonzero degree. There are examples of Hausdorff convergence of Riemannian metrics which increases the volume.

Proof. The proof is similar to the one for Theorem 1 (it suffices to get the required inequality only for small regions in M). We will prove (4) up to an arbitrary $\varepsilon > 0$. Having fixed a point $x \in M$, consider a Banach space $(V^n, \|\cdot\|) = (T_xM, \|\cdot\|_x)$ and a quadratic form $Q = \sum a_i L_i^2$ obtained from Lemma 1 ($i = 1, \dots, N$, $a_i > 0$,

$\sum a_i = n$ and $L_i \in F^*$ where F is the unit sphere of $\|\cdot\|$). Recall that the unit ball of Q is inscribed into F and touches F at some points p_i with $L_i(p_i) = 1$. Let d_Q be the Euclidean metric on V determined by Q. From the inequalities $L_i \le \|\cdot\| \le d_Q$ and the obvious formula

$$\lim_{R\to\infty} (R - d_Q(Rp_i, q)) = L_i(q) = \lim_{R\to\infty} (R - L(Rp_i - q)), \qquad q \in V$$

we have $R - \|Rp_i - q\| \to L_i(q)$ as R goes to infinity. Therefore one can choose large enough R for which

$$\big|R - \|Rp_i - q\| - L_i(q)\big| \le \varepsilon, \qquad q : \|q\| \le 1,\ i \le N \tag{5}$$

(the convergence above is uniform on compact sets).

Identify a neighborhood of x in M with a region in V by a proper local coordinates. We will use the same notations d and d_k for metrics induced on V from d and d_k on M. One may suppose that the norm on $T_0V \cong V$ determined by the Finsler metric d on V coincides with $\|\cdot\| = \|\cdot\|_x$. Let $\delta > 0$ be so small that

$$1 - \frac{\varepsilon}{R+2} \ \le\ \frac{d(p,q)}{\|p-q\|} \ \le\ 1 + \frac{\varepsilon}{R+2}$$

whenever $\|p\|, \|q\| \le R\delta$. Then from (5) we have

$$|R\delta - d(R\delta p_i, q) - L_i(q)| \le 2\varepsilon\delta, \qquad \|q\| \le \delta,\ i \le N$$

Hence for d_k close enough to d

$$|R\delta - d_k(R\delta p_i, q) - L_i(q)| \le 3\varepsilon\delta, \qquad \|q\| \le \delta,\ i \le N$$

Thus we may apply Lemma 4 to the set $\Omega = \{q \in V : \|q\| \le \delta\}$ and Lipschitz-1 (with respect to d_k) functions $B_i(q) = R\delta - d_k(R\delta p_i, q)$, obtaining

$$\operatorname{Vol}(\Omega, d_k) \ge (1 - C(n)\varepsilon) \operatorname{Vol}(\Omega, d_Q) \ge (1 - C(n)\varepsilon) \operatorname{Vol}(\Omega, \|\cdot\|)$$

(the second inequality is implied by $d_Q \ge \|\cdot\|$).

By the standard covering arguments we derive from these estimates for all $x \in M$ that $(1 - C(n)\varepsilon)\operatorname{Vol}(M, d) \le \liminf \operatorname{Vol}(M, d_k)$. Since ε is arbitrary the inequality (4) follows. If it turns out to be equality then for every point $x \in M$ in the above construction we should have $d_Q = \|\cdot\|$ and hence our metric d is actually Riemannian. $\square$

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