ON PARTIALLY HYPERBOLIC DIFFEOMORPHISMS OF 3-MANIFOLDS WITH COMMUTATIVE FUNDAMENTAL GROUP.

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Dedicated to Anatole Katok on his 60th birthday.

Abstract. Let f be a dynamically coherent partially hyperbolic diffeomorphism of a compact three dimensional manifold whose fundamental group is abelian. We show that the action f_* induced by f on the 1-dimensional real homologies, is also partially hyperbolic: it has an eigenvalue of modulus > 1 and an eigenvalue of modulus < 1. In particular, there are no such diffeomorphisms on S^3. The property of dynamical coherence follows from the unique integrability of the central distribution. We also discuss a weaker integrability property of the central distribution.

1. Introduction and main results

Let M be a smooth, connected, compact Riemannian manifold without boundary. A C^1 diffeomorphism $f\colon M \to M$ is said to be *partially hyperbolic* if there are positive real numbers

$$0 < \lambda_1 \le \lambda_2 < \gamma_1 \le 1 \le \gamma_2 < \mu_2 \le \mu_1$$

and, for each $x \in M$, a df-invariant splitting of the tangent space

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^c(x)$$

into subspaces called the *stable, unstable* and *center* subspaces, such that

$$\begin{aligned} df(x)E^a(x) &= E^a(f(x)) && \text{for} \quad a = s, u, c \\ \lambda_1\|v^s\| \le \|df(x)v^s\| &\le \lambda_2\|v^s\| && \text{for} \quad v^s \in E^s(x) \\ \mu_2\|v^u\| \le \|df(x)v^u\| &\le \mu_1\|v^u\| && \text{for} \quad v^u \in E^u(x) \\ \gamma_1\|v^c\| \le \|df(x)v^c\| &\le \gamma_2\|v^c\| && \text{for} \quad v^c \in E^c(x)\,. \end{aligned}$$

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The subspaces $E^s(x)$, $E^u(x)$ and $E^c(x)$ form Hölder continuous stable, unstable and center distributions E^s, E^u and E^c over M which, in general, are not C^1 even if f is C^2 or better [Ano67]. We refer to the direct sums $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ as the center-stable and center-unstable distributions, respectively. Although weaker versions of partial hyperbolicity are meaningful and sometimes are used, we assume throughout this paper that all three distributions E^s, E^u and E^c are nontrivial.

By a *foliation* W of a manifold M we mean a partition of M into C^1 submanifolds $W(x) \ni x$ (called *leaves*) which depend continuously on $x \in M$ in the compact-open C^1 topology. For a foliation W, denote by TW the tangent distribution of W, i.e., the collection of all tangent planes to the leaves of W. A continuous distribution E is *integrable* if there is a foliation W such that $TW = E$.

The stable E^s and unstable E^u distributions are integrable; the corresponding foliations W^s and W^u are called the *stable* and *unstable foliations*, respectively. Moreover, the exponential contraction and expansion imply the uniqueness of integral manifolds: if a C^1 curve is everywhere tangent to E^s, then it lies in one leaf of W^s, and similarly for W^u.

By analogy with ordinary differential equations we say that a continuous k-dimensional distribution E on a manifold M is *uniquely integrable* if there is a foliation W such that every C^1 curve $\sigma\colon \mathbb{R} \to M$ satisfying $\dot\sigma(t) \in E(\sigma(t))$ for all t, is contained in $W(\sigma(0))$ (in particular $TW = E$). Note that unique integrability is stronger than the existence of an integral surface through every point. The latter condition holds for all 1-dimensional distributions, which may however fail to be uniquely integrable.

The integrability of E^c and the integrability of E^{cs} and E^{cu} (which are referred to as *dynamical coherence*) are important assumptions in the theory of stable ergodicity for partially hyperbolic diffeomorphisms (see [PS97], [BPSW01]). In general, the center distribution E^c fails to be integrable (see [Wil98], [Sma67] for a counterexample).

The following two theorems are the main results of this paper.

THEOREM 1.1. *Let M be a compact 3-dimensional manifold whose fundamental group is abelian and let $f\colon M \to M$ be a partially hyperbolic diffeomorphism. Assume that either f is dynamically coherent or the center distribution of f is uniquely integrable. Then the induced map f_* of the first homology group $H_1(M, \mathbb{R})$ is also partially hyperbolic, i.e., it has eigenvalues α_1 and α_3 with $|\alpha_1| > 1$ and $|\alpha_3|^{-1} > 1$.*

It seems that the assumption of unique integrability of the central foliation can be omitted from the formulation of the theorem; however, at the moment the authors do not have a verified argument. We will address this issue elsewhere.

On the other hand, the condition on the fundamental group is essential: the geodesic flow on the unit tangent bundle of a negatively curved surface gives an example of a partially hyperbolic diffeomorphism isotopic to the identity.

The next theorem is an immediate corollary of Theorem 1.1

THEOREM 1.2. *A compact* 3*-dimensional manifold whose fundamental group is finite does not carry a partially hyperbolic diffeomorphism which either is dynamically coherent or has a uniquely integrable center distribution. In particular, there are no such diffeomorphisms on* $\mathbb{S}^3$.

A diffeomorphism f is *robustly transitive* if every diffeomorhism sufficiently close to f in the C^1 topology, is topologically transitive. L. Diaz, E. Pujals and R. Ures studied robustly transitive diffeomorphisms in dimension 3 (see [DPU99]). They showed that such a diffeomorphism $f : M^3 \to M^3$ is generically partially hyperbolic in the sense that $T_xM^3 = E^s \oplus E^{cu}$ (or $T_xM^3 = E^u \oplus E^{cs}$). Assuming, in addition, the integrability of E^{cu}, they proved that $\pi_1(M^3)$ is infinite. It seems that their argument should go through without the assumption of robust transitivity.

A C^0 distribution E on a manifold M is called *weakly integrable* if for each point x there is an immersed complete C^1 manifold $W(x)$ which contains x and is everywhere tangent to E, i.e., $T_yW(x) = E(y)$ for each $y \in W(x)$. We refer to $W(x)$ as an integral manifold of E. Note that *a priori* the integral manifolds $W(x)$ may be self-intersecting and may not form a partition of M.

If the invariant distributions E^c, E^{cs} and E^{cu} of a partially hyperbolic diffeomorphism f are weakly integrable, we call f *weakly dynamically coherent*.

In Section 2 we prove Theorem 1.1. The weak integrability property of the center distribution is discussed in Section 3. In particular we prove that if the center distribution of a partially hyperbolic diffeomorphism f is 1-dimensional, then f is weakly dynamically coherent (Proposition 3.4). We use this fact in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

The proof consists of three steps.

Step 1. We first show that if the center distribution E^c is uniquely integrable, then the center-stable and center-unstable distributions of f are integrable, i.e., f is dynamically coherent.

Proposition 3.4 implies that for each $x \in M$ there is a complete 2-dimensional submanifold $W^{cs}(x)$ which is tangent to E^{cs} and passes through x. Recall that the stable distribution is uniquely integrable, and by our assumption the center distribution is uniquely integrable. It follows that the submanifolds $W^{cs}(x)$ form a partition and therefore a foliation W^{cs} of M called the *center-stable* foliation. The integrability of E^{cu} follows by reversing the time.

Remark. Without the unique integrability assumption, Proposition 3.4 would only imply the weak integrability of E^{cs}, i.e., the existence of a family of complete surfaces (tangent to E^{cs}) with at least one through every point, but the surfaces might branch. It would be enough to approximate E^{cs} by tangent distributions of foliations (and then use a "Gromov-type compactness argument"); it seems that this may be possible by first selecting a "minimum" subset of surfaces tangent to E^{cs}, with still at least one surface through every point, and then separating the surfaces by a small perturbation.

Step 2. Next we show that if $f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ does not have an eigenvalue whose absolute value is greater than one, then there is a compact leaf of W^{cs} which bounds a solid torus.

The classical Novikov Compact Leaf Theorem states that every smooth foliation of $\mathbb{S}^3$ has a compact leaf (see [Nov65]). The theorem has been generalized to C^0 foliations by Solodov (see [Sol82] and [CLN85]). A straightforward inspection of the argument shows that actually it proves the following stronger statement: if a C^0 foliation of a closed 3-manifold admits a closed contractible curve transverse to the foliation, then the foliation has a compact leaf bounding a solid torus. A contractible closed differentiable curve which is transverse to the foliation is called a *transverse contractible cycle*. To prove the statement of Step 2, it is therefore enough to find a transverse contractible cycle for the foliation W^{cs} constructed in Step 1.

By passing to a finite cover we may assume that E^s, E^u and E^c are oriented. We lift the structure we have on M to its universal cover $\widetilde{M}$. Let $\tilde{f} \colon \widetilde{M} \to \widetilde{M}$ be a lift of f. We also use a tilde to denote the lifts of the distributions and foliations associated to f.

Since $\widetilde{M}$ is quasi-isometric to $\pi_1(M)$ which is abelian, $\widetilde{M}$ is quasi-isometric to $H_1(M, \mathbb{R}) = \mathbb{R}^k$ for some k. Thus the volume of balls in $\widetilde{M}$ has polynomial growth.

Assume by contradiction that the absolute values of all eigenvalues of f_* are less than or equal to 1. Then the length of the images of any vector under the iterates of f_* grows sub-exponentially, and therefore so does the diameter of the images of any compact set under the iterates of $\tilde{f}$. Thus the images are contained in a sequence of balls whose volume grows sub-exponentially.

We now apply this observation to a segment I of an unstable leaf $\widetilde{W}^u(x)$. The length of the images $\tilde{f}(I)$ grows exponentially, but the images are contained in a sequence of balls with sub-exponential volume growth. Hence, given any $\epsilon > 0$, one can find a segment of an unstable curve of length > 1 whose endpoints are ϵ-close. Then, for ϵ small enough, one can perturb this segment to close it up preserving its transversality to the leaves of W^{cs}. The image of this closed curve under the covering map is a transverse contractible cycle. By Novikov's theorem, there is a compact leaf of W^{cs} which bounds a solid torus.

Step 3. Finally we show that a closed leaf of W^{cs} cannot bound a solid torus. We need the following proposition.

PROPOSITION 2.1. *Let $f : T^2 \to T^2$ be a diffeomorphism of a 2-torus, and let E^s, E^c be 1-dimensional continuous df-invariant distributions generated by continuous, unit vector fields v^s and v^c with $|df(v^s)| \leq \lambda |v^s|$, $|df(v^c)| \geq \gamma |v^c|$ for some $\lambda < 1$, $\gamma > \lambda$. Then $f_* : \pi_1(T^2) \to \pi_1(T^2)$ has an eigenvalue α with $|\alpha| < 1$.*

Proof. To keep the same terminology, we refer to curves tangent to E^c and E^s as center and stable curves, respectively.

Let $\tilde{f}$ be a lift of f to the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$. We use $\tilde{}$ to denote the lifts to $\mathbb{R}^2$ of E^c, E^s, v^c and v^s. Suppose that all eigenvalues of f_* are ≥ 1 in absolute value. Then the images of any vector under the iterates of f_*^{-1}, and hence the diameter and volume of the images of a ball under the iterates of f^{-1}, grow sub-exponentially. On the other hand, the length of the images of a segment s of a stable curve under the iterates of $\tilde{f}^{-1}$ grows exponentially. Then the same argument as in Step 2 above applied to $f^{-n}(s)$ shows that there exists a simple closed curve σ in $\mathbb{R}^2$ which is almost tangent to $\widetilde{E}^s$ and hence is transverse to $\widetilde{E}^c$. The curve σ bounds a disc in $\mathbb{R}^2$. Since $\tilde{v}^c$ is transverse to σ, its index is 1. Hence $\tilde{v}^c$ must vanish at some point, a contradiction. $\square$

Assume by contradiction that T is a closed leaf of W^{cs} which bounds a solid torus $\mathcal{T}$. Since E^{cs} is continuous and orientable, the volume of such a solid torus is bounded away from 0. Therefore, by switching to another closed leaf and a power of f we may assume that T and $\mathcal{T}$ are invariant under f.

By Proposition 2.1, the action of f on $\pi_1(T)$ is partially hyperbolic. It cannot be extended to a diffeomorphism of $\mathcal{T}$ because the homomorphism i_*, induced by the embedding $i\colon T \to \mathcal{T}$, sends one generator of $\pi_1(T) = \mathbb{Z}^2$ to 0, and hence the image of the corresponding curve in T must be homotopic to its multiple. This contradiction finishes the proof of Theorem 1.1.

3. Weak integrability of the center distribution

We assume as before that all three distributions E^s, E^u and E^c are non-trivial. As we already mentioned, the stable E^s and unstable E^u distributions are uniquely integrable; the integral foliations are the stable and unstable foliations W^s and W^u.

Recall that, in general, the center distribution E^c fails to be integrable (see [Sma67], [Wil98] for a counterexample). However, in several situations described below, due to their dynamical nature, the distributions E^c, E^{cs} and E^{cu} are weakly integrable.

Propositions 3.2 and 3.3 imply that, given a connected component $\mathcal{C}$ of the space of C^1 partially hyperbolic diffeomorphisms of a compact manifold M, either each $f \in \mathcal{C}$ is weakly dynamically coherent or none of them are.

For a unit vector $v \in \mathbb{R}^m$ and a linear subspace $L \subset \mathbb{R}^m$, set $d(v, L) = \min_{w \in L} \|v - w\|$. The following lemma is an immediate consequence of the inequalities governing the contraction and expansion of the tangent vectors by the derivative of a partially hyperbolic diffeomorphism.

LEMMA 3.1. *Let f be a partially hyperbolic diffeomorphism with hyperbolicity constants $0 < \lambda_1 \le \lambda_2 < \gamma_1 \le 1 \le \gamma_2 < \mu_2 \le \mu_1$.*

Then for every $\alpha > 0$ there is $C(\alpha) > 0$ such that

$$\left\| d\big(df^n v, E^{cu}(f^n(x))\big) \right\| \le C(\alpha) \left(\frac{\lambda_2}{\gamma_1}\right)^n \|v\|$$

for every $n > 0$ and every tangent vector $v \in T_x M$ with $d\big(v/\|v\|, E^s(x)\big) \ge \alpha$.

The lemma implies that if L is a linear subspace of $T_x M$ which is transverse to $E^s(x)$ and of complementary dimension, then $df^n L$ is exponentially close to $E^{cu}\big(f^n(x)\big)$.

PROPOSITION 3.2. *Let f be a partially hyperbolic C^1 diffeomorphism of a compact C^1 manifold M. If f is weakly dynamically coherent, then there is a C^1 neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is weakly dynamically coherent.*

Proof. We use sub-indices f and g to denote the invariant distributions of f and g, respectively. We will prove the weak integrability of E_g^{cu}. The weak integrability of E_g^{cs} follows by reversing the time. The integral manifolds of E_g^c can be constructed by intersecting the integral manifolds of E_g^{cu} with the integral manifolds of E_g^{cs}.

Let $0 < \lambda_1 \leq \lambda_2 < \gamma_1 \leq \gamma_2 < \mu_2 \leq \mu_1$ be the hyperbolicity constants of f. To show the weak integrability of E_g^{cu} fix $x \in M$. Denote by $W_f^{cu}(y)$ the integral manifold of E_f^{cu} passing through $y \in M$. For each positive integer n, let $W_n = g^n\big(W_f^{cu}(g^{-n}(x))\big)$. It follows from the hyperbolicity inequalities that the invariant distributions E_g^s, E_g^u and E_g^c depend continuously on g in the C^1 topology. Observe that if g is sufficiently C^1 close to f, then the hyperbolicity constants of g are δ close to those f, and $E_f^{cu}(y)$ is uniformly close to $E_g^{cu}(y)$ and hence is uniformly transverse to $E_g^s(y)$. Lemma 3.1 implies that the tangent space to W_n is exponentially close to E_g^{cu}, i.e.,

$$T_{g^n(y)}\big(g^n\big(W_f^{cu}(g^{-n}(x))\big)\big) \to E_g^{cu}(g^n(y))$$

uniformly in $y \in W_f^{cu}(g^{-n}(x))$ and exponentially in n with rate $(\lambda_2 + \delta)/(\gamma_1 - \delta)$. The sequence W_n is precompact in the compact–open C^1 topology and therefore has a convergent subsequence. The limit is an integral manifold of E_g^{cu} passing through x. □

PROPOSITION 3.3. *Let f_n be a sequence of partially hyperbolic C^1 diffeomorphisms of a compact manifold M. Suppose that*

(1) *$f_n \to g$ in the C^1 topology,*
(2) *each f_n is weakly dynamically coherent,*
(3) *all f_n have the same hyperbolicity constants $0 < \lambda_1 \leq \lambda_2 < \gamma_1 \leq \gamma_2 < \mu_2 \leq \mu_1$.*

Then g is partially hyperbolic and weakly dynamically coherent.

Proof. Let $x \in M$ and let $v_n \in E_{f_n}^u(x)$ be a sequence of unit vectors and assume that $v_n \to v$. Then $\mu_2^n \leq \|df^n v\| \leq \mu_1^n$, and hence $v \in E_g^u(x)$. Similar arguments apply to E^c and E^s. It follows that $\lim E_{f_n}^\alpha(x) = E_g^\alpha(x)$ with $\alpha = u, c, s$, and, in particular, g is partially hyperbolic.

As in the previous proposition we prove the weak integrability of E_g^{cu}. The other two integrability properties follow by reversing the time and taking intersections.

Fix $x \in M$, and let $W_n(x)$ be an integral manifold of $E_{f_n}^{cu}$ passing through x. Since the center-unstable distribution E_g^{cu} depends continuously on g in the C^1 topology, the sequence W_n is precompact in the compact–open C^1 topology and therefore has a convergent subsequence. The limit is an integral manifold of E_g^{cu} passing through x. □

As we mentioned in the proofs of Propositions 3.2 and 3.3, the weak integrability of E_f^{cu} and of E_f^{cs} imply the weak integrability of E_f^c by taking intersections of the integral manifolds. It is not clear whether in general the weak integrability of E_f^c implies weak dynamical coherence. If the center distribution of a partially hyperbolic diffeomorphism is 1-dimensional, it is weakly integrable by the existence theorem of ordinary differential equations.

PROPOSITION 3.4. *Let f be a C^1 partially hyperbolic diffeomorphism of a compact manifold M. Suppose the center distribution E^c of f is 1-dimensional. Then f is weakly dynamically coherent.*

Proof. We will show that E^{cu} is weakly integrable. The weak integrability of E^{cs} follows by reversing the time. Let σ be a complete integral curve of E^c. Let $W_\epsilon(\sigma)$ be an immersed C^1 submanifold of M obtained by ϵ- thickening of σ approximately in the direction of E^u. If E^u is 1-dimensional, such an immersed submanifold $W_\epsilon(\sigma)$ can be constructed by approximating the unstable foliation W^u with a smooth foliation $\widetilde{W}^u$ and drawing the leaf of $\widetilde{W}^u$ of length 2ϵ through each point $x \in \sigma$ so that x is the middle point of the leaf. In the general case such an immersed submanifold $W_\epsilon(\sigma)$ can be constructed by approximating E^u with a smooth distribution $\widetilde{E}^u$ and drawing through each point $x \in \sigma$ all geodesics $\sigma : [0, \epsilon) \to M$ with $\dot\sigma(0) \in \widetilde{E}^u$.

Fix $x \in M$ and a complete integral curve $\sigma \ni x$ of E^c. For each $n > 0$ consider the complete integral curve $\sigma_n = f^{-n}(\sigma)$ of E^c, its ϵ-thickening $W_\epsilon(\sigma_n)$ in the unstable direction and its image $V_n = f^n\big(W_\epsilon(\sigma_n)\big)$ under f^n. The C^1 submanifolds V_n contain σ. Denote by d_s the distance along the stable leaves. Let $y \in \sigma$ and let $z_n \in V_n$ and $z_{n+1} \in V_{n+1}$ be the intersections of a local unstable leaf of W^s in a neighborhood of y with V_n and V_{n+1}. Then $w_n = f^{-n}(z_n)$ and $w_{n+1} = f^{-n}(z_{n+1})$ lie on $W^s\big(f^{-n}(z_n)\big)$ and $d_s(w_n, w_{n+1}) \le C\epsilon$ by the construction of $W_\epsilon(\sigma_k)$. It follows that $d_s(z_n, z_{n+1}) \le C\epsilon\lambda_2^n$, and therefore the submanifolds V_n converge in the C^0 topology. If ϵ is small enough, then, by Lemma 3.1, the tangent planes of $W_\epsilon(\sigma_n)$ are close to $E^c u$, and the tangent planes to V_n are exponentially close to E^{cu}. Therefore the submanifolds V_n converge in the C^1 topology. The limit $W^{cu}(\sigma)$ is an integral manifold of E^{cu} containing σ. It follows immediately from the uniform contraction in the stable direction that $W^{cu}(x, \sigma) = \bigcup_{y\in\sigma} W^u(y)$.

The integral manifold $W^{cu}(\sigma)$ need not be complete. Its boundary consists of unstable leaves. However, it follows from the previous argument that for each integral curve σ of E^c there is a unique integral manifold $W^{cu}(\sigma)$ of E^{cu} containing σ. Let y be a boundary point of $W^{cu}(\sigma)$ and let $\sigma_y : (-\infty, \infty) \to M$ be a complete integral curve of E^c such that $\sigma_y(0) = y$ and $\sigma_y(t) \in W^{cu}(\sigma)$ for $t \in (-\infty, 0)$. The union $W^{cu}(\sigma) \bigcup W^{cu}(\sigma_y)$ is an integral manifold of E^{cu}. A countable repetition of this construction produces a complete integral manifold of E^{cu}.

The above argument is reminiscent of the corresponding constructions in [HPS77]. There is an alternative and more direct (but longer) argument which proves that for every center curve σ the union $\bigcup_{y\in\sigma} W^u(y)$ is a C^1 submanifold of M. It is based on a C^0 interpretation of the commutator of the vector fields tangent to E^c and E^s. If the corresponding center-stable-center-stable rectangle R is closed by a short unstable curve α_u, one gets a contradiction between the growth rate of the area of $f^n(R)$ (which is $\leq (\lambda_2\gamma_2)^n$) and the growth rate of the length of $f^n(\alpha_u)$. □

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