PARTIALLY HYPERBOLIC DIFFEOMORPHISMS OF 3-MANIFOLDS WITH ABELIAN FUNDAMENTAL GROUPS

D. BURAGO AND S. IVANOV

Abstract. We present the first known non-trivial topological obstructions to the existence of partially hyperbolic diffeomorphisms. In particular, we show that the are no partially hyperbolic diffeomorphisms on the three sphere. More generally we show that, for a partially hyperbolic diffeomorphism of a 3-manifold with an Abelian fundamental group, the induced action in the first homology group is partially hyperbolic. This improves the results of [BBI] by dropping the assumption of dynamical coherence.

1. Introduction and main results

Let M be a smooth, connected, compact Riemannian manifold without boundary (a concrete choice of Riemannian metric is of no importance for the sequel). A C^1 diffeomorphism $f\colon M \to M$ is said to be *partially hyperbolic* if there are a df-invariant splitting of the tangent bundle

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^c(x), \qquad x \in M,$$

into C^0 distributions E^s, E^u and E^c (called the *stable, unstable* and *center* distributions), and continuous functions $\lambda, \gamma_1, \gamma_2, \mu : M \to \mathbb{R}$ such that for every $x \in M$,

$$0 < \lambda(x) < \gamma_1(x) \leq 1 \leq \gamma_2(x) < \mu(x)$$

1991 *Mathematics Subject Classification.* 34C40, 34D45, 37C05, 37C70, 37D10, 37D30.

Key words and phrases. partially hyperbolic diffeomorphism; dynamical coherence; stable, unstable, center distributions and foliations; quasi-isometric leaves.

The first author is partially supported by the NSF Grant DMS-0412166. The second author is partially supported by the RFBR grant 05-01-00939.

and

$$\begin{aligned}
&df(x)E^a(x) = E^a(f(x)) && \text{for} \quad a = s, u, c,\\
&\|df(x)v^s\| \le \lambda(x)\|v^s\| && \text{for} \quad v^s \in E^s(x),\\
&\mu(x)\|v^u\| \le \|df(x)v^u\| && \text{for} \quad v^u \in E^u(x),\\
&\gamma_1(x)\|v^c\| \le \|df(x)v^c\| \le \gamma_2(x)\|v^c\| && \text{for} \quad v^c \in E^c(x).
\end{aligned}$$

The distributions E^s, E^u and E^c are Hölder continuous but in general are not C^1 even if f is C^2 or better [Ano67]. We refer to the direct sums $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ as the center-stable and center-unstable distributions, respectively.

In this paper, by a C^0 *foliation with* C^1 *leaves* we mean a continuous foliation W of M whose leaves $W(x)$, $x \in M$, are C^1 and their tangent spaces $T_xW(x)$ depend continuously on $x \in M$. For such a foliation W, we denote by TW the tangent distribution of W, i.e., the collection of all tangent spaces to the leaves of W. Note that a C^0 foliation with C^1 leaves is not necessarily a C^1 foliation (as defined in terms of C^1 charts).

The stable E^s and unstable E^u distributions are integrable in the sense that there exist C^0 foliations W^s and W^u with C^1 leaves (called the *stable* and *unstable foliations*, respectively) such that $TW^s = E^s$ and $TW^u = E^u$. Moreover, the exponential contraction and expansion implies the uniqueness of integral manifolds: if a C^1 curve is everywhere tangent to E^s, then it lies in one leaf of W^s, and similarly for W^u.

By analogy with ordinary differential equations, we say that a continuous k-dimensional distribution E on a manifold M is *uniquely integrable* if there is a C^0 foliation W with C^1 leaves such that every C^1 curve $\sigma\colon \mathbb{R} \to M$ satisfying $\dot\sigma(t) \in E(\sigma(t))$ for all t, is contained in $W(\sigma(0))$.

A partially hyperbolic diffeomorphism f is said to be *dynamically coherent* if the distributions E^c, E^{cs} and E^{cu} are integrable, i.e., tangent to C^0 foliations with C^1 leaves.

In general, the center distribution E^c fails to be integrable (see [Wil98] for a counterexample). It is not known whether the central distribution is uniquely integrable even if it is one-dimensional.

From now on, M is a closed 3-dimensional Riemannian manifold and f is a partially hyperbolic diffeomorphism of M. We assume that all three distributions E^s, E^u and E^c are one-dimensional.

The following two theorems are the main results of this paper.

Theorem 1.1. *There is no partially hyperbolic diffeomorphism on* $\mathbb{S}^3$.

To the best of our knowledge, this result is the first known topological obstruction to the existence of a partially hyperbolic diffeomorphism on a specific three-manifold (see [BW] and references there for a nice discussion of known examples and related topics).

Theorem 1.1 is a particular case of the following theorem:

Theorem 1.2. *Let M be a compact 3-dimensional manifold whose fundamental group is Abelian and let $f: M \to M$ be a partially hyperbolic diffeomorphism. Then the induced map f_* of the first homology group $H_1(M, \mathbb{R})$ is also partially hyperbolic, i.e., it has eigenvalues α_1 and α_2 with $|\alpha_1| > 1$ and $|\alpha_2| < 1$.*

This paper is a continuation of [BBI]. Theorems 1.1 and 1.2 improve the results of [BBI] by removing the assumption of dynamical coherence. Following a suggestion by Christian Bonatti, we also work with a more general notion of partial hyporbolicity than in [BBI]. In particular, we need a differnt proof of Proposition 3.1 asserting the existence of integral submanifolds for E^{cs}.

To work around the lack of dynamical coherence, we prove two technical results which may also be useful for other applications. First, we show that there is a "branching foliation" tangent to E^{cs} (Theorem 4.1). Second, we show that a branching foliation can be approximated by a C^0 foliation with C^1 leaves (cf. Theorem 7.2 and Key Lemma 2.1). For our purposes, the branching foliation and approximations are as good as foliations guaranteed by the dynamical coherence.

The paper is organized as follows. In Section 2 we derive Theorems 1.1 and 1.2 from the Key Lemma 2.1. The proof of the Key Lemma needs some dynamics preliminaries (Section 3), and then the core part of the proof is contained in Sections 4–7. In Sections 4–6 we prove the existence of a branching foliation tangent to E^{cs} (Theorem 4.1) and in Section 7 we approximate a branching foliation by an ordinary foliation (Theorem 7.2).

Acknowledgements: This research has been started jointly with Misha Brin; [BBI] has resulted from this joint work. Misha quit the project not far from the point when it was completed. He requested to be removed from the list of authors. We regretfully did that, but we still consider him as a co-author.

We are very grateful to Christian Bonatti for a tremendous work of thoroughly reading a preliminary version of the paper, pointing out a number of inaccuracies, and helping us with improving the exposition. His list of comments, corrections and suggestions, which was almost as long as that version of the paper, has been of invaluable help to us. Furthermore, some suggestions made by Christian Bonatti allowed us

to formulate and prove the result in a somewhat more general set-up, namely for the point-wise definition of partial hyperbolicity rather than a uniform one.

We thank Mark Levi for making the two figures in this paper.

2. Proofs of Theorems 1.1 and 1.2

Even though Theorem 1.1 follows from Theorem 1.2, we first prove Theorem 1.1 and then show how to modify this proof to obtain Theorem 1.2. Both proofs are slight modifications of arguments from [BBI]. They do not require new ideas and simply show that approximations guaranteed by the Key Lemma below are as good as dynamical coherence assumed in [BBI].

By passing to a finite cover, one can make M, E^s, E^u, and E^c orientable. Throughout the rest of the paper we assume that the manifold and the distributions are oriented.

The following key lemma shows that E^{cs} can be approximated by tangent distributions of foliations.

Key Lemma 2.1. *For every $\varepsilon > 0$ there is a C^0 foliation $\mathcal{A}_\varepsilon$ with C^1 leaves such that the angles between $T\mathcal{A}_\varepsilon$ and E^{cs} are less than ε.*

In fact, in the course of the proof we also show that there is a continuous map $h_\varepsilon\colon M \to M$ which is ε-close in C^0 to the identity and sends every leaf of $\mathcal{A}_\varepsilon$ to a surface tangent to E^{cs} (cf. Theorem 7.2). We do not include this fact in the formulation since we do not use it.

This lemma uses Proposition 3.1 from Section 3, which is an easy generalization of an integrability assertion from [BBI]. The rest of the argument, which is the core of the proof of the Key Lemma, is purely topological. It is however rather long and technical. It occupies Sections 4–7.

In this section we derive Theorems 1.1 and 1.2 from the Key Lemma. We need some preliminaries from the theory of foliations. In this paper, all foliations have C^1 leaves.

First, the classical Novikov Compact Leaf Theorem states that every smooth foliation of $\mathbb{S}^3$ has a compact leaf (see [Nov65]). Moreover there is a compact leaf bounding a Reeb component. A *Reeb component* is a solid torus in which the foliation is homeomorphic to the foliation of $D^2 \times S^1$ by the boundary $\partial D^2 \times S^1$ and the graphs of the functions

$$x \mapsto \text{const} + \frac{1}{1 - |x|^2} \mod 1$$

from the interior of D^2 to S^1. Here $D^2 = \{x \in \mathbb{R}^2 : |x| \le 1\}$ and $S^1 = \mathbb{R}/\mathbb{Z}$.

The original work by S. Novikov assumed high smoothness, however the theory has been generalized to C^0 foliations (see [Sol82] and [CLN85]).

We say that a closed differentiable curve in M is a *transverse contractible cycle* for a 2-dimensional foliation (or distribution) if the curve is transverse to the foliation (or distribution) and homotopic to a point.

There is an easy generalization of Novikov's Theorem asserting that if a C^0 foliation (with C^1-leaves) of a closed 3-manifold admits a transverse contractible cycle, then the foliation has a compact leaf; the compact leaf guaranteed by the theorem is a torus bounding a Reeb component ([CC03], Theorem 9.1.4).

Novikov has also observed that every 1-dimensional foliation W^1 of a compact 3-manifold transverse to a 2-dimensional foliation W^2 with a Reeb component U has a closed leaf. It is easy to see that this is true in the C^0 case. Indeed, we may assume that W^1 is orientable and oriented so that at the boundary of U the positive direction of W^1 points inside U. Let $S \subset U$ be a leaf of W^2. The transversality and the structure of the Reeb component implies that every positively oriented half-leaf of W^1 starting at a point $x \in S$ eventually intersects S again. Denote the first intersection point by $\phi(x)$. Then $\phi : S \to S$ is a continuous map. Furthermore, the image $\phi(S)$ is contained in a compact subset of S. Since S is homeomorphic to $\mathbb{R}^2$, the Brouwer Fixed Point Theorem implies that there is an $x \in S$ such that $\phi(x) = x$. Then the leaf of W^1 containing x is a desired closed leaf.

For the sake of further references we summarize the above statements in the following lemma:

Lemma 2.2. *If a 2-dimensional foliation of a compact 3-manifold admits a transverse contractible cycle, then every 1-dimensional foliation transverse to the foliation has a closed leaf. In particular, every 1-dimensional foliation transverse to a 2-dimensional foliation of S^3 has a closed leaf.*

Modulo the Key Lemma and Lemma 2.2, the proof of Theorem 1.1 is now rather straightforward:

Proof of Theorem 1.1. Apply the Key Lemma to E^{cs} and get a family of foliations $\mathcal{A}_\varepsilon$ approximating E^{cs} (in the sense of the Key Lemma). Then, for a sufficiently small ε, $\mathcal{A}_\varepsilon$ is transverse to E^u. Recall that E^u is uniquely integrable, hence, by Lemma 2.2, E^u has a closed integral curve. Applying iterations of f^{-1} yields arbitrarily short closed integral curves of E^u, a contradiction. $\square$

Proof of Theorem 1.2. Now we show how to modify this argument to prove Theorem 1.2.

Lemma 2.3. *There are no transverse contractible cycles for E^{cs}.*

Proof. Arguing as in the proof of Theorem 1.1, we apply the Key Lemma to approximate E^{cs} by tangent distributions of foliations $\mathcal{A}_\varepsilon$. Note that a transverse contractible cycle for E^{cs}, is also a transverse contractible cycle for all approximating foliations $\mathcal{A}_\varepsilon$ provided that ε is small enough. Hence, by Lemma 2.2, the existence of such a cycle implies that E^u has a closed orbit, a contradiction. □

The rest of the proof essentially repeats the argument in *Step 2* in [BBI]. Assume by contradiction that the absolute values of all eigenvalues of f_* are less than or equal to 1. (If they are greater than or equal to 1, use f^{-1} instead of f.)

We use a tilde to denote the lifts to the universal cover $\widetilde{M}$ of M of f and the distributions and foliations associated to f. Since $\widetilde{M}$ is quasi-isometric to $\pi_1(M)$ which is Abelian, $\widetilde{M}$ is quasi-isometric to $H_1(M,\mathbb{R}) = \mathbb{R}^k$ for some k. Fix a compact fundamental domain D of the deck group $\pi_1(M)$. Then any set of diameter d can be covered by polynomially many (in d) translates of D by elements of the deck group.

Since the absolute values of all eigenvalues of f_* are no greater than 1, the length of the images of any vector under the iterates of f_* grows sub-exponentially, and therefore so does the diameter of the images of any compact set under the iterates of $\tilde{f}$.

We apply this observation to a segment I of an unstable leaf in $\widetilde{M}$. The length of the images $\tilde{f}^n(I)$ grows exponentially, but the image is covered by sub-exponentially many translates of D. Hence, given any $\varepsilon > 0$, one can apply the Pigeon-Hole principle and find a segment of an unstable curve in $\widetilde{M}$ of length > 1 whose endpoints are ε-close. If ε is sufficiently small, one can perturb this segment so that it closes up and remains transverse to E^{cs}. This yields a transverse contractible cycle for E^{cs}, contrary to Lemma 2.3. □

3. Proof of the Key Lemma: Preliminaries

Recall that M and the distributions E^c, E^s and E^u are oriented. By passing to a power of f, we may assume that f preserves all the orientations.

We choose a Riemannian metric on M so that the distributions E^c, E^s and E^u are almost orthogonal, namely that the angle between them is no less than $\pi/2 - 10^{-8}$. We use the same notation E^c, E^s and E^u for the distributions and the corresponding unit vector fields.

By the compactness of M, there is an $r_0 > 0$ (referred to as a *regularity radius*) such that every ball of radius r_0 is covered by a local coordinate system (x, y, z) in which E^c, E^s and E^u differ by no more than 10^{-6} from coordinate vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, respectively. We refer to such coordinate systems as *regular coordinates*. It is easy to see that for every two points x, y within distance at most $r_0/2$ from the center of a regular coordinate system, one has

$$1 - 10^{-5} < \frac{d(x,y)}{d_{coord}(x,y)} < 1 + 10^{-5}$$

where d is the distance in M and d_{coord} is the distance between the corresponding points in $\mathbb{R}^3$.

For convenience, we rescale the metric so that $r_0 = 100$. That is, we assume that every ball of radius 100 is covered by a regular coordinate system. This choice of the scale is used throughout the paper.

We refer to C^1 curves tangent to E^s as *stable curves* or *stable leaves*, and use similar terminology for the center and unstable distributions.

The following proposition is a generalization of [BBI, Proposition 3.4]. The difference with [BBI] is that we start from any curve tangent to E^{cs} and transverse to E^s rather than from a segment of a central curve; we also use a more general definition of partial hyperbolicity, with point-wise inequalities on expansion and contraction rates rather than uniform ones. Hence the proof is based on a slightly different idea.

Proposition 3.1. *Let $I \subset \mathbb{R}$ be an open interval and $\sigma\colon I \to M$ be a C^1 curve tangent to E^{cs} and transverse to E^s. Let Φ^t, $t \in \mathbb{R}$, denote the flow generated by E^s. Then the map $S : I \times \mathbb{R}$ defined by $S(\tau, t) = \Phi^t(\sigma(\tau))$ is a C^0 parameterization of a C^1 immersed 2-dimensional submanifold of M tangent to E^{cs}.*

Proof. Passing to a power of f if necessary, we may assume that $\gamma_1(x) \geq 10\lambda(x)$ and $\mu(x) \geq 10\gamma_2(x)$ for all $x \in M$ where $\lambda, \mu, \gamma_1, \gamma_2$ are from the definition of partial hyperbolicity. Then there is a $\delta_0 > 0$ such that $\gamma_1(x) \geq 5\lambda(y)$ and $\mu(x) \geq 5\gamma_2(y)$ for all $x, y \in M$ such that $d(x, y) \leq \delta_0$. For convenience, we rescale the metric so that $\delta_0 \geq 100$.

Let A denote the maximum expansion factor of f ($A = \max_{x \in M} \mu(x)$) and $\rho_0 = 1/A$. By $\ell(\gamma)$ we denote the length of a curve γ in M.

Obviously S is continuous and locally injective. It suffices to prove that every point $(\tau_0, t_0) \in I \times \mathbb{R}$ has a neighborhood U such that $S(U)$ is an embedded C^1 surface tangent to E^{cs}. Replacing σ by $f^n\sigma$ for a large n, we may assume that $|t_0| < 1$ and the angle between $\dot{\sigma}(\tau_0)$ and E^s is less than 10^{-6}. Note that the latter remains true for all images $f^n\sigma$, $n \geq 0$.

Let n_0 be a positive integer satisfying the following: for every unit-length stable curve γ one has $\ell(f^n(\gamma)) < 10^{-6}\rho_0$ for all $n \geq n_0$. Then choose an ε_0 such that, for every set $X \subset M$ with $diam(X) < 10\varepsilon_0$ and every $n \leq n_0$, one has $diam(f^n X) < \rho_0/10$. Now choose a neighborhood U of (τ_0, t_0) such that $diam(S(U)) < \varepsilon_0$ and $t - t_0 < \varepsilon_0$ for all $(\tau, t) \in U$.

Lemma 3.2. *There exist constants $a, B > 0$ such that the following holds. If γ_c is a central curve starting at a point $p_1 \in S(U)$, and γ_u is an unstable curve connecting the end of γ_c to a point $p_2 \in S(U)$, then*

$$\ell(\gamma_u) \leq B \cdot \ell(\gamma_c)^{1+a}.$$

Proof. Let $p_1 = S(\tau_1, t_1)$ and $p_2 = S(\tau_2, t_2)$ where $(\tau_1, t_1) \in U$ and $(\tau_2, t_2) \in U$. Consider a "pentagon" (a piece-wice C^1 smooth closed curve) composed of γ_c, γ_u, an arc of σ, and two stable curves. The sides of the pentagon (listed in cyclic order) are σ_0, s_1, γ_c, γ_u, s_2. The stable curves s_1 and s_2 connect p_1 to $\sigma(\tau_1)$ and p_2 to $\sigma(\tau_2)$ respectively, and σ_0 is a part of σ between $\sigma(\tau_2)$ and $\sigma(\tau_1)$. Note that the direction in which the sides of the pentagon are traversed do not necessarily agree with the orientations of corresponding leaves.

The assertion of the lemma is trivial if $\ell(\gamma_u) = 0$. Suppose that $\ell(\gamma_u) > 0$ and apply iterations of f to this pentagon until its size becomes comparable with the regularity radius. More precisely, let n be the largest positive integer such that $\ell(f^n\sigma_0) < 1$, $\ell(f^n\gamma_c) < 1$ and $\ell(f^n\gamma_u) < 1$. Then at least one of these lengths exceeds $1/A = \rho_0$, therefore $n > n_0$ by the choice of ε_0. Then $\ell(f^n s_1) < 10^{-6}\rho_0$ and $\ell(f^n s_2) < 10^{-6}\rho_0$.

Consider the pentagon with sides $f^n\sigma_0$, $f^n s_1$, $f^n\gamma_c$, $f^n\gamma_u$, $f^n s_2$. Introduce regular coordinate system covering this pentagon. In these coordinates, $f^n\gamma_u$ is C^1-close to a segment of a z-coordinate line, and $f^n\sigma_0$ and $f^n\gamma_c$ are C^1-close to x-coordinate lines (recall that tangent vectors of σ_0 and hence of $f^n\sigma_0$ are close to E^c). By the choice of n_0, the sides $f^n s_1$ and $f^n s_2$ are very short (of lengths not exceeding $10^{-6}\rho_0$). In addition, the Riemannian structure in these coordinates is 10^{-6}-close to the Euclidean one. It follows that

$$|\ell(f^n\sigma_0) - \ell(f^n\gamma_c)| < 10^{-5}\rho_0$$

and

$$\ell(f^n\gamma_u) < 10^{-5}(\rho_0 + \ell(f^n\sigma_0) + \ell(f^n\gamma_c)).$$

Since one of the three lengths is greater than ρ_0, these inequalities imply that $\ell(f^n\gamma_c) > \rho_0/2$ and $\ell(f^n\gamma_u) < \ell(f^n\gamma_c)$. The latter implies that $\ell(\gamma_u) < 5^{-n}\ell(\gamma_c)$ since $\mu(x) \geq 5\gamma_2(y)$ for all $x \in f^m\gamma_u$, $y \in f^m\gamma_c$, $0 \leq m \leq n$. Observe that

$$n \geq \log_A \frac{\ell(f^n\gamma_c)}{\ell(\gamma_n)} \geq \log_A \frac{\rho_0/2}{\ell(\gamma_n)} = -a(\log \ell(\gamma_c) + b)$$

for some constants $a, b > 0$. Then the inequality $\ell(\gamma_u) < 5^{-n}\ell(\gamma_c) < e^{-n}\ell(\gamma_c)$ implies that

$$\ell(\gamma_u) < e^{a \log \ell(\gamma_c) + ab}\ell(\gamma_c) = B \cdot \ell(\gamma_c)^{1+a},$$

where $B = e^{ab}$. $\square$

Now it is easy to finish the proof of the proposition. Fix a compact set $K \subset S(U)$ and choose $p, q \in K$ sufficiently close to each other. It is easy to see that there is a piecewise C^1 curve starting at p and then traversing three (possibly degenerate) arcs γ_c, γ_u and γ_s (exactly in this order) tangent to E^c, E^u and E^s respectively and of lengths not exceeding $2d(p, q)$.

In regular coordinates, consider the two-plane E passing through p and spanned by $E^s(p)$ and $E^c(p)$. Since E^c and E^s are continuous, the distance $d(q, E)$ from q to the plane satisfies

$$d(q, E) \leq \ell(\gamma_u) + o(\ell(\gamma_c) + \ell(\gamma_u) + \ell(\gamma_s)) = \ell(\gamma_u) + o(d(p, q)),\ d(p, q) \to 0.$$

Observe that γ_s is a part of a stable leaf passing through q, hence $\gamma_s \subset S(U)$ and therefore the end of γ_u lies in $S(U)$. Then the lemma implies that

$$\ell(\gamma_u) \leq B \cdot \ell(\gamma_c)^{1+a} = o(d(p, q)), \qquad d(p, q) \to 0.$$

Thus $d(q, E) = o(d(q, p))$ as $d(p, q) \to 0$.

Now the following easy analytic lemma completes the proof:

Lemma 3.3. *Let $\{E(x)\}_{x \in \mathbb{R}^3}$ be a continuous 2-dimensional distribution in $\mathbb{R}^3$. Let $U \subset \mathbb{R}^2$ be an open region and $S : U \to \mathbb{R}^3$ an injective continuous map such that for every compact $K \subset S(U)$,*

$$d(q, E(p)) = o(|p - q|), \qquad p, q \in K,\ |p - q| \to 0.$$

(Here $E(p)$ is regarded as an affine plane in $\mathbb{R}^3$ passing through p). Then $S(U)$ is a 2-dimensional embedded C^1 submanifold of $\mathbb{R}^3$ tangent to the distribution. $\square$

Since $S|_U$ is continuous and injective, the lemma implies that $S(U)$ is a C^1 embedded submanifold tangent to E^{cs}. $\square$

4. Proof of the Key Lemma I: Constructing a pre-foliation

We prove Key Lemma 2.1 in two steps. First we construct a *pre-foliation* tangent to E^{cs} (its leaves may merge, but they never intersect "essentially"; a formal definition is given below). This is done in Sections 4–6. Then we show that this pre-foliation can be perturbed into a usual foliation. This argument occupies Section 7.

The rest of the paper is independent of the previous sections. It does not use any dynamical arguments and thus we "recycle" a very useful letter f which now does not necessarily denote a diffeomorphism of M.

In the sequel, we utilize only the following topological structure: M is a closed oriented 3-dimensional smooth manifold equipped with continuous transverse oriented 1-dimensional distributions E^s and E^c on M, and the following integrability conditions hold:

(1) E^s is uniquely integrable and has no closed integral curves;

(2) Every C^1 curve γ tangent to the distribution $E^{cs} := E^c \oplus E^s$ and transverse to E^s lies on a C^1 immersed surface tangent to E^{cs} and consisting of whole integral curves of E^s (cf. Proposition 3.1).

As usual we assume that M is equipped with an auxiliary Riemannian metric. As in Section 3, we assume that the metric is chosen so that E^s and E^c are almost orthogonal and the regularity radius is at least 100.

Definition 4.1. A *surface* is a C^1 immersion $F : U \to M$ where U is a connected smooth 2-dimensional manifold (possibly with boundary). The manifold U is called the *domain* of the surface and is denoted by $dom(F)$. We regard U as a Riemannian manifold with a C^0 Riemannian structure induced from M and with the associated length metric.

A surface $F : U \to M$ is said to be *complete* if the induced length metric on U is complete. A surface is *open* if it has no boundary.

We say that a point $a \in U$ is a *lift* of a point $p \in M$ to F if $F(a) = p$. A curve $\tilde{\gamma} : I \to U$ (where I is an interval) is a *lift* of a curve $\gamma : I \to M$ if $\gamma = F \circ \tilde{\gamma}$. Of course, a lift of a curve is uniquely determined by a lift of its starting point. Following traditions of differential geometry, we often abuse notation and make no distinction between γ and $\tilde{\gamma}$. We say that a curve γ *lies on* F if it admits a lift to F.

A *neighborhood* of F is an immersion $\mathcal{F} : U \times \mathbb{R} \to M$ such that $\mathcal{F}(x, 0) = F(x)$ for all $x \in U$. We say that a curve $\gamma : I \to M$ *crosses* F if there is an interval $J \subset I$ such that $\gamma|_J$ can be represented as $\mathcal{F} \circ \tilde{\gamma}$ where $\mathcal{F}$ is a neighborhood of F and $\tilde{\gamma} : J \to U \times \mathbb{R}$ is a curve which intersects both $U \times (0, +\infty)$ and $U \times (-\infty, 0)$.

We say that surfaces F and G have a *topological crossing* if there is a curve which lies on F and crosses G. It is easy to see that this definition is symmetric with respect to F and G.

A *branching foliation* in M is a collection of complete open surfaces tangent to a continuous 2-dimensional distribution such that no two of the surfaces topologically cross and their images cover M.

The main result of this section is the following

Theorem 4.1. *There exists a branching foliation tangent to E^{cs} and invariant under any C^1 diffeomorphism of M which preserves the oriented distributions E^s and E^{cs}.*

The invariance under diffeomorphims is not used in the proof of the Key Lemma. It is however useful for other applications (see [BBI2]).

Definition 4.2. By a *cs-surface* we mean a surface $F : U \to M$ tangent to E^{cs}, consisting of whole stable leaves, and parameterized by a simply connected domain U.

The second condition means that for every $x \in U$, there is a curve $\gamma : \mathbb{R} \to U$ such that $F \circ \gamma$ is a complete integral curve of E^s.

The definition implies that the domain U of F is foliated by lifts of entire stable leaves. We abuse the terminology and use the same term for both the stable leaves in M and their lifts to U. For brevity, we call them *s-lines*. Obviously each boundary component of a cs-surface is a complete s-line.

Let $F : U \to M$ be a cs-surface. A *completion* of F is the map $\overline{F} : \overline{U} \to M$, where $\overline{U}$ is the completion of U with respect to the induced intrinsic metric and $\overline{F}$ is the natural extension of F.

Lemma 4.3. *The completion $\overline{F} : \overline{U} \to M$ of a cs-surface $F : U \to M$ is also a cs-surface. Furthermore, the set $\overline{U} \setminus U$ is contained in the boundary of $\overline{U}$ and consists of entire boundary components.*

Proof. Let $a \in \overline{U}$ and let V be a small neighborhood of a in $\overline{U}$ (more precisely, let V be the ball of radius 10 centered at a with respect to the completed metric). Observe that the set $V \cap U$ is connected (this follows from the fact the metric on U is a length metric). Consider the surface $F_1 = F|_{V \cap U}$. Introduce regular coordinates (x, y, z) in a neighborhood of $p = \overline{F}(a)$ so that p corresponds to the origin of $\mathbb{R}^3$. In these coordinates, F_1 projects injectively to the xy-plane, moreover this projection is C^1-close to F_1. Hence the image of F_1 is the graph $z = h(x, y)$ of a function $h : D \to \mathbb{R}$ (which is C^1-close to zero) where D is a region in the xy-plane.

Since the correspondence between $V \cap U$ and D is bi-Lipschitz, it takes completion to closure. More precisely, within a smaller neighborhood (say, of radius 1) of p, the image of $\overline{F}(V)$ coincides with the graph of $\overline{h} : \overline{D} \to \mathbb{R}$, where $\overline{D} \subset \mathbb{R}^2$ is the closure of D in a neighborhood of the origin, and $\overline{h}$ is the natural extension of h.

Recall that U consists of complete s-lines. Hence D is foliated by the projections (to the xy-plane) of intersections of s-lines with V. These projections are C^1 curves whose tangents are almost parallel to the y-axis and depend continuously on a point of D, and whose endpoints are uniformly separated from p (by distance at least 5). This, together with the fact that D is connected, leaves the following three possibilities for the structure of D near p:

(1) D contains a neighborhood of p in the xy-plane;

(2) the intersection of D with a sufficiently small neighborhood of p is a half-neighborhood bounded by one of the above mentioned curves;

(3) such an intersection is an open half-neighborhood bounded by a limit of such curves.

In the first two cases, the completion procedure does not affect a neighborhood of a. In the third case, the completion adds the limit curve to D. The resulting domain $\overline{D}$ is a C^1 manifold with boundary. In the graph of $\overline{h}$, the added curve corresponds to an interval of the s-line passing through p.

Thus $\overline{U}$ has a natural structure of a C^1 manifold with boundary, and it is obtained from U by adding a number of boundary components. It follows that $\overline{U}$ is simply connected. Obviously the extended map $\overline{F}$ remains a C^1 immersion, and the added boundary components are mapped to (complete) s-lines. Hence $\overline{F}$ is a cs-surface. $\square$

We refer to the boundary components of the completion of a cs-surface F as *edges* of F. An edge is said to be *proper* if it is contained in the surface. The orientation of E^c defines a co-orientation of edges: an edge has *forward co-orientation* if E^c points inside the surface and *backward co-orientation* otherwise.

Let $F : U \to M$ be a cs-surface. Every non-boundary s-line $\ell \subset U$ divides U into two components, which are further referred to as (open) *half-surfaces.* A *closed half-surface* is a union of ℓ and one of the two open half-surfaces that it bounds. Since the distribution E^c is oriented and transverse to s-lines, we can identify the *forward* half-surface as the one which E^c points into. For a point $x \in U$, we use the term "*the forward half-surface from* x" for the forward half-surface with respect to the s-line passing through x. We use the same term "half-surface" for subsets of U and for restrictions of F to these subsets.

Definition 4.4. A *marked surface* is a pair (F, a) where F is a surface and $a \in dom(F)$. The point a is referred to as the *marked point.* We will also say that (F, a) is a *marking of* F. All terms and notation introduced for surfaces are applicable to marked surfaces; in particular, we write $dom(F, a) = dom(F)$.

A *forward surface* is a marked cs-surface whose marked point belongs to an edge of forward co-orientation. Similarly, in a *backward surface* its marked point belongs to an edge of backward co-orientation. If the marked point belongs to the interior of the domain, we say that this is a *passing* surface.

If $A = (F, a)$ is a passing surface, then the *forward half-surface of* A is the closed forward half-surface of F from a, regarded as a marked cs-surface with the same marked point a.

Definition 4.5. Let $\mathcal{A}$ be a collection of cs-surfaces and $p \in M$. We denote by $\mathcal{A}_p$ the set of all markings (F, a) of surfaces F from $\mathcal{A}$ such that $F(a) = p$. By $\mathcal{A}_p^+$, $\mathcal{A}_p^-$ and $\mathcal{A}_p^0$ we denote the sets of forward, backward and passing surfaces from $\mathcal{A}_p$, respectively.

There is a binary relation on $\mathcal{A}_p$, which we call *the geometric order*, defined as follows.

Definition 4.6. Introduce local coordinates (x, y, z) in a neighborhood of p so that the vector field $\frac{\partial}{\partial z}$ is transverse to E^{cs} and the triple $(E^c, E^s, \frac{\partial}{\partial z})$ is positively oriented. Then every marked cs-surface $A \in \mathcal{A}_p$ locally coincides with a graph $z = h_A(x, y)$ where $h_A : D_A \to \mathbb{R}$ is a C^1 function and $D_A \subset \mathbb{R}^2$ is a neighborhood or a half-neighborhood of the origin (a half-neighborhood is bounded by the projection of an s-line to the xy-plane). We say that A is *locally above* B if there is a neighborhood $U \subset \mathbb{R}^2$ of the origin such that $h_A \geq h_B$ in $U \cap D_A \cap D_B$.

Note that if A and B coincide in a neighborhood of p, then A is locally above B and vice versa. The same is true if A is a forward surface and B is a backward surface. The geometric order is transitive on each set $\mathcal{A}_p^0$, $\mathcal{A}_p^+$ and $\mathcal{A}_p^-$ but, in general, is not transitive on $\mathcal{A}_p$.

We say that A is *strictly locally above* B if A is locally above B and B is not locally above A. Equivalently, A is strictly locally above B if $h_A \geq h_B$ in a neighborhood of the origin and $h_A(x_i, y_i) > h_B(x_i, y_i)$ for a sequence $\{(x_i, y_i)\}$ converging to the origin.

We denote by $\mathcal{A}_*$ the set of all markings of all surfaces from $\mathcal{A}$. Equivalently, $\mathcal{A}_* = \bigcup_{p \in M} \mathcal{A}_p$. One can identify $\mathcal{A}_*$ with a disjoint union of the domains of surfaces from $\mathcal{A}$. Then $\mathcal{A}_*$ carries a natural topology and a differential structure. There is a natural "projection" $\pi : \mathcal{A}_* \to M$ given by $\pi(F, a) = F(a)$.

Definition 4.7. By a *patch* we mean a cs-surface whose edges are separated from each other by intrinsic distance at least 1. (The constant 1 here is 1/100 of the regularity radius of the distributions, recall the standing convention made in the previous section.)

Definition 4.8. A *pre-foliation* $\mathcal{A}$ is a collection of patches equipped with a partial order "$>$" on $\mathcal{A}_*$ such that the following axioms are satisfied:

0. $A \in \mathcal{A}_p$ and $B \in \mathcal{A}_q$ are comparable if and only if $p = q$.

1. The order agrees with the geometric order in the following sense: if $A > B$, then A is locally above B.

2. The order does not change if one moves p along the intersection of the surfaces, more precisely, the following holds. Let $\gamma : [0, 1] \to M$ be a C^1 curve tangent to E^{cs} which is either tangent to E^s or transverse to E^s, and let γ_1, γ_2 be lifts of γ to surfaces $F, G \in \mathcal{A}$. Then $(F, \gamma_1(0)) > (G, \gamma_2(0))$ if and only if $(F, \gamma_1(1)) > (F, \gamma_2(1))$.

3. For every C^1-diffeomorphism $f : M \to M$ preserving the oriented distributions E^s and E^{cs}, the structure is f-invariant in the following sense: there is an order-preserving diffeomorphism $f_* : \mathcal{A}_* \to \mathcal{A}_*$ sending $\mathcal{A}_p$ to $\mathcal{A}_{f(p)}$ for every $p \in M$. The map f_* is referred to as an action of f on $\mathcal{A}$.

Axiom 0 implies that for every $p \in M$ the restriction of the order to $\mathcal{A}_p$ is a total order. We denote this restriction by $>_p$.

Axioms 1 and 2 imply that there are no topological crossings between patches from $\mathcal{A}$. Indeed, suppose that surfaces $F, G \in \mathcal{A}$ topologically cross. Then there exists a continuous curve $\gamma : [0, 1] \to M$ admitting lifts γ_1 to F and γ_2 to G such that $(F, \gamma_1(0))$ is locally strictly above $(G, \gamma_2(0)$ but $(F, \gamma_1(1))$ is locally strictly below $(G, \gamma_2(1))$. Then $(F, \gamma_1(0)) > (G, \gamma_2(0)$ and $(F, \gamma_1(1)) < (G, \gamma_2(1))$ by Axiom 1. One can approximate γ_1 by a piecewise C^1 curve $\tilde{\gamma}_1$ with the same endpoints in $dom(F)$, intersecting the same set of s-lines and consisting of C^1 segments each of which is either tangent to E^s or transverse to E^s. Since the surfaces consist of whole s-lines and E^s is uniquely integrable, the curve $F \circ \tilde{\gamma}_1$ admits a lift $\tilde{\gamma}_2$ to G with the same endpoints as γ_2. Applying Axiom 2 to the segments of these curves yields that the order between $(F, \gamma_1(0))$ and $(G, \gamma_2(0)$ is the same as the order between $(F, \gamma_1(1))$ and $(G, \gamma_2(1))$, a contradiction.

Definition 4.9. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-foliations. We say that $\mathcal{B}$ is an *extension* of $\mathcal{A}$ if $\mathcal{B}$ is obtained from $\mathcal{A}$ by enlarging some patches and/or adding some new ones. More precisely, $\mathcal{B}$ extends $\mathcal{A}$ if there is a map $i : \mathcal{A}_* \to \mathcal{B}_*$ which preserves the orders, sends $\mathcal{A}_p$ to $\mathcal{B}_p$ for all

$p \in M$, and commutes with actions of diffeomorphisms (cf. Axiom 3 of Definition 4.8). Such a map i is referred to as an *inclusion map*.

We say that $\mathcal{B}$ *contains* $\mathcal{A}$ if $\mathcal{B}$ is obtained from $\mathcal{A}$ by adding new patches, more precisely, if $\mathcal{A}_* \subset \mathcal{B}_*$ and the natural map $i : \mathcal{A}_* \to \mathcal{B}_*$ satisfies the above requirements.

Our plan is to begin with the empty pre-foliation, and then to extend it step-by-step using the propositions below. (For example, the first step is to build a pre-foliation covering M using the first assertion of Proposition 4.13). Then a direct limit of these step-by-step extensions yields a pre-foliation consisting of complete open surfaces.

The first construction allows us to make larger patches by pasting together adjacent ones.

Definition 4.10. Let $\mathcal{A}$ be a pre-foliation and $p \in M$. We say that $B \in \mathcal{A}_p^+$ is the *upper neighbor* of $A \in \mathcal{A}_p^-$ if $B >_p A$ and for every $C \in \mathcal{A}_p \setminus \{A, B\}$ one has $A >_p C$ iff $B >_p C$.

Of course, a marked patch may have no upper neighbor. The next proposition allows us to extend a pre-foliation by pasting backward patches to copies of their upper neigbors if the latter existed.

Proposition 4.11. *For every pre-foliation $\mathcal{A}$ there is an extension $\mathcal{B}$ of $\mathcal{A}$ (with an inclusion map $i : \mathcal{A}_* \to \mathcal{B}_*$) such that the following holds:*

For every $p \in M$, $A \in \mathcal{A}_p^-$ and $B \in \mathcal{A}_p^+$, if B is the upper neighbor of A and B has only one proper edge, then the marked patch $i(A) \in \mathcal{B}_p$ is a passing one and its forward half-surface is (a copy of) B.

The second proposition allows us to include edges in patches.

Proposition 4.12. *For every pre-foliation $\mathcal{A}$ there exists an extension $\mathcal{B}$ of $\mathcal{A}$ such that all patches from $\mathcal{B}$ are complete (that is, have no non-proper edges).*

The most difficult construction is the one that allows us to add a new patch neighboring a given one (or just creating a patch if there is nothing to extend). We do this simultaneously and in a canonical way at all points $p \in M$ and for all patches "ending at p".

Proposition 4.13. *Let $\mathcal{A}$ be a pre-foliation. Then there is a pre-foliation $\mathcal{B}$ containing $\mathcal{A}$ and such that:*

1. For every $p \in M$, there is a forward patch $B \in \mathcal{B}_p^+$.

2. For every $p \in M$ and $A \in \mathcal{A}_p^-$, there is a forward patch $B \in \mathcal{B}_p^+$ which is the upper neighbor of A in $\mathcal{B}$ and has only one proper edge.

By reversing the orientation of E^c one shows that Propositions 4.11 and 4.13 hold for "backward" extensions as well (that is, with "+" and "−" interchanged).

The proofs of the propositions will follow, but first let us derive a proof of the theorem from the propositions.

Proof of Theorem 4.1. We construct a sequence $\{\mathcal{A}^k\}$ of pre-foliations as follows. First apply Proposition 4.13 to $\mathcal{A} = \varnothing$ and denote the resulting pre-foliation $\mathcal{B}$ by $\mathcal{A}^1$. For $k = 2, 3, \dots$, a pre-foliation $\mathcal{A}^k$ is an extension of $\mathcal{A}^{k-1}$ obtained by applying Propositions 4.12, 4.13 and 4.11 in a row. If k is odd, we apply these propositions as stated. If k is even, we apply the propositions with "+" and "−" interchanged (that is, we alternate forward and backward extensions).

This yields a sequence $\{\mathcal{A}^k\}$ of pre-foliations along with inclusion maps $i : \mathcal{A}^k \to \mathcal{A}^{k+1}$. Let $\mathcal{A}$ be the direct limit of these pre-foliations. Fix a $p \in M$. We are going to prove that $\mathcal{A}$ contains a complete open surface passing through p.

By construction, the first pre-foliation $\mathcal{A}^1$ contains a marked patch $A^1 \in \mathcal{A}^{1+}_p$. Let A^k be the corresponding marked patch in $\mathcal{A}^k$, that is $A^{k+1} = i(A^k)$ where $i : \mathcal{A}^k_* \to \mathcal{A}^{k+1}_*$ is the inclusion map.

The construction of $\mathcal{A}^2$ from $\mathcal{A}^1$ involves three steps. First, Proposition 4.12 adds all edges to the patches from $\mathcal{A}^1$. Let us keep the same notation $\mathcal{A}^1$ for the resulting pre-foliation and A^1 for the corresponding marked patch in it. Second, Proposition 4.13 adds new patches so that the resulting pre-foliation $\mathcal{B}$ contains a marked patch $B \in \mathcal{B}^-_p$ which is the upper neighbor of A^1 with only one proper edge. Finally, Proposition 4.11 pastes B (and upper neighbors of other markings of A^1) to A^1; the resulting surface is A^2. Since the distances between edges of a patch is bounded below by 1, the intrinsic distance from the marked point of A^2 to the union of edges of A^2 is at least 1.

A similar argument and induction in k yield that for every $k \geq 1$, the intrinsic distance from the marked point of A^{2k} to the union of edges of A^{2k} is at least k. Indeed, the construction of $\mathcal{A}^{2k+1}$ from $\mathcal{A}^{2k}$ extends the patch by a distance at least 1 beyond every edge of backward co-orientation, and the construction of $\mathcal{A}^{2k+2}$ does the same for edges of forward co-orientation. Therefore the resulting marked patch in $\mathcal{A}$ is a complete open surface passing through p.

Since p is arbitrary, it follows that complete open surfaces from $\mathcal{A}$ cover the manifold. Then the desired branching foliation can be defined as the set of all complete open surfaces from $\mathcal{A}$. $\square$

Now we proceed with proofs of propositions.

Proof of Proposition 4.11. Consider a doubling $\mathcal{A} \cup \mathcal{A}'$ of $\mathcal{A}$ where $\mathcal{A}'$ is a disjoint copy of $\mathcal{A}$. We refer to patches from $\mathcal{A}'$ as *secondary copies* of corresponding patches from $\mathcal{A}$, and the patches from $\mathcal{A}$ are referred to as *primary copies*.

We define an order on $\mathcal{A} \cup \mathcal{A}'$ as follows. If marked patches $A, B \in (\mathcal{A} \cup \mathcal{A}')_*$ are not copies of the same marked patch from $\mathcal{A}$, the order is inherited from $\mathcal{A}$. If A is a primary copy and B is a secondary copy of the same marked patch, we set $A > B$. Clearly the doubling with this order is a pre-foliation containing $\mathcal{A}$.

We construct the desired pre-foliation $\mathcal{B}$ as a quotient of $\mathcal{A} \cup \mathcal{A}'$. Here by taking a quotient we mean pasting together some pairs of neighboring patches from $\mathcal{A} \cup \mathcal{A}'$. Formally, the quotient is defined as follows. Consider all patches $F \in \mathcal{A}$ satisfying the following conditions:

(a) F has exactly one proper edge ℓ, and this edge has forward co-orientation;

(b) for a point $x \in \ell$, the marked patch (F, x) is the upper neighbor of some marked patch $(G, y) \in \mathcal{A}^-_{F(x)}$.

For every such F, we attach the secondary copy of F to the primary copy of the corresponding patch G by identifying their respective edges to obtain a passing patch. Observe that every edge is involved in at most one identification (since a marked patch may have no more than one upper neighbor), hence the construction yields a collection of surfaces. We denote this collection by $\mathcal{B}$. A primary copy of a patch may have many secondary copies attached to it, but each secondary copy is attached to at most one primary one (since the former has only one proper edge). It follows that the surfaces from $\mathcal{B}$ have simply connected domains, hence they are patches.

We define an order on $\mathcal{B}_*$ so that the quotient map $(\mathcal{A} \cup \mathcal{A}')_* \to \mathcal{B}_*$ (sending every patch to itself or to a patch resulted from a gluing involving this patch) is order-preserving. The correctness of this definition follows from the fact that the merged pairs of surfaces were neighbors with respect to the order on $(\mathcal{A} \cup \mathcal{A}')_*$. It is easy to check that $\mathcal{B}$ with this order is a pre-foliation extending $\mathcal{A}$. $\square$

Proof of Proposition 4.12. To simplify the construction, we perform the completion of patches in two steps: first we add all non-proper edges with forward co-orientation and define the order on the resulting set of marked patches, then we use the same construction to add the edges with backward co-orientation.

Let $\mathcal{B}$ be the set of patches obtained by adding all non-proper edges of forward co-orientation to the patches from $\mathcal{A}$. Let $i : \mathcal{A}_* \to \mathcal{B}_*$ be the natural inclusion map. Let us define the order on $\mathcal{B}_*$. Let $A, B \in \mathcal{B}_p$, $p \in M$, $A \neq B$. If $A, B \in i(\mathcal{A}_p)$, that is, the marked points of A and B do not belong to added edges, we derive the order from that of $\mathcal{A}$.

Namely A and B compare the same way as their ansestors in $\mathcal{A}$ did. This defines a total order on the subset $i(\mathcal{A}_p)$ of $\mathcal{B}_p$.

If the marked point of A belongs to an added edge and B is a forward or passing patch, consider two cases.

Case 1. One of A and B is locally strictly above the other. Then we derive the order between A and B from the geometric order.

Case 2. A coincides with B in a forward-half neighborhood of the marked point. Let $A = (F, x)$ and $B = (G, y)$. Choose interior points x' near x in $dom(F)$ and y' near y in $dom(G)$ such that $F(x') = G(y')$ and define $A > B$ iff $(F, x') > (G, y')$. Note that the order between (F, x') and (G, y') is already defined since x' and y' are interior points.

It is easy to check that the order defined so far is a union of a total order on $i(\mathcal{A}_p)$ and a total order on the set $\mathcal{B}_p^0 \cup \mathcal{B}_p^+$. These two orders agree on the intersection $i(\mathcal{A}_p) \cap (\mathcal{B}_p^0 \cup \mathcal{B}_p^+) = i(\mathcal{A}_p^0 \cup \mathcal{A}_p^+)$. We merge these two orders using a pure set-theoretic construction. Namely, if $A \in \mathcal{B}_p \setminus i(\mathcal{A}_p)$ and $B \in \mathcal{B}_p^-$, define $A > B$ if and only if there exists a $C \in i(\mathcal{A}_p^0 \cup \mathcal{A}_p^+)$ such that $A > C$ and $C > B$ (with respect to the already defined order), and set $B > A$ othewise.

Thus we have defined a total order on every set $\mathcal{B}_p$, $p \in M$. Verifying the axioms from Definition 4.8 is straightforward. $\square$

A proof of Proposition 4.13 occupies sections 5 and 6.

5. A two-dimensional problem

We begin with a discussion of a two-dimensional analog of the Key Lemma. The purpose of this section is twofold. First, it is supposed to motivate the reader and facilitate understanding the sequel. Since the whole solution of this two-dimensional problem is not used later in the paper, we are not aiming at giving a complete and precise argument and leave many details to the reader. The main objective of this section is however to prove technical lemmas summarized in Subsection 5.2, which are quite important ingredient of the proof of the Key Lemma.

There is essentially no difference between the 2-D and 3-D local problems, that is when we want to construct a pre-foliation in one coordinate neighborhood. It turns out however that in two dimensions there is essentially no difference between local and global versions, whereas in 3-D passing from local to global requires quite a bit of work. Therefore we want to prepare local tools first, and we begin with the following model two-dimensional local problem.

Let E be a continuous (not uniquely integrable) 1-dimensional distribution in the standard xy-plane $\mathbb{R}^2$. Since we consider a model local problem we assume that the angle between E and the y-lines is bounded away from zero. Denote $k(x, y) = \frac{v_2(x,y)}{v_1(x,y)}$, where $v = (v_1(x, y), v_2(x, y))$ is a nonzero vector tangent to E at (x, y). Then every complete integral curve of E is the graph $y = f(x)$ of a C^1 function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation $f'(x) = k(x, f(x))$. As the simplest model statement, we want to cover $\mathbb{R}^2$ by integral curves that have no topological crossings. In this context, the graphs of functions f and g have a topological crossing if $f(x_1) < g(x_1)$ and $f(x_2) > g(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

Fix a point $p = (x_0, y_0) \in \mathbb{R}^2$. Define the function $f_+ : [x_0, +\infty)$ as the supremum of all C^1 functions $f : [x_0, +\infty)$ such that $f(x_0) = y_0$ and $f'(x) < k(x, f(x))$ for all $x > x_0$. One can easily show that f_+ is a solution of the equation, moreover, this is the pointwise-minimum solution satisfying $f(x_0) = y_0$. We refer to f_+ as the *lowest forward integral curve* from p.

In the next subsection we will need the following trivial lemma.

Lemma 5.1. *Let $\{y_n\}$ be a non-decreasing sequence in $\mathbb{R}$ converging to $y \in \mathbb{R}$. Let f and f_n be the lowest forward integral curves from (x_0, y) and (x_0, y_n), respectively. Then for every $x \geq x_0$, the sequence $\{f_n(x)\}$ is non-decreasing and it converges to $f(x)$.* □

Observe that, if $f : [x_0, +\infty)$ is a lowest forward integral curve and $x > x_0$, then $f|_{[x,+\infty)}$ is the lowest forward integral curve from $(x, f(x))$. In particular, $f|_{[x,+\infty)}$ is uniquely determined by x and $f(x)$. It follows that lowest forward integral curves have no topological crossings. Indeed, if f_1 and f_2 are lowest forward integral curves and $f_1(x_1) < f_2(x_1)$, $f_1(x_2) > f_2(x_2)$ for some $x_1, x_2 \in dom(f_1) \cap dom(f_2)$, $x_1 < x_2$, then by continuity there is an $x \in [x_1, x_2]$ such that $f_1(x) = f_2(x)$, then $f_1|_{[x,+\infty)} \equiv f_2|_{[x,+\infty)}$, hence $f_1(x_2) = f_2(x_2)$, a contradiction.

Similarly, one can define $f_- : (-\infty, x_0]$ as the infimum of all functions $f : (-\infty, x_0] \to \mathbb{R}$ such that $f(x_0) = y_0$ and $f'(x) < k(x, f(x))$ for all $x < x_0$. (Then f_- is the maximum backward solution.) We say that the union of f_+ and f_- is a *canonical integral curve* through p. An interested reader can check that canonical integral curves have no topological crossings and hence form a pre-foliation of the plane by complete integral curves. We do not use this statement and leave the details to the reader.

Furthermore, every two-dimensional oriented closed manifold with a non-vanishing vector field admits a pre-foliation by integral curves of the vector field. Indeed, the manifold can be covered by a finite collection of "small" coordinate neighborhoods such that in each of them the vector field is almost parallel to the x-axis and the notion of lying above or below agrees in all coordinate neighborhoods. Then one can run the construction described above.

5.1. **Extending a pre-foliation of a plane.** Now we consider a more delicate two-dimensional problem. Suppose that there is a prescribed pre-foliation A of the plane. A pre-foliation of the plane is a family A of solutions of our equation, each defined on an interval of $\mathbb{R}$, equipped with the following additional structure (of orders). For $p = (x, y) \in \mathbb{R}^2$ denote by A_p the set $\{f \in A : x \in dom(f), f(x) = y\}$. For every $p \in \mathbb{R}^2$ there is a total order $\leq_p$ on A_p such that

1. If $f \leq_p g$, then $f(t) \leq g(t)$ for all $t \in dom(f) \cap dom(g)$.
2. If $f, g \in A$ and $f|_I \equiv g|_I$ for some interval $I \subset \mathbb{R}$, then $f \leq_{(x,f(x))} g$ iff $f \leq_{(x',f(x'))}$ for all $x, x' \in I$.

It is easy to see that these requirements imply the following: if $f \leq_p g$ for some p, then $f \leq_q g$ for every common point q of the graphs of f and g. The existence of such a family of orders implies that the elements of A have no topological crossings.

Given a pre-foliation A and a point $p = (x_0, y_0)$, we want to construct a forward solution $f : [x_0, +\infty) \to \mathbb{R}$ with $f(x_0) = y_0$ so that f can be included in the pre-foliation A. Furthermore, assume that the set A_p is split into subsets Σ and $A_p \setminus \Sigma$ so that $\Sigma <_p A_p \setminus \Sigma$. We want f to separate Σ from $A_p \setminus \Sigma$ in the extended pre-foliation.

If the functions from A were defined on the whole $\mathbb{R}$, one could just take the supremum of all functions from Σ. If A is empty, we could use the lowest forward curve as in the previous subsection. With functions defined on intervals (whose lengths are not necessarily bounded away from zero) the construction is more complicated. Loosely speaking, the new function tries to follow the uppermost curve from Σ until it ends; then it switches to the uppermost of the lower curves if any is available at this point; if none is available, it continues as the lowest forward curve until it again meets a curve from A that it could follow and so on. Whereas this description can be easily formalized if A is finite, to handle the general case we construct a desired function as a supremum of *descending curves*, defined as follows.

Definition 5.2. A *descending curve* is a C^1 function $f : [x_0, +\infty) \to \mathbb{R}$ satisfying our equation and equipped with the following structure. There is a partition $x_0 \leq x_1 \leq \cdots \leq x_n$ of $[x_0, +\infty)$ into segments $I_k = [x_k, x_{k+1}]$, $k = 0, 1, \dots, n-1$, and a ray $I_n = [x_n, +\infty)$ and a sequence $f_0, f_1, \dots, f_n \in A \cup \{\varnothing\}$. We refer to the intervals I_k as *f-segments* and to f_k as the *f-label* of I_k. If $f_k = \varnothing$, the corresponding f-segment I_k is said to be *unlabeled*. Furthermore, we require that the following conditions are satisfied:

1. If $f_k \neq \varnothing$, then $I_k \subset dom(f_k)$ and $f|_{I_k} = f_k|_{I_k}$.

2. If I_k is unlabeled, then I_k has nonzero length and $f|_{I_k}$ is (a restriction of) the lowest forward integral curve from $(x_k, f(x_k))$. The ray I_n is unlabeled.

3. For $k \geq 1$, if a segment I_{k-1} is unlabeled and I_k is labelled by a function $f_k \in A$, then the set $I = [x_{k-1}, x_k) \cap dom(f_k)$ is nonempty and $f(x) > f_k(x)$ for all $x \in I$.

4. For $k \geq 1$, if I_{k-1} and I_k are labelled by functions $f_{k-1}, f_k \in A$, then $f_{k-1} \geq_{x_k} f_k$.

We say that a point $x \in [x_0, +\infty)$ is *f-labelled* by $g \in A \cup \{\varnothing\}$ if g is an f-label of a segment I_k containing x. (A point $x \in \{x_1, x_2, \ldots, x_n\}$ may have more than one f-label.) We say that a point is *f-unlabeled* if it is labelled only by $\varnothing$.

Definition 5.3. Let $(X, <)$ be a totally ordered set. We say that a subset $\Sigma \subset X$ is a *section* of $(X, <)$ if for all $g, h \in \Sigma$, if $g \in \Sigma$ and $h \leq_p g$, then $h \in \Sigma$.

Let $p = (x_0, y_0) \in \mathbb{R}^2$, and let $\Sigma \subset A_p$ be a section of the ordered set $(A_p, <_p)$.

Definition 5.4. We say that a descending curve $f : [x_0, +\infty)$ *starts from* (p, Σ) if the following additional conditions are satisfied:

5. $f(x_0) \leq y_0$.

6. If $f(x_0) = y_0$ and the first segment I_0 is labelled by a function $f_0 \in A$, then $f_0 \in \Sigma$.

The set of descending curves starting from (p, Σ) is nonempty since it contains the lowest forward integral curve from p. We define f_+ to be the supremum of all descending curves starting from (p, Σ) and refer to f_+ as the *upper envelope* of (p, Σ). Obviously $f_+(x_0) = y_0$. Since all descending curves are integral curves of v, so is f_+. As a matter of fact, f_+ actually solves the problem formulated at the beginning of this subsection; however, to make this solution more useful in the 3-D case, we equip the upper envelope f_+ with the following additional structure. For every point $q = (x, f_+(x))$ in the graph of f_+, define the set $\Sigma(x) \subset A_q$ as follows: $g \in \Sigma(x)$ iff there is a descending curve h starting from (p, Σ) and such that $h(x) = f_+(x)$ and x is h-labelled by g. We refer to the set $\Sigma(x)$ as the *shadow* of (p, Σ) at x.

Observe that $\Sigma(x_0) = \Sigma$. Indeed, for every $g \in \Sigma$ take h to be the lowest integral curve from p partitioned into a zero-length segment $[x_0, x_0]$ and a ray $[x_0, +\infty)$ labelled by g and $\varnothing$ respectively.

Furthermore, for every $x > x_0$, the set $\Sigma(x)$ is a section of the ordered set $(A_q, \leq_q)$, that is, if $g \in \Sigma(x)$, $f \in A_q$ and $f \leq_q g$, then $f \in \Sigma(x)$. To prove this, choose a descending curve h such that x is labelled by g, add a zero-length segment $[x,x]$ labelled by f into the partition, and replace $h|_{[x,+\infty)}$ by the lowest forward solution. This yields a descending curve for which x is labelled by f, hence $f \in \Sigma(x)$.

Lemma 5.5. *For every $x \in [x_0,+\infty)$, the restriction $f_+|_{[x,+\infty)}$ of f_+ to $[x,+\infty)$ coincides with the upper envelope of $(q,\Sigma(x))$ where $q = (x, f_+(x))$.*

Moreover for every $t > x$, the shadow of $(q,\Sigma(x))$ at t coincides with $\Sigma(t)$.

Proof. The case $x = x_0$ is trivial (recall that $\Sigma(x_0) = \Sigma$), so we assume that $x > x_0$. Let g_+ be the upper envelope of $(q,\Sigma(x))$ and $\Sigma_1(t)$ the shadow of $(q,\Sigma(x))$ at $t \geq x$.

Fix $t \geq x$. We need to prove that $g_+(t) = f_+(t)$ and $\Sigma_1(t) = \Sigma(t)$. This is trivial if $t = x$, so we assume that $t > x$. The proof is divided into two steps.

Step 1. Show that $g_+(t) \geq f_+(t)$, and if $g_+(t) = f_+(t)$, then $\Sigma(t) \subset \Sigma_1(t)$.

Indeed, for a descending curve f starting from (p,Σ), consider the restriction $g = f|_{[x,+\infty)}$ equipped with partitioning and labelling restricted from f. The "restriction of partitioning and labelling" is defined as follows: $[x,+\infty)$ is partitioned into all nonempty intersections $I_k \cap [x,+\infty)$, where $\{I_k\}$ are f-segments, labelled by the respective f-labels of I_k, with one exception: if an unlabeled segment $[x,x]$ appears as an intersection, it should be removed. It is easy to see that g is a descending curve starting from $(q,\Sigma(x))$.

Thus for every descending curve f from (p,Σ) there exists a descending curve g from $(q,\Sigma(x))$ with the same set of labels at t. The desired statement follows.

Step 2. Show that $f_+(t) \geq g_+(t)$, and if $f_+(t) = g_+(t)$, then $\Sigma_1(t) \subset \Sigma(t)$.

It suffices to prove the following statement: for every descending curve g starting from $(q,\Sigma(x))$ and every $\varepsilon > 0$ there exists a descending curve f starting from (p,Σ) such that at least one of the following holds:

(A) $f(t) > g(t)$;

(B) $f(t) = g(t)$ and every g-label at t is also an f-label at t;

(C) t is g-unlabelled and $f(t) > g(t) - \varepsilon$.

Let g_0 be the g-label of the first g-segment. Consider 3 cases.

Case 1: $g(x) < f_+(x)$. Choose a descending curve f starting from (p,Σ) such that $f(x) > g(x)$. Then change f on $[x,+\infty)$ so that $f|_{[x,+\infty)}$ is the (unlabelled) lowest forward integral curves from $(x, f(x))$. If $f > g$ on $(x,+\infty)$, we are done. Otherwise let x' be the first point in $(x,+\infty)$ such that $f(x') = g(x')$. Change f after this point to make it follow g on $[x',+\infty)$. The partitioning and labelling of the new f on $[x',+\infty)$ is defined as the restriction of those of g. Then the new f is a descending curve satisfying Condition (A) if $t < x'$ and (B) if $t \geq x'$.

Case 2: $g(x) = f_+(x)$ and $g_0 \neq \varnothing$. Then $g_0 \in \Sigma(x)$. This means that there exists a descending curve f starting from (p,Σ) such that x is f-labelled by g_0. Change f on $[x,+\infty)$ to make it coincide with g and equip it with the same partitioning and labelling on $[x,+\infty)$. The resulting function satisfies Condition (B) above.

Case 3: $g(x) = f_+(x)$ and $g_0 = \varnothing$. Then g begins with a lowest forward segment $[x,x']$, possibly followed by a segment $[x',x'']$ labelled by a function $g_1 \in A$. Furthermore $dom(g_1)$ contains an interval $[x'-\delta,x']$ and $g > g_1$ on this interval. We may assume that $\varepsilon < g(x'-\delta) - g_1(x'-\delta)$. Then a desired f can be constructed as follows. Choose a descending curve f starting from (p,Σ) so that $f(x)$ is sufficiently close to $f_+(x) = g(x)$. Then change f on $[x,+\infty)$ so that it continues as the lowest forward curve from $(x,f(x))$ until it hits the graph of g_1, then it coincides with (and is labelled by) a segment of g_1 until x', and afterwards it coincides with g (and is partitioned and labelled the same way as g). Lemma 5.1 guarantees that the construction works, namely if $f(x)$ is sufficiently close to $f_+(x)$, then the lowest forward integral curve from $(x,f(x))$ is ε-close to g (and bounded above by g) on $[x,x']$. Hence $f(x'-\delta) > g_1(x'-\delta)$ and therefore f indeed hits the graph of g_1.

The resulting curve f satisfies (C) if $t < x'$ and (B) if $t \geq x'$. ☐

Now we are going to prove that f_+ can be included in the prefoliation, that is, it fits in the orders. For a point $q = (x, f_+(x))$, define a total order $\tilde{>}_q$ on $A_q \cup \{f_+\}$ extending $>_q$ as follows: for $g \in A_q$, set $f_+ \tilde{>}_q g$ if $g \in \Sigma(x)$ and $g \tilde{>}_q f_+$ otherwise. This extension is indeed a total order since $\Sigma(x)$ is a section of the ordered set $(A_q, >_q)$.

For $g,h \in A_q \cup \{f_+\}$ we say that g *majorizes* h at x if one of the two conditions holds:

1. $g(x) > h(x)$, or
2. $g(x) = h(x)$ and $g \tilde{>}_{(x,g(x))} h$.

Lemma 5.6. *The family of total orders $\{\tilde{>}_q\}_{q\in\mathbb{R}^2}$ defined above satisfies the compatibility condition, namely if $g,h \in A_q \cup \{f_+\}$ and h majorizes g at some point x, then the same holds for all x' in the intersection of the domains of g and h.*

Proof. Since the original orders $>_q$ are compatible, it suffices to verify the assertion only in the case when $g = f_+$ or $h = f_+$.

Suppose that $x < x'$ (otherwise interchange x and x'). By Lemma 5.5 we may assume that x is the base point x_0 of f_+ (otherwise consider the restriction $f_+|_{[x,+\infty)}$ instead of f_+).

Case 1: $h = f_+$. In this case, the assumption about g and h means that $g(x) < y_0$ or $g \in \Sigma$. Consider a descending curve f defined as follows: f coincides with g and is labelled by g on $[x_0, x']$, then it continues as the (unlabeled) lowest forward integral curve from $(x', g(x'))$. Then f is a descending curve from (p, Σ). Hence $f_+(x') \geq f(x') = g(x')$ and $g \in \Sigma(x')$ if $f_+(x) = f(x')$. This means that f_+ majorizes g at x'.

Case 2: $g = f_+$. In this case, the assumption about g and h means that either $h(x_0) > y_0$ or $h \in A_p \setminus \Sigma$. We have to prove that $f_+(x') \leq h(x')$ and $h \notin \Sigma(x')$. This is equivalent to the following: if f is a descending curve starting from (p, Σ), then $f(x') \leq h(x')$ and x' is not f-labelled by h.

Let $x_0 \leq \cdots \leq x_n$ be the partitioning associated to f. Let $I_0, \ldots, I_n$ be the corresponding f-segments and $f_0, \ldots, f_n$ the f-labels.

Claim. *For every $k \leq n$ and $t \in I_k \cap dom(h)$ one has $f(t) \leq h(t)$ and moreover $f_k <_{(t,f(t))} h$ if $f(t) = h(t)$ and $f_k \neq \varnothing$.*

Proof of the Claim. The assertion is trivial for $k = 0$ and $t = 0$. We use induction with the following induction step: if the assertion is true for $t = x_k$, then it is true for all $t \in I_k$ and $(k+1, x_{k+1})$ in place of (k, t). Consider the following two cases.

Case A: $f_k \neq \varnothing$. The compatibility of orders implies that the assertion is true for (k, t) for all $t \in I_k$. Substituting $t = x_{k+1}$ yields the following: if $x_{k+1} \in dom(h)$ and $f(x_{k+1}) = h(x_{k+1})$, then $f_k <_q h$ where $q = (x_{k+1}, f(x_{k+1}))$. Then the 4th requirement of the definition of a descending curve implies that $f_{k+1} \leq_q f_k <_q h$ if $f_{k+1} \neq \varnothing$. This finishes the induction step in this case.

Case B: $f_k = \varnothing$. Then $f|_{I_k}$ is a lowest forward integral curve, hence the inequality $f(x_k) \leq h(x_k)$ implies that $f \leq h$ on $I_k \cap dom(h)$, so the assertion is true for all $t \in I_k$. In particular, $f(x_{k+1}) \leq h(x_{k+1})$ if $x_{k+1} \in dom(h)$. Suppose that $f(x_{k+1}) = h(x_{k+1})$ and $f_{k+1} \neq \varnothing$. Then, by the 3rd requirement of the definition of a descending curve, $f_{k+1} < f \leq h$ on an interval of the form $(x_{k+1} - \delta, x_{k+1})$. Hence $f_{k+1} <_q h$ where $q = (x_{k+1}, f(x_{k+1}))$. This finishes the induction step. □

Substituting $t = x'$ and an appropriate k into the Claim finishes the proof. □

The next lemma shows that one can define a non-strict order between different upper envelopes so that it agrees with the geometric order.

Let f_+ and g_+ be upper envelopes of (p, Σ_1) and (q, Σ_2) where $p, q \in \mathbb{R}^2$, Σ_1 is a section of A_p, Σ_2 is a section of A_q, $x \in dom(f_+) \cap dom(g_+)$. We say that f_+ *non-strictly majorizes* g_+ at x if $f_+(x) \geq g_+(x)$ and $\Sigma_2(x) \subset \Sigma_1(x)$ in the equality case. We say that f_+ *strictly majorizes* g_+ at x if $f_+(x) \geq g_+(x)$ and $\Sigma_2(x)$ is a proper subset of $\Sigma_1(x)$ in the equality case.

Lemma 5.7. *Let f_+ and g_+ be as above. Then*

1. If f_+ non-strictly majorizes g_+ at x, then f_+ non-strictly majorizes g_+ at every point $x' > x$ in $dom(f_+) \cap dom(g_+)$.

2. If f_+ strictly majorizes g_+ at x, then f_+ non-strictly majorizes g_+ at every point $x \in dom(f_+) \cap dom(g_+)$.

Proof. 1. By Lemma 5.5, we may assume that $p = (x, f_+(x))$, $q = (x, g_+(x))$, $\Sigma_1 = \Sigma_1(x)$, $\Sigma_2 = \Sigma_2(x)$. Then the definitions imply that every descending curve starting from (p, Σ_1) is also a descending curve starting from (q, Σ_2). Now the lemma follows from the definition of the upper envelope applied to f_+ and g_+ and the definition of the shadow applied to $\Sigma_1(x')$ and $\Sigma_2(x')$.

2. The first part of the lemma covers the case $x' > x$, so we assume that $x' < x$. Suppose that the assertion is false, then g_+ strictly majorizes f_+ at x'. Then by the first part, g_+ non-strictly majorizes f_+ at x, a contradiction. $\square$

Lemma 5.7 implies that f_+ and g_+ are topologically non-crossing. An interested reader can construct a function $f_- : (-\infty, x_0] \to \mathbb{R}$ by switching "up" and "down" and reversing the time, and then form a canonical integral curve by taking the union of f_+ and f_-. Then the reader can verify that such canonical curves indeed form a pre-foliation without topological crossings with functions from A. Since we do not rely on this statement in the sequel, we omit the proof.

5.2. **Pre-foliations of 2-manifolds.** Our constructions in the previous subsection were local and used a certain choice of coordinates. It is easy to see however that they produced canonical objects and therefore can be applied to any manifold. Indeed, consider a one-dimensional an oriented one-dimensional C^0 distribution E on a 2-dimensional oriented C^1 manifold N, and a pre-foliation A tangent to E. (We do not assume that N is compact: actually, we only need a case when N is an open disc.)

That is, A is a family of C^1 curves tangent to E equipped with an additional structure of orders as follows. For every $p \in N$, let A_p denote the set of pairs (γ, t) such that γ is a curve from A, $t \in dom(\gamma)$

and $\gamma(t) = p$. The set A_p is equipped with a linear order $>_p$ satisfying 1-dimensional analogues of axioms 1 and 2 of Definition 4.8:

1. If $(\gamma, t) >_p (\gamma', t')$, then γ is locally (non-strictly) above γ' near p. The notion of being "locally above" is defined with respect to any local coordinate system (x, y) such that the coordinate vector field $\frac{\partial}{\partial y}$ is transverse to E and the pair $(E, \frac{\partial}{\partial y})$ is positively oriented.

2. If two curves γ and γ' from A have a common segment, then the order between γ and γ' at every point of the segment is the same.

Let $p \in N$ and Σ be a section of the ordered set $A_p = (A_p, >_p)$, that is, $a >_p b$ for all $a \in A_p \setminus \Sigma$ and $b \in \Sigma$. Introducing suitable local coordinates near p, one constructs a (local) upper enveloping curve $\gamma_+ : [0, 1) \to N$ as in section 5.1. This curve is equipped with an additional structure of "shadows", namely for every $t \in [0, 1)$ there is a section $\Sigma(t)$ of the ordered set $A_{\gamma(t)}$. By Lemma 5.5, the same construction applied to a point $\gamma(t)$ and a section $\Sigma(t)$, yields the same curve (more precisely, its restriction to $[t, 1)$).

It is easy to see that a local upper enveloping curve does not depend on the choice of coordinates, more precisely, for every two such curves one is an initial segment of the other (and the associated sets $\Sigma(t)$ agree). Now if $\gamma(t)$ has an accumulation point q as $t \to 1$, we can extend γ beyond 1 applying the same construction to local coordinates near q, a point $p' = \gamma(1 - \varepsilon)$ and a section $\Sigma' = \Sigma(1 - \varepsilon)$ for a small enough ε. This argument shows that there is a maximum upper envelope that leaves every compact set as t approaches 1.

It is clear that the assertions of Lemmas 5.5, 5.6, 5.7 hold for upper envelopes constructed on manifolds. In the sequel we use the terms "upper envelope" and "shadows" and Lemmas 5.5, 5.6, 5.7 in this set-up as well as in their local versions. We do not need any details of the above constructions anymore: all we need are these properties of upper envelopes and shadows.

6. Proof of Proposition 4.13

6.1. **Upper enveloping surfaces.** In this subsection we define an upper enveloping surface which is a straightforward analog of the above two-dimensional construction. Since the stable distribution is uniquely integrable, as long as one wants to construct an enveloping surface within one coordinate neighborhood, one could simply take a surface transverse to E^s, solve a two-dimensional problem there as above, and extend the solution along the stable leaves. This yields a cs-surface whose width is not necessarily uniformly bounded from below. Applying the same construction one can extend this surface beyond one of its edges. One can keep repeating this construction and eventually construct a cs-surface containing an entire central curve. The main difficulty however is that this surface may not be complete. Even worse, the orientation of its boundary components may form an obstruction to applying the same construction any further, see Figure 1 on page 33. We will need a new gadget to handle this difficulty, however our nearest goal is to go by the above upper envelope construction as far as possible and obtain a (possibly non-complete) cs-surface with a uniform bound for distances between edges.

Since we cannot construct a complete open surface in one step, we have to take into account all surfaces constructed at the previous steps. This is the reason why in the previous section we discussed the "more delicate" problem with a collection of prescribed solutions.

The key part of the argument is a construction of the forward envelope. The input data is a pre-foliation $\mathcal{A}$, a point $p \in M$ and a subset $\Sigma_0 \subset \mathcal{A}_p$. This subset is a section of the ordered set $(\mathcal{A}_p, \leq_p)$, that is, if $A \in \Sigma_0$ and $B \leq_p A$, then $B \in \Sigma_0$. The output is a forward patch S (denoted by $\sup(p, \Sigma_0)$ and referred to as the upper envelope of (p, Σ_0)) which can be added to $\mathcal{A}_p^+$.

To prove Proposition 4.13, we apply this construction to all points p in M. If $\mathcal{A}_p^- = \varnothing$, we apply the construction to $\Sigma_0 = \varnothing$; otherwise, we apply it to $\Sigma_0 = \{C \in \mathcal{A}_p : C \leq_p A\}$ for each $A \in \mathcal{A}_p^-$. Then, of course, we have to extend the order to all new patches added to $\mathcal{A}$. We first define a new order on newly constructed forward envelopes. Then, at the very end of the proof of Proposition 4.13, we define an order between new patches and patches from $\mathcal{A}$ and verify that every element from $\mathcal{A}_p^-$ has an upper neighbor.

The formal definitions follow. By a *transverse disc* we mean a C^1 embedded 2-dimensional disc $\sigma \subset M$ transverse to E^s. The orientation of E^s canonically defines an orientation of σ. Denote by $E^{cs} \cap \sigma$ the oriented one-dimensional distribution on σ traced by E^{cs}.

Let (S, x) be a marked cs-surface and $S(x) \in \sigma$. By $(S, x) \cap \sigma$ we denote the component of the intersection of S and σ containing x. Formally, this is a curve in $dom(S)$ defined as the connected component of the set $S^{-1}(\sigma)$ containing x. We abuse notation and make no distinction between this curve and its image in σ. Then this intersection is an integral curve of $E^{cs} \cap \sigma$.

We denote by $\mathcal{A} \cap \sigma$ the family of all such intersections of σ with surfaces from $\mathcal{A}$. This family carries the structure of a one-dimensional pre-foliation in σ in the sense of Section 5.2 (with orders naturally induced from those in $\mathcal{A}$).

Let $p \in \sigma$ and Σ be a section of $\mathcal{A}_p$. Denote by $\Sigma \cap \sigma$ the set of all intersections $A \cap \sigma$ where $A \in \Sigma$. Now we can construct the one-dimensional upper envelope for $(p, \Sigma \cap \sigma)$ in σ as in Section 5.2.

By a *forward curve* we mean a C^1 curve γ on a cs-surface such that the velocity $\dot{\gamma}$ is transverse to E^s and oriented in the forward direction, that is, the pair $(\dot{\gamma}, E^s)$ has the same orientation as (E^c, E^s). A degenerate curve (that is, a single point) is also a forward curve. A *backward curve* is a forward curve reparameterized by $t \mapsto -t$.

Definition 6.1. An *upper enveloping surface* is a pair (S, Σ) where S is a cs-surface and Σ is a map assigning a section $\Sigma(x) \subset \mathcal{A}_{S(x)}$ to every point $x \in dom(S)$ such that the following requirements are satisfied.

1. Let $x \in dom(S)$ and σ be a transverse disc containing $S(x)$. Let γ be the forward part (starting from x) of the curve $(S, x) \cap \sigma$. We require that the S-image of γ is contained in the one-dimensional upper envelope for $(S(x), \Sigma(x) \cap \sigma)$ in σ (with respect to the one-dimensional pre-foliation $\mathcal{A} \cap \sigma$). Moreover, for every point $\gamma(t)$ in this curve, we require that the set $\Sigma(\gamma(t)) \cap \sigma$ is the shadow of $(S(x), \Sigma(x) \cap \sigma)$ at $S(\gamma(t))$, cf. Sections 5.1 and 5.2 for the definition of the shadow.

2. If points $x, y \in dom(S)$ lie in the same s-line, then the sets $\Sigma(x)$ and $\Sigma(y)$ agree in the following sense. Let F be a patch from $\mathcal{A}$ and let $a, b \in dom(F)$ lie in one s-line and $F(a) = S(x)$, $F(b) = S(y)$. Then $(F, a) \in \Sigma(x)$ if and only if $(F, b) \in \Sigma(y)$.

3. S has exactly one proper edge, and this edge has forward co-orientation.

4. Every point of $dom(S)$ can be reached from the proper edge by a forward curve.

The 4th requirement implies that for every $x, x' \in dom(S)$ such that x' lies in the open forward half-surface from x, x can be connected to x' by a forward curve. In particular the forward half-surface from x' is contained in that from x.

Lemma 6.2. *Let $p \in M$ and $\Sigma_0 \subset \mathcal{A}_p$ a section of $\mathcal{A}_p$. Then there exist an upper enveloping surface (S, Σ) and a boundary point $o \in dom(S)$ such that*

1. $S(o) = p$ and $\Sigma(o) = \Sigma_0$.

2. The distance in $dom(S)$ from o to any edge except the one passing through o is at least 1.

3. Every s-line in $dom(S)$ contains a point within (intrinsic) distance at most 5 from o.

Proof. Using a regular coordinate system in a neighborhood of p, one can construct a transverse disc $\sigma \ni p$ almost orthogonal to E^s and such that the boundary of σ is separated from p by distance at least 10. Let $\gamma : [0, 5) \to \sigma$ be a unit-speed curve parameterizing the one-dimensional upper envelope for $(p, \Sigma \cap \sigma)$ with respect to the one-dimensional pre-foliation $\mathcal{A} \cap \sigma$, cf. the discussion above. Let S be a surface formed by complete s-leaves passing through points of γ. By Proposition 3.1, S is a cs-surface. Let $o \in dom(S)$ be the point corresponding to p.

Then define a family of sections $\Sigma(x) \subset \mathcal{A}_{S(x)}$, $x \in dom(S)$, in the only way compatible with requirements 1 and 2 of Definition 6.1. Namely, let $\tilde{\gamma}$ be the lift of γ starting from o. If $x \in dom(S)$ lies on $\tilde{\gamma}$, we define $\Sigma(x)$ according to the requirement 1 of the definition, namely so that $\Sigma(x) \cap \sigma$ is the shadow of (p, Σ) at $S(x)$. If x does not lie in $\tilde{\gamma}$, we define $\Sigma(x)$ according to the requirement 2 of the definition, namely so that $\Sigma(x)$ is compatible with $\Sigma(y)$ where y is a point on $\tilde{\gamma}$ lying in the same s-leaf as x.

Requirements 2–4 from Definition 6.1 and assertions 1–3 of Lemma are trivial by construction. It remains to verify the first requirement of the definition. For the original surface σ and a point x lying on $\tilde{\gamma}$, the requirement follows from Lemma 5.5. Then the general case follows from the following simple observation: if $p_1, p_2 \in M$ lie on the same stable leaf and σ_1, σ_2 are surfaces transverse to E^s and passing through p_1 and p_2, respectively, the shift along stable leaves forms an isomorphism of all our structures in some neighborhoods of p_1 and p_2 in σ_1 and σ_2 respectively. □

The next lemma shows that the surface constructed in Lemma 6.2 is locally unique.

Lemma 6.3. *Let (S, Σ) and (S', Σ') be upper enveloping surfaces, $x \in dom(S)$, $x' \in dom(S')$, $S(x) = S'(x')$ and $\Sigma(x) = \Sigma'(x')$. Let γ be a forward curve in S of length ℓ starting from x. Suppose that x' is separated by distance $> \ell$ in $dom(S')$ from any edge of backward co-orientation.*

Then $S \circ \gamma$ admits a lift $\tilde{\gamma}$ to S' starting from x'. Furthermore, $\Sigma(\gamma(t)) = \Sigma'(\tilde{\gamma}(t))$ for all $t \in dom(\gamma)$.

Proof. First assume that γ is contained in a transverse disc σ. Then, by the 1st requirement of Definition 6.1, $S \circ \gamma$ and the forward part of the intersection $(S', x') \cap \sigma$ are both contained in the one-dimensional envelope for $(S(x), \Sigma(x))$ in σ. The intersection $(S', x') \cap \sigma$ ends at the boundary of σ or at an edge of S' with backward co-orientation. Then the assumptions about ℓ imply that this intersection is longer than γ, hence it contains γ. The first assertion follows. The second one follows from the requirement about $\Sigma(\gamma(t))$ of Definition 6.1.

In the general case, divide γ into segments γ_i each of which is contained in a transverse disc. Then the assertions follow by induction in i. $\square$

Corollary 6.4. *Let (S,Σ) and (S',Σ') be upper enveloping surfaces, $x \in dom(S)$, $x' \in dom(S')$, $S(x) = S'(x')$ and $\Sigma(x) = \Sigma'(x')$.*

Then the marked cs-surfaces (S,x) and (S',x') locally coincide in the forward direction. That is, there are arbitrarily small forward half-neighborhoods $U \ni x$ and $U' \ni x'$ such that $S(U) = S'(U')$.

Lemma 6.5. *Let (S,Σ) and (S',Σ') be upper enveloping surfaces, $x \in dom(S)$, $x' \in dom(S')$, $S(x) = S'(x')$.*

Then one of the marked surfaces (S,x) and (S',x') is locally above the other.

Proof. Suppose the contrary. Then the intersections of S and S' with a transverse disc σ topologically cross. By the 1st requirement of Definition 6.1, these sections are one-dimensional upper envelopes in σ. By Lemma 5.7 they cannot cross, a contradiction. $\square$

Lemma 6.6. *Let (S,Σ) and (S',Σ') be upper enveloping surfaces, $x \in dom(S)$, $x' \in dom(S')$. Suppose that $S(x) = S'(x')$ and $\Sigma(x) \subsetneq \Sigma'(x')$.*

Then (S',x') is locally above (S,x).

Proof. Apply Lemma 5.7(2) to sections of S and S' by discs almost orthogonal to E^s. $\square$

Lemma 6.7. *Let (F,Σ_F) and (G,Σ_G) be upper enveloping surfaces, $x, x' \in dom(F)$, $y, y' \in dom(G)$, $x \neq x'$, $y \neq y'$. Assume that there is a forward curve γ in F connecting x to x' and such that $F \circ \gamma$ admits a lift to G connecting y to y'. Suppose that $\Sigma_G(y) \subset \Sigma_F(x)$.*

Then $\Sigma_G(y') \subset \Sigma_F(x')$ and (F,x') is locally above (G,y').

Proof. It suffices to consider the case when γ is contained in a transverse disc. Lemma 5.7(1) applied to the intersections of F and G with this disc yields that $\Sigma_G(y') \subset \Sigma_F(x')$. If $\Sigma_G(y') \subsetneq \Sigma_F(x')$, the second assertion follows from Lemma 6.6.

If $\Sigma_G(y') = \Sigma_F(x')$, then the marked surfaces (F,x') and (G,y') locally coincide. Indeed, by Corollary 6.4 they locally coincide in the forward direction. They also locally coincide in the backward direction since their backward half-neighborhoods are covered by s-lines in M passing though the points of the curve $F \circ \gamma$ near its endpoint. $\square$

Lemma 6.8. *Let (S,Σ) be an upper enveloping surface and F a surface from $\mathcal{A}$. Let $x \in dom(S)$, $a \in dom(F)$, $S(x) = F(a)$. Let γ be a curve in S starting from x and admitting a lift $\tilde\gamma$ to F starting from a. Then*

1. If $(F,a) \in \Sigma(x)$, then (S,x) is locally above (F,a) and $(F,\tilde\gamma(t)) \in \Sigma(\gamma(t))$ for all t.

2. If $(F,a) \notin \Sigma(x)$, then (F,a) is locally above (S,x) and $(F,\tilde\gamma(t)) \notin \Sigma(\gamma(t))$ for all t.

Proof. Observe that γ may be replaced by a piecewise C^1 curve intersecting the same set of s-lines and consisting of segments each of which is transverse to E^s or contained in an s-line. Then the assertion follows from Lemma 5.6 applied to the intersections of S and F with transverse discs. □

Now the reader may forget about one-dimensional envelopes. In the rest of the section we use only the above lemmas and requirements 2–4 of Definition 6.1.

Definition 6.9. Two upper enveloping surfaces (S,Σ) and (S',Σ') are said to be *equivalent* if there is a homeomorphism $\phi : dom(S) \to dom(S')$ such that $S'(\phi(x)) = S(x)$ and $\Sigma'(\phi(x)) = \Sigma(x)$ for all $x \in dom(S)$.

It is easy to see that such a homeomorphism is unique (this follows from the fact that S has a unique proper edge and it is mapped to M injectively).

The next definition introduces objects that will be added to our prefoliations.

Definition 6.10. A *forward envelope* is an upper enveloping surface such that all its edges have forward co-orientation.

Note that every forward envelope is a patch. Indeed, the distance between edges of the same co-orientation is bounded below by the regularity radius of the foliations (which is greater than 1 by our choice of metric, see page 7).

Lemma 6.11. *Let $p \in M$ and $\Sigma_0 \subset \mathcal{A}_p$ a section of $\mathcal{A}_p$. Then there exists a forward envelope (S,Σ) having a boundary point $o \in dom(S)$ such that $S(o) = p$ and $\Sigma(o) = \Sigma_0$.*

Such a forward envelope is unique up to an equivalence in the sense of Definition 6.9.

Proof. Existence. We construct S via a countable extension procedure. At the base step, we apply Lemma 6.2. This gives us an upper enveloping surface (S_1,Σ) and a boundary point $o \in dom(S_1)$ such that $S_1(o) = p$ and $\Sigma(o) = \Sigma_0$. We start from this surface and extend it as follows.

At the nth step, we have a partial upper envelope $S = S_n$ with the same properties as (S_1, Σ). Suppose that some edges have backward co-orientation. We pick a point $x = x_n$ nearest to o (with respect to the intrinsic metric of S) in the union of such boundary components and extend S_n so that the resulting surface S_{n+1} contains S_n and a forward half-ball of radius $1/2$ centered at x does not reach any edges of S_{n+1}.

More precisely, we choose a point $y \in dom(S)$ very close to x (note that $x \notin dom(S)$ but x belongs to the completion of the surface). Then we apply Lemma 6.2 to obtain an upper enveloping surface (S', Σ') and a boundary point $o' \in dom(S')$ such that $S'(o') = S(y)$ and $\Sigma'(o') = \Sigma(y)$. Then we extend S by pasting S' to S so that o' is pasted to y. This would guarantee that (in intrinsic metric) we added a half-ball centered at x of radius at least $1/2$. Note that such a half-ball contains an intrinsic ball of radius $1/5$.

To see that we can indeed paste S' to S, apply Lemma 6.3 to a forward curve $\gamma : [0, 1) \to dom(S)$ connecting y to x. (Formally, connect y to x in the completion of S and remove the endpoint.) We assume that y is so close to x that length$(\gamma) < 1/10$. The lemma implies that the curve $S \circ \gamma$ admits a lift $\gamma' : [0, 1) \to dom(S')$ starting from o' and the maps Σ and Σ' on the two lifts agree.

Observe that γ' can be extended to the closed interval $[0, 1]$. We denote the extension by the same letter γ'. Denote $x' = \gamma'(1)$ and let ℓ' be the s-line in S' passing through x'. The s-line ℓ' divides S' into two components S'_+ and S'_- where S'_- is the one containing o'. By the 3rd assertion of Lemma 6.2, every s-line in S'_- intersects γ'. Since the maps Σ and Σ' are canonically propagated along s-lines (cf. the 2nd requirement of Definition 6.1), the surface S'_- coincides with a sub-surface of S formed by the s-lines intersecting γ (and the maps Σ and Σ' agree). Now we can describe pasting S' to S as follows: add the edge ℓ containing x and then attach S'_+ to S along their edges ℓ' and ℓ. Hence the result is a cs-surface. Thus $S = S_n$ can be extended by adding S' (along with its map Σ') to obtain the next surface S_{n+1}.

Since all cs-surfaces have bounded intrinsic geometry (in the sense that the number of $1/5$-separated points in a ball of radius R is bounded above by a function of R), the distance from o to x_n goes to infinity. Recall that this distance is the distance from o to the union of edges having backward co-orientation. Hence a surface $S = \bigcup_n S_n$ has no such edges. Then it is a desired forward envelope.

Uniqueness. Let (S, Σ) and (S', Σ') satisfy the conditions, and let o and o' be their boundary points corresponding to p. Let $x \in dom(S)$. Connect o to x by a curve $\gamma : [0, 1] \to dom(S)$ consisting of two parts (possibly degenerate): a segment lying on the boundary and a forward curve. The 2nd requirement of Definition 6.1 and Lemma 6.3 imply that this curve admits a lift $\tilde{\gamma}$ to S' starting from o' and $\Sigma(\gamma(t)) = \Sigma'(\tilde{\gamma}(t))$ for all $t \in [0, 1]$. Define $\phi(x) = \tilde{\gamma}(1)$.

This definition does not depend on the choice of γ. Indeed, let γ and γ_1 be two such curves in $dom(S)$. Clearly they intersect the same set of s-lines and are homotopic via a homotopy which moves every point along an s-line. It is straightforward to construct a lift of such a homotopy to S', hence the result.

Thus we have a continuous map $\phi : dom(S) \to dom(S')$ such that $S'(\phi(x)) = S(x)$ and $\Sigma'(\phi(x)) = \Sigma(x)$ for all $x \in dom(S)$. Interchanging S and S' yields an inverse map, hence ϕ is a homeomorphism. □

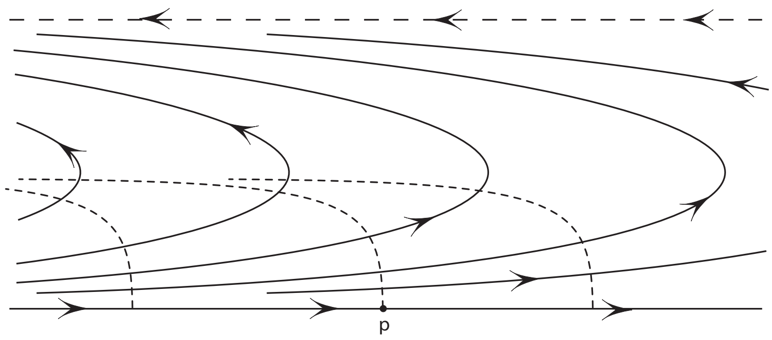

Figure 1. S may fail to be complete

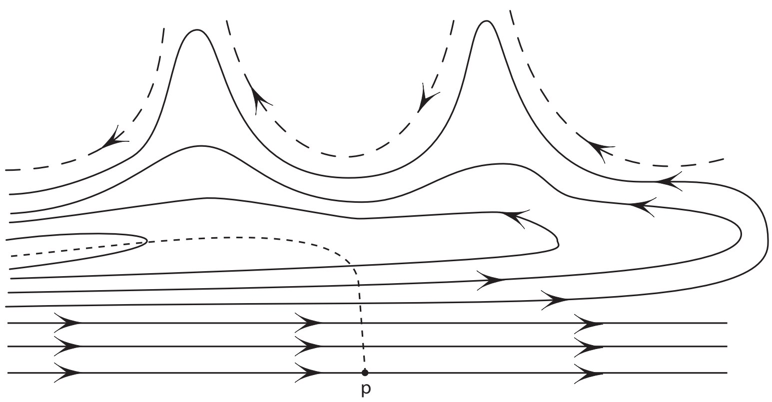

Figure 2. S may have many non-proper edges

Remark. It is very tempting to think of a forward envelope S as an immersed half-plane. However at least *a priori* it is not at all clear that S is complete. Knowing that S is complete would allow us to essentially simplify the proof: Figures 1 and 2 illustrate where the difficulties come from. Figure 1 gives a simplest example suggesting that S may not be complete; Figure 2 shows a general structure of S (we do not know if any of these structures actually occur in any examples). In the figures, solid lines are stable leaves, dotted lines denote unstable leaves, and dashed lines are non-proper edges of S; arrows indicate orientation of the leaves.

6.2. **The order between envelopes.** Let $\mathcal{U}$ denote the set of all forward envelopes modulo the equivalence introduced in Definition 6.9. We are going to define an order on $\mathcal{U}_*$ so that $\mathcal{U}$ is a pre-foliation.

We define the order in three steps. Namely, we introduce three relations $>_i$, $i = 1, 2, 3$. These relations are disjoint (that is, they have pairwise disjoint sets of comparable pairs of envelopes); we define the resulting order as the union $>_1 \cup >_2 \cup >_3$. Each relation $>_i$ satisfies Axioms 1–3 from Definition 4.8 of a pre-foliation.

In this subsection, we use the following notation: $X, Y, Z \in \mathcal{U}_*$, $X = (F, x)$, $Y = (G, y)$, $Z = (H, z)$. We assume that the marked patches X, Y, Z correspond to the same point in M, that is $F(x) = G(y) = H(z)$.

6.2.1. *Intersections.* Let $\gamma_X : [a, b] \to dom(F)$ be a forward curve connecting its proper boundary component to x. (This means that $\gamma_X(a)$ belongs to the boundary and $\gamma_X(b) = x$, cf. the 4th requirement of Definition 6.1). It is possible that $a = b$. Denote by I_X the set of all s-lines intersecting γ and by $D_X \subset dom(F)$ the union of these lines.

Since γ_X is a forward curve, it intersects each s-line from I_X exactly once and therefore defines a bijection between I_X and $dom(\gamma_X) = [a, b]$. This defines a total order and topology on I_X. The bijection between I_X and $[a, b]$ depends on the choice of γ_X but the set I_X and the resulting order and topology on it do not.

The set D_X is homeomorphic to $[a, b] \times \mathbb{R}$ via the homeomorphism $(t, \tau) \mapsto \Psi^\tau(\gamma_X(t))$ where $\{\Psi^\tau\}_{\tau \in \mathbb{R}}$ is the flow generated by the lift of E^s to $dom(F)$. Note that the completion of the surface $F|_{D_X}$ is not necessarily homeomorphic to $[a, b] \times \mathbb{R}$. This surface may have many non-proper edges.

Now we are going to define "the backward component of intersection" of X and Y. It may be easier to follow the sequel if one keeps in mind that, in notations such as D and I with subindexes, the first subindex tells us in which of the domains the object lies.

Let I be the maximum subinterval of $[a,b]$ containing b and such that $F \circ \gamma_X|_I$ admits a lift $\tilde{\gamma}$ to G with $\tilde{\gamma}(b) = y$. (The subinterval I may be a single point.) Let $I_{XY} \subset I_X$ be the set of s-lines intersecting $\gamma_X|_I$ and D_{XY} the union of these lines. The lift $\tilde{\gamma}$ of $\gamma_X|_I$ defines a monotone continuous map $\alpha_{XY} : I_{XY} \to I_Y$ (sending the s-line passing though $\gamma_X(t)$ to the s-line passing through $\tilde{\gamma}(t)$) and a continuous map $\phi_{XY} : D_{XY} \to D_Y$ such that $G(\phi_{XY}(x')) = F(x')$ for all $x' \in D_{XY}$. Clearly D_{XY} is the maximal connected subset of the closed backward half-surface from x for which such a map exists. Note that the maps α_{XY} and ϕ_{XY} do not depend on the choice of γ_X.

The same construction with interchanged x and y yields an interval $I_{YX} = \alpha_{XY}(I_{XY})$ of I_Y, a subset $D_{YX} = \phi_{XY}(D_{XY})$ of D_Y and inverse maps $\alpha_{YX} = \alpha_{XY}^{-1}$, $\phi_{YX} = \phi_{XY}^{-1}$. Thus α_{XY} is a homeomorphism between I_{XY} and I_{YX} and ϕ_{XY} is a homeomorphism between D_{XY} and D_{YX}. We say that $x' \in D_{XY}$ and $y' \in D_{YX}$ are *matching points* if $y' = \phi_{XY}(x')$.

We also need to consider "triple intersections". Define $D_{XYZ} = D_{XY} \cap D_{XZ}$ and $I_{XYZ} = I_{XY} \cap I_{XZ}$.

Observe that one of the intervals I_{XY} and I_{XZ} contains the other since they are subintervals of I_X containing the maximum point of I_X (which corresponds to b when we identify $dom(\gamma_X) = [a,b]$ and I_X). Hence $I_{XYZ} = I_{XY}$ or $I_{XYZ} = I_{XZ}$. The same is true for D_{XY} and D_{XZ}.

Also observe that, if D_{XY} contains the proper edge of the surface, then $I_{XZ} \subset I_{XY}$ and $D_{XZ} \subset D_{XY}$. This follows from the fact that the proper edge corresponds to the minimum of I_X.

Lemma 6.12. *In the above notation,*

$$\phi_{XY}(D_{XYZ}) = D_{YXZ},$$
$$\alpha_{XY}(I_{XYZ}) = I_{YXZ},$$

and

$$\phi_{YZ}(\phi_{XY}(x')) = \phi_{XZ}(x'),$$
$$\alpha_{YZ}(\alpha_{XY}(\ell)) = \alpha_{XZ}(\ell)$$

for all $x' \in D_{XYZ}$, $\ell \in I_{XYZ}$.

Proof. Observe that a point $x' \in dom(F)$ belongs to D_{XYZ} if and only if there is a backward curve (or an s-line) γ connecting x to x' and such that $F \circ \gamma$ admits a lift γ_1 to G starting from y and a lift γ_2 to H starting from z. Observe that in this case γ and γ_2 are lifts of $G \circ \gamma_1$, hence the endpoint $y' = \phi_{XY}(x')$ of γ_1 belongs to D_{YXZ}. Thus $\phi_{XY}(D_{XYZ}) \subset D_{YXZ}$. To prove the opposite inclusion, interchange x and y and recall that $\phi_{YX} = \phi_{XY}^{-1}$. The first assertion follows. To prove the third one, observe that both $\phi_{YZ}(\phi_{XY}(x'))$ and $\phi_{XZ}(x')$ are the endpoint of γ_2. The second and the forth assertions follow from the first and the third one, respectively. $\square$

6.2.2. *The relation $>_1$.* Intuitively, this relation corresponds to "geometric order" when one "surface lies strictly above the other", at least "potentially" (in the sense of Σ). A formal definition follows.

Recall that every upper enveloping surface is equipped with a map Σ as in Definition 6.1. We abuse notations and denote these maps by the same letter Σ for all surfaces. We set $X >_1 Y$ if there exist matching points $x' \in D_{XY}$ and $y' \in D_{YX}$ such that at least one of the following conditions holds:

(a) $\Sigma(y') \subsetneq \Sigma(x')$;

(b) (F, x') is locally strictly above (G, y').

Of course, if one of the opposite conditions holds, then $X <_1 Y$.

Observe that (a) implies that (F, x') is locally (non-strictly) above (G, y') and (b) implies that $\Sigma(y') \subset \Sigma(x')$, cf. Lemma 6.6. Thus, if $X >_1 Y$, then (F, x') is locally above (G, y') and $\Sigma(y') \subset \Sigma(x')$.

Let us prove that the relation $>_1$ is correctly defined, transitive and satisfies Axioms 1–2 from the definition of a pre-foliation.

Correctness. We have to prove that there are no pairs of matching points (x', y') and (x'', y'') (where $x', x'' \in D_{XY}$, $y', y'' \in D_{YX}$) such that (x', y') satisfies (a) or (b) but (x'', y'') satisfies one of the opposite conditions. Suppose the contrary. The 2nd requirement of Definition 6.1 implies that x' and x'' do not belong to one s-line. Assume that x'' lies in the forward half-surface from x' (the other case is similar).

Connect x' to x'' by a forward curve γ. Then $F \circ \gamma$ admits a lift $\tilde{\gamma} = \phi_{XY} \circ \gamma$ to G connecting y' to y''. Then Lemma 6.7 implies that $\Sigma(y'') \subset \Sigma(x'')$ and (F, x'') is locally above (G, y''), a contradiction.

Transitivity. Let $X >_1 Y >_1 Z$. Since the intervals I_{YX} and I_{YZ} have the same maximum point (it is the maximum point of I_Y) their intersection I_{YXZ} coincides with one of them. Assume that $I_{YXZ} = I_{YX}$ (the other case is similar). Then $D_{YXZ} = D_{YX}$.

By the definition of $>_1$, there is a pair of matching points $x' \in D_{XY}$ and $y' \in D_{YX}$ satisfying (a) or (b). Our assumption implies that $y' \in D_{YXZ}$. Consider the point $z' = \phi_{YZ}(y') = \phi_{XZ}(x')$, where the last equality follows from Lemma 6.12. Since $Y >_1 Z$, we have $\Sigma(z') \subset \Sigma(y')$ and (G, y') is locally (non-strictly) above (F, x'). Hence either of the conditions (a) or (b) for the pair (x', y') implies the same assertions for (x', z'). Hence $X >_1 Z$. This finishes the proof of transitivity.

Axiom 1. Recall that Axiom 1 requires that the order agrees with the geometric order, that is, the relation $X >_1 Y$ implies that X is locally (non-strictly) above Y. Let us prove a more general statement which also covers the orders $>_2$ and $>_3$ defined later.

Lemma 6.13. *Every relation $>_*$ on $\mathcal{U}_*$ extending $>_1$ agrees with the geometric order.*

Proof. The relation $X >_* Y$ implies that either $X >_1 Y$ or X and Y are incomparable by $>_1$. In both cases, the definition of $>_1$ implies that $\Sigma(y) \subset \Sigma(x)$ and the marked surface X is locally above Y. Therefore $>_*$ agrees with the geometric order. $\square$

Axiom 2. This axiom requires that the order is preserved along the lifts of curves. Let γ be a curve connecting x to a point $x' \in dom(F)$ such that $F \circ \gamma$ admits a lift $\tilde{\gamma}$ to G connecting y to a point $y' \in dom(G)$. We have to prove that $(F, x') >_1 (G, y')$. We need to consider two cases: when γ is an s-line segment and a forward (or a backward) curve.

Let $x'' \in D_{XY}$ and $y'' \in D_{YX}$ be matching points satisfying (a) or (b) from the definition of $>_1$. If γ is a forward curve or, more generally, x' is in the forward half-surface from x'', then the pair (x'', y'') serves the comparison of (F, x') and (F, y') as well. In the case when x'' and x' are on the same s-line, (a) or (b) is satisfied for (x', y') by the 2nd requirement of Definition 6.1.

Consider the remaining case when x'' is in the forward half-surface from x'. Suppose that the desired conclusion $(F, x') >_1 (G, y')$ does not hold, then $\Sigma(x') \subset \Sigma(y')$. Connect x' to x'' by a forward curve γ_1, and apply Lemma 6.7 to this curve. It yields that $\Sigma(x'') \subset \Sigma(y'')$ and (G, y'') is locally above (F, x''), so neither (a) nor (b) holds for (x'', y'').

Further properties. The next lemma helps us to reduce the number of cases in subsequent arguments.

Lemma 6.14. *Suppose that Y and Z are incomparable by $>_1$, $I_{YX} \subset I_{YZ}$ and $I_{ZXY} \neq I_Z$. Then Z is in the same relation ($>_1$, $<_1$, or incomparable by $>_1$) with X as X is with Y.*

Proof. The assumptions imply that $D_{YX} \subset D_{YZ}$ and $D_{ZXY} \neq D_Z$. Since $D_{YX} \subset D_{YZ}$, we have $D_{YXZ} = D_{YX}$.

Suppose that $X >_1 Y$. Then there are matching points $x' \in D_{XY}$ and $y' \in D_{YX}$ satisfying (a) or (b) from the definition of $>_1$. Let $z' = \phi_{YZ}(y') = \phi_{XZ}(x')$ where the second equality follows from Lemma 6.12. Observe that z' is an interior point of H and $z' \in D_{ZXY} \neq D_Z$. Then, since Y and Z are not comparable, $\Sigma(z') = \Sigma(y')$ and (G, y') locally coincides with (H, z') or with a forward half-surface of (H, z').

Hence each of conditions (a) or (b) for the pair (x', y') implies the same for (x', z').

The case $X <_1 Y$ is similar. The case when X and Y are incomparable follows from the other two cases. $\square$

6.2.3. *The relation* $>_2$. Intuitively, this relation says that one envelope is greater than another if "it is contained in it". Let us proceed with a formal definition in a form convenient for our application.

Let X and Y be incomparable by $>_1$. Suppose that $X \neq Y$ and the interval I_{XY} contains the minimum element $\ell_{XY}^{\min}$. Then define $X <_2 Y$ if $I_{XY} = I_X$ (this means that the minimum of I_{XY} corresponds to the boundary s-line of F, or, in other words, to a when we identify I_X with $[a, b]$ via a parameterization of γ), and $X >_2 Y$ otherwise.

If I_{XY} has no minimum, the order remains undefined. Let us verify the correctness of $>_2$ and transitivity for $>_1 \cup >_2$.

Correctness. We have to verify that the definition is anti-symmetric, that is, if $I_{XY} = I_X$ then $I_{YX} \neq I_Y$ and vise versa.

First suppose that $I_{XY} = I_X$ and $I_{YX} = I_Y$, then $D_{XY} = D_X$ and $D_{YX} = D_Y$. This means that ϕ_{XY} sends the boundary of F to the boundary of G. Let $x' \in D_{XY}$ and $y' \in D_{YX}$ be matching points in the boundaries. Since X and Y are incomparable by $>_1$, we have $\Sigma(x') = \Sigma(y')$. Therefore, by the uniqueness part of Lemma 6.11, F and G are the same surface, hence $X = Y$, a contradiction.

Now suppose that $I_{XY} \neq I_X$ and $I_{YX} \neq I_Y$. This means that the s-lines corresponding to minimum elements of I_{XY} and I_{YX} lie in the interiors of their surfaces. Choose matching points $x' \in D_{XY}$ and $y' \in D_{yx}$ in these s-lines. Lemma 6.5 implies that one of the marked surfaces (F, x') and (G, y') is locally above the other. Moreover it is strictly above, otherwise the surfaces locally coincide, contrary to the maximality of the domains D_{XY} and D_{YX}. Hence X and Y are comparable by $>_1$, contrary to our assumption.

Transitivity. Let $X > Y > Z$ where each sign "$>$" is either the $>_1$ or $>_2$. There are three cases to consider.

Case 1: $X >_2 Y >_1 Z$. The definition of $>_2$ implies that $I_{YX} = I_Y \supset I_{YZ}$ and $I_{XY} \neq I_X$. Applying Lemma 6.14 (with X and Z interchanged) yields that $X >_1 Y$ and hence $X >_1 Z$.

Case 2: $X >_1 Y >_2 Z$. Since $>_1$ is transitive, we may assume that X and Z are incomparable by $>_1$. Then Lemma 6.14 yields that $I_{ZXY} = D_Z$ or $I_{YX} \not\subset I_{YZ}$.

If $I_{ZXY} = I_Z$, we have $I_{ZX} = I_Z$, hence $X >_2 Z$ by definition. If $I_{YX} \not\subset I_{YZ}$, then $I_{YZ} \subset I_{YX}$, hence $I_{YZX} = I_{YZ}$, and then Lemma 6.12

yields that $I_{ZXY} = I_{ZY}$. The relation $Y >_2 Z$ means that $I_{ZY} = I_Z$. Thus $I_{ZXY} = I_Z$, and this case is already done.

Case 3: $X >_2 Y >_2 Z$. We may assume that X and Z are incomparable by $>_1$ (otherwise refer to Case 1). Since $X >_2 Y$, we have $I_{YX} = I_Y$, hence $I_{YXZ} = I_{YZ}$. By Lemma 6.12, this implies that $I_{ZXY} = I_{ZY}$, hence $I_{ZY} \subset I_{ZX} \subset I_Z$. On the other hand, $I_{ZY} = I_Z$ since $Y >_2 Z$. Then $I_{ZX} = I_Z$, hence $X >_2 Z$.

6.2.4. *The relation* $>_3$. This is the most complicated relation, and in order to develop some intuition we begin with an informal description. We want to compare two envelopes X and Y which cannot be compared geometrically nor by their Σ-data, and none of them is contained in the other. Thus we can think of then as two overlapping surfaces, with their proper edges not contained in the common part. Marked points x and y are mapped to the same point p, and D_{XY} and D_{YX} "parameterize the backward intersection". We have curves γ_X and γ_Y connecting x and y with respective proper edges of the surfaces. Choose a point x' on γ_X "very close" to the point where it leaves D_{XY} (that is, close to the minimum of I_{XY}. There is an s-line passing through this point. This s-line meets with the lift of γ_Y to D_{XY}. Recall that all s-lines are oriented. If this meeting point is in the positive direction along this s-line from x', then we say that $X >_3 Y$, and $Y >_3 X$ otherwise. Note that as x' gets closer to the boundary of D_{XY}, the distance from x' along the s-line to the lift of γ_Y goes to infinity, so the resulting relation does not depend on x' provided that it is chosen sufficiently close to the minimum of I_{XY}. Now we proceed with a formal definition.

Recall that the homeomorphism $[a, b] \times \mathbb{R} \to D_X$ is defined by a curve γ_X in section 6.2.1. To make it more invariant, we turn it into a homeomorphism $I_X \times \mathbb{R} \to D_X$ using the natural bijection between I_X and $[a, b] = dom(\gamma_X)$. We regard this homeomorphism as a C^0 coordinate system (t, τ) on D_X where $t \in I_X$, $\tau \in \mathbb{R}$. These coordinates depend on the choice of the curve γ_X, changing this curve yields a change of coordinates of the form

$$(t, \tau) \to (t, \tau + \beta(t))$$

where $\beta : I_X \to \mathbb{R}$ is a continuous function.

In these coordinates, the set D_{XY} corresponds to $I_{XY} \times \mathbb{R}$. The map $\phi_{XY} : D_{XY} \to D_{YX}$ (cf. Section 6.2.1) has the form

$$(t, \tau) \mapsto (\alpha_{XY}(t), \tau + \beta_{XY}(t)) \tag{6.1}$$

where $\alpha_{XY} : I_{XY} \to I_{YX}$ is the order-preserving homeomorphism introduced in section 6.2.1 and $\beta : I_{XY} \to \mathbb{R}$ is a continuous function. Changing coordinates adds a bounded function to β_{XY}.

Let X and Y be incomparable by $>_1$ and $>_2$. Then the interval I_{XY} has no minimum. Denote $t_{XY}^{\inf} = \inf I_{XY}$. Recall that I_{XY} is a maximum left sub-interval of I_X such that $F \circ \gamma_X$ admits a lift to G (with its endpoint at y) when restricted to the corresponding interval of parameters. This lift in coordinates corresponds to a curve in $\mathbb{R}^2$ given by

$$(\alpha_{XY}(t), \beta_{XY}(t)), \qquad t \in I_{XY}.$$

The maximality of I_{XY} implies that this curve has no accumulation points as $t \to t_{XY}^{\inf}$ (the convergence here is from above, that is, $t \downarrow t_{XY}^{\inf}$). Hence $\beta_{XY}(t) \to \pm\infty$ as $t \downarrow t_{xy}^{\inf}$.

We define $X >_3 Y$ if $\beta_{XY}(t) \to +\infty$ as $t \downarrow t_{XY}^{\inf}$ and $X <_3 Y$ otherwise. The identity $\phi_{YX} = \phi_{XY}^{-1}$ and (6.1) implies that $\beta_{YX} = -\beta_{XY} \circ \alpha_{YX}$, hence the relation $>_3$ is correctly defined. Now the order is defined for all pairs $X, Y \in \mathcal{U}_*$ with the same images of their base points x, y in M.

Let us prove that the relation $>_1 \cup >_2 \cup >_3$ is transitive. Suppose the contrary. Since $>_1 \cup >_2$ is transitive, non-transitivity of would imply that there exist $X, Y, Z \in \mathcal{U}_*$ such that

$$X > Y >_3 Z >_3 X$$

where "$>$" is one of $>_1$, $>_2$ and $>_3$. We consider these cases separately.

Case 1: $X >_1 Y >_3 Z >_3 X$. Lemma 6.14 implies that $I_{YX} \not\subset I_{YZ}$ (the condition $I_{ZXY} \neq I_Z$ of Lemma 6.14 is satisfied since the intersections do not have minima). Hence $I_{YXZ} = I_{YZ}$. Let t converge to $t_{YZ}^{\inf}$ from above. Then $\beta_{YZ}(t) \to +\infty$ since $Y >_3 Z$. The relation $\phi_{YX} = \phi_{ZX} \circ \phi_{YZ}$ (cf. Lemma 6.12) means that

$$\beta_{YX}(t) = \beta_{YZ}(t) + \beta_{ZX}(\alpha_{YZ}(t)).$$

It follows that $\beta_{YX}(t) \to +\infty$ (since $Z >_3 X$, the term $\beta_{ZX}(\alpha_{YX}(t))$ cannot converge to $-\infty$). Therefore β_{YX} is undefined at $t_{YZ}^{\inf}$, or, in other words, $t_{YZ}^{\inf} \notin I_{YX}$. Hence $I_{YX} \subset I_{YZ}$, a contradiction.

Case 2: $X >_2 Y >_3 Z >_3 X$. Since $X >_2 Y$, we have $I_{YX} = I_Y$ and hence $t_{YZ}^{\inf} \in I_{YX}$. Then $I_{YXZ} = I_{YZ}$, and the same argument as in Case 1 yields that $\beta_{YX}(t) \to +\infty$ as $t \downarrow t_{YZ}^{\inf}$. Hence $t_{YZ}^{\inf} \notin I_{YX}$, a contradiction.

Case 3: $X >_3 Y >_3 Z >_3 X$. We may assume that $I_{YXZ} = I_{YZ}$ (otherwise interchange x and z and change $+\infty$ to $-\infty$). Then the same argument as in Case 1 yields that $\beta_{YX}(t) \to +\infty$ as $t \downarrow t_{YZ}^{\inf}$. This contradicts the assumption that $X >_3 Y$.

We summarize the results of this subsection in the following

Lemma 6.15. *The set $\mathcal{U}$ of forward envelopes equipped with the relation $>_* := >_1 \cup >_2 \cup >_3$ is a pre-foliation.*

Proof. The above argument shows that $>_*$ is a total order on every set $\mathcal{U}_p$, $p \in M$. It remains to verify axioms 1–3 from Definition 4.8.

Axiom 1 follows from Lemma 6.13. Axiom 2 for the relation $>_1$ is verified in subsection 6.2.2. For the relations $>_2$ and $>_3$, Axiom 2 follows easily from the construction. Indeed, it suffices to verify that the order between marked patches $X = (F, x)$ and $Y = (G, y)$ is preserved when one moves the marked points x and y along a forward curve. More precisely, if the lifts of a forward curve γ to F and G connect x to x' and y to y', respectively, we need to prove that the marked patches $X' = (F, x')$ and $Y' = (G, y')$ are ordered the same way as X and Y. This follows from the fact that the interval I_{XY} is a left subinterval of $I_{X'Y'}$ and the function β_{XY} from the definition of $>_3$ is a restriction of a similar function $\beta_{X'Y'}$.

Axiom 3 follows from the fact that all our structures are defined canonically by the oriented distributions E^s and E^{cs} and the original pre-foliation $(\mathcal{A}, >)$. Hence if a diffeomorphism $f : M \to M$ preserves these structures, the resulting structure $(\mathcal{U}, >_*)$ is also preserved. □

6.3. **Proof of Proposition 4.13.** Recall that our objective is to extend a pre-foliation $\mathcal{A}$ so that for every $p \in M$ the resulting pre-foliation $\mathcal{B}$ contains a marked patch $B \in \mathcal{B}_p^+$. Furthermore, we want $\mathcal{B}_p^+$ to contain neighbors for all backward patches from A_p^-.

We define $\mathcal{B}$ as the union of $\mathcal{A}$ and $\mathcal{U}$ with the order defined as the union of the order of $\mathcal{A}$ and the order $>_*$ on $\mathcal{U}$ extended as follows: if $p \in M$, $A \in \mathcal{A}_p$ and $B \in \mathcal{U}_p$, we set $B > A$ if $A \in \Sigma(X)$ and $A > B$ otherwise.

Let us verify that this order satisfies the requirements of Definition 4.8 of a pre-foliation. Since $\mathcal{A}$ and $\mathcal{U}$ are pre-foliations (cf. Lemma 6.15), we have to check only inter-relations between marked patches from $\mathcal{A}$ and $\mathcal{U}$.

To verify transitivity, first suppose that the order is non-transitive for $A, B \in \mathcal{A}_p$ and $X \in \mathcal{U}_p$, that is, $X >_* A > B >_* X$. The relation $X >_* A$ implies $A \in \Sigma(X)$. Since $\Sigma(X)$ is a section of $\mathcal{A}_p$, it follows that $B \in \Sigma(X)$. Hence $X >_* B$, a contradiction.

Now suppose that the order is non-transitive for $X, Y \in \mathcal{U}_p$ and $A \in \mathcal{A}_p$, that is, $A > X >_* Y > A$. The relation $X >_* Y$ implies that $\Sigma(Y) \subset \Sigma(X)$. Since $Y > A$, we have $A \in \Sigma(Y) \subset \Sigma(X)$, hence $X > A$, a contradiction. Thus the order on $\mathcal{B}$ is transitive and hence a total order on each set $\mathcal{B}_p$.

Axioms 1 and 2 follow from Lemma 6.8. Axiom 3 follows from the same "general nonsense" argument as in the proof of Lemma 6.15.

It remains to verify the additional properties from Proposition 4.13. The existence of a forward patch $B \in \mathcal{B}_p^+$ follows from Lemma 6.11 applied to $\Sigma_0 = \varnothing$. To prove the existence of a forward patch $B \in \mathcal{B}_p^+$ neighboring a given backward patch $A \in \mathcal{A}_p^-$, apply Lemma 6.11 to the set $\Sigma_0 = \{X \in \mathcal{A}_p : X \leq_p A\}$. The resulting patch $B = (S, o)$ is a neighbor of A. Indeed, A is the maximum element of $\Sigma_0 = \Sigma(B)$, hence $B > A$ and there is no $X \in \mathcal{A}_p$ such that $B > X > A$. Suppose that $B > X > A$ for some $X \in \mathcal{U}_p$. Then $\Sigma(X) = \Sigma(B)$ since A is the maximum of $\Sigma(B)$. It follows that B and X are incomparable by $>_1$. Then $X >_2 B$ since I_B is a one-point interval and hence $I_{BX} = I_B$ (cf. the definition of the relation $>_2$). Thus A and B are neighbors in $\mathcal{B}$.

7. Proof of the Key Lemma II: Separation of leaves

Let M be a closed orientable 3-dimensional manifold, E an oriented 2-dimensional continuous distribution and $\mathcal{A}$ a branching foliation tangent to E.

We say that $\mathcal{A}$ is *complete* if it contains all limits of its leaves w.r.t. the compact-open topology.

Remark. There exist non-complete branching foliations. For example, consider a one-dimensional branching foliation of $\mathbb{R}^2$ by the graphs of functions $x \mapsto \pm f(x + \mathit{const})$ where a smooth function $f : \mathbb{R} \to \mathbb{R}$ equals zero on $(-\infty, 0]$ and is strictly monotone on $[0, +\infty)$.

Lemma 7.1. *Every branching foliation $\mathcal{A}$ is contained in a complete one.*

Proof. Consider the set of all limits of surfaces from $\mathcal{A}$ in the compact-open topology. This set is a branching foliation since the conditions that the surfaces are tangent to E and have no topological crossings are preserved by passing to limits. $\square$

Now the following theorem completes the proof of the Key Lemma.

Theorem 7.2. *Let $\mathcal{A}$ be a complete branching foliation and $\varepsilon > 0$. Then there is a C^0 foliation $\mathcal{A}_\varepsilon$ with C^1 leaves such that the angles between $T\mathcal{A}_\varepsilon$ and E are no greater than ε (here we use any auxiliary Riemannian metric on M).*

Furthermore, there is a continuous map $h_\varepsilon \colon M \to M$ such that $dist_{C^0}(h_\varepsilon, id_M) < \varepsilon$ and h_ε sends every leaf of $\mathcal{A}_\varepsilon$ to a leaf of $\mathcal{A}$.

Proof. We will refer to E as the *horizontal* distribution and surfaces tangent to E as horizontal surfaces. Let W be a unit smooth vector field “almost orthogonal” to E (for instance, the angle between W and E is $\frac{1}{1000}$-close to $\pi/2$). We will speak of W as the “vertical” direction.

Let ϕ^t, $t \in \mathbb{R}$, denote the flow along W, that is, $\phi^t : M \to M$ shifts every point by distance t along its trajectory of W.

For each $p \in M$, choose a regular coordinate system at p (cf. Section 3 for the definition of regular coordinates) such that W is the 3rd coordinate vector field $\partial/\partial z$. In this coordinate system, the leaves of $\mathcal{A}$ are almost parallel to the xy-plane. As is the previous sections, we assume that the metric of M is rescaled so that such a coordinate system covers a ball of radius 100 centered at p. Note that ϕ^t adds t to the z-coordinate in such a coordinate system.

We will work with local surfaces Σ whose tangent planes are close to E. It is clear that if Σ passes close to the center of a regular neighborhood, then Σ is the graph of a C^1 function $z = f_\Sigma(x, y)$.

Since M is compact, there is a finite collection $\{p_i\}_{i=1}^k$ such that the unit balls $B(p_i, 1)$ cover M. We refer to the larger balls $B(p_i, 100)$ as *standard neighborhoods* and to the smaller balls $B(p_i, 1)$ as *cores* (of standard neighborhoods).

Fix a standard neighborhood $B(p_i, 100)$ and a segment U_i of the integral curve of W centered at p_i and of length 50. Let $\mathcal{A}_i$ be the collection of pairs (S, x) where S is a surface from $\mathcal{A}$ and $x \in dom(S)$ is such that $S(x) \in U_i$. We refer to elements of $\mathcal{A}_i$ as *marked surfaces.*

Now define a non-strict total order $\geq_i$ on $\mathcal{A}_i$ as follows. Choose $A_1, A_2 \in \mathcal{A}_i$, $A_1 = (S_1, x_1)$, $A_2 = (S_2, x_2)$. Obviously there exists an intrinsic ball $D = B_r(x_1) \subset dom(S_1)$ such that a piece of A_2 is the graph of a C^1 function $f : D \to \mathbb{R}$ in the following sense: the surface $S_1^f : D \to M$ defined by

$$S_1^f(x) = \phi^{f(x)}(S_1(x)), \qquad x \in D,$$

coincides, up to a change of a parameter sending x_1 to x_2, with a region in S_2. Let r be the maximum radius of such a ball (possibly $r = \infty$). Since the surfaces have no topological crossings, the function f does not change sign. We set $A_2 \geq_i A_2$ if $f \geq 0$ and $A_1 \geq_i A_2$ if $f \leq 0$.

Note that it is possible that both inequalities $A_1 \geq_i A_2$ and $A_2 \geq_i A_1$ hold. This means that S_1 and S_2 coincide up to a parameter change (sending x_1 to x_2). In this case we write $A_1 \approx_i A_2$. Observe that $\approx_i$ is an equivalence relation and $\geq_i$ is a total order on the quotient $\bar{\mathcal{A}}_i = \mathcal{A}_i / \approx_i$. We write $A_1 >_i A_2$ if $A_i \geq_i A_2$ and $A_1 \not\approx_i A_2$.

Lemma 7.3. *The ordered set $(\bar{\mathcal{A}}_i, >_i)$ is isomorphic to the interval $(0, 1)$ with the standard order.*

In terms of $\mathcal{A}_i$ (that is, before identifying reparametezations), this means that there is a surjective map $\theta_i : \mathcal{A}_i \to (0, 1)$ such that $\theta_i(A_1) > \theta_i(A_2)$ iff $A_1 >_i A_2$ for all $A_1, A_2 \in \mathcal{A}_i$.

Proof. Recall that the set $\mathcal{A}_i$ regarded with the compact-open topology is pre-compact and hence it contains a countable dense set P. Note that the order $<_i$ agrees with the compact-open topology on $\mathcal{A}_i$ in the following sense: if two sequences $\{A_n\}$ and $\{B_n\}$ in $\mathcal{A}_i$ converge to $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_i$ respectively, and $A_n \geq_i B_n$ for all n, then $A \geq_i B$. Therefore the order $<_i$ on $\bar{\mathcal{A}}_i$ satisfies the following properties:

(1) there is no maximum and no minimum element (since U_i is open);

(2) every increasing bounded sequence has a sharp upper bound (this follows from the fact that our branching foliation is complete).

(3) the countable set $\bar{P} = P/\approx$ separates points in $\bar{\mathcal{A}}_i$, that is, for every $A, B \in \bar{\mathcal{A}}_i$ such that $A <_i B$ there exists a $C \in \bar{P}$ such that $A <_i C <_i B$. Indeed, consider a short vertical segment connecting the surfaces A and B near a point where they diverge. Since P is dense in $\mathcal{A}_i$, there exists $C \in P$ intersecting the interior of this segment, such a C separates A and B.

A total order with these properties is isomorphic to an open interval of $\mathbb{R}$. To prove this, first observe that $\bar{P}$ is isomorphic to a dense subset $Q \subset \mathbb{Q} \cap (0, 1)$. Indeed, add a maximum $+\infty$ and a minimum $-\infty$ to $\bar{P}$ and enumerate the points of $\bar{P}$ as $P_1, P_2, P_3, \ldots$, so that $P_1 = -\infty$ and $P_2 = +\infty$. Then define inductively a sequence of binary rationals r_i. Set $r_1 = 0$ and $r_2 = 1$. Assume that all r_j, $j < i$, are defined. $P_1, P_2, \ldots, P_{i-1}$ divide $\bar{P}$ into $i-2$ intervals, and P_i lies in one of the intervals, say the interval between P_k and P_l $(k, l < i)$. Set $r_i = \frac{r_k + r_l}{2}$, that is, r_i is the midpoint of the segment $[r_k, r_l]$.

Let us show that the set $\{r_i\}_{0<i<\infty}$ is dense. Indeed, r_i's, $i = 1, 2, \ldots, i-1$ partition $[0, 1]$ into $i-2$ segments. Denote the maximal length of these segments by l_i. We want to show that $l_i \to 0$ as $i \to \infty$. When we add r_i, one of the lengths is divided into two equal parts. If l_i does not converge to 0, then there is a segment $[r_i, r_j]$ that was never divided. Therefore there is no element of $\bar{P}$ between P_i and P_j, a contradiction.

Define $\theta_P : \bar{P} \to \mathbb{Q} \cap (0, 1)$ by $\theta_P(P_i) = r_i$. Now for every $A \in \bar{\mathcal{A}}_i$ define

$$\theta(A) = \sup\{\theta_P(B) : B \in P, B <_i A\}.$$

Then θ is an order-preserving bijection from $\bar{\mathcal{A}}_i$ to $(0, 1)$. □

To separate the surfaces of our branching foliation near p_i we will move every point of every surface $A \in \mathcal{A}_i$ upwards (along W) a distance depending on the point and on the parameter $\theta_i(A)$. Adding together these moves for $i = 1, \ldots, k$, we will eliminate all branchings.

We start by considering the marked surfaces $A_1, A_2, A_3 \in \mathcal{A}_i$ such that A_1 lies below the core $B(p_i, \varepsilon_0)$, A_2 lies $2\varepsilon_0$ above the core, and A_3

lies above A_2, and their z-coordinates are separated by at least 2 (see a formal description below). Speaking informally, we pull A_2 up towards A_3 squeezing proportionally the intervals of local leaves of W between A_2 and A_3. Between A_1 and A_2 we increase the z-coordinate by a function which (for fixed x, y) is strictly monotone in the separating parameter θ_i constructed in Lemma 7.3.

The formal description is as follows. Every marked surface $A \in \mathcal{A}_i$ is locally the graph of a function $z = g_A(x, y)$ in our coordinate system. We choose $A_1, A_2, A_3 \in \mathcal{A}_i$ so that $g_{A_1}(0,0) = -2$, $g_{A_2}(0,0) = 2$ and $g_{A_3}(0,0) = 4$. To reflect the dependence of our construction on θ_i from Lemma 7.3, we denote g_A by h_t where $t = \theta_i(A)$. Let $t_j = \theta_i(A_j)$ for $j = 1, 2, 3$.

Choose a positive $\delta < \min(h_{t_3} - h_{t_2})$. Set

$$G_i(x,y,t) = \begin{cases} 0 & \text{if } t \geq t_3 \text{ or } t \leq t_1, \\ \delta \cdot \dfrac{t - t_1}{t_2 - t_1} & \text{if } t \in [t_1, t_2], \\ \delta \cdot \dfrac{h_t(x,y) - h_{t_3}(x,y)}{h_{t_2}(x,y) - h_{t_3}(x,y)} & \text{if } t \in [t_2, t_3]. \end{cases}$$

Let $K(x, y)$ be a C^∞ function such that $K(x, y) = 1$ if $|(x, y)| \leq 2$, and $K(x, y) = 0$ if $|(x, y)| > 10$. Set $G_i^0(x, y, t) = G_i(x, y, t)K(x, y)$. For a marked surface $A = (S, a) \in \mathcal{A}_i$ define a function $F_i^A : dom(S) \to \mathbb{R}$ by

$$F_i^A(q) = \begin{cases} G_i^0(x(S(q)), y(S(q)), \theta_i(A)) & \text{if } dist(a,q) < 20, \\ 0 & \text{otherwise,} \end{cases}$$

where $x(S(q))$ and $y(S(q))$ are the x- and y-coordinates of $S(q)$ in our coordinate system and $dist(a, q)$ is the intrinsic distance in $dom(S)$. (Recall that the intrinsic metric on $dom(S)$ is induced by the immersion $S : dom(S) \to M$.) A surface $S \in A$ may pass near p_i several times (at most countably many) so it may have several markings lying in $\mathcal{A}_i$. These markings correspond to points in the pre-image $S^{-1}(U_i)$, where U_i is a segment of a trajectory of W defined in the beginning of the proof. It is easy to see that these points are separated from one another by intrinsic distance at least 50. We add together all functions F_i^A corresponding to these markings. Namely define $F_i : dom(S) \to \mathbb{R}$ by

$$F_i = \sum_{a \in S^{-1}(U_i)} F_i^{(S,a)}.$$

We may assume that the domains of the surfaces from $\mathcal{A}$ are disjoint, hence we have a function F_i defined on the union if these domains. Observe that F_i is C^1 on every surface because $G_i(x, y, t)$ is C^1 in (x, y) for every fixed t.

Finally let $F = \frac{1}{k}\sum_{i=1}^{k} F_i$.

We perturb every surface $S \in \mathcal{A}$ using F as follows: the new surface S_F is given by the formula $S_F(q) = \phi^{F(q)}(q)$ for all $q \in dom(S)$. Let $\mathcal{A}_F$ denote the set of all perturbed surfaces: $\mathcal{A}_F = \{S_F : S \in \mathcal{A}\}$.

We show that for all $\alpha \in [0, 1]$, $\mathcal{A}_{\alpha F}$ is a foliation. The tangent distributions of these foliations obviously converge to E as $\alpha \to 0$.

First we show that this perturbation preserves the local order of the surfaces along the local leaves of W. Let $S, S' \in \mathcal{A}$, $q \in dom(S)$, $q' \in dom(S')$. Suppose that $S(q)$ and $S'(q')$ lie on the same local leaf of W with $S(q)$ above $S'(q')$. The local order between is preserved if

$$F(q) - F(q') > -dist_W(S(q), S(q'))$$

where $dist_W$ denotes the distance along a leaf of W. For every $i = 1, \ldots, k$, by the definition of F_i, we have

$$F_i(q) - F_i(q') > -dist_W(S(q), S(q')).$$

Averaging these inequalities, we obtain a similar inequality for F and for αF.

If $S(q) = S'(q')$ and the corresponding marked surfaces have different parameters with respect to at least one local order θ_i (say, q is above q'), then

$$F_i(q) > F_i(q')$$

and $F_j(q) \geq F_j(q')$ for all j, by the definition of F_j. Hence $F(q) > F(q')$.

Thus $S_{\alpha F}(q) \neq S'_{\alpha F}(q')$ unless S and S' coincide up to a parameter change sending q to q'. It is obvious from the construction that the surfaces from $\mathcal{A}_{\alpha F}$ cover M. This means that $\mathcal{A}_{\alpha F}$ is a foliation of M.

Since F is C^1 on every surface, the perturbed surfaces are C^1. If α is so small that $\|F\|_{C^1} < \varepsilon$, then the desired continuous map $h_\varepsilon : M \to M$ is defined by returning each point $p \in M$, $p = S_{\{\alpha F\}}(q)$, $q \in dom(S)$, to its original position $S(q)$. □

References

[Ano67] D. V. Anosov. Tangential fields of transversal foliations in y-systems. *Mat. Zametki*, 2:539–548, 1967.

[BW] C. Bonatti, A. Wilkinson. Transitive partially hyperbolic diffeomorphisms on 3-manifolds. *Topology*, 44 (2005), no. 3, 475–508.

[Br] M. Brin. On dynamical coherence. *Ergodic Theory Dynam. Systems* 23 (2003), no. 2, 395–401.

[BBI] M. Brin, D. Burago, S. Ivanov On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group. In *Modern dynamical systems and applications*, 307–312, Cambridge Univ. Press, Cambridge, 2004.

[BBI2] M. Brin, D. Burago, S. Ivanov Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-Torus. In *Journal of Modern dynamical systems and applications*, this volume.

[BPSW01] Keith Burns, Charles Pugh, Michael Shub, and Amie Wilkinson. Recent results about stable ergodicity. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 327–366. Amer. Math. Soc., Providence, RI, 2001.

[CLN85] César Camacho and Alcides Lins Neto. *Geometric theory of foliations*. Birkhäuser Boston Inc., Boston, MA, 1985. Translated from the Portuguese by Sue E. Goodman.

[CC03] Alberto Candel and Lawrence Conlon *Foliations II*. Graduate Studies in Mathematics, vol. 60. Amer. Math. Soc., Providence, RI, 2001.

[DPU99] Lorenzo J. Díaz, Enrique R. Pujals, and Raúl Ures. Partial hyperbolicity and robust transitivity. *Acta Math.*, 183(1):1–43, 1999.

[HPS77] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 583.

[Nov65] S. P. Novikov. The topology of foliations. *Trudy Moskov. Mat. Obšč.*, 14:248–278, 1965.

[PS97] Charles Pugh and Michael Shub. Stably ergodic dynamical systems and partial hyperbolicity. *J. Complexity*, 13(1):125–179, 1997.

[Sma67] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[Sol82] V. V. Solodov. Components of topological foliations. *Mat. Sb. (N.S.)*, 119(161)(3):340–354, 447, 1982.

[Wil98] Amie Wilkinson. Stable ergodicity of the time-one map of a geodesic flow. *Ergodic Theory Dynam. Systems*, 18(6):1545–1587, 1998.

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802
E-mail address: burago@math.psu.edu

Steklov Math. Institute, St. Petersburg, Russia
E-mail address: svivanov@pdmi.ras.ru