TWO COUNTER-EXAMPLES IN LOW DIMENSIONAL LENGTH GEOMETRY

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1. Introduction

One of the key properties of the length of a curve is its *lower semi-continuity*: if a sequence of curves γ_i converges to a curve γ, then $length(\gamma) \leq \liminf length(\gamma_i)$. Here the weakest type of point-wise convergence suffices.

There are higher-dimensional analogs of this semi-continuity for Riemannian (and even Finsler) metrics. For instance, the Besicovitch inequality (see, for instance, [1] and [4]) implies that if a sequence of Riemannian metrics d_i on a manifold M uniformly converges to a Riemannian metric d, then $Vol(M,d) \leq \liminf Vol(M,d_i)$. Furthermore, the same is true if the limit metric is Finsler (where one can use any "reasonable" notion of volume for Finsler manifolds); the proof, though, is more involved (see [2], [7]).

However, we will give an example of an increasing sequence of Riemannian metrics d_i on a 2-dimensional disc D, which uniformly converge to a length metric d on D such that $Area(D,d_i) < \frac{1}{10}$ and $Area(D,d) > 1$ (where by $Area(D,d)$ we mean the 2-dimensional Hausdorff measure). Furthermore, metrics d_i and d can be realized by a uniformly converging sequence of embeddings of D into $\mathbb{R}^3$.

Our motivation for studying the semi-continuity of the surface area functional came from [3], where a more sophisticated Besicovitch-type inequality for Finsler metrics is shown. The proof is essentially Finsler, even though the inequality makes sense for general length spaces.

The counter example undermines a natural approach to proving length-area inequalities for length spaces by means of approximations by Riemannian (more generally, Finsler) metrics. Similar considerations lead to the following question: can every intrinsic metric on a disc be approximated by an increasing sequence of Finsler metrics? There is some evidence suggesting that the answer is likely affirmative in dimension two. However, we will give an example of an intrinsic metric on a 3-dimensional ball such that no neighborhood of the origin admits a Lipschitz bijection to a Euclidean region.

In this elementary exposition we present both counter-examples. Unfortunately, people often choose not to publish the results of research that led to counterexamples rather than proofs of desired theorems; as such, even published counterexamples tend to be forgotten. Hence we cannot be confident in complete novelty of the results. At the very least, we use this paper to raise open problems and embed these problems into a new context.

The paper is organized as follows. In the rest of the Introduction we give rigorous formulations of the results and outline the proofs. Sections 2 and 3 contain proofs

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of Theorems 1 and 2, respectively. Concluding Section 4 contains a brief discussion and several open problems.

Theorem 1. *There exists a length space* (X, d) *and a point* $p \in X$ *satisfying the following properties: 1)* X *is homeomorphic to an open Euclidean ball* $B \subset \mathbb{R}^3$*; 2) No neighborhood* U *of* p *admits a homeomorphism* $\varphi : U \to V \subset \mathbb{R}^3$ *such that* $\forall x, y \in X$*,* $d(x, y) \geq \rho(\varphi(x), \varphi(y))$*, where* ρ *is a Euclidean metric.*

Note that this theorem actually gives us a length metric on a three dimensional Euclidean ball that cannot be approximated from below by any Finsler metric. Indeed, every Finsler metric is locally bi–Lipschitz equivalent to a Euclidean one. As long as the result holds for the Euclidean metric, then it will hold for any Finsler metric, since the Finsler metric is bounded by multiples of the Euclidean metric.

The proof of this first result proceeds as follows: we will modify the standard Euclidean metric by constructing a metric as a limit of metrics which are changed on a countable collection of disjoint tori. These tori are the boundaries of tubular neighborhoods of linked circles; the circles form chains "almost" connecting two fixed points. We will use an estimate on the distance between two linked circles in $\mathbb{R}^3$ via the lengths of the circles. If the metric space we construct could be mapped by a Lipschitz-1 map onto a Finsler disc, this estimate would imply that the distance between the fixed points will be zero, a contradiction.

The second result is the following:

Theorem 2. *The Hausdorff measure* h_2 *is not lower semi-continuous on length metrics on a topological disk with respect to* C^0*–convergence of the distance functions.*

One way to picture the result is as starting with a sequence $\{d_n\}$ of length metrics defined on the same two dimensional disk D. We choose the metrics d_n so that they have a limit d^* and so that the convergence of the l ength spaces (D, d_n) to the limit space (D, d^*) occurs in a C^0 manner. Then the limit of the two dimensional Hausdorff measures $h_2(D, d_n)$ is strictly less than the two dimensional Hausdorff measure $h_2(D, d^*)$.

Actually we prove a somewhat stronger statement. We construct a sequence $\{D_i\}$ of embedded disks in $\mathbb{R}^3$ whose boundaries are all the same Euclidean circle. The result states that the C^0 limit D_0 of these disks exists, is an embedded disk in $\mathbb{R}^3$, and that the limit of the two dimensional Hausdorff measures $h_2(D_i)$ of the disks D_i is strictly less than the two dimensional Hausdorff measure $h_2(D_0)$ of the limit disk D_0. The reader will note that, although we use elements of both interpretations, we favor the former one.

Let us now give an outline for the proof of the second theorem. We construct a surface X as the union of the surface of a neighborhood of an infinite tree inside the unit cube in $\mathbb{R}^3$ and a certain Cantor set on the top face of the cube. The neighborhood of the tree thins as we approach the top of the cube. We show that this surface is homeomorphic to the standard Euclidean disk in $\mathbb{R}^2$. The metrics d_n from the first point of view above will come from the metrics on the disk induced on the surface of certain (finite) parts of the neighborhood of the tree; the limit metric d^* mentioned above will arise from the metric induced on the disk by the metric on the constructed surface X.

2. Proof of The Inability to Approximate From Below by a Finsler Metric

In this section, we prove Theorem 1. We first will construct the space X by constructing a metric on the standard Euclidean ball B in $\mathbb{R}^3$ as a limit of metrics changed on a countable collection of disjoint solid tori.

The tori will arise as tubular neighborhoods of a certain collection of linked Euclidean circles. We will then show that the space X with the new metric d is homeomorphic to B. Finally we will derive a contradiction if both a homeomorphism and a Finsler metric exist as in the statement of the theorem.

Construction 2.1. We now construct X. Let $a, b \in B \subset \mathbb{R}^3$, and choose a sequence of disjoint segments $S_i = [a_i, b_i]$ which are in B and which converge pointwise to $[a, b]$ as $i \to \infty$ in the usual Euclidean metric.

We remark here that the metric we are constructing is not a Riemannian metric. We regard the situation as follows: consider the complement to the segment $[a, b]$. This is an open manifold, and on it we will construct a true Riemannian metric. However, we then want to glue the segment $[a, b]$ back onto the space. What this will amount to is in fact compactifying the space the construction of which we will have just completed. We will also note that adding this key segment $[a, b]$ back into the space does not undermine the compatibility of the metrics; the metrics on the two spaces (the open manifold and $[a, b]$) match.

The next step is to construct a tubular neighborhood around each segment S_i. Inside a portion of these tubular neighborhoods is where we will alter the definition of the metric. Let us explain what will happen. For each S_i, consider a tubular neighborhood T_i of it of radius ϵ_i such that:
- no two tubular neighborhoods of disjoint segments S_i, S_j intersect,
- none of the tubular neighborhoods T_i intersect $[a, b]$,
- all tubular neighborhoods are contained in B, and
- $\epsilon_i \to 0$ as $i \to \infty$.

Given a segment S_i and its tubular neighborhood T_i, we find a sequence of n_i linked Euclidean circles C_i^j contained in T_i. See Figure 1. We desire that the total length of all circles C_i^j be less than 10 times the Euclidean length of the segment S_i. Then to each of the circles C_i^j, associate a tubular neighborhood T_i^j also contained inside T_i such that:
- a_i lies on the first circle C_i^1,
- b_i lies on the last circle $C_i^{n_i}$,
- the radius of each tubular neighborhood T_i^j is $\epsilon_i/8$, and
- no two distinct tubular neighborhoods T_i^j and T_i^k intersect.

We now desire to change the metric on the tubular neighborhood T_i. On the complement $T_i \setminus \cup_j T_i^j$ of the smaller tubular neighborhoods, the metric is the usual Euclidean metric. On each tubular neighborhood T_i^j, we change the metric as follows: first consider the smaller tubular neighborhoods $T_i^j(1) \subset T_i^j(2) \subset T_i^j(3) \subset T_i^j$ around the circle C_i^j of radii $\epsilon_i/32, \epsilon_i/16, 3\epsilon_i/32$ respectively. The plan is to make distances very small when in some close region of the circle C_i^j, but to make the price of reaching this region large by increasing distances around it. See Figure 2 for a sketch of these changes.

To be more specific, on the tubular neighborhood $T_i^j(1)$, we multiply the Euclidean metric by a factor $1/(in_i)$. On $T_i^j(2) \setminus T_i^j(1)$, we multiply the Euclidean metric by a smooth, increasing function f which depends on the distance r from C^i_j. Note that $r \in [\epsilon_i/32, \epsilon_i/16)$. We also require that $f(\epsilon_i/32) = 1/(in_i)$ and that $f(\epsilon_i/16) = 128$. On $T^j_i(3) \setminus T^j_i(2)$, we multiply the Euclidean metric by 128. Finally, on $T^j_i \setminus T^j_i(3)$, we multiply the Euclidean metric by a smooth, decreasing function g depending on the distance r from C^j_i satisfying $g(3\epsilon_i/32) = 128$ and $g(\epsilon_i/8) = 1$. To summarize, on each tubular neighborhood T^j_i, we construct a new metric by multiplying the Euclidean metric by a smooth function h depending on the distance r from C^j_i, such that $h\big|_{[0,\epsilon_i/32]} = 1/(in_i)$, $h\big|_{[\epsilon_i/32,\epsilon_i/16]}$ is increasing, $h\big|_{[\epsilon_i/16,3\epsilon_i/32]} = 128$, $h\big|_{[3\epsilon_i/32,\epsilon_i/8]}$ is decreasing, and $h(\epsilon_i/8) = 1$.

We have defined a new metric d on each tubular neighborhood T_i. On the complement of all tubular neighborhoods in $B \setminus [a, b]$, let d be the usual Euclidean metric. In this way, we have defined a new metric d on all of $B \setminus [a, b]$.

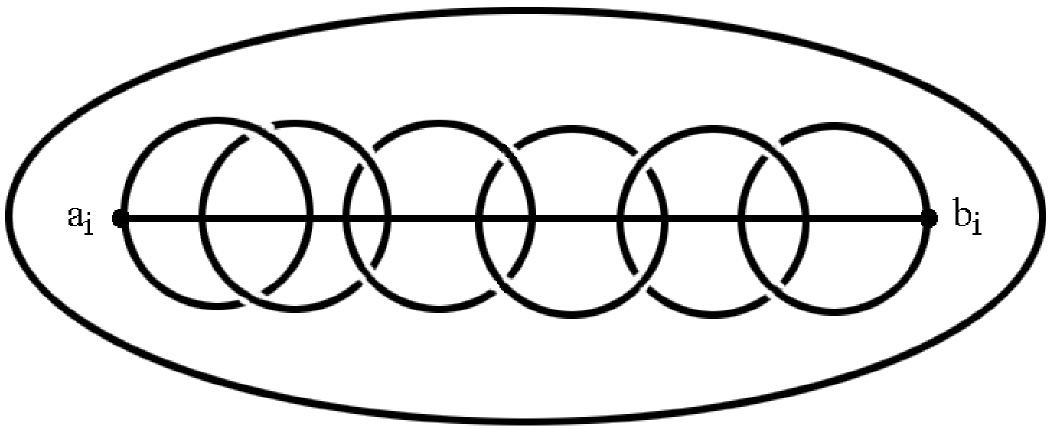

FIGURE 1. A sketch of the tubular neighborhood T_i

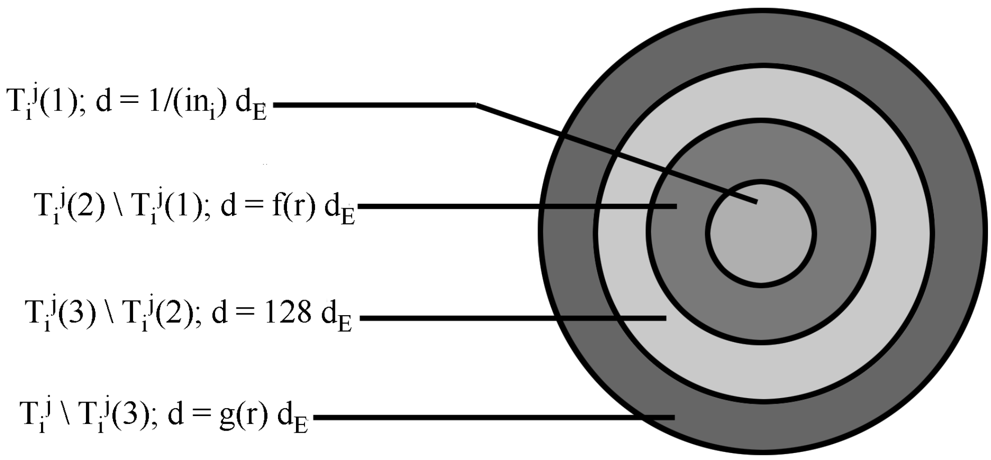

FIGURE 2. The change in metric.

The final step is to glue the segment $[a, b]$ back onto our manifold. This gluing completes the construction of the space X.

We note that gluing the segment $[a, b]$ back onto the manifold is the same as compactifying the manifold. To demonstrate this fact concretely, we need to prove the following

Lemma 2.2. *Given two points $p, q \in [a, b] \subset (X, d)$, the segment $[p, q] \subset [a, b]$ remains the shortest path between p and q.*

Proof. To begin, we note one important property of d:

• On any given tubular neighborhood T_i, d is sandwiched between $1/(in_i)d_E$ from below and $128d_E$ from above.

With this property in mind, let us make an estimate on the length of a segment going through a tubular neighborhood T_i^j. The length of a segment through T_i^j which intersects C_i^j has length

$$2\int_0^{\frac{\epsilon_i}{8}} h(x)\ dx \geq 2\int_0^{\frac{\epsilon_i}{32}} \frac{1}{in_i}\ dx + 2\int_{\frac{\epsilon_i}{16}}^{\frac{3\epsilon_i}{32}} 128\ dx = \frac{\epsilon_i}{16in_i} + 8\epsilon_i.$$

Thus, any shortest path with respect to the Euclidean distance which originally passed through the smaller tubular neighborhood $T_i^j(1)$ will be shorter in the metric d. However, any shortest path with respect to the Euclidean distance which originally passed through every smaller tubular neighborhood of T_i^j will see its length increase drastically. Also, given two points, if the shortest path between them with respect to the Euclidean distance did not pass through every smaller tubular neighborhood of T_i^j, the shortest path between them with respect to d will not pass through every smaller tubular neighborhood of T_i^j either.

Notice, then, that the shortest path between points p, q on the segment $[a, b]$ under the metric d will be along $[a, b]$. $\square$

We have been stating without proof that d is a metric on the space X. We now prove this formally, since d is not bounded away from zero.

Lemma 2.3. *d is a metric on X.*

Proof. To show that d is a metric, we will show that distances under d are finite and that no distinct points are identified as having zero distance with respect to d.

To see that d-distances are finite, it is enough to notice that the segment between two points still has finite distance under d. Therefore, the d-distance between any two points is less than or equal to the length of the segment between those two points under d. The length of this segment under d is no more than 128 times its Euclidean length, and thus, d-distances are finite.

Now we want to show that if $d(p, q) = 0$, then $p = q$. First notice that we did not change the metric on $[a, b] \cup (\cup T_i)^c$. Thus if both p and q are in $[a, b] \cup (\cup T_i)^c$, then there are neighborhoods of p and q in which the shortest path is a segment. This segment must have non-zero length in these neighborhoods, and so $d(p, q) \neq 0$ unless $p = q$.

If p and q are in $\cup T_i$, let $p \in T_j$, $q \in T_k$, where both metrics are finite. At worst, p, q are both in an tubular neighborhood $T_j^l(m)$ for $m = 1, 2$. But if this is true, the distance between p, q is bounded from below by $1/(jn_j)d_E(p, q)$.

The final case to consider is if $p \in [a, b]$, $q \in \cup T_i$. But again, let $q \in T_j$, and then $d(p, q) \geq d_E(p, q)$ as above. $\square$

Our next step is to show that the topology of B is unchanged under the change in metric.

Lemma 2.4. *Given a sequence $\{x_n\}$ in B, it converges under the metric d if and only if it converges under the Euclidean metric d_E.*

Proof. First, assume that a sequence $\{x_n\}$ converges to a point x with respect to the metric d. We must show that $\{x_n\}$ converges to x with respect to the Euclidean metric d_E. There are two cases. If $x \in T_i^j(m)$ for $m = 1, 2$, for some i, j, then for large enough n, $x_n \in T_i^j(m)$ for $m = 1, 2$ also. In this case, $d(x_n, x) \geq 1/(in_i)d_E(x_n, x)$. So, $d(x_n, x) \to 0$ implies $1/(in_i)d_E(x_n, x) \to 0$, which in turn means that $d_E(x_n, x) \to 0$. Thus $\{x_n\}$ converges to x in d_E.

If we assume, on the other hand, that x is not in some $T_i^j(m)$ for $m = 1, 2$, then $d(x_n, x) \geq d_E(x_n, x)$. Thus if a sequence converges in the former metric, it converges in the latter.

Now we must show that if a sequence $\{x_n\}$ converges to x with respect to d_E, then the sequence converges to the same point in (B, d). Since d is bounded from above by $128d_E$, if $d_E(x_n, x) \to 0$, then $d(x_n, x) \to 0$ as well. □

Using the previous lemmas, we note that (X, d) is homeomorphic to (B, d_E). For this particular ball, we have thus proven the first property of Theorem 1.

To prove the second property for this ball B, we now must show that one cannot choose both a (Finsler) metric ρ on B and a homeomorphism $\varphi : X \to B$ as in the theorem. We will assume for the sake of a contradiction that both do exist. To find the contradiction, we require another lemma, one that estimates the distance between linked curves in terms of their lengths.

Lemma 2.5. *There is a constant C such that for any Finsler metric ρ, there is an $\epsilon > 0$ such that given two linked curves γ_1, γ_2 of length less than ϵ, then $\rho(\gamma_1, \gamma_2) \leq C \min(l_\rho(\gamma_1), l_\rho(\gamma_2))$.*

Proof. First, we reduce the problem to looking at constants associated to a Euclidean metric. Then we will show that finding a constant in that case is straightforward. So, given a Finsler metric ρ, choose $\epsilon > 0$ such that every 100 ϵ-ball admits a 10 bi–Lipschitz homeomorphism to a Euclidean region. Having done this, we can now concentrate on the case of a Euclidean metric. Let γ_1, γ_2 be two linked curves the lengths of which are less than ϵ. Assume without loss of generality that γ_1 is the shorter of the two curves. Thus, γ_2 intersects any immersed disc whose boundary is γ_1. Choose a point x on γ_1. Consider the immersed disc D defined by connecting every point of γ_1 to x by a shortest path. Each of these shortest paths must have length less than or equal to $|\gamma_1|/2$. Since γ_2 intersects this immersed disc D (and hence, one of the segments $[x, \gamma_1(t)]$), it must be within $|\gamma_1|/4$ of the boundary $\partial D = \gamma_1$. So we have shown in this case that the constant is $1/4$. The constant for the Finsler metric ρ will then be at most $10/4$. □

With this lemma in hand, we can complete our result for this ball.

Lemma 2.6. *There does not exist both a (Finsler) metric ρ on B and a homeomorphism $\varphi : X \to B$ such that $d(x, y) \geq \rho(\varphi(x), \varphi(y)) \forall x, y \in X$.*

Proof. Assume for the sake of a contradiction that both a metric ρ on B and a homeomorphism φ as above exist. As mentioned, we will need the previous lemma to find the contradiction. As such, if ρ exists and we can show that $d(a_i, b_i) \leq \sum d(C^i_k, C^i_{k+1})$, we will have found a contradiction.

First, note that the length of a circle C^j_i in the metric d is $2\pi/(in_i)$. Then if the assumption above is true, for large enough i, the lengths of the curves are less than the given ϵ. Thus,

$$\rho(\varphi(a_i), \varphi(b_i)) \leq d(a_i, b_i) \leq \sum_{k=1}^{n_i} l(C^i_k) + \sum_{k=1}^{n_i-1} d(C^i_k, C^i_{k+1}) \leq 2\pi\left(\frac{1}{i} + (n_i - 1)C\frac{1}{in_i}\right),$$

which goes to zero as i goes to infinity. But this implies that $\rho(\varphi(a), \varphi(b)) = 0$, a contradiction with the fact that φ was a homeomorphism. □

To prove Theorem 1 in full generality, fix a point $p \in B$ and repeat the described procedure on a sequence of disjoint balls whose radii tend to zero and which converge to p. Since the result is true on each ball in the sequence, it is true for any neighborhood of p.

3. Proof of the Non-Lower-Semicontinuity of the Hausdorff Measure on Length Metrics on a Topological Disk

In this section, we provide a proof of Theorem 2. To give an example of a sequence of spaces which converge to a space but where the limit of the 2-dimensional Hausdorff measure of the spaces is strictly less than the 2-dimensional Hausdorff measure of the limit space, we will first construct an infinite tree in the unit cube in $\mathbb{R}^3$ whose branches tend to the top face of the cube. The surface we construct will be the surface of a certain neighborhood of the tree; the neighborhood will thin as we travel "up" the tree. Also included in the surface will be a Cantor set in the top face of the cube. Proceeding with the proof, we will show that this surface is homeomorphic to the standard 2-disc in $\mathbb{R}^2$. Then, our sequence of spaces will arise from each step of the tree, and we will show the desired claim.

Construction 3.1. In order to determine x and y coordinates of the points on the tree, consider the following Cantor set construction. At step zero, we have the unit interval. At step 1, remove the center interval of length 1/4 (so we have $[0, 3/8] \cup [5/8, 1]$). At step k of the procedure, one removes from the remaining subintervals the 2^{k-1} center intervals such that each of the removed intervals has the same length and the total length of the removed intervals on that step is $1/2^{k+1}$. In this way, the resulting Cantor set will have positive Lebesgue measure. For each of the 2^k endpoints of the removed intervals at step k, we associate the two endpoints of the removed subinterval in step $k+1$ that lie between an endpoint from step k and the closest endpoint from step $k-1$. We also introduce the notation c^i_k to mean the center of the i^{th} removed interval at step k.

Now, we construct the tree. As the initial vertex, choose the point $(1/2, 1/2, 0)$. From this vertex extend 4 edges, one to each point $(x, y, 3/4)$, where $x, y \in \{3/8, 5/8\}$. We then proceed by induction. More specifically, at step k, given a point of the tree $(x, y, 1-1/4^k)$ (where each of x and y is an endpoint from a removed subinterval at step k of the Cantor set construction described above), we add 4 edges from each these points, extending to points $(x_{1,2}, y_{1,2}, 1 - 1/4^{k+1})$, where $x_{1,2}$ are the two points associated to x from step $k+1$ of the Cantor set construction, and similarly for $y_{1,2}$. We then have 4 new points; we extend one edge to each point. In this way, we have constructed a tree T in the unit cube.

Now we will use the tree as a skeleton around which to build a body — a topological surface X. First, around the tree we construct a tube-like neighborhood. Fix a positive $\epsilon < 1$. Around each vertex at height $1 - 1/4^k$, we consider the Euclidean ball of radius $\epsilon/4^{k+2}$. Around each edge which connects vertices at heights $1 - 1/4^k$ and $1 - 1/4^{k+1}$, we consider the tubular neighborhood of radius $\epsilon/4^{k+3}$. Note that the choice of radii of the tubular neighborhoods prevents the intersection of tubular neighborhoods around distinct edges. We require, in regions where the tubular neighborhood around a vertex and tubular neighborhood around an edge intersect, that we adjust our neighborhoods so that they meet smoothly. We also require that the ball around the first vertex (the one at height 0) be a half-sphere. This requirement is present in order to keep our space within the unit cube. It is also because of this requirement, and the fact that tubular neighborhoods cannot thin as we have described, that we drop the term "tubular neighborhood" in favor of "tube-like neighborhood."

Because of the inductive nature of the construction of the tree, we will consider the construction of the topological surface X as a step-by-step process by labelling each successive tube-like neighborhood. Let X_0 be the half-sphere of radius $1/16$ centered at the point $(1/2, 1/2, 0)$ in the unit cube. X_1 is the space defined by the (boundary of the) tube-like neighborhood of the (partial) tree consisting of 5 vertices (1 vertex at height 0 and 4 vertices at height $3/4$) and the edges connecting them. As such, to form X_1, we consider the addition of what we shall call **capped neighborhoods** onto X_0. Each capped neighborhood is the tube-like neighborhood around an edge "capped" by the sphere around the vertex at the end of the edge with higher z coordinate. See Figure 3 for an illustration.

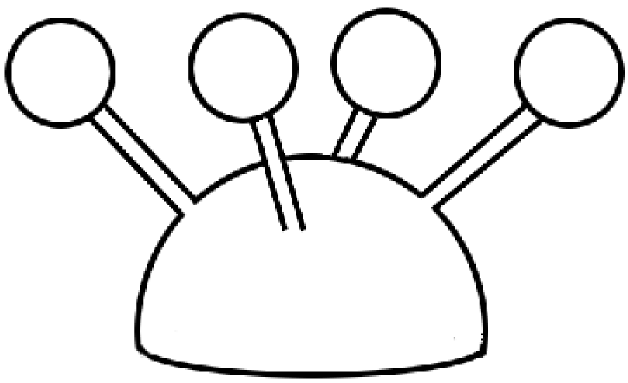

FIGURE 3. A sketch of the space X_1

We proceed inductively. Given the space X_n, we define the space X_{n+1} as the union of X_n, the tube-like neighborhoods around all vertices at height $1 - 1/4^{n+1}$ and the set of tube-like neighborhoods around all edges connecting those vertices to points of X_n. In other words, X_{n+1} is the union of X_n and the 4^{n+1} capped neighborhoods emanating from X_n. In essence, any space X_n are pieces of spheres and cylinders glued together in a particular way.

We make one final addition to complete our definition of the space X, and that is to consider the closure of the union of the spaces X_n above. The limit points we add are at height 1, and we will associate them to points of a certain Cantor set in the disc D.

Now that the construction of X is complete, we show that X is homeomorphic to the two-dimensional disc D. We proceed in two steps. First, we inductively define a sequence of maps (which will in fact be homeomorphisms) between the spaces X_i and the two-dimensional unit disc. We then show that the limit of the sequence of homeomorphisms is itself a homeomorphism.

Lemma 3.2. *Each space X_i is homeomorphic to the two-disc D.*

Proof. To begin, we map X_0 (recall that this is the half-sphere of radius $1/16$ centered at the point $(1/2, 1/2, 0)$ of the unit cube) to the standard unit disc D in $\mathbb{R}^2$ via a homeomorphism F_0. To define the map F_1, consider the unit disc D as divided into five regions: four are smaller discs of radius $1/4$ about the points $(\pm 1/\sqrt{8}, \pm 1/\sqrt{8})$, and the fifth region is the complement of those four discs in the disc D. On that complement, let $F_1 = F_0$. On each of the four smaller discs, define F_1 to be a homeomorphism between one of the four capped tube-like neighborhoods of X_1 to one of the four smaller discs defined previously. We require that at the boundary of the smaller discs, F_1 is smooth. We proceed by induction. Given a homeomorphism F_n between the space X_n and D, define a homeomorphism $F_{n+1} : X_{n+1} \to D$ as follows: consider the smallest 4^n discs of radius $1/4^n$ on which $F_n \neq F_{n-1}$. View each of these discs $\overline{D}$ as consisting of five parts: four smaller discs of radius $1/4^{n+1}$ centered at points which are at distance $1/2^{2n+1}$ from the center of $\overline{D}$ and the complement of those smaller discs. On that complement, let $F_{n+1} = F_n$. Otherwise, define F_{n+1} to be a homeomorphism between one of the capped neighborhoods (whose cap has center at height $1 - 1/4^{n+1}$) and one of the four smaller discs. We again require that F_{n+1} is smooth on the boundary of the four smaller discs. □

Now, we define a map F between the space X and D as the limit of the homeomorphisms F_n above.

Lemma 3.3. *$F : X \to D$ is a homeomorphism.*

Proof. Notice that because F is defined between compact spaces, we need only show that it is injective and that it is continuous to prove that it is a homeomorphism.

First, to show that F is injective, choose two points $x_1 \neq x_2 \in X$. If x_1 or x_2 is in the union $\cup_n X_n$, assume without loss of generality that $x_1 \in X_k$. Then $F = F_k$ in a small neighborhood of x_1, in which case it's a homeomorphism (and hence, injective). Now, assume that x_1 and x_2 are distinct points in $X \setminus (\cup_n X_n)$. Fix two sequences $\{x_1^n\}$ and $\{x_2^n\}$ converging to x_1 and x_2 respectively. For any n, consider the capped neighborhoods in $X_n \setminus X_{n-1}$. The sequence $\{x_1^n\}$ must eventually remain in one of (the extensions of) those specific capped neighborhoods; call this (extension of a) capped neighborhood X_n^1. A similar situation occurs for $\{x_2^n\}$, yielding X_n^2. Since x_1 and x_2 are distinct, for some N, $X_N^1 \neq X_N^2$, and thus the images of the sequence under maps F_k will always be a given distance apart. This implies that the distance between $F(x_1) = \lim_n F(x_1^n)$ and $F(x_2) = \lim_n F(x_2^n)$ must be non-zero. Thus F is injective.

To show that F is continuous, we consider convergent sequences $\{x_n\}$ in X which converge to a point x. We want to demonstrate that the sequence $\{F(x_n)\}$ converges to $F(x)$. If $x \in \cup_n X_n$, then it is in X_k for some k. Hence, $\{x_n\}$ must eventually be in X_{k+1}. Therefore, $F(x_n) = F_{k+1}(x_n)$ for large enough n, and so the sequence of images must converge to $F(x)$. If $x \in X \setminus (\cup_n X_n)$, then we can associate to x a word from the language with 4 symbols. We proceed by induction. Begin by associating to the capped neighborhoods in X_1 the values of $1, 2, 3$, or 4 based on what quadrant the projection of the cap lies in (using the Cartesian coordinate system whose origin is $(1/2, 1/2)$). Then, given an assignment $A_1A_2 \ldots A_N$ to a capped neighborhood in X_N ($A_i \in \{1, 2, 3, 4\}$), we can then assign to each of the four capped neighborhoods stemming from it in X_{N+1} a value $A_1A_2 \ldots A_NA_{N+1}$. The value $A_{N+1} \in \{1, 2, 3, 4\}$ is determined by what quadrant the projection of the cap from the new capped neighborhood is in, using the Cartesian coordinate system whose origin is determined as follows: if the center of the cap is $(x, y, 1 - 1/4^N)$, where x is an endpoint of the i^{th} subinterval and y is an endpoint of the j^{th} subinterval at step N, then the origin is found at (c_N^i, c_N^j). Now, notice that the points in D which are not in the image of any F_n can also be numbered in a similar way. Indeed, F is seen to map points in $X \setminus (\cup_n X_n)$ with associated word $A_1A_2 \ldots$ to points in D with the same associated word. Therefore, images of convergent sequences in X converge in D to the image of the limit. $\square$

Because F is a homeomorphism (as well as all the F_n), we endow X and each X_n with the induced metric from D, induced by the associated homeomorphism. That is, the metric d^* on X is defined by $d^*(x, y) = d_E(F(x), F(y))$, where as usual d_E is the standard Euclidean metric. Similarly, we have metrics d_n on X_n induced by the homeomorphisms $F_n : X_n \to D$.

In order to refer to all X_n and X as D, we need to show that the intrinsic metric d induced from the ambient space determines the same topology on X or X_n as the intrinsic metric d^* or d_n respectively. It will suffice to show the equivalence under d and d^*, and indeed, using only points $x, y \in X \setminus (\cup_n X_n)$. The arguments for at least one point in $\cup_n X_n$ is similar.

Let x, y be two distinct points in $X \setminus (\cup_n X_n)$. We want to find constants C_1 and C_2 such that $d(x, y) \leq C_1 d^*(x, y)$ and $d^*(x, y) \leq C_2 d(x, y)$. Let $x = A_1A_2 \ldots A_N \ldots$ and $y = B_1B_2 \ldots B_N \ldots$, where $A_i, B_i \in \{1, 2, 3, 4\}$ as in the proof of the continuity of F. In addition, let N be the number such that $A_N \neq B_N$ but $A_i = B_i$ for all $i < N$. Then the distance between x and y in the ambient metric is bounded above by the twice the sum of the lengths of half of a great circle on spheres at each of the heights $1 - 1/4^N$ upward plus the sum of the lengths of the segments between the centers of those spheres at each height. One shows that this sum is then strictly less than $4/4^N$. However, the distance in the induced metric $d^*(x, y)$ between the points x and y is no less than the distance between two circles of radius $1/4^{N+1}$ as detailed in the construction of the homeomorphisms F_n. With some calculation, one shows that $d^*(x, y) \geq (1/4^N)/5$. We conclude that $d(x, y) \leq 20d^*(x, y)$.

We now must find the constant C_2. Since x and y have the same representation in 4 digits until the N^{th} term, their images under the homeomorphism F can be no more than $1/4^N$ apart. However, the distance between x and y in X is certainly no less than the twice the length of a segment from x to the center of the sphere at the height where they separate. Namely, $d(x, y) \geq 2/4^N$. Thus, $d^*(x, y) \leq 2d(x, y)$, and we see that the topologies are equivalent under the metrics d and d^*. Since all X_n and X are homeomorphic to D, we now refer to them as D.

To complete the proof of Theorem 2, we now need to show

Lemma 3.4. *The limit of the two-dimensional Hausdorff measures of (D, d_n) is strictly less than the two-dimensional Hausdorff measure of (D, d^*).*

Proof. Recall that the radius of the tubes was a factor of some chosen ϵ. Therefore, the two-dimensional Hausdorff measure of the spaces (D, d_n) can be made as small as one wishes by decreasing the value of ϵ. On the other hand, the two-dimensional Hausdorff measure of (D, d^*) is bounded from below by the Hausdorff measure of the Cantor set which makes up the points of $X \setminus (\cup_n X_n)$, which is strictly positive. Hence with the proper choice of ϵ, $\lim_n h_2(D, d_n) < h_2(D, d^*)$, and the failure of lower semi-continuity of the Hausdorff measure on topological disks is shown. □

4. Discussion of Open Problems

In this concluding section, we would like to discuss several open problems and directions for further research with regard to the two questions that began our paper. These questions are related to problems posed in [5], [6], and [3].

In two dimensions, it seems likely that every length metric is the limit of an increasing sequence of Finsler metrics. The following fact serves as supporting evidence: every two-dimensional disc with a length metric admits a Lipschitz-1 surjection to a Euclidean region. So far, we cannot rule out the situation when this surjection collapses some (connected) parts of the disc to single points, but we have a feeling that this can be fixed.

In higher dimensions, the situation is less clear. One possible question to ask is, what “reasonable” restrictions (if any) can be placed on a class of sequences of Finsler metrics such that all manifolds with length metrics can be obtained as limits of those sequences and the volume (i.e., the Hausdorff measure) is lower semi-continuous?

With regard to the second question, we remark that it is true in any dimension that the Hausdorff measure is lower semi-continuous with respect to uniform convergence if all metrics in question (including the limit metric) are Riemannian. Furthermore, the same is true if the limit metric is Finsler, however the proof is more involved (see [7]). In two dimensions, the lower semi-continuity of symplectic Finsler surface area (a.k.a. the Holmes-Thompson area, [8]) is known for Finsler metrics ([3]). However, it is interesting to note that at this point, the same statement has no answer even in two dimensions if we keep the assumptions of Finsler elements in the sequence and a smooth, Finsler limit metric but consider instead the Hausdorff measure.

References

[1] A. Besicovitch, *On two problems of Loewner*, J. London Math. Soc. **27** (1952), 141–144.

[2] D.Burago, S.Ivanov. *On asymptotic volume of tori*, Geom. Func. Anal. **5** (1995), pp. 800–808

[3] D. Burago and S. Ivanov, *On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume*, Ann. of Math., **156** (2002), 891–914.

[4] Yu. Burago, V. Zalgaller, *Geometric Inequalities*, Springer-Verlag, 1988.

[5] H. Busemann, *Intrinsic area*, Ann. Math. **48** (1947), 234–267.

[6] H. Busemann, G. Ewald, and S. Shephard *Convex bodies and convexity on Grassmann cones. I–IV*, Math. Ann. **151** (1963), 1–41

[7] S. Ivanov, *Gromov-Hausdorff Convergence and volumes of manifolds*, St. Petersburg Math. J., **9** (1998), 945–959.

[8] A. C. Thompson, *Minkowski Geometry*, Encyclopedia of Math and Its Applications, Vol. 63, Cambridge Univ. Press, 1996