A REMARK ON THE GROUP OF PL-HOMEOMORPHISMS IN DIMENSION ONE

DMITRI BURAGO AND SERGEI IVANOV

1. Introduction

We begin with the following definition borrowed from [BIP]:

Definition 1.1. A group G is said to be *bounded* if it is bounded with respect to any bi-invariant metric (that is, as a metric space, it has a finite diameter).

Many groups are unbounded for some obvious reasons. For instance, if a group admits an unbounded homomorphism (or a quasi-morphism, see below) to $\mathbb{R}$, the group is unbounded. Apart from such cases, the problem of deciding whether a certain group is bounded turned out to be a surprisingly difficult task, and the answer is unknown for many nice groups such as the identity component of the group of diffeomorphisms of the two-torus. The main results of [BIP] is boundedness of certain groups of diffeomorphism.

Here we adopt some tools from [BIP] to work with non-smooth homeomorphisms. Namely, we are concerned with the group PL of compactly supported piecewise linear homeomorphisms of $\mathbb{R}$. The group of all PL-homeomorphisms of a segment is unbounded, for it has an obvious homomorphism to $\mathbb{R}$ constructed from derivatives at endpoints (Example 2.2). To stay away from this trivial homomorphism, one can study PL-homeomorphisms of a segment that are identical in some neighborhoods of endpoints. Equivalently, we consider the group $PL = PL^{comp}(\mathbb{R})$ of compactly supported piecewise-linear homeomorphisms of $\mathbb{R}$. The main result of this paper is the following theorem:

Theorem 1.2. *The group PL is bounded.*

In the course of the proof we also obtain some other results about this group; in particular, we describe certain invariants of conjugacy classes.

Note that higher-dimensional analogs of the problem is question remain widely open. We do not know if the identity component of the group of PL-homeomorphisms of the two-torus is bounded. We do not know this for the group of compactly supported PL-homeomorphisms of the plane either.

This paper is closely related to [BIP]. It is organized as follows. In the next section, we borrow some definitions and discussion from [BIP]; we refer the reader to [BIP] for further details and references. Section 3 contains certain versions of technical tools developed in [BIP]. These versions, which may sound somewhat different from their counter-parts used in [BIP], are more suitable for our set-up and, perhaps, in general when we speak about subgroups of diffeomorphism (homeomorphism) groups. Finally, Section 4 is devoted to study of PL.

Partially supported by the NSF Grant DMS-0412166.
Partially supported by RFBR grant 05-01-00939.

2. Preliminaries from [BIP]

An *conjugation-invariant norm* is a function $\nu : G \to [0; +\infty)$ which satisfies the following axioms:

(i) $\nu(1) = 0$;
(ii) $\nu(f) = \nu(f^{-1})\ \ \forall f \in G$;
(iii) $\nu(fg) \le \nu(f) + \nu(g)\ \ \forall f, g \in G$;
(iv) $\nu(f) = \nu(gfg^{-1})\ \ \forall f, g \in G$.

There is an obvious correspondence between bi-invariant (semi)-metrics and conjugation-invariant norms: for a norm ν, one defines a left-invariant metric d_ν on G by $d_\nu(a, b) = \nu(a^{-1}b)$; then conjugacy invariance of ν is equivalent to right-invariance of d_ν.

Thus a group is bounded if and only if every conjugation-invariant norm is bounded.

Convention: by $\tilde{g}$ we denote an element conjugate to g. We also fix the following notations for conjugation: $\mathbb{C}_h(g) = g_h = hgh^{-1}$.

C-generation. We say that a symmetric set $K \subset G$ *c-generates* G in N steps (where N is a positive integer or infinity) if every element $g \in G$ can be represented as a product $g = \tilde{h}_1\tilde{h}_2 \dots \tilde{h}_n$, where $n \le N$ and each $\tilde{h}_i$ is conjugate to some element $h_i \in K$: $\tilde{h}_i = \alpha_i h_i \alpha_i^{-1}$, $\alpha_i \in G$.

If K c-generates G, one defines a norm ν_K: $\nu_K(g)$ is the length of a shortest word representing g and such that each letter is a conjugate to an element from K. Hence if a group is c-generated in infinitely many steps, it is unbounded.

Obviously for any norm ν bounded on K one has $\nu \le c \cdot \nu_K$ where $c = \max \nu(K)$; hence, in a sense, ν_K is a *maximum* norm. In particular, if K is finite (in which case we say that G is finitely c-generated), ν_K is bounded if and only if G is. Note that every simple group (such as groups of smooth diffeomorphisms) is c-generated by any non-trivial symmetric set, for instance by $\{a, a^{-1}\}$, where a is not the identity.

Bounded groups and quasi-morphisms. The following notion of q-norms establishes a link with quasi-morphisms, even though the exact relationship between (the existence of) q-norms and quasi-morphisms still remains rather obscure.

Definition 2.1. Let G be a group. We say that a function $q : G \to \mathbb{R}_+$ is a quasi-norm (abbreviated to q-norm) if it is quasi-subadditive and quasi-conjugacy-invariant, that is, there is a constant c such that:

1. $q(ab) \le q(a) + q(b) + c$;
2. $|q(b^{-1}ab) - q(b)| \le c$

for all $a, b \in G$.

An important example of a q-norm is the absolute value of a quasi-morphism. There is a substantial difference between q-norms and quasi-morphisms. For instance, any unbounded quasi-morphism is unbounded on powers of some element, which is not true for q-norms. Furthermore, a quasi-morphism is bounded on the set of all commutators. Nonetheless, all example of unbounded groups without unbounded quasi-morphisms we are aware of are of the same nature: the elements are represented by transformations of an unbounded space, and the q-norm measures something like "the size of support" of the transformations.

One can see that in fact the existence of an unbounded q-norm implies the existence of an unbounded bi-invariant metric, even though the q-norm is often defined in a more natural way. Hence a group is unbounded if it admits an unbounded q-norm; in other words, the existence of unbounded norms and q-norms are equivalent. Thus without loss of generality in most cases we can restrict ourselves to norms only.

Invariants at fixed points. Finally, the following example explains why we deal with the group of compactly supported PL-homeomorphisms of $\mathbb{R}$ rather than PL-homeomorphisms of a closed segment.

Example 2.2. The group of diffeomorphisms of a segment $[a, b]$ is unbounded. Indeed, for a diffeomorphism ϕ one defines $f(\phi) = \log \phi'(a) + \log \phi'(b)$. Then f is an unbounded homomorphism to $\mathbb{R}$.

More generally, this example explains why whenever one wants to study diffeomorphisms of a manifold with a boundary, it is wise to restrict themselves to diffeomorphisms that are identical in some neighborhood of the boundary.

3. Technical Tools

We need certain analogs of techniques developed in [BIP] that would be applicable to non-smooth maps. Furthermore, the key notions of *portable manifolds* and *core* in [BIP] are applied to a manifold (or, in other words, to the whole group of diffeomorphisms), whereas we may want to work with its subgroups, as well as with certain subgroups of homeomorphism groups. Thus the versions of techniques from [BIP] that we present here seem to be more general.

Let X be a topological space and $f : X \to X$ a homeomorphism. By the *support* of f we mean the set $\mathrm{supp}(f) = closure\{x : f(x) \neq x\}$. Obviously $f \equiv id$ outside $\mathrm{supp}(f)$, $f(\mathrm{supp}(f)) = \mathrm{supp}(f)$, and $\mathrm{supp}(fg) \subset \mathrm{supp}(f) \cup \mathrm{supp}(g)$.

We say that f is *supported in* a region $\Omega \subset X$ if $\mathrm{supp}(f) \subset \Omega$, and f is *compactly supported* if $\mathrm{supp}(f)$ is compact. The set of all compactly supported homeomorphisms is a subgroup; we denote this subgroup by $Homeo^{comp}(X)$.

It is clear that maps supported in a region Ω form a subgroup, and functions with disjoint supports commute. Note also that $\mathrm{supp}(\mathbb{C}_f(g)) = f(\mathrm{supp}(g))$.

Definition 3.1. We say that a subgroup G of homeomorphisms of X *displaces supports* if there is a map $F \in G$ and a region $\Omega \subset X$ such that

1. The sets $\Omega_i = F^i(\Omega)$, $i \in \mathbb{Z}$, are disjoint, and
2. For every finite collection of $h_i \in G$ there exists $k \in G$ such that $k(\bigcup_i \mathrm{supp}(h_i)) \subset \Omega$.

It is easy to check that the first condition follows from a more manageable assumption: there is a region Ω' disjoint with Ω and such that $F(\Omega \cup \Omega') \subset \Omega'$.

Example 3.2. We say that a subgroup G of $Homeo^{comp}(\mathbb{R}^n)$ is *transitive on regions* if for any compact $K \subset \mathbb{R}^n$ and any open $U \subset \mathbb{R}^n$ there exists a $g \in G$ such that $g(K) \subset U$. If G is transitive on regions, then G displaces supports. In particular, the groups of all compactly supported homeomorphisms and diffeomorphisms of $\mathbb{R}^n$ displace supports.

Example 3.3. The groups of compactly supported homeomorphisms and diffeomorphisms of an open annulus displace supports (although it is not transitive on regions).

The main result of this section is the following

Theorem 3.4. *If G displaces supports, then every q-norm q on G is bounded on $[G, G]$.*

Proof. For an element $F \in G$, we say that $g \in G$ is an *F-commutator* if $g = [\tilde{F}, h]$ for some $h \in G$ and some $\tilde{F}$ conjugate to F. Note that all elements of the form $[F, h]$, $[h, F]$, $[F^{-1}, h]$, $[h, F^{-1}]$ and their conjugates are F-commutators. We begin with the following elementary observation:

Lemma 3.5. *Let q be a q-norm on G and $F \in G$. Then q is bounded on the set of all F-commutators.*

Proof. $[\tilde{F}, h]$ is the product of $\tilde{F}$ and $h\tilde{F}^{-1}h^{-1}$; the latter is conjugate to F^{-1}. Hence $q([F, h]) \leq q(F) + q(F^{-1}) + 3c$ where c is a constant from Definition 2.1. □

Fix F and Ω as in Definition 3.1. We will show that the every element from the commutator subgroup $[G, G]$ can be represented as a product of 3 F-commutators. This will imply the Theorem by Lemma 3.5.

We will often consider maps supported in a finite union $\bigcup_0^m \Omega_i$ and such that they can be represented as products of commuting terms each of which is supported in one of the regions Ω_i. Equivalently, we consider products $\prod \mathbb{C}_{F^i}(g_i)$, where $\operatorname{supp}(g_i) \subset \Omega$. Since maps supported in disjoint regions commute, the product of such maps $\prod \mathbb{C}_{F^i}(f_i)$ and $\prod \mathbb{C}_{F^i}(g_i)$ can be computed component-wise:

$$\prod \mathbb{C}_{F^i}(f_i) \cdot \prod \mathbb{C}_{F^i}(g_i) = \prod \mathbb{C}_{F^i}(f_i g_i) \tag{3.1}$$

Lemma 3.6. *Let $g_0, g_1, \dots, g_m \in G$ be such that $\operatorname{supp}(g_i) \subset \Omega$ for all i, and $\prod_0^m g_i = id$. Then the product $g = \prod_0^m \mathbb{C}_{F^i}(g_i)$ is an F-commutator.*

Proof. Let $\phi_i = g_0 g_1 \dots g_i$ for $i = 0, 1, \dots, m$ (note that $\phi_m = id$). Define $\phi = \prod_0^{m-1} \mathbb{C}_{F^i}(\phi_i)$. We will show that $g = [F, \phi^{-1}]$.

Note that $[F, \phi^{-1}] = \mathbb{C}_F(\phi^{-1})\phi$ and

$$\mathbb{C}_F(\phi^{-1}) = \prod_0^{m-1} \mathbb{C}_{F^{i+1}}(\phi_i^{-1}) = \prod_1^m \mathbb{C}_{F^i}(\phi_{i-1}^{-1}).$$

Hence by (3.1)

$$[F, \phi^{-1}] = \mathbb{C}_F(\phi^{-1})\phi = \phi_0 \cdot \prod_1^m \mathbb{C}_{F^i}(\phi_{i-1}^{-1}\phi_i) = \prod_0^m \mathbb{C}_{F^i}(g_i) = g$$

since $\phi_m = id$, $\phi_0 = g_0$ and $\phi_{i-1}^{-1}\phi_i = g_i$ for $i = 1, \dots, m$. □

Remark: the same statement can be proved under the assumption $\prod_m^0 g_i = id$. In this case, g can be represented as $[\phi, F]$ for a suitable ϕ.

Lemma 3.7. *Let $g_1, g_2, \dots, g_m$ be a collection of elements of G supported in Ω. Then $g = \prod_m^1 g_i$ equals an F-commutator times the product $\prod_1^m \mathbb{C}_{F^i}(g_i)$.*

Proof. Introduce $g_0' = g$ and $g_i' = g_i^{-1}$. Note that $\prod_0^m g_i' = id$. Then apply the previous lemma. □

Lemma 3.8. *Any commutator is a product of two F-commutators.*

Proof. Consider a commutator $[f, g]$. Using the second assumption from Definition 3.1, there are conjugates $\mathbb{C}_h f$ and $\mathbb{C}_h g$ whose supports lie in Ω. Therefore without loss of generality we can assume that the supports of f and g lie in Ω. Then by Lemma 3.6, the elements

$$(fg)\mathbb{C}_F(g^{-1})\mathbb{C}_{F^2}(f^{-1})$$

and

$$(f^{-1}g^{-1})\mathbb{C}_F(g)\mathbb{C}_{F^2}(f)$$

are F-commutators. By (3.1), their product equals $[f, g]$. □

Lemma 3.9. *Any product of several commutators can be represented as a product of three F-commutators.*

Proof. Consider $g = \prod_m^1 [f_i, g_i]$. By means of a conjugation, we may assume that all maps f_i and g_i are supported in Ω, By Lemma 3.7, g equals an F-commutator times a product $\prod_1^m \mathbb{C}_{F^i}([f_i, g_i])$, The latter in its turn is equal to the commutator of two products $\prod_1^m \mathbb{C}_{F^i}(f_i)$ and $\prod_1^m \mathbb{C}_{F^i}(g_i)$ by (3.1). By the previous lemma, this commutator equals a product of two F-commutators. Hence g is a product of three F-commutators. □

Now the Theorem follows from Lemma 3.5. □

4. $PL^{comp}(\mathbb{R})$: INVARIANTS OF CONJUGACY CLASSES AND BOUNDEDNESS

Now we proceed with a proof of Theorem 1.2. We derive the theorem from Theorem 3.4. Namely, we show that PL coincides with its commutator subgroup: $[PL, PL] = PL$ (Lemma 4.6 below). Since PL is obviously transitive on regions, this implies that it is bounded.

For $f \in PL$ and $x \in \mathbb{R}$, denote by $f'_+(x)$ and $f'_-(x)$ the right and left derivative of f at x, respectively, and define a cocycle $\chi_f(x) = \frac{f'_+(x)}{f'_-(x)}$. A direct computation shows that

$$(f \circ g)'_\pm(x) = f'_\pm(g(x)) \cdot g'_\pm(x), \qquad (f^{-1})'_\pm(f(x)) = \frac{1}{f'_\pm(x)},$$

and

$$\chi_{f\circ g}(x) = \chi_f(g(x)) \cdot \chi_g(x), \qquad \chi_{f^{-1}}(f(x)) = \frac{1}{\chi_f(x)}.$$

An $x \in \mathbb{R}$ is said to be a *break point* of f if $\chi_f(x) \neq 1$. Every $f \in PL$ has only finitely many break points, and every $f \neq id$ has at least 3 break points (at least two boundary points of $\mathrm{supp}(f)$ at at least one non-fixed break point). An $f \in PL$ is said to be *primitive* if it has exactly 3 breaks. Obviously the inverse of a primitive map is also primitive.

Lemma 4.1. *For every $a, b, c \in \mathbb{R}$, $a < b < c$, and every positive $\lambda \neq 1$, there exists a primitive $f \in PL$ with break points at a, b and c and such that $\chi_f(b) = \lambda$.*

Proof. For every $t \in (a, c)$, let $f_t \in PL$ be the map with break points at a, b, c such that $f(b) = t$. The function $t \mapsto \chi_{f_t}(b)$ is monotone increasing and goes to 0 and ∞ as t goes to a and c, respectively. Hence there is a t such that $\chi_{f_t}(b) = \lambda$. □

Lemma 4.2. *The primitive maps generate PL.*

Proof. We will show that a map $f \in PL$ with n break points can be represented as a product of at most n primitive maps. Let x_0 be a non-fixed break point of f. Then there exist fixed break points $a \in (-\infty, x_0)$ and $b \in (x_0, +\infty)$. Observe that $a < f(x_0) < b$. Using the previous lemma, construct a primitive map g with break points at a, $f(x_0)$ and b such that $\chi_g(f(x_0)) = (\chi_f(x_0))^{-1}$. Consider $f_1 = g \circ f$. It has the same set of break points as f, except for x_0 and possibly a and b. (The break of g at $f(x_0)$ cancels the break of f at x_0.) An induction in the number of break points completes the proof. □

Our next goal is to find some conjugacy invariants in PL. The first trivial observation is that right and left derivatives at fixed points are invariant under conjugations. More precisely, if x is a fixed point of f, then $g(x)$ is a fixed point of $\mathbb{C}_g(f)$ and $(\mathbb{C}_g(f))'_{\pm}(g(x)) = f'_{\pm}(x)$.

Now consider the orbits of f. There are two kinds of orbits: fixed points and monotone (increasing or decreasing) sequences of the form $\{x_i\}_{i\in\mathbb{Z}}$ such that $f(x_i) = x_{i+1}$ for all i.

Let ω be an infinite orbit of f. Define $\chi_f(\omega) = \prod_{x\in\omega} \chi_f(x)$. The product is well-defined since all but finitely many terms in the product equal 1 (recall that there are only finitely many break points). We call $\chi_f(\omega)$ the *characteristic* of ω. The following lemma is a standard assertion for cocycles. It states that the characteristic of an orbit in invariant under conjugations.

Lemma 4.3. *Let $f, g \in PL$, ω an orbit of f, $\omega = \{x_i\}_{i\in\mathbb{Z}}$ where $f(x_i) = x_{i+1}$ for all i. Let $\tilde{f} = \mathbb{C}_g(f)$. Then*

1. $g(\omega) := \{g(x_i)\}_{i\in\mathbb{Z}}$ is an orbit of $\tilde{f}$.

2. $\chi_{\tilde{f}}(g(x_i)) = \chi_f(x_i) \cdot \chi_g(x_{i+1}) \cdot (\chi_g(x_i))^{-1}$.

3. $\chi_{\tilde{f}}(g(\omega)) = \chi_f(\omega)$.

Proof. Recall that $\tilde{f} = \mathbb{C}_g(f) = g \circ f \circ g^{-1}$.

1. $\tilde{f}(g(x_i)) = g(f(x_i)) = g(x_{i+1})$,

2. $\chi_{\tilde{f}}(g(x_i)) = \chi_g(f(x_i)) \cdot \chi_f(x_i) \cdot \chi_{g^{-1}}(g(x_i))$. The first term equals $\chi_g(x_{i+1})$, the third term equals $(\chi_g(x_i))^{-1}$.

3. From the second statement,

$$\chi_{\tilde{f}}(g(\omega)) = \chi_f(\omega) \cdot \prod_i \chi_g(x_{i+1}) \cdot \prod_i (\chi_g(x_i))^{-1}.$$

In this formula, the two products over the orbit cancel each other. □

We say that an infinite orbit ω of f is *essential* if $\chi_f(\omega) \neq 1$. Obviously there are only finitely many essential orbits.

Lemma 4.4. *If a map $f \in PL$ has exactly N essential orbits, then f is conjugate to a map with exactly N non-fixed break points.*

Proof. We say that an orbit is *complicated* if it contains more than one break point. We are going to apply a series of conjugations by primitive maps which eliminates the complicated orbits. Let $\omega = \{x_i\}_{i\in\mathbb{Z}}$ be a complicated orbit, x_0 and x_n be the first and last break points in this orbit. Let a and b be fixed points of f such that $a < x_n < b$. Construct a primitive function g with break points at a, x_n and b with $\chi_g(x_n) = \chi_f(x_n)$. Consider $\tilde{f} = \mathbb{C}_g(f)$ and the orbit $g(\omega)$ of $\tilde{f}$. By the previous lemma, $\chi_{\tilde{f}}(g(x_n)) = \chi_f(x_n)\chi_g(x_n)^{-1} = 1$, and $\chi_{\tilde{f}}(g(x)) =$

$\chi_f(x)$ for $x \notin \{x_{n-1}, x_n\}$. It follows that all break points in $g(\omega)$ are in the set $\{g(x_0), g(x_1), \dots, g(x_{n-1})\}$, so the last break point of the orbit gets closer to the first one. Furthermore, every orbit of $\tilde{f}$ except $g(\omega)$ contains the same number of break points as the corresponding orbit of f, so this conjugation does not introduce new complicated orbits. Applying a similar procedure at most n times, one obtains a map conjugate to f whose orbit corresponding to ω is no longer complicated, and hence the total number of complicated orbits is smaller than that of f.

It follows by induction that there is a map f_1 which is conjugate to f and has no complicated orbits. Every orbit containing one break point is essential, hence f_1 has exactly N non-fixed break points. □

We will apply the lemma only to maps having only one essential orbit. In this case, the lemma implies that the map is conjugate to a primitive one. The next lemma provides a classification of such maps up to a conjugacy.

Lemma 4.5. *The conjugacy class of a map f with one essential orbit is uniquely determined by the right derivative at the leftmost point of supp(f) and the left derivative at the rightmost point of supp(f).*

Proof. By the previous lemma, f is conjugate to a primitive map. Since the right and left derivatives at fixed points are invariant under conjugations, we may assume that f itself is primitive. Let $\text{supp}(f) = [a, b]$. Construct a map $g \in PL$ such that $g([a, b]) = [0, 1]$ and g has no break points in $[a, b]$. Then $\tilde{f} = \mathbb{C}_g(f)$ is again a primitive map, and $supp(f) = [0, 1]$. Such a map $\tilde{f}$ is uniquely determined by its right derivative at 0 and left derivative at 1. The lemma follows. □

Lemma 4.6. *Every primitive element of PL is a commutator. Therefore $[PL, PL] = PL$.*

Proof. Consider a primitive $f \in PL$. Let $supp(f) = [a, b]$, $f'_+(a) = \alpha$, $f'_-(b) = \beta$. We may assume that $\alpha > 1$ and $\beta < 1$ (otherwise consider f^{-1}). Let x_0 be the non-fixed break point of f.

By continuity, there exist $y_0 \in (a, x_0)$ and $y_1 \in (x_0, b)$ such that $f(y_0) = y_1$ and $\frac{f(y_1)-f(y_0)}{y_1-y_0} = 1$. (Indeed, the function $t \mapsto \frac{f(f(t))-f(t)}{f(t)-t}$ takes the value α near a and the value β near b. A point where it takes value 1 is the desired y_0.)

Let $f_1 \in PL$ be the map which coincides with f outside (y_0, y_1) and is linear on $[y_0, y_1]$. Note that $f_1' = 1$ in (y_0, y_1). Observe that $supp(f_1) = [a, b]$, ${f_1}'_+(a) = \alpha$, ${f_1}'_-(b) = \beta$, and both non-fixed break points of f_1 (namely y_0 and y_1) belong to one orbit, Hence by the previous lemma f_1 is conjugate to f, so $f_1 = g \circ f \circ g^{-1}$ for some $g \in PL$. Now consider $f_2 = f_1^{-1} \circ f$. It is a primitive map with $supp(f_2) = [y_0, y_1]$, ${f_2}'_+(y_0) = \alpha$ and ${f_2}'_-(y_1) = \beta$, hence f_2 is conjugate to f. By construction, f_2 is a commutator ($f_2 = [g, f^{-1}]$). Since f is conjugate to a commutator, it is a commutator itself. The first statement of the lemma follows.

Then the second statement follows from the fact that primitive elements generate PL (Lemma 4.2). □

The above lemma and Theorem 3.4 imply Theorem 1.2.

References

[BIP] D. Burago, S. Ivanov and L. Polterovich, *Invariant Norms on Groups of Geometric origin*, preprint.

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, U.S.A.
E-mail address: burago@math.psu.edu

Steklov Institute of Mathematics, Fontanka 27, 191023, St.-Petersburg, Russia
E-mail address: svivanov@pdmi.ras.ru