GROMOV–HAUSDORFF CONVERGENCE AND VOLUMES OF MANIFOLDS

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ABSTRACT. Let $n \geq 2$, M and M_k $(k = 1, 2, \dots)$ be compact Riemannian n-manifolds, possibly with boundaries, and let $\{M_k\}$ converge to M with respect to the Gromov–Hausdorff distance. We prove that $\mathrm{Vol}(M) \leq \liminf_{k\to\infty} \mathrm{Vol}(M_k)$ provided that one of the following holds:

(1) M_k are homotopy equivalent to M, and M admits either a nonzero-degree map onto the torus T^n or an odd-degree map onto $\mathbf{RP}^n$;
(2) $n = 2$, and the Euler characteristics of M_k are uniformly bounded.

For $n \geq 3$ we give examples of convergence in which M and M_k are diffeomorphic to S^n and $\mathrm{Vol}(M_k) \to 0$.

INTRODUCTION

0.1. Let M, M_k $(k = 1, 2, \dots)$ be compact Riemannian manifolds of the same dimension $n \geq 2$. We write $M_k \to M$ if the sequence $\{M_k\}$ converges to M with respect to Gromov–Haudorff distance, cf. §1. Our question is: for what topology types of M an M_k the convergence $M_k \to M$ implies that

$$\mathrm{Vol}(M) \leq \liminf_{k\to\infty} \mathrm{Vol}(M_k)\,? \tag{$*$}$$

We make no assumptions about metrics of the manifolds M_k except that they are Riemannian (in particular, we are not dealing with curvature bounds). By Riemannian metric we mean a length metric (i.e., a distance function) determined by a continuous metric tensor.

0.2. Let us indicate two facts about semi-continuity of the volume that may motivate the above question:

(1) If d and d_k $(k = 1, 2, \dots)$ are Riemannain metrics on the same manifold M and d_k uniformly converge to d (as functions on $M \times M$), then $\mathrm{Vol}(M, d) \leq \liminf \mathrm{Vol}(M, d_k)$. This is true even if d is a Finsler metric (see [2]).
(2) The volume is lower-semicontinuous with respect to the classical Hausdorff distance on the set of compact connected n-dimensional submanifolds of $\mathbf{R}^N$ with a given nonempty boundary.

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(To prove the second statement observe that a smooth submanifold $S \subset \mathbf{R}^N$ admit Lipschitz neighborhood retractions with Lipschitz constats arbitrarily close to 1. Hence submanifolds contained in a sufficiently small neighborhood of S admit maps to S that are identical on the boundary and are almost volume non-increasing. These maps have nonzero degrees and therefore are surjective.)

Note that the convergence of metrics in (1) is a "topologically trivial" case of Gromov–Hausdorff convergence (cf. 1.1 and 1.2). The convergence of submanifolds in (2) does not imply Gromov–Hausdorff convergence of the corresponding Riemannian metrics but the two kinds of convergence are similar in many respects. The proof of (2) sketched above illustrate some ideas that we will utilize in this paper. Similarly to (1), the results of this paper remains true for convergence of Riemannian manifolds to Finsler ones (see 1.6), but for the sake of simplicity of the formulations we restrict ourselves to the pure Riemannian case.

0.3. In general, the Riemannian volume is not semi-continuous with respect to the Gromov–Hausdorff distance. There are simple examples of convergence of two-dimensional closed manifolds for which the inequality $(*)$ fails. For instance, one can compose manifolds M_k from thin tubes of almost zero area so that they approximate suitable fine nets of curves in M (cf. 3.4 and 4.2 for details). However, the genus of manifolds M_k constructed in such a way grows infinitely as $M_k \to M$.

We will study the question of semi-continuity of volume under the assumption that the topology of the manifolds M_k remains bounded or fixed. For example, does $(*)$ hold when all the M_k are homeomorphic to M? (Compare with (1) above.) It turns out that the answer to this last question depend on the topology of M: it is negative in general but there exist topology (and even homotopy) types of manifolds within which the volume is semi-continuous.

0.4. We say that a continuous map between closed manifolds *has nonzero degree* if it induces a nontrivial homomorphism of the higher homology groups with coefficients in either $\mathbf{Z}$ or $\mathbf{Z}_2$ (we use the notation $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$). If this homomorphism is nontrivial for $\mathbf{Z}_2$, we say that the map *has odd degree*. We will prove the following

Theorem 2.4. *Let M and M_k $(k = 1, 2, \dots)$ be homotopy equivalent closed n-dimensional Riemannian manifolds. Let M admit either a nonzero-degree map onto the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ or an odd-degree map onto the projective space $\mathbf{RP}^n$. Then the convergence $M_k \to M$ implies that*

$$\mathrm{Vol}(M) \le \liminf_{k\to\infty} \mathrm{Vol}(M_k).$$

On the other hand, the semi-continuity of volume can be violated for Riemannian manifolds homeomorphic to the three-sphere (and therefore for spheres of any dimension $n \ge 3$, products of these speres to any manifolds etc.):

Theorem 4.3. *For any Riemannian metric d on S^3 there exists a sequence $\{d_k\}_{k=1}^\infty$ of Riemannian metrics on S^3 such that $(S^3, d_k) \to (S^3, d)$ and $\mathrm{Vol}(S^3, d_k) \to 0$ as $k \to \infty$.*

Three is the minimal dimension for which such examples exist. In the two-dimensional case, assuming that the geni of the manifolds (surfaces) M_k are uniformly bounded, one can give a complete description of the structure of a manifold M_k that is sufficiently close to a given limit manifold M. This description is given by Theorem 3.2. Roughly speaking, it states that M_k can be obtained from M by a combination of two procedures: a small perturbation of the metric (as in (1) in 0.2 above) and topological transformations within domains of small diameter (i.e., attaching a number of small handles and films, and making small holes if manifolds with boundary are allowed). We will then derive

Corollary 3.3. *Let M and M_k ($k = 1, 2, \dots$) be compact two-dimensional Riemannian manifolds (possibly with boundaries) such that $\sup_k |\chi(M_k)| < \infty$ where χ denotes the Euler characteristic. Then the convergence $M_k \to M$ implies that*

$$\mathrm{Vol}(M) \le \liminf_{k\to\infty} \mathrm{Vol}(M_k).$$

In other words, the two-dimensional Riemannian volume (i.e., the area) is lower semi-continuous on any class of two-dimensional manifolds representing a finite number of topology types.

Remarks. 1. In Theorem 2.4 it is essential that the manifolds M_k are homotopy equivalent to M. This condition cannot be replaced, e.g., by a requirement that the M_k are mutually homotopy equivalent and admit nonzero-degree maps onto the torus or $\mathbf{RP}^n$. Counterexamples can be easily constructed in a way similar to the proof of Theorem 4.3.

2. On the other hand, the assumptions about the topology of M in Corollary 3.3 can be weakened. The arguments of §3 that are essential for this corollary can be easily adopted to the case when M is an arbitrary cell complex. (In fact, such a complex is necessarily two-dimensional, see 3.4.2.) The requirement that the metric of M is Riemannian can also be weakened, see 1.6. It seems reasonable to conjecture that Corollary 3.3 remains true without any topological or metric assumptions about the limit space M (for some suitable generalization of the area to non-Riemannian spaces).

The proofs of Theorem 2.4 and Corollary 3.3 are based upon the following fact (Theorem 1.5): for the inequality ($*$) to hold it is sufficient that some "almost isometric" maps $\varphi_k \colon M_k \to M$ have nonzero degrees (cf. §1 for definitions). This fact also allows to prove semi-continuity of the volume under certain metric restrictions, cf. 1.3. In the other parts of the proofs (§§2, 3) we only study topological properties of almost isometric maps (which may be of interest on its own, see e.g. remark 3.4.3). In doing this we no longer rely on the fact that the metrics of M and M_k are Riemannian, but it is important that they are *length metrics*, i.e., that the distance between every two points equals the length of the shortest curve joining them.

0.5. Notations. "By default" the distance function of a metric space will be denoted by d. By $U_\varepsilon(A)$ we denote the ε-neighborhood of a set A in a metric space, and by $\mathrm{dist}(A, B)$ the infimum of distances between points of two sets A and B.

A *graph* is a finite one-dimensional cell complex, its zero-dimensional cells are called *vertices* and one-dimensional cells are called *edges.* We denote the set of vertices of a graph Γ by $V(\Gamma)$.

§1. Almost isometries and volumes

The Gromov–Hausdorff distance between two metric spaces X and Y, that we denote by $d_H(X, Y)$, is defined as follows (cf. [7]): $d_H(X, Y) < \varepsilon$ if and only if there exists a metric space Z and two sets $X', Y' \subset Z$ such that X' is isometric to X, Y' is isometric to Y, and the Hausdorff distance between X' and Y' in Z is less than ε. The last condition means that $X' \subset U_\varepsilon(Y')$ and $Y' \subset U_\varepsilon(X')$.

The distance d_H is a metric on the "space" of isometry classes of compact metric spaces. Let $\{X_k\}_{k=1}^\infty$ be a sequence of metric spaces. By definition, $X_k \to X$ if $d_H(X_k, X) \to 0$. Below we reformulate this condition in terms of maps between spaces.

1.1. Definition. Let X and Y be metric spaces, $\varphi\colon X \to Y$ be a (possibly discontinuous) map, and $\varepsilon > 0$. We say that φ is an *ε-isometry* if the following two conditions hold:

(1) $f(X)$ is an ε-net in Y;
(2) $|d(f(x), f(x')) - d(x, x')| < \varepsilon$ for all $x, x' \in X$.

The infimum of those ε for which φ is an ε-isometry will be called the *error* of φ and denoted by $E(\varphi)$.

Note that for any maps $\varphi_1, \varphi_2\colon X \to Y$ one has $|E(\varphi_1) - E(\varphi_2)| < 2d_{\sup}(\varphi_1, \varphi_2)$ where $d_{\sup}(\varphi_1, \varphi_2) := \sup_{x\in X} d(\varphi_1(x), \varphi_2(x))$. It is easy to see (cf. [7]) that

(1) if $d_H(X, Y) < \varepsilon$ then there exists a (2ε)-isometry $\varphi\colon X \to Y$;
(2) if there is an ε-isometry $\varphi\colon X \to Y$ then $d_H(X, Y) < 2\varepsilon$.

Hence a convergence $X_k \to X$ takes place if and only if there exists a sequence of maps $\varphi_k\colon X_k \to X$ with $E(\varphi_k) \to 0$. We call such a sequence of maps a *sequence of almost isometries.*

If the topology of the limit space is good enough then almost isometries can be made continuous:

1.2. Proposition. *Let $X_k \to X$ and let X be a compatc metric space homeomorphic to a neighborhood retract of a Euclidean space. Then there exists a sequence of almost isometries $\varphi_k\colon X_k \to X$ in which all maps φ_k are continuous.*

Proof. Let $i\colon X \to \mathbf{R}^n$ be an inclusion map, $U \subset \mathbf{R}^n$ be a neighborhood of the set $i(X)$, $p\colon U \to i(X)$ be a retraction. Pick any sequence of almost isometries $f_k\colon X_k \to X$ and define $\varepsilon_k = E(f_k)$. For every k construct a finite ε_k-net $S_k \subset X_k$ and define a map $i_k\colon X_k \to \mathbf{R}^n$ by

$$i_k(x) = \frac{\sum_{y\in S_k} w_k(d(x,y)) \cdot i(f_k(y))}{\sum_{y\in S_k} w_k(d(x,y))}$$

where $w_k\colon [0, \infty) \to \mathbf{R}$ is an arbitrary continuous function which is positive on $[0, 2\varepsilon_k)$ and equals zero on $[2\varepsilon_k, \infty)$. Clearly i_k is well defined and continuous. Let x be an arbitrary point of X_k. For all $y \in S_k$ such that $w_k(d(x, y)) > 0$ one has $d(f_k(x), f_k(y)) < 3\varepsilon_k$. Thus

$$|i_k(x) - i(f_k(x))| \le \sup\{|i(y') - i(f_k(x))| : y' \in U_{3\varepsilon_k}(f_k(x)) \subset X\}.$$

Since i is an equicontinuous map this implies that

$$\sup_{x\in X_k} |i_k(x) - i(f_k(x))| \xrightarrow[k\to\infty]{} 0.$$

Hence for all sufficiently large k there is a (continuous) map $\varphi_k = i^{-1} \circ p \circ i_k\colon X_k \to X$ and the distance between φ_k and $f_k = i^{-1} \circ p \circ (i \circ f_k)$ goes to zero as $k \to \infty$.

Therefore $E(\varphi_k) \to 0$. For the remaining values k (there are finitely many of them) one may let φ_k be arbitrary continuous maps from X_k to X. $\square$

1.3. Remark. In general, there is no similar sequence of continuous maps $\varphi'_k\colon X \to X_k$ with $E(\varphi'_k) \to 0$. For example, let X be the standard two-dimensional sphere, X_k be homeomorphic to the torus and obtained from X by "attaching" a handle of diameter less than $1/k$. Then $X_k \to X$, but any continuous map $\varphi\colon X \to X_k$ has error at least π because it maps some pair of opposite points of the sphere to one point in the torus.

The existence of "inverse" continuous almost isometries can be assured by imposing some metric restrictions. Let X and the X_k have bounded dimensions and be *uniformly locally contractible*, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that any ball of radius δ in any of these spaces can be contracted within a ball of radius ε. Then, as shown in [9], X_k are homotopy equivalent to X for all large enough k. In fact, the homotopy equivalences can be realized by pairs of maps $\varphi_k\colon X \to X_k$ and $\varphi'_k\colon X_k \to X$ whose errors tend to zero.

This fact and Theorem 1.5 imply that the volume is lower semi-continuous on any class of closed Riemannian manifolds of the same dimension satisfying the above uniform local contractibility condition.

1.4. We will restrict ourselves to the case when the converging and the limit spaces are compact Riemannian manifolds (possibly with boundaries). All manifolds are assumed connected and having the same dimension $n \geq 2$.

Let M and M' be two such manifolds, $\varphi\colon M' \to M$ a continuous map, $U \subset M$ an open set such that $U \cap \partial M = \varnothing$. We say that φ *has nonzero degree over* U if $\varphi(\partial M') \cap U = \varnothing$ and for every point $x \in U$ the induced homomorphism

$$\varphi_*\colon H_n(M', \partial M') \to H_n(M, M \setminus \{x\})$$

of homology groups is nontrivial for some coefficient group. (It makes sense to take $\mathbf{Z}$ as a coefficient group for orientable manifolds and $\mathbf{Z}_2$ for non-orientable ones.)

A map $\varphi\colon M' \to M$ has nonzero degree if and only if it has nonzero degree over $M \setminus \partial M$ is the above sense. The notion of degree applies well to manifolds with singular boundaries, in particular, to any compact domains in manifolds (we will utilize such ones in the proof of Theorem 1.5).

1.5. Theorem. *Let M and M_k $(k = 1, 2, \dots)$ be compact n-dimensional Riemannian manifolds* (*possibly with bondaries*) *such that $M_k \to M$, and let $U \subset M$ be an open set such that $U \cap \partial M = \varnothing$. Assume that there is a sequence of continuous almost isometries $\varphi_k\colon M_k \to M$ which have nonzero degree over U for all large enough k. Then*

$$\mathrm{Vol}(U) \leq \liminf_{k\to\infty} \mathrm{Vol}(\varphi_k^{-1}(U)) \leq \liminf_{k\to\infty} \mathrm{Vol}(M_k).$$

Proof. Let $\overline{U}$ denote the closure of U. Fix an $\varepsilon > 0$ and assume that U is almost isometric to a small cube $(\delta I)^n = [0, \delta]^n \subset \mathbf{R}^n$ in the sense that there is a diffeomorphism $f\colon \overline{U} \to (\delta I)^n$ such that

$$(1+\varepsilon)^{-1} d(x, y) \leq |f(x) - f(y)| \leq d(x, y)$$

for all $x, y \in \overline{U}$. Then $\mathrm{Vol}(U) \leq \delta^n (1+\varepsilon)^n$. To estimate the volumes of the sets $\varphi_k^{-1}(U)$ from below we will use the Besikovitch inequality [1] in the following generalized form (cf. [6]):

Besikovitch Inequality. *Let V be a compact Riemannian n-manifold with (possibly singular) boundary and let $f\colon V \to I^n$ be a map having nonzero degree. Then*

$$\operatorname{Vol}(V) \ge \prod_{i=1}^{n} \operatorname{dist}(f^{-1}(F_i), f^{-1}(F_i')),$$

where F_i and F_i' denote the i-th pair of opposite faces of I^n.

For each k let $U_k = \varphi_k^{-1}(U)$ and consider the map $f_k = f \circ \varphi_k$ from $\overline{U}_k$ to $(\delta I)^n$. It has nonzero degree and increases the distances by at most $E(\varphi_k)$. In particular, for every pair of opposite faces of $(\delta I)^n$ the distances between their f_k-preimages in $\overline{U}_k$ is no less than $\delta - E(\varphi_k)$. The Besikovitch inequality implies that $\operatorname{Vol}(U_k) \ge (\delta - E(\varphi_k))^n$ whenever $E(\varphi_k) < \delta$, and therefore

$$\liminf_{k\to\infty} \operatorname{Vol}(U_k) \ge \delta^n \ge (1+\varepsilon)^{-n} \operatorname{Vol}(U).$$

Now let $U \subset M$ be any open set. One can cover U, up to an arbitrarily small volume, by a number of disjoint sets that are almosts isometric to cubes in the sense specified in the beginning of the proof. Adding up the above inequalities for those sets we obtain that

$$\operatorname{Vol}(U) \le (1+\varepsilon)^n \liminf_{k\to\infty} \operatorname{Vol}(\varphi_k^{-1}(U)).$$

Since ε is arbitrary, Theorem 1.5 follows. □

1.6. Finsler limits. Theorem 1.5, along with Theorem 2.4 and Corollary 3.3, remain true in the case when the limit space M is a Finsler manifold (for any definition of Finsler volume assuring that the volume is monotonous with respect to metric). Moreover if the limit metric is not Riemannian then the inequality of Theorem 1.5 is strict. This has been proved in [2] for uniform convergence of metrics on the same manifold. The proof in [2] is based upon an estimate of volume in terms of distances similar to Besikovich inequality. With little changes, that proof works for general case as well.

It is an intriguing question whether Theorem 1.5 holds for convergence of Finsler manifolds (or at least for uniform convergence of Finsler metrics). The answer may depend on the definition of volume. There are several natural generalizations of the Riemannian volume to Finsler manifolds, among which are the Hausdorff measure and the projection of the simplectic volume from the unit tangent bundle. For the later definition of volume, a proof or a counterexample to the analog of Theorem 1.5 might be helpful for understanding the Finsler tori without conjugate points, cf. [3], [2].

§2. Lifting curves

2.1. Let $n \ge 2$, M and M_k ($k = 1, 2, \dots$) be compact n-manifolds equipped with length metrics. Every two points in such a manifold can be joined with a curve whose lenght equals the distance between the endpoints. Suppose that $M_k \to M$ and let a sequence of almost isometries $\varphi_k\colon M_k \to M$ be fixed. Throughout §2 and §3 we will ingnore the dependencies on φ_k in notations and statements.

We say that a point $\tilde p \in M_k$ is an *ε-lift* of a point $p \in M$ (where ε is a positive number) if $d(\varphi_k(\tilde p), p) < \varepsilon$. We say that a map $\tilde f\colon X \to M_k$ is an *ε-lift* of a map $f\colon X \to M$ if $\tilde f(x)$ is an ε-lift of $f(x)$ for every $x \in X$. (Here X is an arbitrary set.) By *ε-lift* of a set $X \subset M$ we mean an ε-lift of the inclusion map $i_X\colon X \to M$. If $E(\varphi_k) < \varepsilon$ then every point of M clearly admit an ε-lift to M_k. Observe also that for any ε-lift with values in M_k, an ε-lift with values in $M_k \setminus \partial M_k$ can be obtained by a small variation.

The following lemma allows us to construct lifts of one-dimensional subsets of M. This lemma does not rely on the fact that M is a manifold.

2.2. Lemma. 1. *Let $\gamma\colon [a,b] \to M$ be a curve, $\varepsilon > E(\varphi_k)$, $\tilde p$ and $\tilde q$ be ε-lifts to M_k of the points $\gamma(a)$ and $\gamma(b)$. Then there is a rectifiable curve $\tilde\gamma\colon [a,b] \to M_k$ joining $\tilde p$ to $\tilde q$ and being a (7ε)-lift of γ.*

2. *Let $\Gamma \subset M$ be an embedded graph. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $E(\varphi_k) < \delta$ then any δ-lift of $V(\Gamma)$ to M_k can be extended to an ε-lift of Γ to M_k which is a topological embedding.*

Proof. 1. Divide $[a,b]$ by points $a = t_0 < t_1 < \cdots < t_n = b$ so that the diameters of the intervals $\gamma([t_i, t_{i+1}])$ of γ are less than ε. Let $\tilde\gamma(t_0) = \tilde p$ and $\tilde\gamma(t_n) = \tilde q$. For every $i = 1, \dots, n-1$ let $\tilde\gamma(t_i)$ be any ε-lift of $\gamma(t_i)$. On every interval $[t_i, t_{i+1}]$ define $\tilde\gamma$ to be a shortest path between $\tilde\gamma(t_i)$ and $\tilde\gamma(t_{i+1})$. The length of this shortest path is $d(\tilde\gamma(t_i), \tilde\gamma(t_{i+1})) < 4\varepsilon$. Hence for every $t \in [t_i, t_{i+1}]$ we have $d(\tilde\gamma(t), \tilde\gamma(t_i)) < 4\varepsilon$, so

$$\begin{aligned} d(\varphi_k(\tilde\gamma(t)), \gamma(t)) &\le d(\varphi_k(\tilde\gamma(t)), \varphi_k(\tilde\gamma(t_i))) + d(\varphi_k(\tilde\gamma(t_i)), \gamma(t)) \\ &< d(\tilde\gamma(t), \tilde\gamma(t_i)) + 2\varepsilon + d(\gamma(t_i), \gamma(t)) < 4\varepsilon + 2\varepsilon + \varepsilon = 7\varepsilon\,. \end{aligned}$$

2. One may assume that all edges of Γ are not loops and that any two vertices of Γ are joined by at most one edge. Denote by ε_0 the minimal possible distance between two disjoing sub-graphs of Γ. For a $\delta > 0$ let $\theta(\delta)$ denote the maximal possible diameter of a simple curve contained in Γ, having the distance between endpoints no greater than δ, and containing at most one vertice of Γ. Clearly $\theta(\delta) \to 0$ as $\delta \to 0$.

Let $\delta > 0$ be small enough, $E(\varphi_k) < \delta$, and let $\psi\colon V(\Gamma) \to M_k$ be a δ-lift of $V(\Gamma)$. Let us first construct a self-disjoint lift of a single edge of Γ. Parameterize the edge as a curve $\gamma\colon [0,1] \to M$. By the first part of the lemma, γ has a (7ε)-lift $\tilde\gamma\colon [0,1] \to M_k$ with $\tilde\gamma(0) = \psi(\gamma(0))$ and $\tilde\gamma(1) = \psi(\gamma(1))$. Consider the class of curves $s\colon [0,1] \to M_k$ such that for every $t \in [0,1]$ either $s(t) = \tilde\gamma(t)$ or there is an interval $[a,b] \ni t$ on which s is constant and $s(t) = \tilde\gamma(a) = \tilde\gamma(b)$. This class of curves is closed in C^0 and hence contains a curve of minimal length. This minimal curve obviously joins $\tilde\gamma(0)$ to $\tilde\gamma(1)$, is self-disjoint, and is an ε_1-lift of γ for $\varepsilon_1 = 7\delta + \theta(14\delta)$. Any constant intervals that this lift may have can be got rid of by a slight variation of the parameterization.

Applying the above construction to all edges gives an ε_1-lift of Γ which is injective on every edge. Let $p \in V(\Gamma)$, $\tilde p = \psi(p)$, let $\gamma_1, \dots, \gamma_m$ be the edges of Γ emanating from p, and $\tilde\gamma_1, \dots, \tilde\gamma_m$ be their ε_1-lifts that we have constructed. Then all intersections of the curves $\tilde\gamma_i$ are contained within a neighborhood $U = U_{\varepsilon_2}(\tilde p)$ where $\varepsilon_2 = \theta(2\varepsilon_1) + 2\delta$. One may assume that $\varepsilon_1 + \varepsilon_2 < \varepsilon_0/10$. Then lifts of other edges and vertices of the graph have no points in U. For $i = 1, \dots, m$, denote by $\tilde p_i$ the point through which the curve $\tilde\gamma_i$ leaves U for the last time. Replace initial intervals of the curves $\tilde\gamma_i$ between $\tilde p$ and $\tilde p_i$ by simple curves lying in $U \cup \{\tilde p_1, \dots, \tilde p_m\}$

and having no common interior points. This is possible because M_k is a manifold of dimension $n \geq 2$ and U is open and connected (recall that U is a length metric ball). The modification deals with curve intervals having distance ε_2 between endpoints, so the resulting curves are ε_3-lifts of the curves γ_i for $\varepsilon_3 = \theta(\varepsilon_2 + 2\varepsilon_1) + \varepsilon_2 + 2\delta$. Having applied this construction to all vertices of the graph we obtain its ε_3-lift which is an embedding. Observing that $\varepsilon_3 \to 0$ as $\delta \to 0$ completes the proof. □

2.3. Corollary. *If the maps φ_k are continuous then for all large enough k they induce surjective homomorphisms of the fundamental groups.*

Proof. Since M is compact and locally simply connected there is an $\varepsilon > 0$ such that any two ε-close curves in M with the same edpoints are homotopic. Let k be so large that $E(\varphi_k) < \varepsilon/7$. Pick a $\tilde{p} \in M_k$ and let $p = \varphi_k(\tilde{p})$. By Lemma 2.2, any loop in M with endpoints at p admits an ε-lift to M_k with endpoints at $\tilde{p}$. The image of that lift is homotopic to the initial loop. □

Corollary 2.3 allows to derive the semi-continuity of the volume in cases when epimorphisms of fundamental groups can only be induced by maps having nonzero degree. The following theorem is an example of statement obtained this way.

2.4. Theorem. *Let M and M_k ($k = 1, 2, \dots$) be homotopy equivalent closed Riemannian n-manifolds. Let M admit a nonzero-degree map onto the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ or an odd-degree map onto the projective space $\mathbf{RP}^n$. Then the convergence $M_k \to M$ implies that*

$$\mathrm{Vol}(M) \leq \liminf_{k\to\infty} \mathrm{Vol}(M_k).$$

Proof. In view of Theorem 1.5, Proposition 1.2 and Corollary 2.3, it is sufficient to prove the following statement: if a manifold M' is homotopy equivalent to a manifold M satisfying the conditions of the theorem, and a map $\varphi\colon M' \to M$ induces an epimorphism of the fundamental groups, then φ has nonzero degree.

1. Suppose there is a map $f\colon M' \to T^n$ having nonzero degree (the existence of such a map is a homotopy invariant). Consider the diagram

$$\begin{array}{ccccc} H_1(M';\mathbf{Z}) & \xleftarrow{\;h'\;} & \pi_1(M') & \xrightarrow{\;f_\#\;} & \pi_1(T^n) \\ \big\downarrow{\scriptstyle\varphi_*} & & \big\downarrow{\scriptstyle\varphi_\#} & & \\ H_1(M;\mathbf{Z}) & \xleftarrow{\;h\;} & \pi_1(M) & & \end{array}$$

(where h and h' are Hurewich homomorphisms). The maps h and $\varphi_\#$ are epimorphisms, so is φ_*. Observe that $H_1(M', \mathbf{Z})$ and $H_1(M; \mathbf{Z})$ are two isomorphic finitely generated abelian groups, so any epimorphism between them is an isomorphism. Thus

$$\ker \varphi_\# \subset \ker(\varphi_* \circ h') = \ker h' = [\pi_1(M'), \pi_1(M')].$$

On the other hand, $\ker f_\# \supset [\pi_1(M'), \pi_1(M')]$ because $\pi_1(T^n)$ is an abelian group. So there exists a homomorphism $g\colon \pi_1(M) \to \pi_1(T^n)$ such that $g \circ \varphi_\# = f_\#$. Since T^n is an aspherical space, g is induced by some continuous map $\bar{f}\colon M \to T^n$ with $\bar{f} \circ \varphi \sim f$. Therefore φ induces a nontrivial homomorphism of n-dimensional homologies whenever f does.

2. Suppose there is a map $f_1\colon M' \to \mathbf{RP}^n$ having odd degree. Define $f = i \circ f_1$ where i is the standard inclusion of $\mathbf{RP}^n$ into $\mathbf{RP}^\infty$. Then f induces a nontrivial

homomorphism $f_*\colon H_n(M';\mathbf{Z}_2) \to H_n(\mathbf{RP}^\infty;\mathbf{Z}_2) \simeq \mathbf{Z}_2$. The rest of the proof goes as in the first part, with $\mathbf{RP}^\infty$ in place of T^n. $\square$

2.5. Remark. One can see from the above proof that the statement of Theorem 2.4 holds for any manifold M that admits a continuous map $f\colon M \to X$ to some aspherical space X with abelian group $\pi_1(X)$ such that the induced map $f_*\colon H_n(M) \to H_n(X)$ is nontrivial for some coefficient group. N. Yu. Netsvetaev observed that the statement of the theorem can also be proved for a manifold M for which there exist $n = \dim M$ cohomology classes in $H^1(M)$ with nonzero $\cup$-product.

2.6. A queston. Does the statement of Theorem 2.4 hold for any aspherical manifold M? If so, does it hold for any essential M (cf. [6])?

§3. Convergence of two-dimensional manifolds

Throughout this section all manifolds are assumed two-dimensional and possibly having boundaries. We denote by $g(M)$ the genus of a manifold M, by $|\partial M|$ the number of its boundary components, and by $\chi(M)$ its Euler characteristic.

3.1. Definition. Let M and M' be two-dimensional manifolds. We say that a continuous map $\varphi\colon M' \to M$ is an *almost homeomorphism* if there is a finite set $P \subset M \setminus \partial M$ such that

(1) φ maps $\varphi^{-1}(M \setminus P)$ onto $M \setminus P$ as a homeomorphism;
(2) for every $p \in P$ the inverse image $\varphi^{-1}(p)$ is either a boundary component of M' or a two-dimensional submanifold bounded (in M') by a simple closed curve.

Note that any almost homeomorphism between closed manifolds has degree ± 1.

3.2. Theorem. *Let M and M_k $(k = 1, 2, \dots)$ be compact two-dimensional manifolds with length metrics such that $M_k \to M$ and $\sup_k g(M_k) < \infty$. Then there is a sequence of almost isometries $\varphi_k\colon M_k \to M$ that are almost homeomorphisms for all large enough k.*

The proof of this theorem is contained in sections 3.5 and 3.7–3.10. In fact, we will show that any sequence of almost isometries can be approximated by a sequence of almost homemorphisms. In section 3.6 we outline a plan of the proof and its main ideas.

3.3. Corollary. *Let M and M_k $(k = 1, 2, \dots)$ be compact two-dimensional Riemannian manifolds (possibly with boundaries) such that $\sup_k |\chi(M_k)| < \infty$. Then the convergence $M_k \to M$ implies that*

$$\mathrm{Vol}(M) \le \liminf_{k\to\infty} \mathrm{Vol}(M_k).$$

Proof. Suppose the contrary. Then one may assume that there exists a limit $\lim_{k\to\infty} \mathrm{Vol}(M_k) < \mathrm{Vol}(M)$. The condition $\sup_k |\chi(M_k)| < \infty$ is equivalent to that both $g(M_k)$ and $|\partial M_k|$ are uniformly bounded. Let $\varphi_k\colon M_k \to M$ $(k = 1, 2, \dots)$ be almost isometries given by Theorem 3.2. For each k define

$$Q_k = \{p \in M \setminus \partial M : \varphi_k^{-1}(p) \text{ contains a boundary component of } M_k\}.$$

Every set Q_k consists of at most $N = \sup_k |\partial M_k|$ points. Passing to a subsequence one can achieve that the set $Q = \bigcup_k Q_k$ contains at most N accumulation points and hence its closure $\overline{Q}$ is countable. Every almost homeomorphism φ_k has nonzero degree over $M \setminus (\partial M \cup \overline{Q})$, thus by Theorem 1.5 we have

$$\lim_{k\to\infty} \operatorname{Vol} M_k \geq \operatorname{Vol}(M \setminus (\partial M \cup \overline{Q})) = \operatorname{Vol} M.$$

This is a contradiction. □

3.4. Remarks. 1. In Corollary 3.3, the requirement that the geni and the numbers of boundary components are uniformly bounded is essential (moreover it is the weakest topological condition possible). Indeed, any Riemannian manifold M can be aproximated by embedded graphs (cf. 4.2). One can embed these graphs to $\mathbf{R}^3$ and let M_k be smoothed boundaries of their tubular neighborhoods, thus obtaining an example of convergence with $\operatorname{Vol}(M_k) \to 0$. If $|\partial M_k|$ is allowed to grow infinitely, one can let M_k be neighborhoods of those graphs in M.

2. In the same manner, a sequence of manifolds M_k with $g(M_k) \to \infty$ can be equipped with Riemannian metrics so as to converge to any prescribed compact length metric space. On the other hand, if $\sup g(M_k) < \infty$ then the topological dimension of the limit cannot be greater than 2. Indeed, the limit space cannot contain complete graphs with very large number of vertices, otherwise Lemma 2.2 would imply that such graphs are embeddable to M_k.

3. Let the topology types of manifolds M and M_k be given. How to determine whether $\{M_k\}$ can converge to M? If $\sup g(M_k) < \infty$ then by Theorem 3.2 the existence of almost homeomorphisms from M_k to M for all large enough k is necessary. This condition is obviously sufficient as well. It is equivalent to the following: $|\partial M_k| \geq |\partial M|$ and either $g(M_k) \geq g(M)$ while M and M_k are of the same orientability, or M is orientable, M_k is not, and $g(M_k) \geq 2g(M) + 1$. In particular, orientable manifolds cannot converge to a non-orientable one, and closed manifolds cannot converge to a manifold with a nonempty boundary.

3.5. Let M and M_k $(k = 1, 2, \dots)$ satisfy the assumptions of Theorem 3.2. Define $g = \sup_k g(M_k) + 1$. To prove the theorem it is sufficient to show that for any $\varepsilon > 0$, for all large enouh k there are ε-isometries $\varphi'_k\colon M_k \to M$ that are almost homeomorphisms. We start by fixing some sequence of continuous almost isometries $\varphi_k\colon M_k \to M$.

All curves that we consider throughout the proof are assumed self-disjoint. We freely identify such curves with corresponding subsets of M and M_k. By *properly ebedded* curve we mean a closed curve that has a connected (possibly empty) intersection with the manifold's boundary. We call a curve *dividing* if it is properly embedded and have disconnected complement.

3.6. The proof of Theorem 3.2 contains many technical details, so we first present a simplified argument upon which the proof is based. It also shows how we utilize the conditions that the manifolds M_k have bounded geni and their metrics are length ones.

Fix sufficiently many (at least g) disjoint discs in M. Then, for a large enough k, construct in M_k "lifts" (in the sence of 2.1 and 2.2) of the boundaries of these discs. These lifts are closed simple curves in M_k. Since the number of these curves is greater than the genus of M_k, some (sub)collection of them divides M_k into two components. Since the metric in M_k is a length one, points of different components that are distant from the dividing curves are also distant from one another. The φ_k-images of the components must also possess such a property because φ_k is an almost isometry. This easily implies that the image of one of the components is contained in a small neighborhood of one of the discs in M and the image of the other is contained in a small neighborhood of the complement of the same disc. In particular, the dividing collection consists of only one curve (cf. Lemma 3.7 for details). It follows that φ_k has a nonzero degree over some domain inside the disc, and for closed manifolds this implies that φ_k has nonzero degree over M. (Note that this is sufficient to prove the semi-continuity of the volume.)

In addition to the above considerations, a simple combinatorical argument can be used to construct an almost homeomorphism which is close to φ_k. This construction is given in section 3.10. For M having boundary, we also need the fact (Lemma 3.9) that every boundary component of M admits a "lift" which is a boundary component of M_k. Note that this fact is not trivial: it implies in particular that closed manifolds (two-dimensional, of bounded geni, and with length metrics) cannot converge to a manifold with boundary.

3.7. Lemma. *Let $\gamma_1, \dots, \gamma_m$ be disjoint dividing curves in M. For every $\varepsilon > 0$ there is a $\delta > 0$ such that: if $E(\varphi_k) < \delta$ and if properly embedded curves $\tilde\gamma_1, \dots, \tilde\gamma_m$ in M_k are δ-lifts of $\gamma_1, \dots, \gamma_m$, respectively, then*

1. *if the union of the curves $\tilde\gamma_i$ divides M_k, at least one of these curves is a dividing one;*
2. *if $m \ge g$, then at least one of the curves $\tilde\gamma_i$ is a dividing one;*
3. *if γ_i divides M into sets V and W and $\tilde\gamma_i$ divides M_k into sets $\tilde V$ and $\tilde W$, then either $\varphi_k(\tilde V) \subset U_\varepsilon(V)$ and $\varphi_k(\tilde W) \subset U_\varepsilon(W)$ or $\varphi_k(\tilde V) \subset U_\varepsilon(W)$ and $\varphi_k(\tilde W) \subset U_\varepsilon(V)$.*

Proof. One may assume that the distances between the curves γ_i are greater than 3ε. Then the curves $\tilde\gamma_i$ are disjoint provided $\delta < \varepsilon$. For each i, draw two curves γ_i' and γ_i'' in the ε-neighborhood of γ_i so that they lie toward different sides of γ_i and separate γ_i from $M \setminus U_\varepsilon(\gamma_i)$. (If $\gamma \cap \partial M \ne \varnothing$ then one of the curves γ_i' and γ_i'' is not closed but joins two boundary points.) We will show that (1)–(3) hold for $\delta < \min_i \operatorname{dist}(\gamma_i, \gamma_i' \cup \gamma_i''))/5$.

Note that (2) follows from (1) because $g(M_k) < g$. In a proof of (1) we may assume that $\{\tilde\gamma_1, \dots, \tilde\gamma_m\}$ is a minimal collection of curves that divides M_k. Then these curves divide M_k into two sets $\tilde V$ and $\tilde W$, and $\partial\tilde V = \partial\tilde W = \tilde\gamma_1 \cup \dots \cup \tilde\gamma_m$. Define $V' = U_\delta(\varphi_k(\tilde V))$ and $W' = U_\delta(\varphi_k(\tilde W))$. We have $V' \cup W' = M$ and $\gamma_i \subset V' \cap W'$ for $i = 1, \dots, m$.

The curves γ_1' and γ_1'' split M into three sets X, Y and Z, such that $\partial X = \gamma_1'$, $\partial Y = \gamma_1''$, and $U_{5\delta}(\gamma_1) \subset Z \subset U_\varepsilon(\gamma_1)$. We claim that either $V' \subset X \cup Z$ and $W' \subset Y \cup Z$ or $V' \subset Y \cup Z$ and $W' \subset X \cup Z$. Suppose the contrary, for example, assume that $V' \cap X \ne \varnothing$ and $W' \cap X \ne \varnothing$. Then $V' \cap \gamma_1' \ne \varnothing$ and $W' \cap \gamma_1' \ne \varnothing$ since V' and W' are connected. Hence $V' \cap W' \cap \gamma_1' \ne \varnothing$. This means that there are points $p \in \tilde V$ and $q \in \tilde W$ such that $d(\varphi_k(p), x) < \delta$ and $d(\varphi_k(q), x) < \delta$ for some point x on γ_1'. For these points we have $d(p, q) < 3\delta$. On the other hand, the facts that the metric of M_k is a length one and $\varphi_k(\partial\tilde V) \subset U_\delta(\bigcup \gamma_i)$ imply that

$$d(p,q) \ge \operatorname{dist}(p, \partial\tilde V) + \operatorname{dist}(q, \partial\tilde W) > 2\operatorname{dist}(\gamma_1', \gamma_1 \cup \dots \cup \gamma_m) - 6\delta \ge 4\delta$$

with a contradiction. Therefore we have $V' \subset X \cup Z$ and $W' \subset Y \cup Z$ up to a change of notation. For $m = 1$ this gives the statement (3) of the lemma. If $m > 1$, similar inclusions must also hold for the partition of M by γ_2' and γ_2'', but this is impossible. This proves the statement (1). □

3.8. Corollary. *Let $\varepsilon > 0$ and γ be a dividing curve in M. Then for every large enough k there is a dividing ε-lift of γ in $M_k \setminus \partial M_k$.*

Proof. Construct g disjoint dividing curves that are $(\varepsilon/2)$-close to γ. By the statement (2) of Lemma 2.2, for a large enogh k these curves admit properly embedded $(\varepsilon/2)$-lifts in $M_k \setminus \partial M_k$. By the statement (2) of Lemma 3.7, one of these lifts is a dividing curve. □

3.9. Lemma. *Let $\varepsilon > 0$. Then for every large enough k there exists an ε-lift of ∂M in M_k which maps ∂M homeomorphly onto a union of several boundary components of M_k.*

Proof. Let γ be a component of ∂M. Fix some retraction $\pi\colon V_0 \to \gamma$ where V_0 is a neighborhood of γ in M. Let $U \subset V_0$ be a smaller neighborhood of γ. We will first prove that for any large enough k there is a boundary component $\tilde\gamma \subset \partial M_k$ such that $\varphi_k(\tilde\gamma) \subset U$ and the map $\pi \circ \varphi_k|_{\tilde\gamma}\colon \tilde\gamma \to \gamma$ has nonzero degree. Construct a dividing curve $\gamma_0 \subset U$ such that the map $\pi|_{\gamma_0}\colon \gamma_0 \to \gamma$ has degree ± 1. Let $\tilde\gamma_0 \subset M_k \setminus \partial M_k$ be a dividing σ-lift of γ_0 (cf. 3.8) for σ so small that the loop $\varphi_k \circ \tilde\gamma_0$ is homotopic to γ_0 and the statement (3) of Lemma 3.7 assures that $\varphi_k(\tilde U) \subset U$ where $\tilde U$ is the closure of one of the components of $M_k \setminus \tilde\gamma_0$. Consider the map $\pi \circ \varphi_k\colon \tilde U \to \gamma \simeq S^1$. The degree of its restriction on $\tilde\gamma_0$ is ± 1, hence this degree is nonzero for at least one of the components of the set $\partial M_k \cap \tilde U = \partial \tilde U \setminus \tilde\gamma_0$. This component is the desired $\tilde\gamma$.

Now fix an orientation on γ and pick a cyclically ordered collection of points $x_1, \dots, x_N \in \gamma$ so that $N > 100g$ and the points $\{x_i\}$ split γ into intervals of diameter less than $\varepsilon/10g$. Let δ be so small that all nonzero distances between those intervals are greater than 10δ. Construct a dividing curve $\gamma_1 \subset M \setminus \partial M$ which is δ-close to γ. Pick a $\sigma > 0$ such that $U_\sigma(\gamma) \subset V_0$ and $d(\pi(x), x) < \operatorname{dist}(\gamma, \gamma_1)/10$ for all $x \in U_\sigma(\gamma)$. Let k be large enough, $\tilde\gamma_1 \subset M_k \setminus \partial M_k$ be a dividing σ-lift of γ_1 (cf. 3.8), $\tilde\gamma$ be a component of ∂M_k for which $\varphi_k(\tilde\gamma) \subset U_\sigma(\gamma)$ and the composition $\varphi := \pi \circ \varphi_k|_{\tilde\gamma}\colon \tilde\gamma \to \gamma$ has nonzero degree (see above). Let $\tilde V$ be the component of $M_k \setminus \tilde\gamma_1$ containing $\tilde\gamma$.

Choose an orientation on $\tilde\gamma$ so that the degree of φ is positive. Then one can find a cyclically ordered collection of points $y_1, \dots, y_N \in \tilde\gamma$ such that $\varphi(y_i) = x_i$ for all i. For points p and q on γ we denote by $[p, q]$ the interval of γ that goes from p to q in accordance with the orientation. We will prove that every point of $[y_i, y_{i+1}]$ is an ε-lift of any point of $[x_i, x_{i+1}]$ (the indices here are taken modulo N). To do that, it suffices to show that $\varphi([y_i, y_{i+1}])$ contains less than $10g$ of points $\{x_j\}$.

Suppose the contrary, e.g., let $\varphi([y_{N-1}, y_N])$ contain points $x_1, \dots, x_m$ where $m = 4g$. For each $i = 1, \dots, m$ find a point $y_i' \in [y_{N-1}, y_N]$ such that $\varphi(y_i') = x_i$. One may assume that $E(\varphi_k) < \sigma$. Then

$$d(y_i, y_i') < \sigma < \operatorname{dist}(\tilde\gamma, \tilde\gamma_1) \le \operatorname{dist}(\{y_i\} \cup \{y_i'\}, \tilde\gamma_1) < \delta + 2\sigma < 2\delta.$$

Therefore one can construct curves $r_i, s_i, s_i' \subset U_{2\delta}(\{y_i\} \cup \{y_i'\})$ and points $z_i, z_i' \in \tilde\gamma_1$ $(z_i \ne z_i')$ so that r_i joins y_i to y_i', s_i joins y_i to z_i, s_i' joins y_i' to z_i', and r_i, s_i and s_i'

have no common internal points with one another, with $\tilde{\gamma}_1$ and with ∂M_k. Since these curves are close to the points y_i and y_i', they do not cross similar curves constructed for other values of i.

Let Γ denote the graph formed by the curves $\tilde{\gamma}$, $\tilde{\gamma}_1$, r_i, s_i and s_i' $(1 \le i \le m)$. This graph is embedded into $\tilde{V}$ and its cycles $\tilde{\gamma}$ and $\tilde{\gamma}_1$ are contained in $\partial\tilde{V}$. Let us show that the existence of such a graph contradicts to that $g(\tilde{V}) < g$. We may assume that $\partial\tilde{V}$ consists of only two components, $\tilde{\gamma}$ and $\tilde{\gamma}_1$. The graph Γ has $4m$ vertices (namely, the points $y_i, y_i' \in \tilde{\gamma}$ and $z_i, z_i' \in \tilde{\gamma}_1$, $1 \le i \le m$) and $7m$ edges of which $4m$ ones are contained in $\partial\tilde{V}$. And it contains at most two cycles of length 2 or 3 (any such cycle must contain y_1 or y_m). Thus the number of components into which $\tilde{V}$ is divided by Γ does not exceed $(2 \cdot 7m - 4m + 4)/4 = \frac{5}{2}m + 1$. Hence

$$\chi(\tilde{V}) \le 4m - 7m + \tfrac{5}{2}m + 1 = 1 - m/2 = 1 - 2g.$$

Contrary to this, $\chi(\tilde{V}) \ge 2 - 2g$ when $g(\tilde{V}) < g$ and $|\partial\tilde{V}| = 2$.

We have proved that a suitable parameterization of $\tilde{\gamma}$ is an ε-lift of γ. To finish the proof construct such lifts for all components of ∂M. □

3.10. Proof of Theorem 3.2. Having fixed an $\varepsilon_0 > 0$ pick a sufficiently fine triangulation of M (the exact requirements to the fineness will be clear from the sequel). The boundary of every triangle must be a properly embedded curve (see 3.5 for definition). We denote by Γ the one-dimensional skeleton of the triangulation. We call a *polyhedron* any domain in M that is homeomorphic to disc and bounded by a properly embedded curve composed from edges of Γ. Find a positive $\varepsilon < \varepsilon_0$ such that $U_{10\varepsilon}(M \setminus T) \ne M$ for any triangle T. Pick a $\delta = \delta(\varepsilon) > 0$ for which the statement of Lemma 3.7 holds for any collection $\{\gamma_i\}$ of curves composed from edges of Γ. For k large enough the lemmas 3.9 and 2.2 allow us to construct a δ-lift $\psi_k \colon \Gamma \to M_k$ such that $\psi_k(\partial M) \subset \partial M_k$, $\psi_k(\Gamma \setminus \partial M) \subset M_k \setminus \partial M_k$, and ψ_k is an embedding.

We call a triangle T *suitable* if $\psi_k(\partial T)$ divides M_k. For such T Lemma 3.7, part (3), implies that one of the components of $M_k \setminus \psi_k(\partial T)$ is mapped by φ_k into $U_\varepsilon(T)$. We call that component the *lift* of T and denote it by $\tilde{T}$. Find a maximal collection of disjoint non-suitable triangles. By Lemma 3.7, part (2), this collection contains at most $g - 1$ triangles. If the triangulation is fine enough then these triangles, wherever they are, can be included in the interior of a union of disjoint polygons $P_1, \dots, P_m$ $(m < g)$ whose diameters do not exceed ε_0. Note that all the triangles in $M \setminus \bigcup P_i$ are suitable and also have diameters no greater than ε_0. Now we exclude the triangles contained in $\bigcup P_i$ from the list of suitable triangles. Instead, if $\psi_k(\partial P_i)$ is a dividing curve then we call a polygon P_i suitable and define its lift $\tilde{P}_i$ in the same way as for triangles.

Let M' denote the closure of the union of all suitable trangles and polygons, M_k' denote the closure of the union of their lifts. By the choice of ε, the lifts of different suitable triangles and polygons cannot contain one another. Thus these lifts are disjoint and form the same combinatorical structure as the respective triangles and polygons do. In particular, $\partial M_k' \setminus \partial M_k = \psi_k(\partial M' \setminus \partial M)$. This implies that $M' = M$ and $M_k' = M_k$. Indeed, otherwise we have $\partial M_k' \setminus \partial M_k \ne \varnothing$ and hence the ψ_k-images of the boundaries of non-suitable polygons divide M_k, which contradicts to Lemma 3.7, part (1).

Now construct an almost homeomorphism $\varphi_k' \colon M_k \to M$ which is close to φ. Define $\varphi_k'|_{\psi_k(\Gamma)} = \psi_k^{-1}$. On the lift of every triangle $T \subset M \setminus \bigcup P_i$ define φ_k' to be an almost homeomorhphism from $\tilde{T}$ to T that extend $\psi_k^{-1}|_{\partial\widetilde{T}}$ (for example, contract everything but a narrow strip along $\partial\tilde{T}$ into one point). The same is to be done for the polygons P_i. The resulting map φ'_k is an almost homeomorphism and its distance from φ_k is at most $\varepsilon_0+\varepsilon$. Since ε_0 is arbitrary, the theorem follows. □

§4. Examples

In this section we give examples of convergence of three-dimensional spheres in which the semi-continuity of volume fails. The construction can be easily extended to spheres S^n of any dimension $n\geq 3$, furthermore, examples for $n>3$ can be obtained from ones for $n=3$ by taking a suspension and smoothing. The main idea of our construction is in Lemma 4.1.

By a *disc with holes* we mean a three-dimensional disc D^3 from which there are removed interiors of several smaller discs that are separated away from one another and from ∂D^3.

4.1. Lemma. *Let M be a disc with holes and d be a Riemannian metric on M. Then there exists a sequence of Riemannian metrics $\{d_k\}_{k=1}^\infty$ on S^3 such that $(S^3,d_k)\to(M,d)$ and $\operatorname{Vol}(S^3,d_k)<2\operatorname{Vol}(M,d)$ for all k.*

Proof. Let M have m boundary components. Denote these components by $F_1,\dots,F_m$. Pick an $\varepsilon>0$ and construct smooth disjoint curves $\gamma_1,\dots,\gamma_m\subset M$ in such a way that

1. for every $i<m$ the curve γ_i joins F_i to F_{i+1}, while γ_m starts at F_m and ends at an interior point of M;
2. the curves γ_i do not meet ∂M except at endpoints;
3. γ_m is an ε-net in (M,d).

Then, for a sufficiently small $\delta>0$, consider the set $M_\delta=M\setminus U_\delta(\bigcup\gamma_i)$ and denote by d_δ its induced length metric. As $\delta\to 0$, the metrics d_δ converge uniformly to the induced length metric of the set $\bigcup_{\delta>0}M_\delta=M\setminus\bigcup\gamma_i$, and that metric in its turn coincides with the restriction of d because M is three-dimensional. Thus the spaces (M_δ,d_δ) converge to (M,d) as $\delta\to 0$. Furthermore M_δ is homeomorphic to D^3 when δ is small.

Let δ be so small that $d_H(M_\delta,M)<\varepsilon$ and $M_\delta\simeq D^3$. Consider the doubling of M_δ, i.e., the space $S_\delta=M_\delta\cup M'_\delta$ where M'_δ is an isometric copy of M_δ attached to M_δ by means of the natural isometry of their boundaries. (The distance in S_δ between $x\in M_\delta$ and $x'\in M'_\delta$ is defined to be $\inf_{y\in\partial M_\delta}\{\operatorname{dist}(x,y)+\operatorname{dist}(x',y)\}$.) The space S_δ is homeomorphic to S^3 and its metric can be made Riemannian by smoothing near ∂M_δ (with an arbitarily small change of the distances and the volume). Moreover $\operatorname{Vol}(S_\delta)=2\operatorname{Vol}(M_\delta)<2\operatorname{Vol}(M)$.

The construction implies that ∂M_δ is an ε-net in M'_δ. Thus $d_H(S_\delta,M_\delta)\leq\varepsilon$, and hence $d_H(S_\delta,M)<2\varepsilon$. Since ε is arbitrary, the lemma follows. □

We will need the following technical fact:

4.2. Lemma. *For every compact lenght metric space X and every $\varepsilon>0$ there is a graph $\Gamma\subset M$ such that the inclusion $\Gamma\hookrightarrow X$ is an ε-isometry with respect to the induced length metric of Γ.*

Proof. Pick a finite ε-net S in X. Join every pair of points of S by a shortest path and denote by Γ_0 the union of those paths. Let $S'\subset\Gamma_0$ be a finite $(\varepsilon/8)$-net with respect to the induced length metric of Γ_0. For each pair of points $x \in S$, $y \in S'$ with $d(x,y) < \varepsilon/8$ draw a shortest path (in X) joining x to y. Let Γ be the union of Γ_0 and these new paths, d_Γ be the induced length metric on Γ. Then the inclusion of (Γ, d_Γ) into (X,d) is an ε-isometry. To prove this, consider any two points $x, y \in \Gamma$. Let x_1 be a point of S' closest to x with respect to the metric d_Γ, x_2 be a point of S closest to x_1 with respect to d, and let $y_1 \in S'$ and $y_2 \in S$ be constructed in a similar way for y. The the distances $d_\Gamma(x,x_1)$, $d_\Gamma(x_1,x_2)$, $d_\Gamma(y,y_1)$ and $d_\Gamma(y_1,y_2)$ are no greater than $\varepsilon/8$, and $d_\Gamma(x_2,y_2) = d(x_2,y_2)$. Hence $d_\Gamma(x,y) \le d(x,y) + \varepsilon$, which is the desired relation.

It is easy to show that the shortest paths in the above construction can be chosen so that the intersection of any two of them, if nonempty, is either a point or an interval. Then the resulting set Γ is a graph. □

4.3. Theorem. *For every Riemannian metric d on S^3 there is a sequence $\{d_k\}_{k=1}^{\infty}$ of Riemannian metrics on S^3 such that $(S^3, d_k) \to (S^3, d)$ and $\mathrm{Vol}(S^3, d_k) \to 0$ as $k \to \infty$.*

Proof. By removing suitable neighborhoods of some point one can approximate the space (S^3, d) by its subsets diffeomorphic to D^3. Hence to prove the theorem it suffices to approximate any prescribed metric on the standard three-disc $B \subset \mathbf{R}^3$ by spheres of arbitrarily small volume. Lemma 4.1 allows to construct the approximating metrics on discs with holes instead of spheres.

Let d be a Riemannian metric on B and $\varepsilon > 0$. Split B into small cells by three families of planes parallel to the coordinate ones so that any straight segment contained in a single cell has length (with respect to d) no greater than ε. Then, using Lemma 4.2, find a graph $\Gamma \subset B$ whose inclusion into (B,d) is an ε-isometry with respect to its length metric. One may assume that every cell contains at least one vertice of Γ and that the edges of Γ are composed from straight segments. Include each of those segments into a planar section of B. Let X denote the union of all those sections and the faces of all cells, and let X be equipped with its induces length metric. It is easy to see that Γ is a (10ε)-net in X, thus X well approaches (B,d).

The set $X \subset B$ is a union of planar discs that split B into convex domains. A proper small neighborhood of X, with its induced length metric, is the desired example of a disc with holes that well approaches (B,d) and have arbitrarily small volume. □

4.4. Remarks. The constructions from Lemma 4.1 and Theorem 4.3 can be thought of as a way to construct a metric on a given manifold (the 3-sphere in our case) such that some prescribed map (the projection of the sphere to disc) is an almost isometry with respect to that metric. These constructions easily extend to other manifolds, provided there are maps with relatively simple singularities (for example, one may allow a ramification over a set of codimension 2 in addition to the projection structure).

It would be intersting to find out which homotopy types of maps can be realized by sequences of almost isometries. For two-dimensional manifolds, the answer is given by Theorem 3.2. For higher dimensions, however, it is unclear whether there are any restrictions except those from Corollary 2.3.

Another question is, given a convergence realized by almost isometries of zero degree, is it always possible to modify the metrics so that they converge to the same limit but their volumes tend to zero? The construction given in the proof of Theorem 4.3 is quite flexible, and perhars some its version can serve for this general case as well. If so, then the problem of semi-continuity of the volume for given topology completely reduces (by means of Theorem 1.5) to the study of degrees of almost isometries.

References

1. A. S. Besikowiĉ, *On two problems of Loewner*, J. London Math. Soc. **27** (1952), 141–144.
2. D. Burago, S. Ivanov, *On asymptotic volume of tori*, Geom. Funct. Anal. **5** (1995), 800–808.
3. C. Croke, B. Kleiner, *On tori without conjugate points*, Invent. Math. **120** (1995), 241–257.
4. W. Derrick, *A weighted volume-diameter inequality for n-cube*, J. Math. Mech. **18** (1968), 453–472.
5. ———, *A volume-diameter inequality for n-cube*, Analysi Math. **22** (1969), 1–36.
6. M. Gromov, *Filling Riemannian manifolds*, J. Diff. Geom. **18** (1983), 1–147.
7. M. Gromov, J. Lafontaine, P. Pansu, *Structures métriques pour les variétés riemanniennes*, Cedic, Paris, 1981.
8. K. Grove, P. Petersen, J.-Y. Wu, *Geometric finiteness theorems via controlled topology*, Invent. Math. **99** (1990), 205–213; *Erratum*, Invent. Math. **104** (1991), 221–222.
9. P. Petersen, *A finiteness theorem for metric spaces*, J. Diff. Geom. **31** (1990), 387-395.

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