Lectures on Coherent Configurations (update June, 2024)

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Preface

Originating in the late 1960s, now the theory of coherent configurations has become the central part of algebraic combinatorics understood in the spirit of the Bannai–Ito monograph [**10**]. One of the main goals of this theory is to provide a common method to study symmetries of (mostly finite) algebraic, geometrical, and combinatorial objects. It is therefore not so surprising that a rich source of coherent configurations is the permutation group theory providing a natural way to deal with automorphism groups of the objects. This line goes back to the method of Schur rings introduced by I. Schur (1933) and then developed by H. Wielandt in 1960s. The Schur rings can be considered as a special case of coherent configurations defined by D. Higman (1971) (see [**63**]) as a tool to study permutation groups via analyzing invariant binary relations; in particular, the coherent configuration associated with a permutation group is formed by the minimal invariant binary relations. In this way, it was proved that some permutation representations of classical groups are uniquely determined by subdegrees, and a sporadic simple group (the Higman–Sims group) was found. It should be also mentioned that the proof of a tight upper bound for the order of a uniprimitive group, found by L. Babai (1981), is also based on using coherent configurations.

Another source of coherent configuration is closely related with the Graph Isomorphism Problem: to find a most efficient algorithm to test isomorphism of finite graphs. In 1968, B. Weisfeiler and A. Leman described a rather simple procedure (called now the Weisfeiler–Leman algorithm), which given a graph constructs a matrix algebra that keeps all the information on the automorphism group of the graph. This algebra has a uniquely determined linear basis consisting of $\{0,1\}$-matrices and the binary relations underlying these matrices form a coherent configuration. The use of the Weisfeiler–Leman algorithm for graph isomorphism testing forced researchers to study these coherent configurations. The results obtained in this way by S. Evdokimov and I. Ponomarenko in 2000s show that this algorithm tests isomorphism correctly in various classes of graphs. A culmination of the coherent configuration approach to the Graph Isomorphism Problem became an outstanding result of L. Babai (2015) giving a quasipolynomial algorithm testing isomorphism of arbitrary graphs.

The last but not the least source of coherent configurations is the association scheme theory originated in 1950s in papers of R. Bose concerning designs in statistics and studied by F. Delsarte in 1970s in connection with coding theory. In some sense, association schemes form the commutative part of the coherent configuration theory. Probably, the most impressive results here were obtained for a special class of association schemes that arise from distance-regular graphs. There are some signs (see [**1**]) that non-commutative analogs of these schemes correspond to the spherical buildings. This is not so surprising in view of theory of the Coxeter schemes developed by P.-H. Zieschang (1995) and showing that the spherical buildings of finite order can naturally be considered as (homogeneous) coherent configurations.

Each of the above mentioned topics is a living and constantly developing field of mathematics, presented in a series of remarkable monographs concerning permutation groups [**23, 33**], association schemes [**10, 128**], and distance-regular graphs [**17**]. Although in each of them coherent configurations are mentioned in one way or another, none of them can serve as a complete introduction to the whole theory. On the other hand, some parts of the theory are distributed in a number of overlapping papers with different motivations and notation. Thus our primary goal is to create a comprehensive and self-contained introduction to the theory of coherent configurations.

Our approach to the theory of coherent configurations emphasizes on two interrelating basic problems common for all of the above mentioned topics. These problems have never been discussed in detail in monographs and our secondary goal is to fill this gap. In the following two paragraphs we explain the problems separately.

As already noted above, each permutation group defines a coherent configuration. The reverse statement is not true and this is not so obvious. As H. Wielandt wrote in [**126**, p.54] "Schur had conjectured for a long time that every Schur ring is determined by a suitable permutation group"; the same story happened to "cellular algebras" [1] introduced by B. Weisfeiler and A. Leman (but this fallacy was overcome in a year). However, it turns out that overwhelming majority of the coherent configurations are not *schurian*, i.e., they do not come from permutation groups. It seems quite interesting and important to find a "border" between schurian and non-schurian cases, or in other words, a place of the group theory within the theory of coherent configurations; in what follows, we refer to this as the *schurity problem.* For example, the solution to this problem in the class of Coxeter schemes and in the class of symmetric schemes of rank 3 is nothing else than the Tits theorem on spherical buildings and the classification of the rank 3 groups, respectively.

Any coherent configuration $\mathcal{X}$ comes with a natural set of invariants, namely, the intersection number array. When $\mathcal{X}$ is associated with a permutation group K, it is just the structure constant tensor of the centralizer algebra of K with respect to the linear basis of primitive idempotents under the Hadamard multiplication. We say that $\mathcal{X}$ is *separable* if the intersection

[1] They are essentially the adjacency algebras of coherent configurations.

number array determines $\mathcal{X}$ up to isomorphism. Among such configurations, one can see the coherent configurations of classical distance-regular graphs and designs, of groups uniquely determined by their character tables, of spherical buildings of finite order and rank at least 3, etc. Furthermore, the isomorphism of any graph, the coherent configuration of which is separable, with any other graph can be efficiently tested by the Weisfeiler–Leman algorithm. All we said shows the importance of the *separability problem*, which consists in, roughly speaking, identifying separable coherent configurations.

Keeping in mind the schurity and separability problems, we cannot cover in this book many interesting parts of the theory of coherent configurations. In particular, we only touch on issues such as structure theory, representation theory, connections with the Graph Isomorphism Problem; we refer the reader to Zieschang's monograph on homogeneous coherent configurations [**128**] and to the exhaustive surveys [**55**] on representation theory, and [**108**] on a logical approach to testing graph isomorphism. On the other hand, trying to give a self-contained and detailed introduction to coherent configurations, we provide each chapter with a set of exercises. Considered as a part of the main text, they enable us to get much deeper understanding of the whole theory. Let us briefly go through the chapters of the book.

Chapter 1 is a little bit eclectic collection of notation and concepts concerning binary relations, matrices, and permutation groups that are used throughout the text. Some of the notation are not standard and some of the concepts are not widespread. It is assumed that the readers are familiar with the basics of graph theory, linear algebra, and group theory.

Chapter 2 lays the foundation of the whole theory. We start with exact definitions and then analyze the structure of an arbitrary coherent configuration. The Galois correspondence between the coherent configurations and permutation groups provides us with numerous examples of schurian coherent configurations. At this point the schurity problem is introduced. Remaining within the framework of algebra, we establish a one-to-one correspondence between the coherent configurations and coherent algebras, which are the matrix algebras closed under the Hermitian conjugation and the Hadamard multiplication. This enables us to illustrate the concept of an algebraic isomorphism playing a key role in the separability problem. The schurity and separability problems remain nontrivial for a historically first class of coherent configurations, namely, the Cayley schemes which are in one-to-one correspondence with the Schur rings. This is made even more obvious when we pass to coherent configurations associated with finite geometries: projective and affine planes, and designs. In the last part of this chapter, we introduce the concept of the coherent closure, which can be considered as a functor from the category of graphs to the category of coherent configurations. This naturally leads to distance-regular and strongly regular graphs which are discussed in the context of coherent configurations.

In Chapter 3, we present basic constructions and techniques for studying coherent configurations. The most of them are dual to that of permutation groups in the sense of the Galois correspondence. For example, the concept of quotient corresponds to taking the permutation group induced by the action of a group on the classes of an invariant equivalence relation, the direct sum and tensor product correspond to the direct product of permutation groups acting on the disjoint union and Cartesian product of underlying sets, respectively, and the coherent closure extension with respect to a set of points corresponds to the pointwise stabilizer of this set in a permutation group. The latter enables us to introduce the concept of the base of a coherent configuration and establish an upper bound for the base number of a primitive coherent configuration (the Babai theorem). The situation with wreath product is a little bit complicated and we consider several constructions, including the canonical and generalized wreath product of coherent configurations, and exponentiation of a coherent configuration by a permutation group. We are particularly interested in conditions under which the resulting coherent configuration is schurian or separable. The last two sections provide us with introductions to multidimensional constructions corresponding to actions of a permutation group on the Cartesian powers of the underlying set, and to representation theory of coherent configurations over the complex number field.

Chapter 4 contains a collection of results from different parts of the theory. We start with quasiregular coherent configurations; they are schurian and separable only locally. The developed theory is used for constructing coherent configurations which are so far away from being schurian or/and separable as far as it is possible at all. On the other hand, the same theory is used to get a group-theoretical characterizations of homogeneous coherent configurations with at most two different valencies. In studying the problem of separability for cyclotomic schemes over a finite field, we show a way to apply a classification of Schur rings over a cyclic group obtained by K. H. Leung and S. H. Man in [**89, 90**]. Each of the two last topics includes as a special case the schemes of prime degree. In this connection, we present all known results on these schemes including the Hanaki–Uno theorem. The last section concerns the Weisfeiler–Leman method and the connections of it with the Graph Isomorphism Problem, with the first order logic with bounded number of variables, and with theory of multidimensional extensions of coherent configurations and algebraic isomorphisms.

The exercises given at the end of each chapter vary in complexity from very simple to moderately difficult. In the latter case, we provide a reference to a paper containing full proof (however, the presence of such a reference does not mean that the exercise is really difficult). Including the full proof for each exercise would increase the size of the book at least by half. Therefore, we decided to collect all the proofs in a separate text and provide free access to it at (`http://www.pdmi.ras.ru/~inp/coherent_configurations.answers.pdf`). Some of the exercises, as well as some statements in the main text, are of a computational nature. Therefore, it is very useful for the readers to make calculations using the computer package GAP [**51**]. We also recommend an additional free package COCO2P intended for calculations with association schemes and coherent configurations [**84**].

A large part of this book appeared in more than twenty joint papers of Sergei Evdokimov and the second author in 1990-2000s. Sergei suddenly passed away in 2016. However, his influence on the content of this book is hard to overestimate. Unfortunately, some important results obtained by himself have not been published in the form of papers and can only be found in his habilitation thesis written in Russian [**35**]. These results include algebraic foundations of coherent configurations, constructing the multidimensional extensions of the wreath product, and establishing a relationship between coherent configuration and factorization of polynomials over a finite field. Together with the second author, Sergei gave a course on algebraic combinatorics in St. Petersburg State University (St. Petersburg, Russia, 1999). The notes of the course and two other courses on coherent configurations given by the second author in Institute for Advanced Studies in Basic Science (Zanjan, Iran, 2004), and by both authors in Central China Normal University (Wuhan, China, 2017) are used in preparing this text.

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CHAPTER 1

Preliminaries

Three basic languages using in studying coherent configurations come from graphs, matrix algebras, and groups. This gives some freedom to choose the most relevant way to describe a concrete situation, and makes difficulties in developing a general theory. In fact, this explains why at present, there is no one generally accepted system of notation and concepts in the coherent configuration theory. The system adopted in our book is based on a combination of suitable terminology and notation used in standard texts on graph theory and permutation groups. In this chapter, we try to list them in a more or less compact form.

Let us start with general notation. In what follows, Ω denotes a finite set of order $|\Omega| = n$. The elements of Ω are written by lowercase Greek letters and called the *points*. For a collection T of subsets of Ω, we set $T^\natural = T \setminus \{\varnothing\}$.

Given a positive integer m, the Cartesian m-power of Ω is denoted by Ω^m. The diagonal of the latter is set to be

$$\mathrm{Diag}(\Omega^m) = \{(\alpha, \ldots, \alpha) \in \Omega^m : \ \alpha \in \Omega\},$$

and for brevity, we put $1_\Omega = \mathrm{Diag}(\Omega^2)$ and write 1 instead of 1_Ω where this cannot lead to a misunderstanding. The ith coordinate of an m-tuple $\alpha \in \Omega^m$ is denoted by α_i; in particular, $\alpha = (\alpha_1, \ldots, \alpha_m)$. The concatenation of the m-tuples α and β is defined by $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$.

The symmetric group consisting of all permutations of elements of Ω is denoted by $\mathrm{Sym}(\Omega)$, or by $\mathrm{Sym}(n)$ if $\Omega = \{1, \ldots, n\}$. Similar notation $\mathrm{Alt}(\Omega)$ and $\mathrm{Alt}(n)$ are used for the alternating group.

As usual, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ are the ring of integers and the field of rational and complex numbers, respectively. The ring of integers modulo n and the finite field of order n are denoted by $\mathbb{Z}_n$ and $\mathbb{F}_n$, respectively.

We use standard notation $\mathrm{GL}(n, q)$, $\mathrm{AGL}(n, q)$, $\mathrm{PSL}(n, q)$, and $\mathrm{P\Gamma L}(n, q)$ for general linear, affine general linear, projective special linear, and projective semilinear groups of dimension n over field $\mathbb{F}_q$. Cyclic group of order n and dihedral group of order $2n$ are denoted by C_n and D_{2n}, respectively.

For any two objects $\mathcal{X}$ and $\mathcal{Y}$ of the same category (e.g., graphs, abstract or permutation groups, etc.), the set of all isomorphisms from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathrm{Iso}(\mathcal{X}, \mathcal{Y})$.

Logarithms are always of base 2.

1.1 Relations and graphs

Let $s \subseteq \Omega^2$ be a (binary) relation on Ω. It is said to be *reflexive* if $s \supseteq 1_\Omega$, *symmetric* if s equals the *transpose relation*

$$s^* = \{(\beta,\alpha) :\ (\alpha,\beta) \in s\},$$

irreflexive if $s \cap 1_\Omega = \varnothing$, and *antisymmetric* if $s \cap s^* = \varnothing$. The *support* $\Omega(s)$ of s is defined to be the union of the sets

$$\Omega_-(s) = \{\alpha \in \Omega :\ \alpha s \neq \varnothing\} \quad \text{and} \quad \Omega_+(s) = \{\alpha \in \Omega :\ \alpha s^* \neq \varnothing\},$$

where

$$\alpha s = \{\beta \in \Omega :\ (\alpha,\beta) \in s\}$$

is the *neighborhood* of α in s. Any point of αs is called an s-neighbor of α.

For a pair of sets $\Delta, \Gamma \subseteq \Omega$, the *restriction* of s to the product $\Delta \times \Gamma$ is defined by

$$s_{\Delta,\Gamma} = s \cap (\Delta \times \Gamma),$$

and denoted by s_Δ if $\Delta = \Gamma$. The image of s with respect to a bijection f from Ω to another set is denoted by

$$s^f = \{(\alpha^f, \beta^f) :\ (\alpha,\beta) \in s\}.$$

For relations r and s (not necessarily on the same set), we consider two different products, namely the *composition*

$$r \cdot s = \{(\alpha,\beta) \in \Omega_-(r) \times \Omega_+(s) :\ \alpha r \cap \beta s^* \neq \varnothing\},$$

which is a relation on the union of the supports, and the *tensor product*

$$(1.1.1)\quad r \otimes s = \{((\alpha,\alpha'),(\beta,\beta')) \in (\Omega(r) \times \Omega(s))^2 :\ (\alpha,\beta) \in r,\ (\alpha',\beta') \in s\},$$

which is a relation on the Cartesian product of the supports. Both products are associative and distributive with respect to the union.

We extend the above notation for a set S of relations on Ω. Namely, S^* is defined to be the set of all s^*, $s \in S$, the neighborhood αS of $\alpha \in \Omega$ in S is defined to be the union of all αs, $s \in S$. The restriction $S_{\Delta,\Gamma}$ with $\Delta, \Gamma \subseteq \Omega$ consists of all nonempty $s_{\Delta,\Gamma}$, $s \in S$; we also set $S_\Delta := S_{\Delta,\Delta}$. For a bijection f, the set of all s^f, $s \in S$, is denoted by S^f. The composition $S \cdot T$ and product $S \otimes T$, where T is a set of relations on Ω, are defined to be, respectively, the sets of all $s \cdot t$ and all $s \otimes t$, $s \in S$, $t \in T$.

Let s be a relation on Ω. For points α and α', the notation $\alpha \xrightarrow{s} \alpha'$ means that there is an *s-path* connecting α and α', i.e., a sequence $\alpha_0, \alpha_1, \ldots, \alpha_d$ of points ($d \geq 0$) such that $\alpha_0 = \alpha$, $\alpha_d = \alpha'$, and

$$\alpha_{i+1} \in \alpha_i s, \qquad i = 0, 1, \ldots, d-1.$$

The number d is called the *length* of the path. This path is said to be *closed* if $\alpha_0 = \alpha_d$. The relation s is said to be *strongly connected* if $\alpha \xrightarrow{s} \alpha'$ for any $\alpha, \alpha' \in \Omega(s)$.

A relation $e \subseteq \Omega^2$ is said to be *transitive* if the conditions $\beta \in \alpha e$ and $\gamma \in \beta e$ imply $\gamma \in \alpha e$ for all points α, β, and γ. If this relation is also symmetric, we say that e is a *partial equivalence relation* on Ω. In this case, the support $\Omega(e)$ is a disjoint union of subsets called the *classes* of e; the corresponding partition is denoted by Ω/e. Moreover, the union

$$e = \bigcup_{\Delta \in \Omega/e} \Delta \times \Delta$$

is also disjoint. If e is also reflexive, then $\Omega(e) = \Omega$ and hence e is an equivalence relation on Ω. In particular, any partial equivalence relation e is an equivalence relation on $\Omega(e)$.

With each relation $s \subseteq \Omega^2$, we associate two natural partial equivalence relations, the *equivalence closure* and *radical* defined, respectively, as follows:

$$\langle s \rangle = \min\{e : \ s \subseteq e\} \quad \text{and} \quad \mathrm{rad}(s) = \max\{e : \ e \cdot s = s \cdot e = s\}, \tag{1.1.2}$$

where in both cases e runs over the partial equivalence relations on Ω with $\Omega(e) = \Omega(s)$. Equivalent and more constructive definitions are given in Exercises 1.4.1 and 1.4.3. One can see that $\mathrm{rad}(s) \subseteq \langle s \rangle$ with equality if and only if s is an equivalence relation on Ω.

Let e be a partial equivalence relation on Ω. The quotient of $s \subseteq \Omega^2$ modulo e is a relation on Ω/e defined as follows:

$$s_{\Omega/e} = \{(\Delta, \Gamma) \in (\Omega/e)^2 : \ s_{\Delta,\Gamma} \neq \varnothing\}. \tag{1.1.3}$$

Assume that e is an equivalence relation on Ω. Then the classes of e are the sets αe, $\alpha \in \Omega$, and the mapping

$$\pi_e : \Omega \to \Omega/e, \ \alpha \mapsto \alpha e \tag{1.1.4}$$

is a surjection. It induces a surjection $s \mapsto s_{\Omega/e}$ between the relations on Ω and on Ω/e. This induced surjection is also denoted by π_e. Thus, $s_{\Omega/e} = \pi_e(s)$ and

$$\pi_e(1_\Delta) = 1_{\pi_e(\Delta)} \quad \text{and} \quad \pi_e(s^*) = \pi_e(s)^*$$

for all $\Delta \subseteq \Omega$ and all s.

A relation $s \subseteq \Omega^2$ is said to be *thin* or a *matching* if $|\alpha s| \leq 1$ and $|\alpha s^*| \leq 1$ for all $\alpha \in \Omega$. In this case, the mapping

$$f_s : \Omega_-(s) \to \Omega_+(s) \tag{1.1.5}$$

taking a point α to the unique s-neighbor of α, is a bijection; in particular, if, in addition, $\Omega_-(s) = \Omega_+(s) = \Omega$, then f_s is a permutation of Ω, $f_{s^*} = (f_s)^{-1}$, and s is the graph of the permutation $g = f_s$, i.e.,

$$s = s_g = \{(\alpha, \alpha^g) : \ \alpha \in \Omega\}. \tag{1.1.6}$$

Under a (directed) *graph* on Ω, we mean a pair $\mathfrak{X} = (\Omega, D)$ with $D \subseteq \Omega^2$; the elements of

$$\Omega = \Omega(\mathfrak{X}) \quad \text{and} \quad D = D(\mathfrak{X})$$

are called the *vertices* and *arcs* of $\mathfrak{X}$, respectively. The graph $\mathfrak{X}$ is said to be *undirected*, *loopless*, or *tournament* if, respectively, D is symmetric, irreflexive, or antisymmetric and $D \cup D^* = \Omega^2 \setminus 1_\Omega$. The vertices α and β of an undirected graph $\mathfrak{X}$ are called *adjacent* if both (α, β) and (β, α) are arcs of $\mathfrak{X}$. The subgraph of $\mathfrak{X}$ induced by a set $\Delta \subseteq \Omega$ is defined to be the graph $\mathfrak{X}_\Delta = (\Delta, D_\Delta)$.

The graph $\mathfrak{X}$ is said to be *strongly connected* if so is the relation D. A (connected) *component* of $\mathfrak{X}$ is defined to be a class of the partial equivalence relation $\langle D \rangle$. The graph $\mathfrak{X}$ is said to be *connected* if it has exactly one connected component. The *distance* between α and β in $\mathfrak{X}$ is defined to be the smallest number $d = d(\alpha, \beta)$ such that there exists a D-path of length d connecting α and β. The maximal distance between two vertices is called the *diameter* of $\mathfrak{X}$.

The *valency* of a vertex α of $\mathfrak{X}$ is defined to be the cardinality $|\alpha D|$ of the neighborhood of α in D. A graph $\mathfrak{X}$ is said to be *regular* of valency d if the valency of each vertex equals d; when $d = 3$, the graph $\mathfrak{X}$ is said to be *cubic*. A tournament $\mathfrak{X}$ is said to be *doubly regular* if every two of its vertices have exactly $(n-3)/4$ common neighbors (recall that $n = |\Omega|$).

Let $s = s_g$, where $g \in \mathrm{Sym}(\Omega)$ is an n-cycle. A graph $\mathfrak{X}$ is called a *directed cycle* (respectively, *undirected cycle*) if the arc set of $\mathfrak{X}$ coincides with s (respectively, $s \cup s^*$). An undirected graph is said to be *complete* or a *clique* if any two of its vertices are adjacent, and *empty* if the arc set of it is empty.

The automorphism group $\mathrm{Aut}(\mathfrak{X})$ of the graph $\mathfrak{X}$ consists of all permutations $k \in \mathrm{Sym}(\Omega)$ such that $D^k = D$. The graph $\mathfrak{X}$ is said to be *vertex-transitive* (respectively, *arc transitive*) if $\mathrm{Aut}(\mathfrak{X})$ acts transitively on the vertex set (respectively, the arc set) of $\mathfrak{X}$.

Sometimes it will be convenient to consider colored structures. Under a *coloring* of a set Ω, we mean any function $c : \Omega \to \mathbb{N}$. The *color* of $\alpha \in \Omega$ with respect to the coloring c is defined to be $c(\alpha)$. The elements of the same color form a *color class* of c; the number of color classes (or equivalently, the colors of c) is denoted by $|c|$. Thus the color classes of c form a partition $\mathcal{P}_c$ of the set Ω into $|c|$ classes.

A graph $\mathfrak{X}$ together with a coloring $c_\mathfrak{X} : D(\mathfrak{X}) \to \mathbb{N}$ of the arc set is said to be a *colored graph* if for any (α, α), $(\beta, \gamma) \in D(\mathfrak{X})$,

$$\beta \neq \gamma \quad \Rightarrow \quad c_\mathfrak{X}(\alpha) \neq c_\mathfrak{X}(\beta, \gamma),$$

where $c_\mathfrak{X}(\alpha) = c_\mathfrak{X}(\alpha, \alpha)$.

Two colored graphs are said to be *isomorphic* if there exists a bijection of their vertex sets preserving the colors of arcs. Any such bijection is called an *isomorphism* of these graphs. The group of all isomorphisms of $\mathfrak{X}$ to itself is denoted by $\mathrm{Aut}(\mathfrak{X})$ and called the *automorphism group* of $\mathfrak{X}$. A colored graph $\mathfrak{X}'$ is called a *subgraph* of $\mathfrak{X}$ if

$$\Omega(\mathfrak{X}') \subseteq \Omega(\mathfrak{X}), \qquad D(\mathfrak{X}') \subseteq D(\mathfrak{X}), \qquad c_{\mathfrak{X}'} = c_{\mathfrak{X}}|_{D(\mathfrak{X}')},$$

and an *induced subgraph* if $D(\mathfrak{X}')$ equals the restriction of $D(\mathfrak{X})$ to $\Omega(\mathfrak{X}')$.

1.2 Matrices and algebras

The linear space of all complex rectangular matrices with rows and columns indexed by the elements of sets Ω and Ω' is denoted by

$$\mathrm{Mat}_{\Omega,\Omega'} = \mathrm{Mat}_{\Omega,\Omega'}(\mathbb{C}).$$

When $\Omega = \Omega'$, this space becomes the standard algebra $\mathrm{Mat}_\Omega = \mathrm{Mat}_\Omega(\mathbb{C})$ of all $n \times n$ complex matrices. We set I_Ω and J_Ω to be the identity and the all one matrices in Mat_Ω, respectively; the all one matrix in $\mathrm{Mat}_{\Omega,\Omega'}$ is denoted by $J_{\Omega,\Omega'}$. The transposed matrix of A is denoted by A^T. For $\Omega = \{1,\ldots,n\}$, we set $\mathrm{Mat}_n = \mathrm{Mat}_\Omega(\mathbb{C})$, $I_n = I_\Omega$, and $J_n = J_\Omega$.

The *Hadamard product* $A \circ B$ of the matrices $A, B \in \mathrm{Mat}_\Omega$ is defined by the formula

$$(A \circ B)_{\alpha,\beta} = A_{\alpha,\beta} B_{\alpha,\beta}$$

for all $\alpha, \beta \in \Omega$. The *Kronecker product* $A_1 \otimes A_2$ of the matrices $A_1 \in \mathrm{Mat}_{\Omega_1}$ and $A_2 \in \mathrm{Mat}_{\Omega_2}$ is the matrix in $\mathrm{Mat}_{\Omega_1 \times \Omega_2}$ defined by

$$(A_1 \otimes A_2)_{\alpha,\beta} = (A_1)_{\alpha_1,\beta_1} (A_2)_{\alpha_2,\beta_2},$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$.

The *adjacency matrix* of a relation $s \subseteq \Omega^2$ is defined to be a $\{0,1\}$-matrix $A_s \in \mathrm{Mat}_\Omega$ such that

$$(A_s)_{\alpha,\beta} = 1 \quad \Leftrightarrow \quad (\alpha,\beta) \in s.$$

The adjacency matrix of a graph is defined to be the adjacency matrix of its arc set.

The *permutation matrix* P_k of a permutation $k \in \mathrm{Sym}(\Omega)$ is defined to be the adjacency matrix of the graph of k. Thus, P_k is the $\{0,1\}$-matrix in Mat_Ω such that

$$(P_k)_{\alpha,\beta} = \begin{cases} 1, & \text{if } \alpha^k = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

The linear space over the field $\mathbb{C}$ spanned by the elements of Ω is denoted by $\mathcal{L}_\Omega$. Thus,

$$\mathcal{L}_\Omega = \big\{ \sum_{\alpha\in\Omega} a_\alpha \alpha : \ a_\alpha \in \mathbb{C} \ \text{ for all } \ \alpha \big\}.$$

Any of its elements is a formal linear combination of points of Ω with complex coefficients. In particular, for any $\Delta \subseteq \Omega$, the formal sum

$$\underline{\Delta} = \sum_{\alpha\in\Delta} \alpha$$

is considered as a vector of $\mathcal{L}_\Omega$, and the point $\alpha \in \Omega$ is represented by the vector $\underline{\{\alpha\}}$.

It is assumed that the readers are familiar with basics on linear representations of algebras and groups, at least in the amount of the first two chapters of [28]. Very briefly, let $\mathcal{A}$ be a subalgebra of Mat_Ω. Then the linear space $\mathcal{L}_\Omega$ can be treated as an $\mathcal{A}$-module. If $\mathcal{A}$ is semisimple, i.e., has no nontrivial nilpotent two-sided ideal, then $\mathcal{L}_\Omega$ is decomposed into the direct sum of irreducible $\mathcal{A}$-modules. Each irreducible module $\mathcal{L}$ affords an irreducible representation

$$\pi : \mathcal{A} \to \mathrm{End}(\mathcal{L}).$$

The (irreducible) character associated with π is defined to be the complex-valued function taking a matrix $A \in \mathcal{A}$ to its trace

$$\mathrm{tr}(A) = \sum_{\alpha \in \Omega} A_{\alpha,\alpha}.$$

The character does not depend on the choice of a representation in a class of equivalent representations of $\mathcal{A}$. For the irreducible characters, the orthogonality relations hold; in particular, the irreducible characters of non-equivalent representations are orthogonal with respect to the standard scalar product.

1.3 Abstract and permutation groups

It is assumed that the readers are familiar with basics on group theory including such things as the Sylow theorems, composition series, automorphism groups, and basics on permutation groups at least in the amount of the first two chapters of [33].

Let G be a finite group. The standard notation $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ are used for the groups of all and inner automorphisms of G, respectively. Let $g \in G$. For a group $H \leq G$, the subsets gH and Hg are called, respectively, left and right cosets of H in G. We define the permutations

$$g_{left} : x \mapsto g^{-1}x,\ x \in G \quad \text{and} \quad g_{right} : x \mapsto xg,\ x \in G,$$

corresponding to multiplication by g^{-1} from the left and multiplication by g from the right. The groups

$$G_{right} = \{g_{right} :\ g \in G\} \quad \text{and} \quad G_{left} = \{g_{left} :\ g \in G\}$$

form respectively, the right and left regular representations of G in $\mathrm{Sym}(G)$.

The group ring of G is denoted by $\mathbb{C}G$. It is obviously closed under taking componentwise inverse and multiplication defined by

$$\left(\sum a_g g\right)^{-1} = \sum a_g g^{-1} \quad \text{and} \quad \left(\sum a_g g\right) \circ \left(\sum b_g g\right) = \sum a_g b_g g,$$

where the sums are taken over all $g \in G$ and all the a_g and b_g are complex numbers. For any set $X \subseteq G$, the quantity

$$\underline{X} = \sum_{x \in X} x$$

is treated as an element of $\mathbb{C}G$.

As usual, $G \times K$ and $G \rtimes K$ denote the direct product of the groups G and K, and a semidirect product of G by K, respectively. Note that in the latter case, the group $G \rtimes K$ is not defined uniquely and depends on action of K on G. For example, if the action is given by the mapping

$$(g, k) \mapsto g^k, \quad g \in G,\ k \in K,$$

then the multiplication of the pairs (g_1, k_1) and (g_2, k_2) in $G \rtimes K$ is defined by formula

$$(g_1, k_1)(g_2, k_2) = (g_1 g_2^{k_1^{-1}}, k_1 k_2).$$

Assume that $K \leq \mathrm{Sym}(\Omega)$ for a set Ω. To define the *wreath product* $G \wr K$ of G by K, let us consider the pairs (f, k), where $f : \Omega \to G$ is a function and $k \in K$. The natural action of the group K on the set G^Ω consisting of the functions f that is given by

$$f^k(\alpha) := f(\alpha^{k^{-1}}), \quad \alpha \in \Omega,\ k \in K,$$

defines a multiplication on the pairs (f, k), as follows:

$$(f_1, k_1)(f_2, k_2) = (f_1 f_2^{k_1^{-1}}, k_1 k_2), \quad f_1, f_2 \in G^\Omega,\ k_1, k_2 \in K.$$

Now the set $G \wr K$ of all the pairs (f, k) is a group with respect to this operation. Clearly, it is isomorphic to the semidirect product $G^\Omega \rtimes K$ of the base group G^Ω by the group K.

Let $G = C_n$. Then every automorphism of G is defined by raising to a power coprime to n (see Exercise 1.4.17). Furthermore,

$$\mathrm{Aut}(G) = \prod_{p \in \mathcal{P}} \mathrm{Aut}(G_p),$$

where $\mathcal{P}$ is the set of primes dividing n and G_p is the Sylow p-subgroup of G. Let $n = p^k$ for a prime p and $k \geq 1$. If p is odd, then $\mathrm{Aut}(G)$ is isomorphic to $C_{(p-1)p^{k-1}}$. If $p = 2$, then $\mathrm{Aut}(G)$ is trivial for $k = 1$, and isomorphic to $C_2 \times C_{2^{k-2}}$ for $k \geq 2$.

Let K be a permutation group on a set Ω, i.e., $K \leq \mathrm{Sym}(\Omega)$. The number $n = |\Omega|$ is called the degree of K. The set of all orbits of K (the K-orbits for brevity) is denoted by

$$\mathrm{Orb}(K) = \mathrm{Orb}(K, \Omega).$$

A relation s on Ω is said to be K-*invariant* if $s^k = s$ for all $k \in K$. The setwise and pointwise stabilizers of a set Δ in K are defined to be the permutation groups

$$K_{\{\Delta\}} = \{k \in K :\ \Delta^k = \Delta\} \quad \text{and} \quad K_\Delta = \{k \in K :\ \alpha^k = \alpha \text{ for all } \alpha \in \Delta\},$$

respectively. If $\Delta = \{\alpha, \beta, \ldots\}$, then the pointwise stabilizer K_Δ is also written as $K_{\alpha,\beta,\ldots}$. The restriction of K to Δ is defined to be the group

$$K^\Delta = \{k^\Delta :\ k \in K_{\{\Delta\}}\},$$

where $k^\Delta \in \mathrm{Sym}(\Delta)$ is the permutation induced by k.

The group $K \leq \mathrm{Sym}(\Omega)$ is said to be

- *transitive* if $\mathrm{Orb}(K) = \{\Omega\}$;
- *semiregular* if $K_\alpha = 1$ for all $\alpha \in \Omega$;
- *regular* if K is transitive and semiregular;
- *quasiregular* if K^Δ is regular for all $\Delta \in \mathrm{Orb}(K, \Omega)$;
- $1/2$-*transitive* if all the orbits of K are of the same cardinality;
- $3/2$-*transitive* if K is transitive and K_α is $1/2$-transitive on $\Omega \setminus \{\alpha\}$;
- 2-*transitive* if the componentwise action of K on $\Omega^2 \setminus 1_\Omega$ is transitive.

A transitive group K is said to be *primitive* if the only equivalence relations invariant with respect to K are 1_Ω and Ω^2, and *imprimitive* otherwise.

Let $K' \leq \mathrm{Sym}(\Omega')$. A bijection $f : \Omega \to \Omega'$ is called a *permutation group isomorphism* from K onto K' if

$$K' = \{f^{-1}kf : \; k \in K\}.$$

The set of all permutation isomorphisms from the group K onto K' is denoted by $\mathrm{Iso}(K, K')$.

The direct product $K_1 \times K_2$ of two permutation groups $K_1 \leq \mathrm{Sym}(\Omega_1)$ and $K_2 \leq \mathrm{Sym}(\Omega_2)$ has two natural actions: on the disjoint union $\Omega_1 \cup \Omega_2$,

$$\alpha^{(k_1,k_2)} = \begin{cases} \alpha^{k_1}, & \text{if } \alpha \in \Omega_1, \\ \alpha^{k_2}, & \text{if } \alpha \in \Omega_2, \end{cases} \tag{1.3.1}$$

and on the Cartesian product $\Omega_1 \times \Omega_2$,

$$(\alpha_1, \alpha_2)^{(k_1,k_2)} = (\alpha_1^{k_1}, \alpha_2^{k_2}). \tag{1.3.2}$$

The first of these actions is always intransitive, whereas the second one is transitive if and only if each factor is transitive.

A similar situation takes place for the wreath product $G \wr K$ of the groups $G \leq \mathrm{Sym}(\Delta)$ and $K \leq \mathrm{Sym}(\Omega)$. Namely, the *imprimitive action* of $G \wr K$ is defined on the set $\Delta \times \Omega$ so that for any $(f, k) \in G \wr K$,

$$(\delta, \alpha)^{(f,k)} = (\delta^{f(\alpha)}, \alpha^k), \quad \delta \in \Delta, \; \alpha \in \Omega.$$

In this case, the base group G^Ω acts on the disjoint union of $|\Omega|$ copies of Δ.

The *primitive action* or *exponentiation* $G \uparrow K$ is defined on the set Ω^Δ of the functions $\lambda : \Omega \to \Delta$ by

$$\lambda^{(f,k)}(\delta) = \lambda(\delta^{k^{-1}})^{f(\delta^{k^{-1}})}, \qquad \delta \in \Omega. \tag{1.3.3}$$

In this case, the base group G^Ω acts on the Cartesian product of $|\Omega|$ copies of Δ.

1.4 Exercises

1.4.1 For a set S of relations on Ω, denote by S^∞ the union of all finite compositions $r \cdot s \cdots$ with $r, s, \ldots$ belonging to S. Then given $s \subseteq \Omega^2$,

$$\langle s \rangle = \{1_{\Omega(s)}, s, s^*\}^\infty. \tag{1.4.1}$$

1.4.2 Let $s \subseteq \Omega^2$. Then the points α and α' belong to the same class of the partial equivalence $\langle s \rangle$ if and only if $\alpha \stackrel{s\cup s^*}{\rightarrow} \alpha'$.

1.4.3 Let $s \subseteq \Omega^2$. Then $\mathrm{rad}(s)$ is equal to the largest partial equivalence relation e on $\Omega(s)$ for which

$$s = \bigcup_{\substack{\Delta,\Gamma\in\Omega/e:\\ \Delta\times\Gamma\subseteq s}} \Delta \times \Gamma. \tag{1.4.2}$$

1.4.4 Let e be an equivalence relation on Ω. Then the mapping π_e induces a surjection from the set of (partial) equivalence relations on Ω that contain e to the set of (partial) equivalence relations on Ω/e.

1.4.5 Let $e \subseteq \Omega^2$ be an equivalence relation and s a relation on Ω/e. Then

$$e \cdot \pi_e^{-1}(s) \cdot e = \pi_e^{-1}(s).$$

In particular, $e \subseteq \mathrm{rad}(\pi_e^{-1}(s))$.

1.4.6 Let r and s be thin relations on Ω. Then so are the relations s^* and $r \cdot s$. Furthermore, if t is a thin relation on Δ, then $s \otimes t$ is a thin relation on $\Omega \times \Delta$.

1.4.7 The mapping $s \mapsto A_s$ defines a one-to-one correspondence between the relations on Ω and $\{0,1\}$-matrices of Mat_Ω.

1.4.8 Given relations $r, s \subseteq \Omega^2$,

(1) $A_{r^*} = (A_r)^T$;
(2) $A_{r\cap s} = A_r \circ A_s$;
(3) $A_{r\cup s} = A_{r\setminus s} + A_{s\setminus r} + A_{r\cap s}$; in particular, $A_{r\cup s} = A_r + A_s$ if $r \cap s = \varnothing$;
(4) $|\alpha r \cap \beta s^*| = (A_r A_s)_{\alpha,\beta}$ for all $\alpha, \beta \in \Omega$.

1.4.9 For any relations r and s, we have $A_{r\otimes s} = A_r \otimes A_s$.

1.4.10 For any permutations $k, k' \in \mathrm{Sym}(\Omega)$,

$$P_{kk'} = P_k P_{k'}. \tag{1.4.3}$$

In particular, $P_{k^{-1}} = (P_k)^{-1}$, and the mapping $k \mapsto P_k$ is a linear representation of the group $\mathrm{Sym}(\Omega)$.

1.4.11 For a relation $s \subseteq \Omega^2$ and a permutation $k \in \mathrm{Sym}(\Omega)$,

$$A_{s^k} = P_k^{-1} A_s P_k. \tag{1.4.4}$$

1.4.12 For any relation $s \subseteq \Omega^2$,

$$A_s \alpha = \underline{\alpha s^*}, \qquad \alpha \in \Omega. \tag{1.4.5}$$

1.4.13 For any group G,

$$\langle G_{left}, G_{right}\rangle = G_{left}\,\mathrm{Inn}(G) = G_{right}\,\mathrm{Inn}(G).$$

1.4.14 For any group G, the mapping

$$\tau : \mathbb{C}G \to \mathrm{Mat}_G(\mathbb{C}), \qquad g \mapsto P_{g_{left}}, \tag{1.4.6}$$

is an algebra monomorphism. Moreover,

(1) $\tau(1) = I_G$ and $\tau(\underline{G}) = J_G$;
(2) $\tau(\xi^{-1}) = \tau(\xi)^T$ for all $\xi \in \mathbb{C}G$;
(3) $\tau(\xi \circ \eta) = \tau(\xi) \circ \tau(\eta)$ for all $\xi, \eta \in \mathbb{C}G$.

1.4.15 For any group G and any set $X \subseteq G$,

$$\tau(\underline{X}^{-1}) = A_s$$

where

$$s = \{(g, xg) : x \in X, g \in G\}. \tag{1.4.7}$$

This relation is G_{right}-invariant. The mapping

$$\rho : X \mapsto s \tag{1.4.8}$$

is a bijection between the subsets of G and the G_{right}-invariant relations on G. The inverse of ρ is defined by formula

$$\rho^{-1}(s) = \alpha s \tag{1.4.9}$$

where α is the identity of G.

1.4.16 Let G be a group, and let ρ be the mapping from Exercise 1.4.15. Then for any sets $X, Y \subseteq G$,

(1) $\rho(X) = 1_G$ if and only if X consists of the identity of G;
(2) $\rho(X) = G \times G$ if and only if $X = G$;
(3) $\rho(X^{-1}) = \rho(X)^*$;
(4) $\rho(X) \subseteq \rho(Y)$ if and only if $X \subseteq Y$;
(5) $\langle\rho(X)\rangle = \rho(\langle X\rangle)$;
(6) $X \le G$ if and only if $\rho(X)$ is an equivalence relation and the classes of $\rho(X)$ are the cosets of X in G;
(7) $\mathrm{rad}(\rho(X)) = \rho(\mathrm{rad}((X))$ with $\mathrm{rad}(X) = \{g \in G : gX = Xg = X\}$.

1.4.17 For an abelian group G of order n, the center of $\mathrm{Aut}(G)$ consists of all mappings

$$\sigma_m : G \to G,\ \ g \mapsto g^m, \tag{1.4.10}$$

where m is coprime to n.

1.4.18 The identity element of the wreath product $G \wr K$ is the pair $(f_1, 1)$, where the function f_1 takes any element to the identity of G. The element inverse to (f, k) is given by $(f, k)^{-1} = ((f^k)^{-1}, k^{-1})$.

1.4.19 Let e be a partial equivalence relation on Ω. Suppose that e is invariant with respect to a group $K \leq \mathrm{Sym}(\Omega)$. Then the natural action of K on Ω/e induces the homomorphism $k \mapsto k^{\Omega/e}$ from K to $\mathrm{Sym}(\Omega/e)$ with the image and kernel equal to

$$K^{\Omega/e} = \{k^{\Omega/e} : \ k \in K\} \quad \text{and} \quad K_e = \bigcap_{\Delta \in \Omega/e} K_{\{\Delta\}}, \tag{1.4.11}$$

respectively.

1.4.20 Any abelian permutation group is quasiregular, and is regular if and only if it is transitive.

1.4.21 A normal subgroup of a transitive group is 1/2-transitive.

CHAPTER 2

Basics and examples

Most of the concepts studied in the first three sections of the present chapter are well known in the theory of permutation groups. In many cases, it is somehow clear how to translate the relevant parts of the theory into the language of coherent configurations. In the other three sections, we give examples from algebra, geometry, and combinatorics that illustrate the introduced concepts.

2.1 Coherent configurations

In this section, the main definitions and concepts concerning coherent configurations are introduced. Some parts of the presented material can be found in many papers, e.g., [**48, 65, 67**], and monographs, e.g., [**123, 128**]. However, our notation and terminology are different from those used there. The main reason is that in most of these sources, the subject was focused on special cases of coherent configurations, like association schemes, or on the adjacency algebras of them. Nevertheless, we try to maintain continuity, at least at the level of concepts.

2.1.1 Rainbows

Let S be a partition of the Cartesian square $\Omega^2 = \Omega \times \Omega$. Each element of S is a binary relation, and the set of all unions of these relations is denoted by $S^{\cup}$.

Definition 2.1.1. *A pair $\mathcal{X} = (\Omega, S)$ is called a rainbow on Ω if*
(CC1) $1_\Omega \in S^{\cup}$*;*
(CC2) $S^* = S$*.*

The numbers $|\Omega|$ and $\mathrm{rk}(\mathcal{X}) := |S|$ are called the *degree* and *rank* of $\mathcal{X}$, respectively. The rainbow $\mathcal{X}$ is associated with a complete colored graph $\mathfrak{X} = \mathfrak{X}(\mathcal{X})$ defined by

$$\Omega(\mathfrak{X}) = \Omega \quad \text{and} \quad D(\mathfrak{X}) = \Omega^2 \quad \text{and} \quad \mathcal{P}_{c_\mathfrak{X}} = S,$$

where $c_\mathfrak{X}$ is a coloring of Ω^2; any such coloring is said to be *standard* and is denoted also by $c_\mathcal{X}$. In particular, the colors of loops are different from the colors of the other arcs (the condition (CC1)) and the permutation $(\alpha, \beta) \mapsto (\beta, \alpha)$ of Ω^2 preserves the set of color classes (the condition (CC2)).

Two obvious examples are the *trivial* and *discrete* rainbows $\mathcal{T}_\Omega$ and $\mathcal{D}_\Omega$, respectively; they are also denoted by $\mathcal{T}_n$ and $\mathcal{D}_n$ if the set Ω is not essential. The trivial rainbow is of rank at most 2; in this case, S consists of 1_Ω and (if $n \geq 2$) its complement in Ω^2. The discrete rainbow is of rank n^2: in this case, S consists of all n^2 singleton relations $\{(\alpha, \beta)\}$ with $\alpha, \beta \in \Omega$.

The elements of the sets $S = S(\mathcal{X})$ and $S^{\cup} = S(\mathcal{X})^{\cup}$ are called the *basis relations* and *relations* of the rainbow $\mathcal{X}$, respectively. The unique basis relation containing a pair (α, β) is denoted by $r(\alpha, \beta) = r_{\mathcal{X}}(\alpha, \beta)$. In particular,

$$r(\beta, \alpha)^* = r(\alpha, \beta).$$

From the conditions (CC1) and (CC2), it follows that every basis relation is either reflexive[1] (and then is contained in 1_Ω) or irreflexive, and either symmetric or antisymmetric. A rainbow is called *symmetric* (respectively, *antisymmetric*) if all (respectively, all irreflexive) basis relations are symmetric (respectively, antisymmetric). Sometimes, a relation $s \in S^{\cup}$ is considered as a (*basis*) *graph of* s, i.e., as a graph with vertex set $\Omega(s)$ and arc set s. In this sense, the relation s is said to be connected, regular, etc.

Example 2.1.2. *The trivial and discrete rainbows are symmetric and antisymmetric, respectively. They are equal if and only if $n = 1$. The irreflexive basis graph of a trivial rainbow is complete.*

There is a natural partial order $\leq$ on the set of all rainbows on Ω with the smallest and greatest elements equal to $\mathcal{T}_\Omega$ and $\mathcal{D}_\Omega$, respectively. Namely, for rainbows $\mathcal{X}$ and $\mathcal{X}'$ on Ω, set

$$\mathcal{X} \leq \mathcal{X}' \iff S^{\cup} \subseteq (S')^{\cup},$$

where $S = S(\mathcal{X})$ and $S' = S(\mathcal{X}')$. In this case, $\mathcal{X}'$ is called an *extension* or *fission* of $\mathcal{X}$ and $\mathcal{X}$ is called a *fusion* of $\mathcal{X}'$. It is easily seen that $\mathcal{X} \leq \mathcal{X}'$ only if $|S| \leq |S'|$, and if and only if every basis relation of $\mathcal{X}$ is a union of basis relations of $\mathcal{X}'$.

For any $\Delta \subseteq \Omega$, the set S_Δ forms a partition of Δ^2. This partition obviously satisfies the conditions (CC1) and (CC2). Therefore, for any rainbow $\mathcal{X}$ on Ω, the pair

$$\mathcal{X}_\Delta = (\Delta, S_\Delta)$$

is also a rainbow; it is called the *restriction* of $\mathcal{X}$ to Δ.

The concept of a rainbow is probably too general to hope that the theory could be rich in content. However, it becomes such if we add a coherence condition.

Definition 2.1.3. *A rainbow $\mathcal{X}$ is called a coherent configuration if*

(CC3) *for any $r, s, t \in S$, the number $c_{rs}^t = |\alpha r \cap \beta s^*|$ does not depend on the choice of $(\alpha, \beta) \in t$.*

In other words, c_{rs}^t is equal to the number of the r-neighbors of α that are s^*-neighbors of β, i.e., the number of triangles of the form depicted in Fig. 2.1 below with fixed points α and β.

[1]More exactly, reflexive on its support.

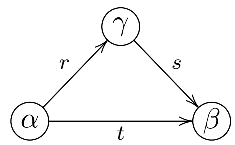


FIGURE 2.1 A triangle corresponding to c_{rs}^t.

Two obvious examples of coherent configurations are the trivial and discrete rainbows. In the former case, the only nontrivial number $c_{rs}^t = n-2$ is obtained for $n \geq 2$ and the irreflexive relation r equal to both s and t, whereas in the latter case the number c_{rs}^t equals 0 unless

$$r = \{(\alpha, \gamma)\}, \quad s = \{(\gamma, \beta)\}, \quad t = \{(\alpha, \beta)\},$$

for some points α, β, and γ. The coherent configurations of degree at most 7 were classified in [**88**] without computer; for computer classification of small coherent configurations, see [**56, 85**].

Proposition 2.1.4. *Let $\mathcal{X}$ be a rainbow and $S = S(\mathcal{X})$. Then the set $S^{\cup}$ is closed under taking complement, union, and intersection. If $\mathcal{X}$ is a coherent configuration, then $S^{\cup}$ is also closed under the composition.*

Proof. The statements on the complement, union, and intersection follow from the definition of $S^{\cup}$. Now let $\mathcal{X}$ be a coherent configuration and $r, s \in S^{\cup}$. Without loss of generality, we may assume that $r, s \in S$. If a relation $t \in S$ intersects the composition $r \cdot s$, then by the condition (CC3) we have $c_{rs}^t \neq 0$. Consequently, all the pairs of t lie in this composition. Therefore, $t \subseteq r \cdot s$. Thus,

$$r \cdot s = \bigcup_{t:\ c_{rs}^t \neq 0} t$$

and hence $r \cdot s \in S^{\cup}$, as required. □

By Proposition 2.1.4, the composition $r \cdot s$ of the relations r and s of the coherent configuration $\mathcal{X}$ is a union of basis relations. The set of these relations is denoted by rs and is called the *complex product* or, briefly, *product* of r and s. In particular, if $r, s \in S$, then

$$rs = \{t \in S :\ c_{rs}^t \neq 0\}.$$

This product is associative (statement (4) of Exercise 2.7.6) but can be equal to the empty set. One can easily verify that

$$rs = r's' \quad \Leftrightarrow \quad r \cdot s = r' \cdot s'$$

for all relations r' and s' of $\mathcal{X}$.

Definition 2.1.5. *A set $\Delta \subseteq \Omega$ is called a fiber of the rainbow $\mathcal{X}$ if $1_{\Delta} \in S$. The set of all fibers of $\mathcal{X}$ is denoted by $F(\mathcal{X})$.*

Obviously,

$$1 \leq |F(\mathcal{X})| \leq |\Omega|$$

with equalities for the trivial and discrete coherent configurations, respectively. Since S is a partition of Ω^2, the condition (CC1) implies that the union

$$\Omega = \bigcup_{\Delta \in F(\mathcal{X})} \Delta$$

is disjoint. If $\mathcal{X}$ is a coherent configuration, then this partition induces (see Proposition 2.1.6 below) a partition of S into the sets

$$S_{\Delta,\Gamma} = \{s \in S : \ s \subseteq \Delta \times \Gamma\}, \tag{2.1.1}$$

where Δ and Γ are fibers of $\mathcal{X}$ (this notation agrees with the notation from Section 1.1). However, these sets do not necessarily form a partition of S for a general rainbow.

Proposition 2.1.6. *Let $\mathcal{X}$ be a coherent configuration. Then*

$$\Delta \times \Gamma \in S^{\cup} \quad \textit{for all } \Delta, \Gamma \in F,$$

where $S = S(\mathcal{X})$ and $F = F(\mathcal{X})$. In particular, the union

$$S = \bigcup_{\Delta,\Gamma \in F} S_{\Delta,\Gamma} \tag{2.1.2}$$

is disjoint.

Proof. Let $\Delta, \Gamma \in F$. Clearly, $\Omega^2 \in S^{\cup}$ and hence

$$1_\Delta \cdot \Omega^2 = \Delta \times \Omega \quad \text{and} \quad \Omega^2 \cdot 1_\Gamma = \Omega \times \Gamma$$

are relations of the coherent configuration $\mathcal{X}$, see Proposition 2.1.4. By the same proposition, this implies that

$$\Delta \times \Gamma = (\Delta \times \Omega) \cap (\Omega \times \Gamma)$$

is a relation of $\mathcal{X}$. This proves the first statement and shows that $\Delta \times \Gamma$ contains each basis relation intersecting it. Thus equality (2.1.2) holds. □

Corollary 2.1.7. *For each basis relation s of a coherent configuration, there exist uniquely determined fibers Δ and Γ such that $s \subseteq \Delta \times \Gamma$.*

The set of all unions of fibers of the rainbow $\mathcal{X}$ is denoted by $F(\mathcal{X})^{\cup}$; any element of this set is called a *homogeneity set* of $\mathcal{X}$. From formula (2.1.2), it follows that if $\mathcal{X}$ is a coherent configuration, then for any two homogeneity sets Δ and Γ the union

$$S_{\Delta,\Gamma} := \bigcup_{\substack{\Delta' \in F,\\ \Delta' \subseteq \Delta}} \bigcup_{\substack{\Gamma' \in F,\\ \Gamma' \subseteq \Gamma}} S_{\Delta',\Gamma'}$$

is disjoint.

Furthermore, any relation $s \in S_\Delta$ belongs to $S_{\Delta',\Gamma'}$ for some fibers Δ' and Γ' contained in Δ. Therefore for all $\alpha \in \Delta$, the set αs is either empty or contained in Δ. Thus,

$$|\alpha r \cap \beta s^* \cap \Delta| = |\alpha r \cap \beta s^*| = c_{rs}^t$$

for all $r, s, t \in S_\Delta$ and all $\alpha, \beta \in \Delta$ such that $r(\alpha, \beta) = t$. This implies that the condition (CC3) is satisfied for the rainbow $\mathcal{X}_\Delta$ and hence it is a coherent configuration on Δ. If the latter set is a fiber, then $\mathcal{X}_\Delta$ is called the *homogeneous component* of $\mathcal{X}$ (associated with Δ).

Example 2.1.8. *Let $\mathcal{X}$ be a rainbow on the set $\Omega = \Delta \cup \Gamma$, where $\Delta = \{1, 2\}$ and $\Gamma = \{3, 4, 5\}$. Set*

$$S_{\Delta,\Delta} = S(\mathcal{T}_\Delta), \quad S_{\Gamma,\Gamma} = S(\mathcal{T}_\Gamma), \quad S_{\Delta,\Gamma} = \{\Delta \times \Gamma\}, \quad S_{\Gamma,\Delta} = \{\Gamma \times \Delta\}.$$

Then $\mathcal{X}$ is a coherent configuration of degree 5 and rank 6. It has two fibers Δ and Γ, and the homogeneous components associated with them are $\mathcal{T}_\Delta$ and $\mathcal{T}_\Gamma$, respectively.

Definition 2.1.9. *A coherent configuration $\mathcal{X}$ on Ω is said to be homogeneous, or an association scheme, or, briefly, a scheme if the set $F(\mathcal{X})$ is a singleton, or equivalently, if $1_\Omega \in S(\mathcal{X})$.*

In the homogeneous case, the only reflexive basis relation is 1_Ω, and the set of all other basis relations is denoted by $S(\mathcal{X})^\#$. In particular, the trivial coherent configuration is always homogeneous, whereas the discrete one is homogeneous if and only if $n = 1$.

Let Δ and Γ be two distinct fibers of a coherent configuration $\mathcal{X}$. Then the set S contains at least four elements: 1_Δ, 1_Γ, and two distinct antisymmetric relations, one from $S_{\Delta,\Gamma}$ and another from $S_{\Gamma,\Delta}$. This shows that any symmetric coherent configuration as well each non-discrete coherent configuration of rank at most 4 is homogeneous.

2.1.2 Intersection numbers

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. The nonnegative integers in the condition (CC3) are called the *intersection numbers* of $\mathcal{X}$. We may interpret the intersection number c_{rs}^t as the number of all triangles in Fig. 2.1 with vertices $\alpha, \beta, \gamma \in \Omega$ that have a fixed base $(\alpha, \beta) \in t$, and the other two sides (α, γ) and (γ, β) belong to r and s, respectively. This interpretation immediately implies that

$$c_{rs}^t = c_{s^*r^*}^{t^*}, \qquad r, s, t \in S. \tag{2.1.3}$$

To find more relations connecting the intersection numbers, let $s \in S$. By Corollary 2.1.7, there exists a fiber Δ such that $s \in S_{\Delta,\Omega}$.

Definition 2.1.10. *The number $n_s = c_{ss^*}^{1_\Delta}$ is called the valency of s.*

Clearly, the valency of any reflexive basis relation equals 1. Interpreting this intersection number as the number of triangles in Fig. 2.1 with $\alpha = \beta$, we immediately see that the valency equals the number of s-neighbors of the point $\alpha \in \Delta$ in the relation s, i.e.,

$$n_s = |\alpha s|, \qquad \alpha \in \Delta,\ s \in S_{\Delta,\Omega}. \tag{2.1.4}$$

This shows that the valency of a vertex in the basis graph of s equals n_s. It follows that the number $|s|$ of arcs of this graph is equal to the sum of the numbers $|\alpha s|$, where α runs over Δ. Consequently,

$$|s| = n_s|\Delta|, \qquad s \in S_{\Delta,\Omega}. \tag{2.1.5}$$

In general, one cannot assign the valency to a nonbasis relation s of a coherent configuration, because the number $|\alpha s|$ depends on the point $\alpha \in \Omega_-(s)$. However, if the coherent configuration is homogeneous, then $\Omega_-(s) = \Omega$ and the valency of s can be defined as the sum of the valencies of the basis relations contained in s.

Example 2.1.11. *Let s be a basis relation of the coherent configuration in Example 2.1.8. Then*

$$n_s = \begin{cases} 1, & \text{if } s \in S_{\Delta,\Delta}, \\ 1 \text{ or } 2, & \text{if } s \in S_{\Gamma,\Gamma}, \\ 3, & \text{if } s \in S_{\Delta,\Gamma}, \\ 2, & \text{if } s \in S_{\Gamma,\Delta}. \end{cases}$$

Of course, the valencies of the trivial scheme of degree n are 1 and $n-1$, whereas all the valencies of the discrete coherent configuration are equal to 1.

Proposition 2.1.12. *Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. Then*

$$\sum_{\substack{s\in S:\\ \alpha s\neq\varnothing}} n_s = n, \qquad \alpha\in\Omega, \tag{2.1.6}$$

$$\sum_{t\in S} c_{st}^r = n_s, \qquad r,s\in S, \tag{2.1.7}$$

$$\sum_{t\in S} n_t c_{rs}^t = n_r n_s, \qquad r,s\in S,\ rs\neq\varnothing, \tag{2.1.8}$$

$$|t|\,c_{rs}^{t^*} = |r|\,c_{st}^{r^*} = |s|\,c_{tr}^{s^*}, \qquad r,s,t\in S. \tag{2.1.9}$$

Proof. Since S is a partition of Ω^2, the nonempty sets αs, $s\in S$, form a partition of Ω. Note that $\alpha s\neq\varnothing$ if and only if $\alpha\in\Omega_-(s)$. Thus equality (2.1.6) follows from (2.1.4).

To prove equality (2.1.7), let $r,s\in S$, and let $\alpha,\beta\in\Omega$ be such that $r=r(\alpha,\beta)$. Since the nonempty sets βt^*, $t\in S$, form a partition of Ω, we obtain

$$\sum_{t\in S} c_{st}^r = \sum_{t\in S} |\alpha s\cap\beta t^*| = \Big|\bigcup_{t\in S}(\alpha s\cap\beta t^*)\Big| = |\alpha s| = n_s$$

provided that $\alpha\in\Omega_-(a)$.

To prove equalities (2.1.8) and (2.1.9), we consider the triangles (α,β,γ) satisfying some of the following conditions:

$$(\alpha,\beta)\in r, \quad (\beta,\gamma)\in s, \quad (\gamma,\alpha)\in t. \tag{2.1.10}$$

In the former case, we count in two ways the amount of the triangles with fixed point $\alpha\in\Omega_-(r)$ and arbitrary β, γ satisfying the first two conditions in (2.1.10); in the latter case, we count the triangles satisfying all the conditions in (2.1.10), but this time in three different ways. □

The identities for the intersection numbers can be simplified a little if the coherent configuration $\mathcal{X}$ is homogeneous. In this case, in formula (2.1.5), we have $\Delta=\Omega$, which immediately implies the first two statements of the corollary below. The other two statements follow from identities (2.1.6) and (2.1.9), respectively.

Corollary 2.1.13. *If the coherent configuration $\mathcal{X}$ is homogeneous, then*

$$|s| = n_s n = n_{s^*} n, \qquad s\in S, \tag{2.1.11}$$

$$n_s = n_{s^*}, \qquad s\in S, \tag{2.1.12}$$

$$\sum_{s\in S} n_s = n, \tag{2.1.13}$$

$$n_t c_{rs}^{t^*} = n_r c_{st}^{r^*} = n_s c_{tr}^{s^*}, \qquad r,s,t\in S. \tag{2.1.14}$$

More relations between the intersection numbers of a coherent configuration can be found in Exercise 2.7.6.

An important invariant expressed via the intersection numbers is the *indistinguishing number* of a relation $s \in S$, which is defined as follows:

$$c(s) = \sum_{t \in S} c_{tt^*}^s. \tag{2.1.15}$$

It counts how many isosceles triangles have a fixed element of s as the base (here the term "isosceles" means that two sides of the triangle belong to the same basis relation). One can see that $c(s) = n$ if and only if the basis relation s is reflexive.

Under an *equivalenced* or $3/2$-*homogeneous* scheme of valency k we mean a one for which the valency of any irreflexive basis relation is equal to k. In the following statement, we find the arithmetical mean of the numbers $c(s)$, $s \in S^\#$, for an equivalenced scheme.

Lemma 2.1.14. *Let (Ω, S) be an equivalenced scheme of valency $k \geq 1$. Then*

$$\sum_{s \in S^\#} c(s) = (k-1)\,|S^\#|.$$

Proof. Set

$$T = \{(\alpha, \beta, \gamma) \in \Omega^3 : \ \beta \neq \gamma, \ r(\alpha, \beta) = r(\alpha, \gamma)\}.$$

Then for any β and γ, the number of all α for which $(\alpha, \beta, \gamma) \in T$ is equal to $c(s)$, where $s = r(\beta, \gamma)$. For a fixed s, this gives $c(s)\,|s|$ triples belonging to T. However, $|s| = nk$ by formula (2.1.11) and the assumption. Thus,

$$|T| = \sum_{s \in S^\#} c(s)|s| = nk \sum_{s \in S^\#} c(s).$$

On the other hand, given $\alpha \in \Omega$ and $s \in S^\#$, there are exactly $n_s(n_s - 1)$ triples $(\alpha, \beta, \gamma) \in T$ such that $r(\alpha, \beta) = s$. Therefore,

$$|T| = n \sum_{s \in S^\#} n_s(n_s - 1) = n\,(k-1)k\,|S^\#|.$$

Thus the required statement follows from the above two formulas. □

2.1.3 Parabolics

Among the relations of a coherent configuration $\mathcal{X} = (\Omega, S)$, the partial equivalence relations are of special interest. The following definition goes back to D. Higman [68].

Definition 2.1.15. *An equivalence (respectively, a partial equivalence) relation of a coherent configuration $\mathcal{X}$ is called a parabolic (respectively, partial parabolic) of $\mathcal{X}$. The set of partial parabolics of $\mathcal{X}$ is denoted by $E = E(\mathcal{X})$.*

The set E always contains *trivial parabolics* 1_Ω and Ω^2. They are the only partial parabolics of the trivial coherent configuration $\mathcal{T}_\Omega$. On the other hand, the set $E(\mathcal{D}_\Omega)$ consists of all partial equivalence relations on Ω.

Example 2.1.16. *Let e be a partial equivalence relation on Ω, each class of which is a homogeneity set of the coherent configuration $\mathcal{X}$; for example, the classes of e could be the fibers of $\mathcal{X}$. Then $e \in E$ by Proposition 2.1.6.*

Let $e \in E$ and $\Delta \in \Omega/e$. Recall that the set S_Δ consists of nonempty relations s_Δ with $s \in S$; note that each such s is contained in e, because e is a union of basis relations of the coherent configuration $\mathcal{X}$. Furthermore, if $\alpha, \beta \in \Delta$ and $r, s \in S$ are such that r_Δ and s_Δ are not empty, then

$$\alpha r_\Delta \cap \beta (s_\Delta)^* = \alpha r \cap \beta s^*. \tag{2.1.16}$$

It follows that the cardinality of the set on the left-hand side equals c_{rs}^t, where $t = r(\alpha, \beta)$. Note that the relation t_Δ is nonempty. Therefore, the rainbow $\mathcal{X}_\Delta$ satisfies the condition (CC3) and is a coherent configuration. Certainly, the restriction of $\mathcal{X}$ to a homogeneity set Δ is obtained as a special case with $e = \Delta \times \Delta$.

In the following statement, we introduce an important numeric invariant of a partial parabolic that will be used in Subsection 3.1.2 to define a quotient coherent configuration modulo parabolic.

Proposition 2.1.17. *For any $e \in E$, $s \in S$, and $\Delta, \Gamma \in \Omega/e$, we have*

$$s_{\Delta,\Gamma} = \varnothing \quad \text{or} \quad |s_{\Delta,\Gamma}| = \sum_{u,v \subseteq e} \sum_{w \in S} c_{us}^w c_{wv}^s,$$

where the external sum is taken over $u, v \in S$. In particular, the number

$$n_e(s) = |s_{\Delta,\Gamma}|$$

does not depend on the classes Δ and Γ for which the relation $s_{\Delta,\Gamma}$ is not empty.

Proof. Let $(\alpha, \beta) \in s_{\Delta,\Gamma}$. Then the number $|s_{\Delta,\Gamma}|$ is equal to the number of distinct pairs $(\gamma_1, \gamma_2) \in \Omega^2$ such that

$$(\alpha, \gamma_1) \in e, \quad (\gamma_1, \gamma_2) \in s, \quad (\gamma_2, \beta) \in e.$$

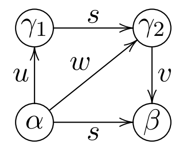


FIGURE 2.2 The configuration in Proposition 2.1.17.

For a fixed $\gamma_2 \in \Gamma$, the number of ways to choose γ_1 equals the sum of c_{us}^w with $w = r(\alpha, \gamma_2)$, where $u = r(\alpha, \gamma_1)$ runs over the basis relations contained in e.

On the other hand, the number of ways to choose γ_2 equals the sum of c_{wv}^s, where $v = r(\gamma_2, \beta)$ runs over the basis relations contained in e (see Fig. 2.2). Thus the total number of the pairs (γ_1, γ_2) is equal to

$$|s_{\Delta,\Gamma}| = \sum_{v \subseteq e} \sum_{w \in S} c_{wv}^s \Big(\sum_{u \subseteq e} c_{us}^w \Big) = \sum_{u,v \subseteq e} \sum_{w \in S} c_{us}^w c_{wv}^s,$$

and we are done. □

The join $e = e_1 \vee e_2$ of the partial equivalence relations e_1 and e_2 on Ω is defined to be the equivalence closure $\langle e_1 \cup e_2 \rangle$. Clearly,

$$\Omega(e) = \Omega(e_1) \cup \Omega(e_2)$$

and each class of e_i is contained in some class of e $(i = 1, 2)$. The following statement shows that if the coherent configuration $\mathcal{X}$ is homogeneous, then E is a lattice with respect to intersection and join; the minimal and maximal elements of it are 1_Ω and Ω^2.

Proposition 2.1.18. *The set E is closed under intersections and joins, and contains the partial equivalence relations $\langle s \rangle$ and* $\mathrm{rad}(s)$ *for each $s \in S^\cup$.*

Proof. Since the intersection and join of two partial equivalence relations are also partial equivalence relations, the part concerning intersections, joins, and $\langle s \rangle$ is an easy consequence of Proposition 2.1.4 and formula (1.4.1). Next, in accordance with Exercise 1.4.3,

$$\mathrm{rad}(s) = e(s) \cap e(s^*)$$

for all $s \subseteq \Omega^2$, where $e(s)$ and $e(s^*)$ are the partial equivalence relations defined in statement (1) of Exercise 2.7.8. Now if s is a relation of $\mathcal{X}$, then the same statement implies that $e(s), e(s^*) \in E$. Hence, $\mathrm{rad}(s) \in E$. □

Corollary 2.1.19. $E = \{\langle s \rangle : \ s \in S^\cup\}$.

A set $T \subseteq S$ is said to be *closed* if the union of all relations lying in T is a partial parabolic of $\mathcal{X}$. In particular, if $e \in E$ then the set $\{s \in S : \ s \subseteq e\}$ is closed. It can easily be verified that the set T is closed if and only if the set of basis relations in $(TT^*)^\natural$ coincides with T.

The concept of a closed set plays a key role in the theory of homogeneous coherent configurations developed in [**128**]. In our book, we prefer to deal with partial parabolics which up to the language give the same concept.

Definition 2.1.20. *A nonempty partial parabolic e of $\mathcal{X}$ is said to be indecomposable if e is not a disjoint union of two nonempty partial parabolics of $\mathcal{X}$; otherwise e is said to be decomposable.*

The parabolic Ω^2 is always indecomposable, and 1_Ω is indecomposable if and only if $\mathcal{X}$ is homogeneous.

Every partial parabolic $e \in E$ is uniquely represented as a disjoint union of indecomposable partial parabolics; each of them is called an *indecomposable component* of e. Indeed, if e' is an indecomposable component of e and

$$e = e_1 \cup \cdots \cup e_k$$

is a decomposition of e into indecomposable partial parabolics e_i, then e' equals the union of the partial parabolics $e' \cap e_i \in E$ (Proposition 2.1.4). But this is possible only if e' coincides with a certain e_i, $1 \leq i \leq k$.

Lemma 2.1.21. *The supports of distinct indecomposable components of a partial parabolic are disjoint.*

Proof. Let e_1 and e_2 be indecomposable components of a partial parabolic of a coherent configuration. Assume that there exists a point

$$\alpha \in \Omega(e_1) \cap \Omega(e_2).$$

Then obviously $1_\alpha \subseteq e_1 \cap e_2$. By the uniqueness of the indecomposable components, this implies that $e_1 = e_2$. □

Theorem 2.1.22. *Let $e \in E$ be an indecomposable partial parabolic. Then*

(1) *if $s \in S$, then $s \subseteq e$ if and only if $s_\Delta \neq \varnothing$ for each $\Delta \in \Omega/e$;*
(2) *the classes of e have the same cardinality;*
(3) *$|\Delta \cap \Lambda| = |\Gamma \cap \Lambda|$ for all $\Delta, \Gamma \in \Omega/e$ and any $\Lambda \in F$ intersecting both Δ and Γ.*

Proof. The sufficiency in statement (1) is obvious. To prove the necessity, we assume on the contrary that there exists $s \subseteq e$ such that $s_\Delta = \varnothing$ for some $\Delta \in \Omega/e$.

Denote by Ω' the union of all $\Gamma \in \Omega/e$ such that $s_\Gamma = \varnothing$. Then $\Delta \subseteq \Omega'$ and e is the disjoint union of two nonempty partial equivalence relations $e_1 = e_{\Omega'}$ and $e_2 = e_{\Omega(e)\setminus\Omega'}$. However,

$$\Omega(e) \setminus \Omega' = \Omega(e \cdot s \cdot e)$$

is a homogeneity set (Exercise 2.7.4). Therefore so is Ω'. Consequently, both e_1 and e_2 belong to E. Thus the partial parabolic e is decomposable, a contradiction.

To prove statement (2), let Δ and Γ be classes of e. Take any fiber $\Lambda \in F$ intersecting Δ, say in a point α. Then $s = 1_\Lambda$ being a basis relation, is contained in e. By statement (1), this implies that $s_\Gamma \neq \varnothing$, i.e., Γ intersects Λ, say in a point β. Thus,

$$\Delta = \alpha T \quad \text{and} \quad \Gamma = \beta T,$$

where T is the set of all relations belonging to $S_{\Lambda,\Omega}$ and contained in e. Since $|\alpha t| = n_t = |\beta t|$ for all $t \in T$, we conclude that

$$|\Delta| = |\alpha T| = \sum_{t \in T} |\alpha t| = \sum_{t \in T} n_t = \sum_{t \in T} |\beta t| = |\beta T| = |\Gamma|,$$

as required.

To prove statement (3), we note that the coherent configuration $\mathcal{X}_\Lambda$ is homogeneous. By Lemma 2.1.21 this implies that any parabolic of $\mathcal{X}_\Lambda$ is indecomposable. In particular, so is e_Λ. Since $\Delta \cap \Lambda$ and $\Gamma \cap \Lambda$ are classes of e_Λ, we are done by statement (2). □

It should be noted that statement (3) of Theorem 2.1.22 remains true if the partial parabolic e is decomposable.

The statement below is formally a consequence of Lemma 2.1.21 and statement (2) of Theorem 2.1.22. In fact, it has been verified in the last part of the proof of that theorem.

Corollary 2.1.23. *Every partial parabolic e of a homogeneous coherent configuration is an indecomposable parabolic. Moreover, each class of e is of cardinality n_e; in particular, $|\Omega/e|$ divides $|\Omega|$.*

2.1.4 Thin relations

Among the basis relations of a coherent configuration, one always find thin relations (see p. 3), for example, all reflexive relations are thin. In general, the more thin basis relations we find, the more the coherent configuration looks like a group. In the limit case, when all the basis relations are thin, the theory of coherent configurations reduces to group theory.

Let r and s be basis relations of a coherent configuration $\mathcal{X} = (\Omega, S)$. From Proposition 2.1.4, we know that the composition $r \cdot s$ is a relation of $\mathcal{X}$. Assume that s is thin. Then the relation $s \cdot s^*$ is reflexive. If the composition contains a relation $t \in S$, then this implies that

$$t \cdot s^* \subset (r \cdot s) \cdot s^* = r \cdot (s \cdot s^*) = r.$$

Since $r \in S$, this implies that $t \cdot s^* = r$ and hence $r \cdot s = t$ is also a basis relation. The same argument works if r is thin and $s \in S$. Thus we arrive at the following useful statement.

Lemma 2.1.24. *The composition of two basis relations one of which is thin is either basis or empty relation.*

Denote by $S_1 = S_1(\mathcal{X})$ the set of all thin relations $s \in S^{\cup}$ with full support, which means that $\Omega_-(s) = \Omega_+(s) = \Omega$; in particular, the bijection f_s defined by formula (1.1.5) is a permutation of Ω. Note that in view of the condition (CC1), the set S_1 always contains 1_Ω and hence is nonempty.

Theorem 2.1.25. *Let $\mathcal{X}$ be a coherent configuration. Then*

1. *for each thin $r \in S$, there exists $s \in S_1$ such that $r \subseteq s$;*
2. *S_1 is a group with respect to the composition;*
3. *the mapping $S_1 \to \mathrm{Sym}(\Omega)$, $s \mapsto f_s$ is a group monomorphism;*
4. *if $T \leq S_1$, then the union of all relations in T belongs to E.*

Proof. Let $r \in S_{\Delta,\Gamma}$, where Δ and Γ are fibers of $\mathcal{X}$, and let Λ be the complement of $\Delta \cup \Gamma$ in Ω. Set s to be $r \cup 1_\Lambda$ or $r \cup r^* \cup 1_\Lambda$ depending on whether $\Delta = \Gamma$ or not. Then $s \in S_1$ and $r \subseteq s$. This proves statement (1).

Next for each $s \in S_1$ the mapping f_s is a permutation of Ω. Furthermore, $1_\Omega \in S_1$ and

$$f_{r^*} = f_r^{-1}, \quad f_{r \cdot s} = f_r f_s \qquad \text{for all } r, s \in S_1.$$

This proves statement (2) and shows that the mapping in statement (3) is a homomorphism. The injectivity of it follows from the fact that s is the graph of f_s.

Finally, denote by t the union of all relations in a set $T \leq S_1$. Then obviously $T \cdot T^* = T$ and hence $t = \langle t \rangle$. Thus statement (4) follows from Proposition 2.1.18. □

A parabolic of $\mathcal{X}$ is said to be *thin* if it is the union of some relations belonging to S_1. From statement (2) of Theorem 2.1.25, we obtain the following statement.

Lemma 2.1.26. *The join of thin parabolics of a coherent configuration is also thin.*

From Lemma 2.1.26, it follows that there exists the largest thin parabolic of a coherent configuration $\mathcal{X}$ that is the join of all thin parabolics. It is called the *thin radical parabolic* of $\mathcal{X}$. Clearly, the thin radical parabolic is equal to the union of all relations of S_1.

Definition 2.1.27. *The set of all basis relations contained in the thin radical parabolic of $\mathcal{X}$ is called the thin radical of $\mathcal{X}$.*

In particular, the thin radical of a scheme $\mathcal{X}$ coincides with $S_1(\mathcal{X})$. In general case this is not true, because the support of some thin basis relations can be different from Ω.

Statement (3) of Theorem 2.1.25 defines a mapping from the coherent configurations to permutation groups. In general, it is neither injective nor surjective. On the other hand, as we will see below, it induces a functor from the category of finite groups to the category of coherent configurations. The image of this functor consists of regular schemes defined as follows.

Definition 2.1.28. *A scheme $\mathcal{X}$ is called regular if $S(\mathcal{X}) = S_1(\mathcal{X})$.*

The following obvious statement gives two more equivalent definitions of the regular scheme.

Theorem 2.1.29. *For a scheme $\mathcal{X}$, the statements below are equivalent:*

(1) *$\mathcal{X}$ is regular;*
(2) *$n_s = 1$ for all $s \in S$;*
(3) *$|S| = |\Omega|$;*
(4) *the thin radical parabolic of $\mathcal{X}$ equals Ω^2.*

Statement (3) of Theorem 2.1.25 shows that every regular coherent configuration can be considered as an abstract group. Conversely, if G is a group, then for any $g \in G$, the graph s_g of the permutation g_{right} (see (1.1.6)) is a thin relation on $\Omega := G$. Since

$$1_\Omega = s_1 \quad \text{and} \quad (s_g)^* = s_{g^{-1}} \quad \text{and} \quad s_g \cdot s_h = s_{gh}$$

for all $g, h \in G$, the pair $\mathcal{X} = (\Omega, S)$ is a regular coherent configuration. Thus the group theory is really "embedded" to the theory of coherent configurations.

Definition 2.1.30. *A coherent configuration $\mathcal{X}$ is said to be semiregular if $n_s = 1$ for all relations $s \in S$.*

Every discrete coherent configuration is semiregular. The class of all semiregular coherent configurations is obviously closed under taking fissions and restrictions. Furthermore, every regular coherent configuration is semiregular and each homogeneous component of a semiregular coherent configuration is regular. Further properties of semiregular coherent configurations are in Exercise 2.7.13.

2.2 Galois correspondence

In this section, we consider the Galois correspondence between the coherent configurations and permutation groups, which was explicitly established by M. Klin and his colleagues in 1970s (for the historical remarks, we refer to the survey [**49**]). This correspondence expresses the basic principle of our methodology in the study of coherent configurations and leads to the first of the two main problems that are discussed in this book, namely, the schurity problem.

2.2.1 Isomorphisms

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be rainbows.

Definition 2.2.1. *A bijection $f : \Omega \to \Omega'$ is called a combinatorial isomorphism or, briefly, an isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$ if*

$$S' = S^f.$$

The set of all of them is denoted by $\mathrm{Iso}(\mathcal{X}, \mathcal{X}')$*. Two rainbows $\mathcal{X}$ and $\mathcal{X}'$ are said to be combinatorially isomorphic or isomorphic if* $\mathrm{Iso}(\mathcal{X}, \mathcal{X}') \neq \varnothing$*.*

For any bijection f from Ω to another set, one can define a rainbow $\mathcal{X}^f = (\Omega^f, S^f)$. Clearly, $\mathcal{X}^f$ is a rainbow isomorphic to $\mathcal{X}$.

Example 2.2.2. *Let $\mathcal{X}$ be a coherent configuration, $s \in S_1$, and f_s a permutation of Ω defined in* (1.1.5)*. Then*

$$r^{f_s} = s^* \cdot r \cdot s$$

is a basis relation of $\mathcal{X}$ for all $r \in S$ (Lemma 2.1.24). It follows that $S^{f_s} = S$ and hence $f_s \in \mathrm{Iso}(\mathcal{X}, \mathcal{X})$ and $\mathcal{X}^{f_s} = \mathcal{X}$.

It is easily seen that $\mathrm{Iso}(\mathcal{X}) = \mathrm{Iso}(\mathcal{X}, \mathcal{X})$ is a permutation group on Ω. In a natural way it acts also on the set S and the kernel of this action is a normal subgroup of $\mathrm{Iso}(\mathcal{X})$. It is denoted by $\mathrm{Aut}(\mathcal{X})$ and called the *automorphism group* of $\mathcal{X}$; any element of this group is called the *automorphism* of $\mathcal{X}$. Thus,

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : \ s^f = s \text{ for all } s \in S\}.$$

This group is obviously equal to the intersection of the automorphism groups of basis graphs of $\mathcal{X}$.

Definitely, it would be more natural to consider $\mathrm{Iso}(\mathcal{X})$ rather than $\mathrm{Aut}(\mathcal{X})$ as the automorphism group of $\mathcal{X}$, but here we follow a long tradition in accordance with which a rainbow is treated as a colored graph.

One can easily check that

$$\mathrm{Iso}(\mathcal{T}_\Omega) = \mathrm{Aut}(\mathcal{T}_\Omega) = \mathrm{Sym}(\Omega)$$

and

$$\mathrm{Iso}(\mathcal{D}_\Omega) = \mathrm{Sym}(\Omega) \quad \text{and} \quad \mathrm{Aut}(\mathcal{D}_\Omega) = \{\mathrm{id}_\Omega\}, \tag{2.2.1}$$

where id_Ω is the identity permutation of Ω.

2.2.2 Coherent configuration of a permutation group

For every permutation group $K \le \mathrm{Sym}(\Omega)$, one can define a natural partition S of the set Ω^2 into the *orbitals* or 2-*orbits* of K, i.e., the orbits in the induced action of K on Ω^2: a permutation $k \in K$ takes a pair (α, β) to the pair (α^k, β^k). Thus,

$$S = \mathrm{Orb}(K, \Omega^2).$$

It is easily seen that the pair (Ω, S) satisfies the conditions (CC1) and (CC2).

Furthermore, any relation $t \in S$ consists of the pairs $(\alpha, \beta)^k$ with $k \in K$ and fixed $(\alpha, \beta) \in t$. Therefore,

$$|\alpha r \cap \beta s^*| = |(\alpha r \cap \beta s^*)^k| = |\alpha^k r^k \cap \beta^k (s^*)^k| = |\alpha^k r \cap \beta^k s^*|$$

for all $r, s \in S$, which proves the condition (CC3). Thus the rainbow (Ω, S) is a coherent configuration.

Definition 2.2.3. *The pair*

$$\mathrm{Inv}(K) = \mathrm{Inv}(K, \Omega) = (\Omega, \mathrm{Orb}(K, \Omega^2))$$

is called the coherent configuration associated with the group $K \le \mathrm{Sym}(\Omega)$.

The coherent configurations associated with the groups $\mathrm{Sym}(\Omega)$ and $\{\mathrm{id}_\Omega\}$ are equal to $\mathcal{T}_\Omega$ and $\mathcal{D}_\Omega$, respectively.

Cyclotomic schemes. Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with q elements, and let M be a subgroup of its multiplicative group $\mathbb{F}^\times$. Denote by K a subgroup of $\mathrm{Sym}(\mathbb{F})$ consisting of the permutations

$$x \mapsto a + bx, \; x \in \mathbb{F},$$

where $a \in \mathbb{F}$ and $b \in M$. The group K is transitive and the 2-orbits of K are the relations

$$s_u := (0, u)^K = \{(\alpha, \beta) \in \mathbb{F} \times \mathbb{F} : \; \beta - \alpha \in Mu\},$$

where $u \in \mathbb{F}$. It follows that the irreflexive basis relations of the scheme

$$\mathrm{Cyc}(M, \mathbb{F}) := \mathrm{Inv}(K, \mathbb{F})$$

associated with the group K are exactly the relations s_u, which are in one-to-one correspondence with the cosets Mu of the group M in $\mathbb{F}^\times$. We say that $\mathrm{Cyc}(M, \mathbb{F})$ is a *cyclotomic scheme* over the field $\mathbb{F}$. The rank of this scheme is one more than the index of the group M in $\mathbb{F}^\times$.

Let

$$m = |M| \quad \text{and} \quad m' = \frac{q-1}{m}.$$

Denote by ξ a primitive element of the field $\mathbb{F}$. Then the group M consists of the elements $\xi^{m'i}$, where i runs over the set $I = \{0, 1, \dots, m-1\}$. Now if α is the zero of $\mathbb{F}$ and $u, v, w \in \mathbb{F}^\times$, then

$$\alpha s_u = \{u\xi^{m'i} : \; i \in I\} \quad \text{and} \quad w s_v^* = \{w - v\xi^{m'j} : \; j \in I\}.$$

It follows that the intersection number $c_{s_u s_v}^{s_w} = |\alpha s_u \cap w s_v^*|$ is equal to the number of pairs $(i,j) \in I^2$ with

$$u\xi^{m'i} = w - v\xi^{m'j}.$$

The explicit evaluation of these integers called the cyclotomic numbers is a hard number-theoretic problem (see [**91**, p.247]).

The following result, first proved by R. McConnel in [**94**], shows, in particular, that the automorphism group of a nontrivial cyclotomic scheme is solvable. The proof of this result goes beyond the material of this book; the references to several known proofs can be found in survey paper [**81**, Sec. 9].

Theorem 2.2.4. *The automorphism group of a nontrivial cyclotomic scheme over a finite field* $\mathbb{F}$ *is contained in the one-dimensional affine semi-linear group*

$$\mathrm{A\Gamma L}(1,\mathbb{F}) = \{\alpha \mapsto a + \alpha^\sigma b,\ \alpha \in \mathbb{F}:\ a \in \mathbb{F},\ b \in \mathbb{F}^\times,\ \sigma \in \mathrm{Aut}(\mathbb{F})\}. \tag{2.2.2}$$

Assume that the order q of the field $\mathbb{F}$ is odd. Then the multiplicative group $\mathbb{F}^\times$ is cyclic of even order and hence contains a subgroup M of index 2. It follows that the cyclotomic scheme $\mathrm{Cyc}(M,\mathbb{F})$ contains a basis relation

$$s = \{(\alpha,\beta) \in \mathbb{F} \times \mathbb{F}:\ \beta - \alpha \text{ is a square in } \mathbb{F}\}.$$

If $q = 1 \pmod 4$, then the element -1 is a square in $\mathbb{F}$ and the relation s is symmetric. The (undirected) basis graph of s is called a *Paley graph*; for $q = 5$, this graph is isomorphic to the pentagon. If $q = 3 \pmod 4$, then the relation s is antisymmetric and the (directed) basis graph of s is called a *Paley tournament*.

Permutation groups via coherent configurations. Let us return to the general case. A number of concepts concerning permutation groups can be expressed in terms of the associated coherent configurations. In the following statement, we consider several of them.

Proposition 2.2.5. *Let* $K \le \mathrm{Sym}(\Omega)$ *and* $\mathcal{X} = \mathrm{Inv}(K,\Omega)$. *Then*

1. $\mathrm{Orb}(K,\Omega) = F(\mathcal{X})$;
2. $\mathrm{Orb}(K,\Omega^2) = S(\mathcal{X})$;
3. $\mathrm{Orb}(K_\alpha,\Omega) = \{\alpha s:\ s \in S(\mathcal{X})\}^\natural$, *where* $\alpha \in \Omega$;
4. *if* $K' \le \mathrm{Sym}(\Omega')$ *and* $\mathcal{X}' = \mathrm{Inv}(K',\Omega')$, *then* $\mathrm{Iso}(K,K') \subseteq \mathrm{Iso}(\mathcal{X},\mathcal{X}')$.

Proof. Statement (1) follows from the obvious fact that

$$\Delta \in \mathrm{Orb}(K,\Omega) \quad \Leftrightarrow \quad 1_\Delta \in \mathrm{Orb}(K,\Omega^2).$$

Statements (2) and (4) are direct consequences of the definition of the coherent configuration associated with group.

To prove statement (3), let $\Delta \in \mathrm{Orb}(K_\alpha,\Omega)$. Then any two pairs in $\{\alpha\} \times \Delta$ belong to the same orbital. Therefore, it suffices to check that for any $s \in S$,

$$\alpha \in \Omega_-(s) \quad \Rightarrow \quad \alpha s \in \mathrm{Orb}(K_\alpha, \Omega).$$

However, any two pairs (α, β) and (α, β') in the relation s belong to the same orbital of K. It follows that $(\alpha, \beta)^k = (\alpha, \beta')$ for some $k \in K$. Then

$$\alpha^k = \alpha \quad \text{and} \quad \beta^k = \beta'.$$

Consequently, $k \in K_\alpha$ and αs is contained in an orbit of K_α. On the other hand, $(\alpha s)^{K_\alpha} \subseteq \alpha s$, because $s^K = s$. Thus, αs is equal to that orbit. □

The first two statements of Corollary 2.2.6 below are straightforward consequences of Proposition 2.2.5; the third one follows from statements (3) and (4) of Exercise 2.7.17. In what follows, a coherent configuration is said to be *half-homogeneous* if all of its fibers have the same cardinality. In particular, any semiregular coherent configuration is half-homogeneous.

Corollary 2.2.6. *Let* $K \leq \mathrm{Sym}(\Omega)$ *and* $\mathcal{X} = \mathrm{Inv}(K, \Omega)$. *Then*

(1) K *is* m*-transitive with* $m = 1/2$ *(respectively,* $m = 1,\ 3/2,\ 2$*) if and only if* $\mathcal{X}$ *is half-homogeneous (respectively, homogeneous, equivalenced, trivial);*
(2) K *is regular (respectively, semiregular, quasiregular*[2]*) if and only if* $\mathcal{X}$ *is regular (respectively, semiregular, quasiregular);*
(3) K *is a* p*-group (respectively, a group of odd order) if and only if* $|s|$ *is a* p*-power for all* $s \in S$ *(respectively,* $\mathcal{X}$ *is antisymmetric).*

Coherent configuration on cosets. It is well known that any transitive group K is permutation isomorphic to the group induced by the action of K on the right cosets of a point stabilizer by right multiplication. In what follows, we are interested in the coherent configuration associated with the induced permutation group and establish a connection between this configuration and $\mathrm{Inv}(K)$ for arbitrary (not necessarily transitive) group K.

In the notation of Proposition 2.2.5, let $\Lambda \subseteq \Omega$ be a set intersecting each fiber of the coherent configuration $\mathcal{X}$ in exactly one point. Given $\lambda, \mu \in \Lambda$, denote by $S_{\lambda,\mu}$ the set $S_{\Delta,\Gamma}$ for which $\lambda \in \Delta$ and $\mu \in \Gamma$, and put

$$\Omega_\Lambda = \bigsqcup_{\lambda \in \Lambda} K/K_\lambda,$$

where K/K_λ is the set of right cosets of K_λ in K. Taking here the disjoint union, we mean that the groups K_λ are not necessarily distinct subgroups of K.

For each point $\alpha \in \Omega$, there exists a unique point $\lambda_\alpha \in \Lambda$ lying in the same fiber as α. By statement (1) of Proposition 2.2.5, there is a permutation $k = k_\alpha$ belonging to K such that

$$(\lambda_\alpha)^k = \alpha, \quad \alpha \in \Omega.$$

[2]Quasiregular coherent configurations are defined in Exercise 2.7.19.

Note that for a fixed α, the coset $K_{\lambda_\alpha}k$ does not depend on the choice of the permutation k, and the mapping

$$f_\Lambda : \Omega \to \Omega_\Lambda,\ \alpha \mapsto K_{\lambda_\alpha} k_\alpha \tag{2.2.3}$$

is a bijection.

Let $s \in S_{\lambda,\mu}$. Fix a point $\beta \in \lambda s$. Then the permutations k_γ can be chosen so that

$$(k_\beta)^{-1} k_\gamma \in K_\lambda \quad \text{for all } \gamma \in \lambda s.$$

It immediately follows that

$$(\lambda s)^{f_\Lambda} = \{K_\mu k_\gamma :\ \gamma \in \lambda s\} = \{C \in K/K_\mu :\ C \subseteq K_\mu k_\beta K_\lambda\}. \tag{2.2.4}$$

The left-hand side and hence the right-hand side does not depend on the choice of β. Thus the basis relation $s \in S_{\lambda,\mu}$ is associated with the double coset

$$D_s = K_\mu k_s K_\lambda,$$

where $k_s = k_\beta$. This double coset is also treated as a subset of Ω_Λ, i.e., the elements of D_s are the right cosets of the group K_μ that are contained in D_s. Put

$$D_\Lambda(K) = \bigsqcup_{\lambda,\mu\in\Lambda} \{K_\lambda k K_\mu :\ k \in K\}.$$

Theorem 2.2.7. *In the above notation, the group K acts on the set Ω_Λ by right multiplications, and*

(1) $f_\Lambda \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y})$, *where* $\mathcal{X} = \mathrm{Inv}(K, \Omega)$ *and* $\mathcal{Y} = \mathrm{Inv}(K, \Omega_\Lambda)$*;*
(2) $F(\mathcal{Y}) = \{K/K_\lambda :\ \lambda \in \Lambda\}$*;*
(3) *the mapping* $S \to D_\Lambda(K)$, $s \mapsto D_s$ *is a bijection;*
(4) *if* $\lambda, \mu \in \Lambda$ *and* $s \in S_{\lambda,\mu}$, *then* $n_s = \frac{|D_s|}{|K_\mu|}$ *and* $n_{s^*} = \frac{|D_s|}{|K_\lambda|}$.

Proof. By the definition of $f = f_\Lambda$, we have

$$f(\alpha^k) = f(\alpha)^k, \quad \alpha \in \Omega,\ k \in K.$$

Therefore, f is a permutation isomorphism from the group $K \le \mathrm{Sym}(\Omega)$ onto $K \le \mathrm{Sym}(\Omega_\Lambda)$. Thus statement (1) follows from statement (4) of Proposition 2.2.5.

The group K_{λ_α} does not depend on the point α belonging to a fixed fiber of $\mathcal{X}$. Therefore the image of this fiber with respect to f_Λ coincides with the set K/K_λ of all right cosets of K_λ in K, where $\lambda = \lambda_\alpha$. This proves statement (2).

To prove statement (3), it suffices to verify that

$$D_s \cap D_{s'} = \varnothing$$

for all distinct $s, s' \in S$. Without loss of generality, we may assume that $s, s' \in S_{\lambda,\mu}$ for suitable $\lambda, \mu \in \Lambda$. Now if $k \in D_s \cap D_{s'}$, then $D_s = D_{s'}$ and the relations s and s' have a common pair $(\lambda, f^{-1}(K_\mu k))$. But then $s = s'$.

Finally, the mapping f is a bijection. By formula (2.2.4) this implies that for any $s \in S_{\lambda,\mu}$,

$$n_s = |\lambda s| = |(\lambda s)^f| = |\{C \in K/K_\mu :\ C \subseteq K_\mu k_\beta K_\lambda\}| = \frac{|D_s|}{|K_\mu|},$$

which proves statement (4). □

The Galois correspondence. Let $\mathcal{X}'$ and $\mathcal{X}$ be rainbows on Ω. Assume that $\mathcal{X} \le \mathcal{X}'$. Then each automorphism of $\mathcal{X}'$ is also an automorphism of $\mathcal{X}$. Therefore,

$$\mathcal{X} \le \mathcal{X}' \quad \Rightarrow \quad \operatorname{Aut}(\mathcal{X}) \ge \operatorname{Aut}(\mathcal{X}'), \tag{2.2.5}$$

i.e., the mapping Aut from the set of all rainbows on Ω to the set of permutation groups on Ω reverses the partial order.

On the other hand, the larger permutation group has always smaller orbits. It follows that if K and K' are permutation groups on Ω, then

$$K \le K' \quad \Rightarrow \quad \operatorname{Inv}(K) \ge \operatorname{Inv}(K'). \tag{2.2.6}$$

Therefore, the mapping Inv from the set of all permutation groups on Ω to the set of coherent configurations on Ω also reverses the partial order.

Theorem 2.2.8. *The mappings $\mathcal{X} \mapsto \operatorname{Aut}(\mathcal{X})$ and $K \mapsto \operatorname{Inv}(K)$ form a Galois correspondence between the coherent configurations on Ω and the permutation groups on Ω, i.e., together with (2.2.5) and (2.2.6),*

$$\mathcal{X} \le \operatorname{Inv}(\operatorname{Aut}(\mathcal{X})), \qquad K \le \operatorname{Aut}(\operatorname{Inv}(K)). \tag{2.2.7}$$

In particular,

$$\operatorname{Aut}(\mathcal{X}) = \operatorname{Aut}(\operatorname{Inv}(\operatorname{Aut}(\mathcal{X}))), \qquad \operatorname{Inv}(K) = \operatorname{Inv}(\operatorname{Aut}(\operatorname{Inv}(K))). \tag{2.2.8}$$

Proof. Every basis relation of a coherent configuration $\mathcal{X}$ is an $\operatorname{Aut}(\mathcal{X})$-invariant and hence is the union of some 2-orbits of the group $\operatorname{Aut}(\mathcal{X})$. This implies that

$$S(\mathcal{X}) \subseteq \operatorname{Orb}(\operatorname{Aut}(\mathcal{X}), \Omega^2)^\cup = S(\operatorname{Inv}(\operatorname{Aut}(\mathcal{X}), \Omega))^\cup,$$

where the equality follows from statement (1) of Exercise 2.7.17. This proves the first of inclusions in (2.2.7).

The second inclusion holds, because every basis relation of the coherent configuration $\operatorname{Inv}(K)$ is obviously K-invariant. The first equality in (2.2.8) follows from the first inclusion in (2.2.7) and formula (2.2.5):

$$\operatorname{Aut}(\mathcal{X}) \ge \operatorname{Aut}(\operatorname{Inv}(\operatorname{Aut}(\mathcal{X}))) \ge \operatorname{Aut}(\mathcal{X}).$$

The second equality in (2.2.8) is proved similarly. □

Definition 2.2.9. *A coherent configuration $\mathcal{X}$ is said to be schurian if*

$$\mathcal{X} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X})),$$

and a group $K \leq \mathrm{Sym}(\Omega)$ *is said to be* 2*-closed if*

$$K = \mathrm{Aut}(\mathrm{Inv}(K)).$$

Thus the schurian coherent configurations and 2-closed permutation groups are exactly the Galois closed objects with respect to the Galois correspondence from Theorem 2.2.8. By the general properties of a Galois correspondence [**106**, Chap.11], the mappings

$$\mathcal{X} \mapsto \mathrm{Aut}(\mathcal{X}) \quad \text{and} \quad K \mapsto \mathrm{Inv}(K)$$

are mutually inverse one-to-one correspondences between schurian coherent configurations and the 2-closed groups, and between the 2-closed groups and schurian coherent configurations, respectively.

The Galois correspondence from Theorem 2.2.8 is nontrivial in the sense that there are many non-schurian coherent configurations and non-2-closed permutation groups. In the following two sections, we construct the corresponding examples.

2.2.3 Schurian coherent configurations

Computer computations show that all coherent configurations of degree at most 13 are schurian and up to isomorphism, there is a unique non-homogeneous coherent configuration of degree 14 which is not schurian, see [**85**]. It should be noted that the smallest degree of non-schurian scheme equals 15 and again the corresponding example is unique up to isomorphism: it is an antisymmetric scheme of rank 3 (for the explicit construction of this scheme, we refer to Section 4.5).

The smallest non-schurian coherent configuration [**85**]. The group

$$K = \mathrm{SL}(2,3) = Q_8 \rtimes C_3,$$

where Q_8 is the quaternion group, has a permutation representation of degree 14 with two orbits of cardinalities 6 and 8. Namely, let Ω_1 and Ω_2 be the sets of right cosets of K by cyclic subgroups of order 4 and 3, respectively,

$$|\Omega_1| = 6 \quad \text{and} \quad |\Omega_2| = 8.$$

Then K acts (intransitively) by right multiplications on the set $\Omega = \Omega_1 \cup \Omega_2$.

The coherent configuration $\mathcal{X} = \mathrm{Inv}(K, \Omega)$ is of degree 14, rank 12, and has exactly two fibers Ω_1 and Ω_2. The homogeneous components $\mathcal{X}_{\Omega_1}$ and $\mathcal{X}_{\Omega_2}$ of $\mathcal{X}$ are non-symmetric schemes of rank 4 with valencies belonging to the multisets

$$\{1,1,2,2\} \quad \text{and} \quad \{1,1,3,3\}.$$

There are four more basis relations of $\mathcal{X}$: two in S_{Ω_1,Ω_2} each of valency 4, and two in S_{Ω_2,Ω_1} each of valency 3.

Denote by T the set obtained from $S(\mathcal{X})$ by replacing the two basis relations of valency 2 (in S_{Ω_1,Ω_1}) by their union. Then $\mathcal{Y} = (\Omega, T)$ is a coherent configuration of degree 14, rank 11, and two homogeneous components of rank 3 and 4. Since

$$\mathcal{Y} < \mathcal{X} \quad \text{and} \quad \mathrm{Aut}(\mathcal{Y}) = \mathrm{Aut}(\mathcal{X})^3,$$

this coherent configuration is non-schurian.

Theorem 2.2.10. *For a coherent configuration $\mathcal{X}$ on Ω, the following statements are equivalent:*

1. *$\mathcal{X}$ is schurian;*
2. *there exists a group $K \le \mathrm{Sym}(\Omega)$ such that $\mathcal{X} = \mathrm{Inv}(K)$;*
3. *there exists a group $K \le \mathrm{Aut}(\mathcal{X})$ such that $\mathrm{Orb}(K) = F(\mathcal{X})$ and $\mathrm{Orb}(K_\alpha) = \{\alpha s : \ s \in S\}^\natural$ for all $\alpha \in \Omega$.*

Proof. The implication (1) ⇒ (2) follows for $K = \mathrm{Aut}(\mathcal{X})$ from the definition of schurian coherent configuration. The implication (2) ⇒ (3) follows from statements (1) and (3) of Proposition 2.2.5.

[3] This equality can be verified with the help of the computer package COCO2P [**84**].

To prove the implication (3) $\Rightarrow$ (1), let $s \in S(\mathcal{X})$. Then for each pair $(\alpha,\beta) \in s$, the assumption of statement (3) implies that

$$s = \bigcup_{\gamma \in \Omega_-(s)} \{\gamma\} \times \gamma s = \bigcup_{k \in K} \{\alpha^k\} \times \alpha^k s^k = \bigcup_{k \in K} (\{\alpha\} \times \alpha s)^k = (\alpha,\beta)^K,$$

It follows that $s \in \mathrm{Orb}(K,\Omega^2) = \mathrm{Orb}(\mathrm{Aut}(\mathcal{X}),\Omega^2)$. Thus,

$$S(\mathcal{X}) = \mathrm{Orb}(\mathrm{Aut}(\mathcal{X}),\Omega^2)$$

and hence $\mathcal{X} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$ as required. □

Schurian coherent configuration are usually studied in the frames of permutation group theory. However, as we will see later, most of coherent configurations are non-schurian and because of this the theory of coherent configurations is sometimes called "group theory without groups" (see [**10**, p.i]). In fact, finding a border between schurian and non-schurian coherent configurations presents one of the most fundamental problems in this theory.

Schurity problem. Given a class $\mathcal{K}$ of coherent configurations identify all schurian coherent configurations in $\mathcal{K}$.

One more motivation for the schurity problem comes from the Graph Isomorphism Problem, which is a computational problem of determining whether two finite graphs are isomorphic (see also Subsection 4.6.1). At this point, we mention that this problem is equivalent to finding the orbits of the automorphism group of a given coherent configuration. However, if the coherent configuration is schurian, then the orbits coincide with fibers (statement (1) of Proposition 2.2.5), which can be found efficiently if the coherent configuration is explicitly given. (Of course, the latter argument does not help to test isomorphism of schurian coherent configurations.)

In general, the schurity problem (even for a class consisting of only one coherent configuration) can be quite difficult. The following statement gives a complete solution to it for the class of regular schemes (for semiregular coherent configurations, see Exercise 2.7.35).

Theorem 2.2.11. *Let $\mathcal{X} = (\Omega, S)$ be a regular scheme. Then S is a group with respect to the composition. Moreover, for a fixed $\alpha \in \Omega$, the mapping*

$$f : \Omega \to S,\ \beta \mapsto r(\alpha,\beta)^*$$

is a bijection satisfying the following conditions:

(1) $S(\mathcal{X}^f) = S_{left}$;
(2) $\mathrm{Aut}(\mathcal{X}^f) = S_{right}$;
(3) $\mathcal{X}^f = \mathrm{Inv}(S_{right}, S)$.

In particular, every regular scheme is schurian and the automorphism group of it is regular.

Proof. The regularity of $\mathcal{X}$ implies that $S = S_1(\mathcal{X})$. Thus, S is a group with respect to the composition by statement (2) of Theorem 2.1.25. Since also $r(\alpha, \beta) = r(\alpha, \beta')$ if and only if $\beta = \beta'$, the mapping f is a bijection. To prove statement (1), let $s \in S$. Then

$$s_{left} = \{(x, s^* \cdot x) : \ x \in S\}.$$

For each $x \in S$, let β and γ be such that $\beta^f = x$ and $\gamma^f = s^* \cdot x$. Then by the definition of f, we have

$$(\alpha, \beta) \in x^* \quad \text{and} \quad (\alpha, \gamma) \in x^* \cdot s.$$

Consequently,

$$(\beta, \gamma) \in x \cdot x^* \cdot s = s.$$

This shows that the f-preimage of s_{left} is equal to s, as required.

The group S_{right} centralizes the group S_{left}. By statement (1), this implies that

$$S_{right} \leq \operatorname{Aut}(\mathcal{X}^f).$$

On the other hand, the orbits of the group $\operatorname{Aut}(\mathcal{X})_\alpha$ are contained in the sets αs, $s \in S$. Since the scheme $\mathcal{X}$ is regular, these sets and hence the orbits are singletons. Consequently, the groups $\operatorname{Aut}(\mathcal{X})_\alpha$ and $\operatorname{Aut}(\mathcal{X})$ are trivial and of order n, respectively. Thus,

$$n = |S_{right}| \leq |\operatorname{Aut}(\mathcal{X}^f)| = |\operatorname{Aut}(\mathcal{X})| = n,$$

and hence $\operatorname{Aut}(\mathcal{X}^f) = S_{right}$, which proves statement (2).

To prove statement (3), it suffices to note that the schemes $\mathcal{X}^f$ and $\operatorname{Inv}(S_{right})$ have the same rank (namely, n), whereas

$$\mathcal{X}^f \leq \operatorname{Inv}(\operatorname{Aut}(\mathcal{X}^f)) = \operatorname{Inv}(S_{right})$$

by statement (2). □

2.2.4 Closed permutation groups

It is more convenient to discuss the 2-closedness concept in more general setting. The theory presented below was developed by H. Wielandt in [**126**] in the framework of the method of invariant relations that he used to study permutation groups.

The m-closure. Let m be a positive integer. In the induced action of a group $K \le \mathrm{Sym}(\Omega)$ on the Cartesian power Ω^m, a permutation $k \in K$ takes an m-tuple $\alpha = (\alpha_1, \dots, \alpha_m)$ to the m-tuple

$$\alpha^k = (\alpha_1^k, \dots, \alpha_m^k).$$

Any orbit of this action is called an *m-orbit* of K.

Definition 2.2.12. *Two permutation groups K and L on the same set Ω are said to be m-equivalent if*

$$\mathrm{Orb}(K, \Omega^m) = \mathrm{Orb}(L, \Omega^m). \tag{2.2.9}$$

Every m-ary relation on Ω that is invariant with respect to the group K is obviously the disjoint union of some m-orbits of K. Therefore, the groups K and L are m-equivalent if and only if they have the same set of invariant m-ary relations.

Example 2.2.13. *All transitive (respectively, 2-transitive) groups on the same set are 1-equivalent (respectively, 2-equivalent). The groups K and L are 2-equivalent if and only if* $\mathrm{Inv}(K, \Omega) = \mathrm{Inv}(L, \Omega)$.

The projection of an m-tuple $\alpha \in \Omega^m$ to the coordinates belonging to a set $I \subseteq \{1, \dots, m\}$, is denoted by $\mathrm{pr}_I(\alpha)$. It is easily seen that

$$\mathrm{pr}_I(\alpha)^k = \mathrm{pr}_I(\alpha^k), \quad k \in \mathrm{Sym}(\Omega). \tag{2.2.10}$$

This immediately implies that the projections of an m-orbit of a group K with respect to the set I of cardinality l, is an l-orbit of K. It follows that any two m-equivalent groups are also l-equivalent for all $1 \le l \le m$.

Let S be a set of relations of arbitrary arities on the set Ω. The *automorphism group* of S is defined to be the group

$$\mathrm{Aut}(S) = \{k \in \mathrm{Sym}(\Omega) : \ s^k = s, \ s \in S\}. \tag{2.2.11}$$

In particular, if S consists of one binary relation, then $\mathrm{Aut}(S)$ equals the automorphism group of the corresponding graph, whereas if $S = S(\mathcal{X})$ for a rainbow $\mathcal{X}$, then $\mathrm{Aut}(S) = \mathrm{Aut}(\mathcal{X})$. Obviously,

$$S \subseteq T \ \Rightarrow \ \mathrm{Aut}(S) \ge \mathrm{Aut}(T), \tag{2.2.12}$$

and the automorphism group of S is the largest subgroup of $\mathrm{Sym}(\Omega)$ with respect to which all the relation of S are invariant.

Definition 2.2.14. *The m-closure of $K \le \mathrm{Sym}(\Omega)$, is defined to be the group*

$$K^{(m)} = \mathrm{Aut}(\mathrm{Orb}(K, \Omega^m)),$$

or equivalently, the automorphism group of the set of all K-invariant relations of arity m.

In particular, the 1-closure of K equals the direct product of the groups $\mathrm{Sym}(\Delta)$ with $\Delta \in \mathrm{Orb}(K)$ (statement (1) of Exercise 2.7.23), whereas the 2-closure of K is equal to the automorphism group of the coherent configuration $\mathrm{Inv}(K)$. The following statement shows that taking the m-closure is indeed a closure operator.

Theorem 2.2.15. *Let $K \le \mathrm{Sym}(\Omega)$ and $m \ge 1$. Then*

(1) $K \le K^{(m)}$;
(2) *K and $K^{(m)}$ are m-equivalent;*
(3) *$K^{(m)} = L^{(m)}$ whenever K and L are m-equivalent.*

Proof. Statement (1) is obvious. By statement (1), any m-orbit s of the group $K^{(m)}$ is a union of some m-orbits of the group K. On the other hand, s cannot be larger than an orbit of K by the definition of m-closure. Thus, $K^{(m)}$ and K have the same m-orbits, which proves statement (2).

Finally, if the groups K and L are m-equivalent, then equality (2.2.9) holds and hence

$$K^{(m)} = \mathrm{Aut}(\mathrm{Orb}(K, \Omega^m)) = \mathrm{Aut}(\mathrm{Orb}(L, \Omega^m)) = L^{(m)}$$

which proves statement (3). □

Definition 2.2.16. *The group K is said to be m-closed if $K^{(m)} = K$.*

By the above remark on the 1-closure, the 1-closed groups are the intransitive direct products of the symmetric groups (taken in the natural action). For $m = 2$, our definition coincides with that given earlier because of the following natural characterization of m-closed groups as the automorphism groups of m-ary relations.

Theorem 2.2.17. *A permutation group $K \le \mathrm{Sym}(\Omega)$ is m-closed if and only if there exists a set S of m-ary relations on Ω such that $K = \mathrm{Aut}(S)$.*

Proof. The necessity follows for $S = \mathrm{Orb}(K, \Omega^m)$. To prove the sufficiency, let $K = \mathrm{Aut}(S)$, where S is a set of m-ary relations on Ω. Then every relation of S is K-invariant. On the other hand, by statement (2) of Theorem 2.2.15,

$$\mathrm{Orb}(K^{(m)}, \Omega^m) = \mathrm{Orb}(K, \Omega^m).$$

Thus every relation of S is also $K^{(m)}$-invariant. Therefore,

$$\mathrm{Aut}(S) \ge K^{(m)}.$$

Now by statement (1) of Theorem 2.2.15,

$$K^{(m)} \ge K = \mathrm{Aut}(S) \ge K^{(m)}$$

implying $K = K^{(m)}$. □

Corollary 2.2.18. *A permutation group K is 2-closed if and only if K is the automorphism group of a coherent configuration.*

From Theorem 2.2.15, it follows that $K^{(m)}$ is the largest subgroup in the class of groups m-equivalent to K. In particular, each class of the m-equivalence relation contains exactly one m-closed group and this group equals the m-closure of any group in this class. If such a class consists of exactly one group, then this group is equal to $K = K^{(m)}$; in this case, the group K is said to be *m-isolated*, and is uniquely determined by the set of m-orbits.

Example 2.2.19. *The dihedral group $K = D_{2n}$ acting on n-points is 2-isolated. Indeed, K is the automorphism group of an indirected cycle with n vertices. Therefore, K is 2-closed, and acts regularly on the $2n$ arcs of the cycle. No proper subgroup of K can have the latter property. Thus, K is 2-equivalent to no proper subgroup.*

The base number. A sufficient condition for a group $K \leq \mathrm{Sym}(\Omega)$ to be m-closed can be formulated in terms of the base number of K that is defined below.

Definition 2.2.20. *A set $\Delta \subseteq \Omega$ is called a base of K if the pointwise stabilizer*

$$K_\Delta = \{k \in K : \ \alpha^k = \alpha \text{ for all } \alpha \in \Delta\}$$

of Δ in K is trivial.

Let $\Delta = \{\alpha_1, \ldots, \alpha_b\}$. The group K_Δ is equal to the stabilizer of the point $\alpha = (\alpha_1, \ldots, \alpha_b)$ in the action of K on Ω^b. Therefore, Δ is a base of K if and only if the b-orbit α^K is regular and faithful. In particular, the order of K equals the cardinality of this orbit and hence

$$|K| \leq n^{b(K)}, \tag{2.2.13}$$

where

$$b(K) = \min\{|\Delta| : \ \Delta \text{ is a base of } K\}$$

is the *base number* of K. Clearly,

$$0 \leq b(K) \leq n - 1$$

with equalities if $K = \{\mathrm{id}\}$ and $K = \mathrm{Sym}(\Omega)$, respectively. The following theorem was proved by H. Wielandt [**126**, Theorem 5.12].

Theorem 2.2.21. *A permutation group having a base of cardinality b is $(b + 1)$-closed.*

Proof. Let $\{\alpha_1, \dots, \alpha_b\}$ be a base of a group $K \le \mathrm{Sym}(\Omega)$. By statement (1) of Theorem 2.2.15, it suffices to verify that

$$K^{(b+1)} \le K.$$

To this end, let $k \in K^{(b+1)}$. Since the $(b+1)$-orbits of the groups K and $K^{(b+1)}$ coincide, for any $\alpha \in \Omega$ there exists permutation $k_\alpha \in K$ such that

$$(\alpha_1, \dots, \alpha_b, \alpha)^k = (\alpha_1, \dots, \alpha_b, \alpha)^{k_\alpha}. \tag{2.2.14}$$

It follows that for each $\beta \in \Omega$,

$$(\alpha_i)^{k_\beta} = (\alpha_i)^k = (\alpha_i)^{k_\alpha}, \quad i = 1, \dots, b.$$

Therefore,

$$k_\alpha k_\beta^{-1} \in K_{\alpha_1, \dots, \alpha_b} = \{\mathrm{id}\}.$$

Consequently, the permutation $k_0 := k_\alpha$ does not depend on $\alpha \in \Omega$. By formula (2.2.14), this implies that

$$\alpha^k = \alpha^{k_0} \quad \text{for all } \alpha \in \Omega.$$

Thus, $k = k_0 \in K$, as required. □

Corollary 2.2.22. *Any permutation group with base number b is $(b+1)$-closed.*

Let K be a semiregular permutation group. Then any point of the underlying set forms a base of K. Thus the following statement is an immediate consequence of Corollary 2.2.22.

Corollary 2.2.23. *Any semiregular permutation group is 2-closed.*

The group $K = \mathrm{Sym}(n)$ is 1-closed but $b(K) = n - 1$. Therefore, the bound in Theorem 2.2.21 can be arbitrarily far from the minimal number m for which the group K is m-closed. On the other hand, Corollary 2.2.23 shows that the bound is attained for semiregular groups.

From formula (2.2.10), it follows that given $l \le m$, the induced action of the group $K^{(m)}$ on the set Ω^l leaves each l-orbit of K fixed (as a set). Therefore,

$$l \le m \quad \Rightarrow \quad K^{(m)} \le K^{(l)}.$$

In other words, the greater is m, the less is $K^{(m)}$. However, $K^{(m)}$ always contains K (statement (1) of Theorem 2.2.15). Thus by Theorem 2.2.21, the series of the m-closures collapses to K when $m \ge b + 1$.

Corollary 2.2.24. *For any permutation group K of degree n and base number b,*

$$K^{(1)} \ge K^{(2)} \ge \cdots \ge K^{(b+1)} = K = \cdots = K^{(n)}.$$

Non-closed groups from MDS-codes. Using Corollary 2.2.23, one can easily find a 2-closed permutation group which is not 1-closed. However, examples of $(m+1)$-closed groups which are not m-closed for $m > 1$ are not so obvious.

Example 2.2.25. *For $m = 2$, one can take the group of degree 6, given by*

$$K = \langle (1,2)(3,4), (3,4)(5,6) \rangle. \tag{2.2.15}$$

A straightforward check shows that $\{1,3\}$ is a base of K. Therefore, K is 3-closed by Theorem 2.2.21. However, K is not 2-closed, because

$$|\operatorname{Aut}(\operatorname{Inv}(K))| = 8 \quad \textit{and} \quad |K| = 4.$$

The above example is a special case of a general construction producing a permutation group from an MDS-code. In accordance with [**93**, Chap. 11], such a code can be thought as an m-dimensional subspace U of an n-dimensional linear space $V = (\mathbb{F}_q)^n$ that satisfies the following condition:

$$\dim(U_I) = \dim(U) \quad \text{for all } I \subseteq \{1,\dots,n\},\ |I| = m, \tag{2.2.16}$$

where $U_I = \{\mathrm{pr}_I(u) :\ u \in U\}$.

The additive group of U has a natural action on the disjoint union Ω of n copies of $\mathbb{F} = \mathbb{F}_q$: given an element $u = (u_1,\dots,u_n)$ belonging to U, and an element α belonging to the ith copy Ω_i,

$$\alpha^u = \alpha + u_i.$$

This action defines an elementary abelian group $K \le \mathrm{Sym}(\Omega)$ such that

$$\mathrm{Orb}(K,\Omega) = \{\Omega_1,\dots,\Omega_n\} \quad \text{and} \quad K^{\Omega_i} \cong \mathbb{F}^+.$$

One can see that the group defined by formula (2.2.15) is obtained for $n = 2$, $q = 2$, $m = 1$, and $U = \mathrm{Diag}(\mathbb{F}^2)$.

Proposition 2.2.26. *In the above notation, suppose that $m < n$. Then the group K is $(m+1)$-closed but not m-closed.*

Proof. Denote by L be the intransitive direct product of the groups K^{Ω_i}, $i = 1,\dots,n$. Then

$$L \le \mathrm{Sym}(\Omega) \quad \text{and} \quad \mathrm{Orb}(L) = \mathrm{Orb}(K).$$

In view of condition (2.2.16),

$$K^{\Omega_I} = L^{\Omega_I} \quad \text{for all } I \subseteq \{1,\dots,n\},\ |I| = m, \tag{2.2.17}$$

where Ω_I is the union of Ω_i with $i \in I$.

Every m-orbit of K or L is contained in the product $\Omega_{i_1} \times \cdots \times \Omega_{i_m}$, where $i_j \in \{1,\dots,n\}$ for all j. Therefore from formula (2.2.17), it follows that K and L are m-equivalent. Taking into account that $m < n$ and hence $K < L$, we conclude that

$$K < L \leq L^{(m)} = K^{(m)},$$

i.e., K is not m-closed.

On the other hand, condition (2.2.16) implies that the restriction homomorphism $K \to K^{\Omega_I}$ with $I = \{1, \ldots, m\}$ is an isomorphism. Taking into account that the group K^{Ω_i}, being abelian and transitive, is regular for all i, we have

$$K_{\alpha_1, \ldots, \alpha_m} = \{\mathrm{id}\}$$

for any points $\alpha_i \in \Omega_i$ with $i \in I$. Thus, $\{\alpha_1, \ldots, \alpha_m\}$ is a base of K, and hence K is $(m + 1)$-closed by Theorem 2.2.21. □

2.3 Coherent algebras

With any permutation group K, one can associate the algebra of all complex matrices centralizing the permutation matrices of the elements of K. This algebra is called a centralizer algebra of K; it is closed under the Hermitian conjugation and is a unitary algebra with respect to both ordinary and Hadamard multiplications. Following D. Higman [**66**], every matrix algebra satisfying these conditions is said to be coherent; the cellular algebras introduced by B. Weisfeiler and A. Leman in [**124**] differ from the coherent algebras: the algebra with respect to matrix multiplication is not necessarily unitary.

There is a one-to-one correspondence between the coherent configurations and coherent algebras that takes schurian coherent configurations to the centralizer algebras of permutation groups, and symmetric schemes to the well-known Bose–Mesner algebras. In the present section, we introduce the coherent algebras and establish the above correspondence. This enables us to define another type of isomorphisms for coherent configurations that leads to the separability problem.

2.3.1 Adjacency algebra

Let $\mathcal{X} = (\Omega, S)$ be a rainbow. The subset of the full matrix algebra Mat_Ω, defined by

$$\mathcal{M} = \mathcal{M}(\mathcal{X}) = \{A_s : \ s \in S\},$$

consists of $\{0,1\}$-matrices, which are pairwise orthogonal with respect to the Hadamard multiplication (statement (2) of Exercise 1.4.8). Therefore, $\mathcal{M}$ is a linear basis of the subspace spanned by the matrices of $\mathcal{M}$,

$$\mathcal{A} = \mathrm{Adj}(\mathcal{X}) = \mathrm{Span}_{\mathbb{C}}\,\mathcal{M}.$$

Definition 2.3.1. *The linear basis $\mathcal{M}$ of the space $\mathcal{A}$ is called the standard basis of the rainbow $\mathcal{X}$.*

The rainbow axioms imply that the linear space $\mathcal{A}$ contains the identity matrix I_Ω, the all one matrix J_Ω, and is closed under transposition. The following statement is straightforward.

Proposition 2.3.2. *The mapping $s \mapsto A_s$, $s \in S^\cup$, defines a one-to-one correspondence between the relations of $\mathcal{X}$ and $\{0,1\}$-matrices of $\mathcal{A}$. This mapping takes S onto $\mathcal{M}$. For any rainbow $\mathcal{X}'$ on Ω,*

$$\mathcal{X} \le \mathcal{X}' \quad \Leftrightarrow \quad \mathrm{Adj}(\mathcal{X}) \le \mathrm{Adj}(\mathcal{X}'). \tag{2.3.1}$$

From now on, we assume that $\mathcal{X}$ is a coherent configuration. Then by statement (4) of Exercise 1.4.8 for all $r, s \in S$,

$$A_r\,A_s = \sum_{t\in S} c_{rs}^t A_t. \tag{2.3.2}$$

This means that the linear space $\mathcal{A}$ is an algebra of dimension $|S|$ and the intersection numbers of $\mathcal{X}$ are the structure constants of $\mathcal{A}$ with respect to the standard basis $\mathcal{M}$.

Definition 2.3.3. *The matrix algebra $\mathcal{A}$ is called the adjacency algebra of the coherent configuration $\mathcal{X}$.*

It is easily seen that

$$\mathrm{Adj}(\mathcal{D}_\Omega) = \mathrm{Mat}_\Omega \quad \text{and} \quad \mathrm{Adj}(\mathcal{T}_\Omega) = \mathrm{Span}_{\mathbb{C}}\{I_\Omega, J_\Omega\}.$$

It should be noted that exactly in the same way, one can define the adjacency algebra of a coherent configuration over any field, and even over a ring; for details, see [**55**].

Example 2.3.4. *Let G be a group and $\mathcal{X} = \mathrm{Inv}(G_{right})$. Each basis relation of $\mathcal{X}$ is of the form*

$$s_g = \{(x, g^{-1}x) : \ x \in G\}$$

for some $g \in G$, and the standard basis $\mathcal{M}(\mathcal{X})$ consists of the permutation matrices

$$A_{s_g} = P_{g_{left}}.$$

It follows that the monomorphism defined by equality (1.4.3) induces an algebra isomorphism from $\mathbb{C}G$ to $\mathrm{Adj}(\mathcal{X})$. Thus the adjacency algebra of $\mathcal{X}$ is isomorphic to the group algebra of G.

Now assume that the coherent configuration $\mathcal{X}$ is associated with a permutation group K on Ω. Then for any $s \in S$ and all $k \in K$, we have $s^k = s$, which implies by formula (1.4.4) that

$$A_s = P_k^{-1} A_s P_k.$$

Since any matrix of the algebra $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ is a linear combination of the basis matrices A_s, this shows that $\mathcal{A}$ is a subalgebra of the *centralizer algebra* of the group K, i.e., the algebra of all $A \in \mathrm{Mat}_\Omega$ that commute with each permutation matrix P_k, $k \in K$.

Conversely, for any such matrix A,

$$A_{\alpha,\beta} = (P_k^{-1} A P_k)_{\alpha,\beta} = A_{\alpha^k,\beta^k}$$

for all $\alpha, \beta \in \Omega$ and $k \in K$. Therefore, A is a linear combination of the matrices A_s, where s runs over the set $\mathrm{Orb}(K, \Omega^2) = S$, i.e., $A \in \mathcal{A}$. This proves the following statement.

Proposition 2.3.5. *The adjacency algebra of the coherent configuration associated with a permutation group coincides with the centralizer algebra of this group.*

It should also be noted that if the group K is transitive and H is a point stabilizer of K, then the bijection $s \mapsto D_s$, $s \in S$, defined in statement (3) of

Theorem 2.2.7, induces a linear isomorphism $A_s \mapsto D_{s^*}$ from the adjacency algebra of the coherent configuration $\mathrm{Inv}(K)$ to the algebra of double cosets of HkH, $k \in K$. Exercise 2.7.20 shows that this linear isomorphism is an algebra antiisomorphism.

The following characterization of the adjacency algebras in the class of all matrix algebras immediately follows from the definition of coherent configuration and Exercise 1.4.8.

Theorem 2.3.6. *The adjacency algebra $\mathcal{A}$ of a coherent configuration on Ω is coherent, i.e., satisfies the following conditions:*

(A1) *$\mathcal{A}$ is closed under the Hermitian conjugation;*
(A2) *$\mathcal{A}$ is an algebra with identity I_Ω with respect to the matrix multiplication;*
(A3) *$\mathcal{A}$ is an algebra with identity J_Ω with respect to the Hadamard multiplication.*

Let $\mathcal{A}$ be a coherent subalgebra of Mat_Ω. With respect to the Hadamard multiplication, $\mathcal{A}$ is a commutative algebra. It is also semisimple, because contains no nonzero nilpotent elements. Therefore, there is a unique unity decomposition

$$J_\Omega = E_1 + \cdots + E_d, \tag{2.3.3}$$

where d is the dimension of $\mathcal{A}$ and the summands are pairwise orthogonal idempotents.

Each matrix E_i, being an idempotent with respect to the Hadamard multiplication, is a $\{0,1\}$-matrix. Therefore E_i is the adjacency matrix of a relation s_i on Ω. Set

$$S = \{s_i : \ i = 1, \ldots, d\}.$$

Theorem 2.3.7. *The pair $\mathcal{X} = (\Omega, S)$ is a coherent configuration, and $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$.*

Proof. All the matrices J_Ω, $E_1, \ldots, E_d$ are $\{0,1\}$-matrices. In view of formula (2.3.3), this implies that the set S forms a partition of Ω^2. The identity matrix I_Ω is an idempotent of $\mathcal{A}$ and hence is the sum of some idempotents E_i. Therefore, the condition (CC1) holds. The condition (CC2) follows from the condition (A1).

Finally, since the matrices $E_1, \ldots, E_d$ form a linear basis of the algebra $\mathcal{A}$, we have

$$E_i E_j = \sum_{k=1}^{d} c_{ij}^k E_k$$

for all i, j and integers c_{ij}^k. In accordance with statement (4) of Exercise 1.4.8, we have

$$c_{ij}^k = |\alpha s_i \cap \beta s_j^*|.$$

This proves the condition (CC3). Thus, $\mathcal{X}$ is a coherent configuration. The fact that $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ is obvious from the construction. □

Corollary 2.3.8. *A unitary algebra $\mathcal{A} \subseteq \mathrm{Mat}_\Omega$ is coherent if and only if it has a linear basis consisting of $\{0,1\}$-matrices, the sum of which equals J_Ω, that is closed under transposition.*

Theorems 2.3.6 and 2.3.7 establish a one-to-one correspondence between the coherent configurations on Ω and coherent subalgebras of Mat_Ω. This gives us a freedom to use combinatorial or algebraic language in studying coherent configurations. In the rest of this section, we present examples showing interaction between these languages. Other examples appear in Section 3.6 concerning the representation theory of the adjacency algebras.

The adjacency algebra $\mathcal{A}$ of the coherent configuration $\mathcal{X}$ is commutative (with respect to the matrix multiplication) if and only if $A_r A_s = A_s A_r$ for all $r, s \in S$, or equivalently, if

$$c_{rs}^t = c_{sr}^t, \qquad r, s, t \in S.$$

In this case, the coherent configuration $\mathcal{X}$ is said to be *commutative*. Clearly, the trivial scheme is commutative, whereas the discrete coherent configuration of degree at least 2 is not.

Proposition 2.3.9. *Let $\mathcal{X}$ be a coherent configuration. Then*

$$\mathcal{X} \text{ is symmetric} \quad \Rightarrow \quad \mathcal{X} \text{ is commutative} \quad \Rightarrow \quad \mathcal{X} \text{ is homogeneous.}$$

Proof. Set $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$. Assume that $\mathcal{X}$ is symmetric. Then $\mathcal{A}$ consists of symmetric matrices. It follows that for all $A, B \in \mathcal{A}$,

$$AB = A^T B^T = (BA)^T = BA,$$

which proves the first implication.

To prove the second implication, let $s \in S_{\Delta,\Gamma}$, where Δ and Γ are distinct fibers of $\mathcal{X}$. Then obviously,

$$I_\Delta A_s = A_{1_\Delta \cdot s} = A_s \quad \text{and} \quad A_s I_\Delta = A_{s \cdot 1_\Delta} = 0,$$

where $I_\Delta \in \mathrm{Mat}_\Omega$ is the adjacency matrix of 1_Δ. It follows that I_Δ and A_s do not commute. Thus if $\mathcal{X}$ is commutative, then it is homogeneous. □

Our second example is presented by the *Wielandt principle*. It enables us to construct relations of a coherent configuration from a given matrix of its adjacency algebra. This principle appears as Proposition 22.3 in book [**125**], as a special case. Some applications of the Wielandt principle can be found in Exercises 2.7.24 and 2.7.25.

Theorem 2.3.10. *Let $\mathcal{X}$ be a coherent configuration on Ω. Then for any matrix $A \in \mathrm{Adj}(\mathcal{X})$ and any function $f : \mathbb{C} \to \{0,1\}$, the relation*

$$s_f(A) = \{(\alpha, \beta) \in \Omega^2 : \ f(A_{\alpha,\beta}) = 1\} \tag{2.3.4}$$

belongs to $S(\mathcal{X})^{\cup}$.

Proof. Since $\mathcal{M}(\mathcal{X})$ is a linear basis of the algebra $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$, there exist complex numbers c_s, $s \in S$, such that

$$A = \sum_{s \in S} c_s A_s.$$

All linear combinations of the basis matrices A_s belong to $\mathcal{A}$. Therefore, the algebra $\mathcal{A}$ contains the matrix

$$A^f := \sum_{s \in S} f(c_s) A_s.$$

It is a $\{0,1\}$-matrix by the choice of f. Consequently, $A^f = A_r$ for some relation r on Ω. This relation belongs to $S(\mathcal{X})^{\cup}$ by Proposition 2.3.2. Since for all $\alpha, \beta \in \Omega$,

$$f(A_{\alpha,\beta}) = 1 \quad \Leftrightarrow \quad f(c_{r(\alpha,\beta)}) = 1 \quad \Leftrightarrow \quad (A^f)_{\alpha,\beta} = 1 \quad \Leftrightarrow \quad (A_r)_{\alpha,\beta} = 1,$$

we conclude that $s_f(A) = r$ belongs to $S(\mathcal{X})^{\cup}$. □

Our third example concerns the intersection of coherent configurations on Ω. A naive idea to define this concept via the intersection of the sets of basis relations does not work, because in this way one cannot get a partition of Ω^2. However, it is easy to define the intersection of coherent configurations via the intersection of their adjacency algebras. Let us discuss this approach in detail.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be coherent configurations on Ω. The adjacency algebras of them are coherent by Theorem 2.3.6. It is easily seen that the intersection of these algebras satisfies the conditions (A1), (A2), and (A3) and hence is also a coherent algebra on Ω. By Theorem 2.3.7, it is the adjacency algebra of a certain coherent configuration on Ω. It is denoted by $\mathcal{X}_1 \cap \mathcal{X}_2$ and called the *intersection* of $\mathcal{X}_1$ and $\mathcal{X}_2$. Thus,

$$\mathrm{Adj}(\mathcal{X}_1 \cap \mathcal{X}_2) = \mathrm{Adj}(\mathcal{X}_1) \cap \mathrm{Adj}(\mathcal{X}_2). \tag{2.3.5}$$

The following statement gives an equivalent definition of the intersection $\mathcal{X}_1 \cap \mathcal{X}_2$ in terms of relations of $\mathcal{X}_1$ and $\mathcal{X}_2$.

Theorem 2.3.11. *For any coherent configurations* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *on the same set,*

$$S(\mathcal{X}_1 \cap \mathcal{X}_2)^{\cup} = S(\mathcal{X}_1)^{\cup} \cap S(\mathcal{X}_2)^{\cup}. \tag{2.3.6}$$

In particular, $\mathcal{X}_1 \cap \mathcal{X}_2$ *is the largest common fusion of* $\mathcal{X}_1$ *and* $\mathcal{X}_2$.

Proof. By Proposition 2.3.2, the set $S(\mathcal{X}_1 \cap \mathcal{X}_2)^{\cup}$ is in a one-to-one correspondence with the set of $\{0,1\}$-matrices of the algebra $\mathrm{Adj}(\mathcal{X}_1 \cap \mathcal{X}_2)$. This set is obviously equal to the intersection of the set of $\{0,1\}$-matrices of $\mathrm{Adj}(\mathcal{X}_1)$ and the set of $\{0,1\}$-matrices of $\mathrm{Adj}(\mathcal{X}_2)$. Using Proposition 2.3.2

again, we obtain equality (2.3.6). Now the second statement of the theorem follows from this equality, because

$$\mathcal{X} \le \mathcal{X}_1 \quad \text{and} \quad \mathcal{X} \le \mathcal{X}_2 \qquad \Rightarrow \qquad S(\mathcal{X})^{\cup} \subseteq S(\mathcal{X}_1)^{\cup} \cap S(\mathcal{X}_2)^{\cup}$$

for any coherent configuration $\mathcal{X}$. □

At this point, one can define the lattice structure on the set of coherent configurations on Ω. Namely, the meet and join of $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined to be the intersection $\mathcal{X}_1 \cap \mathcal{X}_2$, and the intersection of all common fissions of $\mathcal{X}_1$ and $\mathcal{X}_2$ (note that the discrete configuration on Ω is one of these fissions), respectively. The minimal and maximal elements of the lattice are, of course, the trivial and discrete configurations. Formula (2.3.5) shows that the mapping $\mathcal{X} \mapsto \mathrm{Adj}(\mathcal{X})$ is, in fact, a lattice isomorphism.

The following statement shows that the intersection is partially preserved under the Galois correspondence between the coherent configurations and permutation groups.

Theorem 2.3.12. *For any groups $K_1, K_2 \le \mathrm{Sym}(\Omega)$,*

$$\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2) = \mathrm{Inv}(\langle K_1, K_2 \rangle), \tag{2.3.7}$$

and for any coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$ on the same set,

$$\mathrm{Aut}(\mathcal{X}_1 \cap \mathcal{X}_2) \ge \langle \mathrm{Aut}(\mathcal{X}_1), \mathrm{Aut}(\mathcal{X}_2) \rangle. \tag{2.3.8}$$

Proof. To prove the second statement, let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two coherent configuration on the same set. Then $\mathcal{X}_1 \cap \mathcal{X}_2 \le \mathcal{X}_1$ and $\mathcal{X}_1 \cap \mathcal{X}_2 \le \mathcal{X}_2$ (Theorem 2.3.11). In view of the Galois correspondence (formula (2.2.5)), this implies that

$$\mathrm{Aut}(\mathcal{X}_1 \cap \mathcal{X}_2) \ge \mathrm{Aut}(\mathcal{X}_1) \quad \text{and} \quad \mathrm{Aut}(\mathcal{X}_1 \cap \mathcal{X}_2) \ge \mathrm{Aut}(\mathcal{X}_2)$$

which proves inclusion (2.3.8). Substituting the coherent configurations associated with groups $K_1, K_2 \le \mathrm{Sym}(\Omega)$ to this inclusion, we obtain

$$\mathrm{Aut}(\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2)) \ge \langle \mathrm{Aut}(\mathrm{Inv}(K_1)), \mathrm{Aut}(\mathrm{Inv}(K_2)) \rangle \ge \langle K_1, K_2 \rangle,$$

where the latter inclusion follows from the right-hand side of formula (2.2.7). Again by the Galois correspondence (formula (2.2.6)), this yields

$$\mathrm{Inv}(\mathrm{Aut}(\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2))) \le \mathrm{Inv}(\langle K_1, K_2 \rangle).$$

However, by the left-hand side of formula (2.2.7), we have

$$\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2) \le \mathrm{Inv}(\mathrm{Aut}(\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2))).$$

Thus from the last two formulas, we obtain the inclusion

$$\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2) \le \mathrm{Inv}(\langle K_1, K_2 \rangle).$$

Conversely, both K_1 and K_2 are contained in $\langle K_1, K_2\rangle$ and hence

$$\mathrm{Inv}(K_1) \geq \mathrm{Inv}\langle K_1, K_2\rangle \quad \text{and} \quad \mathrm{Inv}(K_2) \geq \mathrm{Inv}\langle K_1, K_2\rangle.$$

Thus the required inclusion follows from the Galois correspondence and Theorem 2.3.11. □

Note that inclusion (2.3.8) can be strict. Indeed, let p be a prime, and let $K \leq \mathrm{Sym}(p)$ be a 2-transitive Frobenius group of order $p(p-1)$. Then

$$K = \langle K_1, K_2\rangle,$$

where K_1 and K_2 are cyclic subgroups of K of the orders p and $p-1$, respectively. By formula (2.3.7),

$$\mathrm{Inv}(K_1) \cap \mathrm{Inv}(K_2) = \mathrm{Inv}(\langle K_1, K_2\rangle) = \mathrm{Inv}(K) = \mathcal{T}_p, \tag{2.3.9}$$

where the latter equality holds by the 2-transitivity of K.

On the other hand, the group K_1 is regular and the group K_2 acts regularly on $p-1$ points. Consequently, they are 2-closed by Theorem 2.2.21. Thus,

$$K_1 = \mathrm{Aut}(\mathrm{Inv}(K_1)) \quad \text{and} \quad K_2 = \mathrm{Aut}(\mathrm{Inv}(K_2)). \tag{2.3.10}$$

Now set

$$\mathcal{X}_1 = \mathrm{Inv}(K_1) \quad \text{and} \quad \mathcal{X}_2 = \mathrm{Inv}(K_2).$$

Then equalities (2.3.9) and (2.3.10) imply respectively that

$$|\,\mathrm{Aut}(\mathcal{X}_1 \cap \mathcal{X}_2)| = |\,\mathrm{Aut}(\mathcal{T}_p)| = p!$$

and

$$|\langle \mathrm{Aut}(\mathcal{X}_1), \mathrm{Aut}(\mathcal{X}_2)\rangle = |K| = p(p-1).$$

Thus, $\langle \mathrm{Aut}(\mathcal{X}_1), \mathrm{Aut}(\mathcal{X}_2)\rangle$ is a proper subgroup of $\mathrm{Aut}(\mathcal{X}_1 \cap \mathcal{X}_2)$ for $p \geq 5$.

Corollary 2.3.13. *The intersection of schurian coherent configurations is schurian.*

Proof. Follows from formula (2.3.7). □

2.3.2 Algebraic isomorphisms

One more type of isomorphisms of coherent configurations comes from matrix algebra isomorphisms of the corresponding adjacency algebras. More precisely, let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be coherent configurations.

Definition 2.3.14. *A bijection*

$$\varphi : S \to S', \ s \mapsto s' \tag{2.3.11}$$

is called an algebraic isomorphism from $\mathcal{X}$ *to* $\mathcal{X}'$ *if*

$$c_{rs}^t = c_{r's'}^{t'} \quad \textit{for all } r, s, t \in S. \tag{2.3.12}$$

The set of all such algebraic isomorphism φ is denoted by $\operatorname{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. The coherent configurations $\mathcal{X}$ and $\mathcal{X}'$ are said to be *algebraically isomorphic* if this set is not empty. Clearly,

$$\operatorname{Aut}_{\mathrm{alg}}(\mathcal{X}) := \operatorname{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X})$$

is a permutation group on S; its elements are called *algebraic automorphisms* of $\mathcal{X}$. Obviously, the group $\operatorname{Aut}_{\mathrm{alg}}(\mathcal{T}_\Omega)$ is trivial.

Example 2.3.15. *Let* $\mathcal{X}$ *be a commutative scheme. Then from* (2.1.3)*, it follows that the bijection* $\varphi : s \mapsto s^*$ *is an algebraic automorphism of* $\mathcal{X}$*.*

Example 2.3.16. *Let* Δ *and* Γ *be classes of an indecomposable partial parabolic of a coherent configuration* $\mathcal{X} = (\Omega, S)$*. Then by statement* (1) *of Theorem* 2.1.22*, the mapping*

$$\varphi_{\Delta,\Gamma} : S_\Delta \to S_\Gamma, \ s_\Delta \mapsto s_\Gamma$$

is a bijection. Equalities (2.3.12) *follow from formula* (2.1.16)*. Thus,*

$$\varphi_{\Delta,\Gamma} \in \operatorname{Iso}_{\mathrm{alg}}(\mathcal{X}_\Delta, \mathcal{X}_\Gamma).$$

An algebraic isomorphism $\varphi \in \operatorname{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ is said to be induced by a bijection $f : \Omega \to \Omega'$ if

$$\varphi(s) = s^f \quad \text{for all } \ s \in S.$$

In this case, f takes S to S' and hence is an isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

Conversely, each isomorphism $f \in \operatorname{Iso}(\mathcal{X}, \mathcal{X}')$ obviously preserves the intersection numbers and hence induces an algebraic isomorphism

$$\varphi_f : S \to S', \ s \mapsto s^f.$$

It should be emphasized that the set

$$\operatorname{Iso}(\mathcal{X}, \mathcal{X}', \varphi) := \{f \in \operatorname{Iso}(\mathcal{X}, \mathcal{X}') : \ \varphi_f = \varphi\}$$

may be empty. This is exactly the case when the algebraic isomorphism φ is induced by no bijection. Such a situation is quite common (see Section 2.5) and leads to the separability problem discussed below.

In general, the mapping

$$\mathrm{Iso}(\mathcal{X}) \to \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}),\ f \mapsto \varphi_f, \tag{2.3.13}$$

is a group homomorphism which is an epimorphism if and only if the set $\mathrm{Iso}(\mathcal{X}, \mathcal{X}, \varphi)$ is not empty for all $\varphi \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$. The kernel of this homomorphism coincide with the automorphism group of $\mathcal{X}$, because

$$\mathrm{Iso}(\mathcal{X}, \mathcal{X}, \mathrm{id}_S) = \mathrm{Aut}(\mathcal{X}).$$

To see how the algebraic isomorphisms relate with isomorphisms of adjacency algebras, let $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ and $\mathcal{A}' = \mathrm{Adj}(\mathcal{X}')$. Every bijection (2.3.11) induces a bijection

$$\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X}'),\ A_s \mapsto A_{s'}$$

between the standard basises of $\mathcal{A}$ an $\mathcal{A}'$. This bijection is extended to a linear space isomorphism between these algebras; it is convenient to denote this isomorphism again by φ. Note that it obviously respects the Hadamard multiplication.

If the initial bijection is an algebraic isomorphism, then φ preserves the structure constants with respect to the standard basis (see formula (2.3.12)). Thus,

$$\varphi : \mathcal{A} \to \mathcal{A}',\ \sum_{s \in S} a_s A_s \mapsto \sum_{s' \in S'} a_s A_{s'} \tag{2.3.14}$$

is an algebra isomorphism with respect to the matrix multiplication. This proves the necessity in the following statement.

Proposition 2.3.17. *Let $\mathcal{X}$ and $\mathcal{X}'$ be coherent configurations. Then a bijection $\varphi : S(\mathcal{X}) \to S(\mathcal{X}')$ is an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ if and only if*

$$\varphi(AB) = \varphi(A)\varphi(B) \quad \text{and} \quad \varphi(A \circ B) = \varphi(A) \circ \varphi(B) \tag{2.3.15}$$

for all $A, B \in \mathrm{Adj}(\mathcal{X})$.

Proof. By the above remark, it suffices to verify the sufficiency only. Assume that equalities (2.3.15) hold. The right-hand side formula implies that φ takes the primitive idempotents of the algebra $\mathrm{Adj}(\mathcal{X})$ with respect to the Hadamard multiplication, to those of the algebra $\mathrm{Adj}(\mathcal{X}')$. Therefore,

$$\mathcal{M}(\mathcal{X})^{\varphi} = \mathcal{M}(\mathcal{X}').$$

Thus the mapping $S(\mathcal{X}) \to S(\mathcal{X}')$, $s \mapsto s'$ such that $\varphi(A_s) = A_{s'}$, is a bijection. In view of formula (2.3.2), the left-hand side formula in (2.3.15) implies that this bijection satisfies condition (2.3.12). Thus, φ is an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$. □

Let s be a relation of the coherent configuration $\mathcal{X}$, i.e., $s = s_1 \cup \cdots \cup s_k$ for some $k \geq 1$ and basis relations s_i, $i = 1, \ldots, k$. Then

$$\varphi(s) := \varphi(s_1) \cup \cdots \cup \varphi(s_k) \tag{2.3.16}$$

is a relation of the coherent configuration $\mathcal{X}'$. It is easily seen that the mapping $s \mapsto \varphi(s)$ is a bijection from $S^{\cup}$ onto $(S')^{\cup}$. The main properties of this bijection are presented in several statements below; some other properties are given in Exercises 2.7.29–2.7.32.

Proposition 2.3.18. *For any* $r, s \in S^{\cup}$,

1. $r \subseteq s$ *implies* $\varphi(r) \subseteq \varphi(s)$;
2. $\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s)$;
3. s *is reflexive if and only if so is* $\varphi(s)$;
4. $\varphi(s^*) = \varphi(s)^*$.

Proof. Obviously, $r \subseteq s$ if and only if $A_r \circ A_s = A_r$. By the right-hand side of formula (2.3.15), this implies statement (1). In the rest of the proof, we may assume that $r, s \in S$. Now statement (2) immediately follows from the fact that $r \cdot s$ is the union of all $t \in S$ for which $c_{rs}^t \neq 0$. Next,

$$s \subseteq 1_\Omega \quad \Leftrightarrow \quad A_s A_s = A_s \text{ and } A_s \circ A_s = A_s.$$

Thus statement (3) is a consequence of Proposition 2.3.17. Finally, statement (4) follows from statement (3) and statement (1) of Exercise 2.7.6. □

Remark 2.3.19. *Proposition* 2.3.17 *together with statement* (4) *of Proposition* 2.3.18 *show that the algebraic isomorphisms are nothing else than the coherent algebra isomorphisms.*

Corollary 2.3.20. $n_s = n_{\varphi(s)}$ *for all* $s \in S$.

Let us define the image of a fusion $\mathcal{Y}$ of the coherent configuration $\mathcal{X}$ with respect to the algebraic isomorphism φ by

$$\mathcal{Y}^\varphi = (\Omega', \varphi(T)),$$

where $T = S(\mathcal{Y})$. By statement (1) of Exercise 2.7.29, the set $\varphi(T)$ forms a partition of Ω', and statements (3) and (4) of Proposition 2.3.18 show that $\mathcal{Y}^\varphi$ is a rainbow.

For any basis relations r, s, and t of $\mathcal{Y}$, and any points $\alpha', \beta' \in \Omega'$, the number

$$|\alpha' r' \cap \beta' s'| = \sum_{u \in S, u \subseteq r} \sum_{v \in S, v \subseteq s} c_{uv^*}^w$$

depends neither on the relation $w \subseteq t$, nor on the pair $(\alpha', \beta') \in t'$. This proves the following statement.

Corollary 2.3.21. *In the above notation,* $\mathcal{Y}^\varphi$ *is a coherent configuration and the restriction of* φ *to* $S(\mathcal{Y})$ *is an algebraic isomorphism from* $\mathcal{Y}$ *to* $\mathcal{Y}^\varphi$.

Although the algebraic isomorphism φ is not defined on the points of $\mathcal{X}$, one can extend φ to homogeneity sets of $\mathcal{X}$. Namely, in view of statement (3) of Proposition 2.3.18, for any homogeneity set Δ of $\mathcal{X}$ the relation $\varphi(1_\Delta)$ is reflexive. It follows that it equals $1_{\Delta'}$ for some homogeneity set Δ' of $\mathcal{X}'$.

Denoting the set Δ' by Δ^φ, we have

$$\varphi(1_\Delta) = 1_{\Delta^\varphi}, \qquad \Delta \in F(\mathcal{X})^\cup.$$

Since φ induces a bijection between the reflexive relations of $\mathcal{X}$ and $\mathcal{X}'$, the mapping $\Delta \mapsto \Delta^\varphi$ is a bijection from $F^\cup$ onto $(F')^\cup$ that takes $F = F(\mathcal{X})$ to $F' = F(\mathcal{X}')$.

Proposition 2.3.22. *For all* $\Delta, \Gamma \in F^\cup$,

(1) $\varphi(S_{\Delta,\Gamma}) = S'_{\Delta^\varphi,\Gamma^\varphi}$;
(2) $|\Delta| = |\Delta^\varphi|$.

Proof. By statement (2) of Proposition 2.3.18, for all $s \in S$,

$$\varphi(1_\Delta \cdot s \cdot 1_\Gamma) = 1_{\Delta^\varphi} \cdot \varphi(s) \cdot 1_{\Gamma^\varphi}.$$

It follows that

$$\varphi(S_{\Delta,\Gamma}) \subseteq S'_{\Delta^\varphi,\Gamma^\varphi}.$$

The reverse inclusion is obtained in the same way with φ replaced by φ^{-1}. This proves statement (1).

Next, set $\Delta' = \Delta^\varphi$. From formula (2.1.13) applied for the coherent configurations $\mathcal{X}_\Delta$ and $\mathcal{X}'_{\Delta'}$, it follows that

$$|\Delta| = \sum_{s \in S_\Delta} n_s \quad \text{and} \quad |\Delta'| = \sum_{s' \in S'_{\Delta'}} n_{s'}.$$

Since $\varphi(S_\Delta) = S'_{\Delta'}$ by statement (1), and $n_s = n_{s'}$ by Corollary 2.3.20, we conclude that $|\Delta| = |\Delta'|$. □

Corollary 2.3.23. *For any* $s \in S^\cup$,

(1) $|s| = |\varphi(s)|$;
(2) $\Omega_\pm(s)^\varphi = \Omega_\pm(\varphi(s))$; *in particular,* $\Omega(s)^\varphi = \Omega(\varphi(s))$.

Proof. Let $\Delta \in F^\cup$ be such that $s \in S_{\Delta,\Omega}$. Then by Proposition 2.3.22, we have

$$s' \in S'_{\Delta',\Omega'} \quad \text{and} \quad |\Delta| = |\Delta'|,$$

where $s' = \varphi(s)$ and $\Delta' = \Delta^\varphi$. By formula (2.1.5) and Corollary 2.3.20, this implies that

$$|s| = |\Delta| \cdot n_s = |\Delta'| \cdot n_{s'} = |s'|.$$

This proves statement (1). Statement (2) follows from statement (1) of Proposition 2.3.22. □

Corollary 2.3.24. *Algebraically isomorphic coherent configurations have the same degree, rank, and the number of fibers.*

Let Δ be a homogeneity set of a coherent configuration $\mathcal{X}$. From statement (1) of Proposition 2.3.22, it follows that the algebraic isomorphism φ induces a bijection

$$\varphi_\Delta : S_\Delta \to S'_{\Delta'},\ s_\Delta \mapsto \varphi(s)_{\Delta'},$$

which is obviously an algebraic isomorphism from $\mathcal{X}_\Delta$ to $\mathcal{X}'_{\Delta'}$; it is called the *restriction* of φ to Δ. Thus the homogeneous components of $\mathcal{X}$ are in one-to-one correspondence with those of $\mathcal{X}'$, and the algebraic isomorphism φ induces by restriction the algebraic isomorphisms between the corresponding homogeneous components of $\mathcal{X}$ and $\mathcal{X}'$.

Let e be a relation of the coherent configuration $\mathcal{X}$. From Corollary 2.1.19 and statement (2) of Exercise 2.7.29, it follows that e is a partial parabolic of $\mathcal{X}$ if and only if $\varphi(e)$ is a partial parabolic of $\mathcal{X}'$. This proves the first part of the following proposition.

Proposition 2.3.25. *The algebraic isomorphism φ induces a bijection from $E = E(\mathcal{X})$ onto $E' = E(\mathcal{X}')$. Moreover, for any $e \in E$,*

(1) *φ induces a bijection between the indecomposable components of e and $\varphi(e)$;*
(2) *e is indecomposable if and only if so is $\varphi(e)$;*
(3) $|\Omega/e| = |\Omega'/\varphi(e)|$.

Proof. The induced bijection $\varphi : E \to E'$ takes any representation of e as a disjoint union of nonempty partial parabolics contained in E to the corresponding representation of e' with the same number of the summands. This proves statement (1) and hence statement (2).

To prove statement (3), we may assume that e and e' are indecomposable. By statement (2) of Corollary 2.3.23, we have $\Omega(e)^\varphi = \Omega(e')$ and hence

$$|\Omega(e)| = |\Omega(e')| \tag{2.3.17}$$

by statement (2) of Proposition 2.3.22. Since all the classes of e have the same cardinality, and the same is true for e' (statement (2) of Theorem 2.1.22), it suffices to verify that there exist $\Delta \in \Omega/e$ and $\Delta' \in \Omega'/e'$ such that

$$|\Delta| = |\Delta'|.$$

But this is an immediate consequence of the fact that in accordance with Exercise 2.7.31 and Corollary 2.3.24, the coherent configurations $\mathcal{X}_\Delta$ and $\mathcal{X}'_{\Delta'}$ are algebraically isomorphic. □

2.3.3 Algebraic fusions

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration, and let Φ be a group of algebraic automorphisms of $\mathcal{X}$. Given $s \in S$, set

$$s^\Phi = \bigcup_{\varphi \in \Phi} \varphi(s).$$

Clearly, the relations r^Φ and s^Φ have nonempty intersection if and only if r and s belong to the same orbit of Φ acting on S, and in the latter case $r^\Phi = s^\Phi$. Therefore the set

$$S^\Phi = \{s^\Phi : \ s \in S\}$$

forms a partition of Ω^2. Furthermore, statements (3) and (4) of Proposition 2.3.18 imply that the pair

$$\mathcal{X}^\Phi = (\Omega, S^\Phi)$$

is a rainbow. Note that $\mathcal{X}^\Phi \le \mathcal{X}$ with equality if and only if the group Φ is trivial.

Lemma 2.3.26. *The rainbow $\mathcal{X}^\Phi$ is a coherent configuration.*

Proof. By Theorem 2.3.7, it suffices to verify that the linear space

$$\mathcal{A}^\Phi = \mathrm{Adj}(\mathcal{X}^\Phi)$$

is a coherent algebra with the standard basis $\{A_{s^\Phi} : \ s \in S\}$. Since $\mathcal{A}^\Phi$ is closed under the Hermitian conjugation and the Hadamard multiplication, we only need to verify that it is closed also with respect to matrix multiplication.

By Proposition 2.3.17, the group Φ acts as an automorphism group of the algebra $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$, where the action is as in formula (2.3.14). In particular, for any $s \in S$,

$$(A_s)^\Phi := \sum_{\varphi \in \Phi} A_{\varphi(s)} = |\Phi_s| A_{s^\Phi},$$

where Φ_s is the stabilizer of s in Φ. It follows that for any $\varphi \in \Phi$,

$$\Big(|\Phi_s| A_{s^\Phi}\Big)^\varphi = \Big(\sum_{\psi \in \Phi} A_{\psi(s)}\Big)^\varphi = \sum_{\psi \in \Phi} A_{\psi\varphi(s)} = \sum_{\psi' \in \Phi} A_{\psi'(s)} = |\Phi_s| A_{s^\Phi},$$

where $\psi' = \psi\varphi$. Consequently, the linear space $\mathcal{A}^\Phi$ consists of all matrices of $\mathcal{A}$ that are fixed by any element of the group Φ,

$$\mathcal{A}^\Phi = \{A \in \mathcal{A} : \ A^\Phi = A\}.$$

Since the right-hand side is obviously closed under the matrix multiplication, we are done. $\square$

Definition 2.3.27. *The coherent configuration $\mathcal{X}^{\Phi}$ is called the algebraic fusion of $\mathcal{X}$ with respect to the group Φ.*

In a special case when we are given a group $K \leq \mathrm{Iso}(\mathcal{X})$ such that

$$\Phi = \Phi_K = \{\varphi_f : \ f \in K\},$$

the coherent configuration $\mathcal{X}^K := \mathcal{X}^{\Phi_K}$ is called the algebraic fusion of $\mathcal{X}$ with respect to K. In this case, K normalizes $\mathrm{Aut}(\mathcal{X})$ (Exercise 2.7.18 for $\mathcal{X} = \mathrm{Inv}(K)$). Therefore, each 2-orbit of the group $\mathrm{Aut}(\mathcal{X})K$ equals the union of the relations

$$(\alpha, \beta)^{\mathrm{Aut}(\mathcal{X})k} = (\alpha, \beta)^{k\,\mathrm{Aut}(\mathcal{X})}$$

where $\alpha, \beta \in \Omega$ and $k \in K$. Since this union is a 2-orbit of $\mathrm{Aut}(\mathcal{X})K$, we arrive at the following statement.

Proposition 2.3.28. *Let $\mathcal{X}$ be schurian and $K \leq \mathrm{Iso}(\mathcal{X})$. Then*

$$\mathcal{X}^K = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X})K, \Omega). \tag{2.3.18}$$

In particular, the algebraic fusion $\mathcal{X}^K$ is schurian.

Now let $\mathcal{X} = \mathcal{D}_\Omega$. Then the group $\mathrm{Aut}(\mathcal{X})$ is trivial and formula (2.3.18) implies that

$$(\mathcal{D}_\Omega)^K = \mathrm{Inv}(K, \Omega). \tag{2.3.19}$$

This shows that every schurian coherent configuration on Ω is the algebraic fusion of the discrete configuration $\mathcal{D}_\Omega$.

2.3.4 Separable coherent configurations

An old problem in permutation group theory is to characterize a given permutation group up to isomorphism by its combinatorial invariants, e.g., by subdegrees [**61**, **62**].[4] Similar problems arise in combinatorics where one would like to characterize up to isomorphism a combinatorial structure, for instance a design, by its parameters, see, e.g., [**13**]. To deal with problems of this type, it is convenient to give the following definition.

Definition 2.3.29. *A coherent configuration $\mathcal{X}$ is said to be separable with respect to a class $\mathcal{K}$ of coherent configurations if for any $\mathcal{X}' \in \mathcal{K}$, every algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ is induced by an isomorphism, i.e.,*

$$\mathrm{Iso}(\mathcal{X},\mathcal{X}',\varphi) \ne \varnothing \quad \textit{for all } \mathcal{X}' \in \mathcal{K} \textit{ and } \varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{X}').$$

Certainly, $\mathcal{X}$ is separable with respect to any class not containing a coherent configuration algebraically isomorphic to $\mathcal{X}$. If the class $\mathcal{K}$ consists only of $\mathcal{X}$, then the separability of $\mathcal{X}$ with respect to $\mathcal{K}$ is equivalent to the surjectivity of the homomorphism (2.3.13).

Separability problem. Given any two classes $\mathcal{K}_1$ and $\mathcal{K}_2$ of coherent configurations identify all coherent configurations in $\mathcal{K}_1$ that are separable with respect to $\mathcal{K}_2$.

We are often interested in the case where $\mathcal{K}_1$ consists of a single coherent configuration and $\mathcal{K}_2$ is the class of all coherent configurations.

Definition 2.3.30. *A coherent configuration separable with respect to the class of all coherent configurations is said to be separable.*

Thus a separable coherent configuration is determined up to isomorphism by the intersection numbers (more exactly, by the tensor of them). In this sense, aforementioned results [**61**, **62**] show that the coherent configurations of certain permutation groups are separable with respect to the class of schurian coherent configurations.

Example 2.3.31. *The trivial and discrete coherent configurations are separable, because they are determined up to isomorphism by the degree and rank, which are preserved by algebraic isomorphisms* (*Corollary* 2.3.24).

In accordance with computations made by A. Hanaki (2017), every scheme $\mathcal{X}$ of degree up to 14 is separable and there are several non-separable schemes of degree 15 (the smallest degree for which there exist algebraically isomorphic but not isomorphic schemes is equal to 16). Among them, one can find a unique antisymmetric scheme of degree 15 and rank 3, which was mentioned as the non-schurian scheme of the smallest degree.

[4] The subdegrees of a transitive group K are the cardinalities of the orbits of a point stabilizer of K.

Example 2.3.32. *Let $\mathcal{X}$ be the coherent configuration of degree* 14 *that was used in Subsection* 2.2.3 *to construct the smallest non-schurian coherent configuration. Denote by φ the transposition in* $\mathrm{Sym}(S)$ *interchanging two basis relations of valency* 2 *in* S_{Ω_1,Ω_2}. *A computer calculation shows that*

$$\varphi \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}).$$

This implies that the algebraic fusion $\mathcal{X}^{\langle\varphi\rangle}$ is the smallest non-schurian coherent configuration. It follows that φ is not induced by a bijection (*Proposition* 2.3.28). *Thus the coherent configuration $\mathcal{X}$ is not separable.*

At present, the coherent configuration in Example 2.3.32 is the only known non-separable coherent configuration of degree at most 14. It would also be interesting to find infinitely many non-schurian coherent configurations that are separable.

The question of whether a given coherent configuration is separable seems to be very difficult. For example, the scheme of conjugacy classes of a group (Example 2.4.3) is separable only if this group is determined up to isomorphism by the character table (Exercise 3.7.58).

In the following sections, we will meet many examples of separable and non-separable coherent configurations and find some sufficient conditions for a coherent configuration to be separable. At this point, we solve the separability problem in the case when $\mathcal{K}_1$ and $\mathcal{K}_2$ are the classes of regular and all coherent configurations, respectively (for the semiregular coherent configurations, see Exercise 2.7.35).

Theorem 2.3.33. *Every regular scheme is separable.*

Proof. Let $\mathcal{X} = (\Omega, S)$ be a regular scheme, $\mathcal{X}' = (\Omega', S')$ a coherent configuration, and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. By Corollary 2.3.24,

$$|\Omega| = |\Omega'| \quad \text{and} \quad |S| = |S'|.$$

The regularity of $\mathcal{X}$ implies that $|\Omega| = |S|$, which yields

$$|\Omega'| = |\Omega| = |S| = |S'|.$$

Consequently, $\mathcal{X}'$ is a regular scheme (Theorem 2.1.29). By Theorem 2.2.11, we may assume that $\Omega = G$, $\Omega' = G'$, and

$$S = G_{left} \quad \text{and} \quad S' = G'_{left}, \tag{2.3.20}$$

where G and G' are groups of the same order. Hence, $\varphi : G_{left} \to G'_{left}$ is a group isomorphism (statement (2) of Proposition 2.3.18). It induces a group isomorphism $f : G \to G'$ defined by the equality

$$\varphi(g_{left}) = f(g)_{left}, \qquad g \in G.$$

This isomorphism obviously induces φ, and by virtue of (2.3.20) takes S to S'. Thus, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$. Since φ was arbitrary, the scheme $\mathcal{X}$ is separable. □

Corollary 2.3.34. *Let* G *and* G' *be groups. Then the schemes* $\mathrm{Inv}(G_{right})$ *and* $\mathrm{Inv}(G'_{right})$ *are algebraically isomorphic if and only if the groups* G *and* G' *are isomorphic.*

2.4 Cayley schemes and Schur rings

Two opposite poles in the world of coherent configurations are represented by the coherent configurations with trivial automorphism group and schurian coherent configurations. An intermediate position is occupied by the coherent configurations admitting a regular automorphism group. From combinatorial and algebraic points of view they can be considered as the Cayley schemes (also called the translation schemes in [**17**]) and the Schur rings (also called S-rings in [**125**]), respectively. This section provides an introduction in the theory of these objects.

Definition 2.4.1. *Let G be a group. A coherent configuration $\mathcal{X} = (\Omega, S)$ is called a Cayley scheme over G if*

$$\Omega = G \quad \textit{and} \quad \mathrm{Aut}(\mathcal{X}) \geq G_{right}. \tag{2.4.1}$$

Conditions (2.4.1) imply that the group $\mathrm{Aut}(\mathcal{X})$ is transitive. Therefore the coherent configuration $\mathcal{X}$ is homogeneous. Each relation $s \in S$ is the arc set of a *Cayley graph* over G,

$$\mathrm{Cay}(G, X) = (G, \{(x, y) \in G^2 : \ yx^{-1} \in X\}),$$

where X is the neighborhood of the identity of G in the relation s. Thus a Cayley scheme over G can be considered as a special partition of the Cartesian product G^2 into Cayley graphs over G.

In view of the Galois correspondence, the second inclusion in (2.4.1) implies that the Cayley scheme $\mathcal{X}$ has a regular fission which is the scheme associated with the group G_{right},

$$\mathrm{Inv}(G_{right}, G) \geq \mathcal{X}. \tag{2.4.2}$$

This fission is the largest element in the inherited partial order on the set of all Cayley schemes over G. The smallest one is obviously the trivial scheme $\mathcal{T}_G$.

It follows from definition that the Galois correspondence in Theorem 2.2.8 induces a Galois correspondence between the Cayley schemes over G and the permutation groups on G that contain G_{right} as a subgroup.

Example 2.4.2. *Any cyclotomic scheme over a field $\mathbb{F}$ is a Cayley scheme over the additive group $\mathbb{F}^+$ of this field.*

This example is easily generalized to the schemes $\mathrm{Inv}(K, G)$, where G is the additive group of an n-dimensional linear space over a field $\mathbb{F}$ and $K \leq \mathrm{AGL}(n, \mathbb{F})$ contains G as a subgroup.

Example 2.4.3. *For any group G, the coherent configuration $\mathcal{X} = \mathrm{Inv}(K, G)$ with $K = \langle G_{right}, G_{left}\rangle$ is a Cayley scheme over G.*

In the above examples, the stabilizer of the identity of G in K is a subgroup of $\mathrm{Aut}(G)$. This situation can easily be generalized as follows.

Definition 2.4.4. *A cyclotomic scheme over a group G associated with a group $M \leq \mathrm{Aut}(G)$ is defined to be the scheme*

$$\mathrm{Cyc}(M, G) := \mathrm{Inv}(G_{right}M, G). \tag{2.4.3}$$

Note that Example 2.4.3 is obtained as a special case with $M = \mathrm{Inn}(G)$ (Exercise 1.4.13).

The relations of a Cayley scheme $\mathcal{X}$ over a group G admit a description in terms of this group. Namely, let $s \in S^{\cup}$. Inclusion (2.4.2) implies that s is the union of the relations g_{left} for some $g \in G$ (see statement (1) of Theorem 2.2.11 for $f = \mathrm{id}_G$). Since the element g is a unique neighbor of the identity of G in g_{left}, the set X_s of all of these g coincides with the neighborhood of the identity in s. Thus,

$$s = \bigcup_{g \in X_s} g_{left} \quad \text{and} \quad A_s = \sum_{g \in X_s} P_{g_{left}}. \tag{2.4.4}$$

In particular, if the group G is abelian, then the matrices $P_{g_{left}}$ commute with each other. But then so do the matrices A_s, which implies that the algebra $\mathrm{Adj}(\mathcal{X})$ is commutative.

Proposition 2.4.5. *Every Cayley scheme over an abelian group is commutative.*

Let $\mathcal{X}$ and $\mathcal{X}'$ be Cayley schemes over groups G and G', respectively.

Definition 2.4.6. *A group isomorphism $f : G \to G'$ is called a Cayley isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ if $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$; in this case, $\mathcal{X}$ and $\mathcal{X}'$ are said to be Cayley isomorphic. The set of all Cayley isomorphisms from $\mathcal{X}$ to $\mathcal{X}'$ is denoted by $\mathrm{Iso}_{cay}(\mathcal{X}, \mathcal{X}')$.*

Certainly,

$$\mathrm{Iso}_{cay}(\mathcal{X}, \mathcal{X}') \subseteq \mathrm{Iso}(\mathcal{X}, \mathcal{X}').$$

The reverse inclusion is not true because an isomorphism do not necessarily sends the identity element of G to the identity element of G'. The group

$$\mathrm{Iso}_{cay}(\mathcal{X}) := \mathrm{Iso}_{cay}(\mathcal{X}, \mathcal{X})$$

contains a normal subgroup

$$\mathrm{Aut}_{cay}(\mathcal{X}) := \mathrm{Aut}(\mathcal{X}) \cap \mathrm{Iso}_{cay}(\mathcal{X}).$$

Any element of this subgroup is called the *Cayley automorphism* of the Cayley scheme $\mathcal{X}$. One can see that

$$\mathrm{Aut}_{cay}(\mathcal{X}) \leq \mathrm{Aut}(G).$$

Example 2.4.7. *Let $\mathcal{X} = \mathrm{Inv}(G_{right})$ and $\mathcal{X}' = \mathrm{Inv}(G'_{right})$. Since any isomorphism between $\mathcal{X}$ and $\mathcal{X}'$ preserves the composition of basis relations* (*statement* (2) *of Proposition* 2.3.18), *Theorem* 2.2.11 *implies that*

$$\mathrm{Iso}_{cay}(\mathcal{X}, \mathcal{X}') = \mathrm{Iso}(G, G').$$

For $\mathcal{X} = \mathcal{X}'$, this shows that

$$\mathrm{Iso}_{cay}(\mathcal{X}) = \mathrm{Aut}(G)$$

and the group $\mathrm{Aut}_{cay}(\mathcal{X})$ is trivial.

A homogeneous coherent configuration can be isomorphic to Cayley schemes over non-isomorphic groups: for example, $\mathcal{T}_n$ is isomorphic to the Cayley scheme $\mathcal{T}_G$ over any group G of order n. A *Cayley representation* of a scheme $\mathcal{X}$ with respect to a group G is defined to be an isomorphism from $\mathcal{X}$ to a Cayley scheme over G. Two such representations $f_{\mathcal{Y}} : \mathcal{X} \to \mathcal{Y}$ and $f_{\mathcal{Z}} : \mathcal{X} \to \mathcal{Z}$ are *equivalent* if $s^{f_{\mathcal{Y}}\sigma} = s^{f_{\mathcal{Z}}}$ for all $s \in S(\mathcal{X})$ and some $\sigma \in \mathrm{Aut}(G)$. All Cayley representations of $\mathcal{T}_n$ with respect to a fixed group of order n are obviously equivalent. The following statement goes back to the Babai lemma [**4**, Lemma 3.1].

Theorem 2.4.8. *The classes of equivalent Cayley representations of a scheme $\mathcal{X}$ with respect to a group G are in one-to-one correspondence with the conjugacy classes of regular subgroups of $\mathrm{Aut}(\mathcal{X})$ that are isomorphic to G.*

Proof. Let $\mathcal{X}$ be a scheme on Ω. For every Cayley representation $f_{\mathcal{Y}} \in \mathrm{Iso}(\mathcal{X}, \mathcal{Y})$, we set

$$G_{\mathcal{Y}} = f_{\mathcal{Y}} G_{right} f_{\mathcal{Y}}^{-1}.$$

Obviously, $G_{\mathcal{Y}}$ is a regular subgroup of $\mathrm{Aut}(\mathcal{X})$. If two Cayley representations $f_{\mathcal{Y}}$ and $f_{\mathcal{Z}}$ are equivalent, then there exists an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $s^{f_{\mathcal{Y}}\sigma} = s^{f_{\mathcal{Z}}}$ for all $s \in S(\mathcal{X})$. In particular,

$$(G_{\mathcal{Y}})^{f_{\mathcal{Y}}\sigma f_{\mathcal{Z}}^{-1}} = (G_{right})^{f_{\mathcal{Z}}^{-1}} = G_{\mathcal{Z}}. \tag{2.4.5}$$

and the permutation $f_{\mathcal{Y}}\sigma f_{\mathcal{Z}}^{-1}$ is an automorphism of $\mathcal{X}$. Thus this permutation conjugates the groups $G_{\mathcal{Y}}$ and $G_{\mathcal{Z}}$ inside $\mathrm{Aut}(\mathcal{X})$ (see (2.4.5)). Conversely, any automorphism of $\mathcal{X}$ that conjugates $G_{\mathcal{Y}}$ and $G_{\mathcal{Z}}$ can be written in the form $f_{\mathcal{Y}}\sigma f_{\mathcal{Z}}^{-1}$ for some $\sigma \in \mathrm{Aut}(G)$. □

Any partial parabolic of a Cayley scheme is a parabolic, and all the parabolics are parametrized by some subgroups of the underlying group. Namely, let e be a parabolic of a Cayley scheme $\mathcal{X}$ over a group G. Denote by X the class of e that contains the identity element of G, and set

$$H = \{g \in G : \ Xg = X\}. \tag{2.4.6}$$

Then H is a subgroup of G isomorphic to the setwise stabilizer of X in G_{right}. Furthermore, $H \subseteq X$ as X contains the identity element of G.

On the other hand, any $x \in X$ belongs to the group H: indeed, x_{right} being an automorphism of $\mathcal{X}$, preserves the parabolic e and takes the identity element of G to $x \in X$. Therefore,

$$X = X^{x_{right}} = X\,x.$$

It follows that $x \in H$ and hence $X \subseteq H$.

Thus, X coincides with H, and so is a subgroup of G. Since the group G_{right} is transitive, any class of e is of the form Hg for some $g \in G$. This proves the following useful statement.

Proposition 2.4.9. *The classes of any parabolic of a Cayley scheme over a group G are the right cosets of a certain subgroup of G.*

To formulate the next result known as the Schur theorem on multipliers, we need some preparation. Let G be a group of order n. Given an integer m coprime to n, the mapping σ_m defined in (1.4.10) is a permutation of the elements of G. The number m is called the *multiplier* of a Cayley scheme $\mathcal{X}$ over G if

$$\sigma_m \in \mathrm{Iso}(\mathcal{X}),$$

or equivalently, if $s^{\sigma_m} \in S$ for all $s \in S$, where $S = S(\mathcal{X})$. One can see that the set of all multipliers modulo n, is a subgroup of the multiplicative group of the ring $\mathbb{Z}_n$.

Theorem 2.4.10. *Let $\mathcal{X}$ be a Cayley scheme over an abelian group G of order n. Then any integer m coprime to n is a multiplier of $\mathcal{X}$. In particular,*

$$\sigma_m \in \mathrm{Iso}_{cay}(\mathcal{X}).$$

Proof. Let $s \in S$ and $m \in \mathbb{Z}$. Since $s^{\sigma_{m_1 m_2}} = s^{\sigma_{m_1}\sigma_{m_2}}$ for all integers m_1 and m_2 coprime to n, without loss of generality we may assume that $m = p$ is a prime.

We make use of the well-known fact that the coefficient at $x_1^{a_1} \cdots x_d^{a_d}$ of the polynomial $(x_1 + \cdots + x_d)^p$ in the variables $x_1, \ldots, x_d$ equals the multinomial number

$$\binom{p}{a_1, \ldots, a_d} = \frac{p!}{a_1! \cdots a_d!},$$

where the a_i are nonnegative integers whose sum is p. By the primality of p, this number is divided by p only if one of the a_i is equal to p.

Using the right-hand side of formula 2.4.4, we obtain

$$(2.4.7)\qquad (A_s)^p = (\sum_{g \in X_s} A_g)^p \equiv \sum_{g \in X_s} (A_g)^p = \sum_{g \in X_s} A_{g^p} = A_{s^{\sigma_p}} \pmod p,$$

where X_s is the neighborhood of the identity of G in the relation s and A_g is the permutation matrix corresponding to the left multiplication by g^{-1}.

It follows that

$$s^{\sigma_p} = s_f(A_s^p),$$

where f is the function such that $f(x) = 0$ if x is not integer or p divides x, and $f(x) = 1$ otherwise, and the operator s_f is defined by (2.3.4). By the Wielandt principle (Theorem 2.3.10), this implies that

$$s^{\sigma_p} \in S^{\cup}.$$

We claim that $s^{\sigma_p} \in S$. Indeed, otherwise $\mathcal{X}$ has a basis relation t contained in s^{σ_p}. Denote by p' the inverse of p in $\mathbb{Z}_n^{\times}$. Then $(p', n) = 1$ and $pp' = 1 \pmod{n}$. Consequently,

$$t^{\sigma_{p'}} \subsetneq s^{\sigma_{pp'}} = s.$$

However, by the first part of the proof, $t^{\sigma_{p'}} \in S^{\cup}$. Thus, $s \notin S$, a contradiction. □

One can use Theorem 2.4.10 to get a direct proof that every Cayley scheme over a (cyclic) group of prime order p is isomorphic to a cyclotomic scheme over the field $\mathbb{F}_p$. However, we deduce this fact in Section 4.5 from a more general statement.

In accordance with Corollary 2.2.18, the automorphism groups of the Cayley schemes $\mathcal{X}$ over a group G are exactly the 2-closed permutation groups containing G_{right} as a subgroup. When this subgroup is normal in $\operatorname{Aut}(\mathcal{X})$,

$$G_{right} \trianglelefteq \operatorname{Aut}(\mathcal{X}),$$

we say that $\mathcal{X}$ is a *normal* Cayley scheme over the group G. Obviously, any regular Cayley scheme is normal.

Example 2.4.11. *A nontrivial cyclotomic scheme over a finite field* $\mathbb{F}$ *is a normal Cayley scheme over the additive group of* $\mathbb{F}$ *(Theorem 2.2.4).*

In general, a normal Cayley scheme is neither schurian nor separable. A lot of examples can be found in the class of affine schemes defined later in Subsection 2.5.2. On the other hand, any normal Cayley scheme over a cyclic group is schurian, and separable with respect to the class of all Cayley schemes over a cyclic group (Exercise 4.7.38). The following statement shows, in particular, that any normal schurian Cayley scheme is cyclotomic (note that the converse is not true).

Theorem 2.4.12. *A Cayley scheme* $\mathcal{X}$ *over a group* G *is normal if and only if*

$$\operatorname{Aut}(\mathcal{X}) \leq G_{right} \operatorname{Aut}(G).$$

Proof. The sufficiency is obvious. Conversely, we assume that $\mathcal{X}$ is normal. Let k be an automorphism of $\mathcal{X}$. Since $\operatorname{Aut}(\mathcal{X})$ contains G_{right}, without loss of generality we may assume that k leaves the identity α of the group G fixed.

The normality of $\mathcal{X}$ implies that k induces by conjugation an automorphism of G_{right}. Hence there exists $\sigma \in \mathrm{Aut}(G)$ such that

$$k^{-1} g_{right} k = (g^{\sigma})_{right} \quad \text{for all } \ g \in G. \tag{2.4.8}$$

The images of the point α with respect to the permutations on the left- and right-hand sides are equal to

$$(\alpha^{k^{-1}} g)^k = (\alpha g)^k = g^k \quad \text{and} \quad \alpha g^{\sigma} = g^{\sigma},$$

respectively. By formula (2.4.8), this implies that $g^k = g^{\sigma}$ for all $g \in G$. It follows that $k = \sigma$ belongs to $\mathrm{Aut}(G)$, as required. □

Corollary 2.4.13. *The automorphism groups of the normal Cayley schemes over a group G are exactly the 2-closed subgroups of $G_{right}\,\mathrm{Aut}(G)$.*

Concluding the discussion of Cayley schemes, we note that among them one can find many non-schurian and non-separable, see Example 2.6.15 and Exercise 2.7.61.

The rest of the section is devoted to the Schur rings, which provide a very convenient tool simplifying computations with adjacency algebras. To explain the method of Schur rings introduced in [**115**], let us fix a group G with identity element α.

Definition 2.4.14. *A subring $\mathfrak{A}$ of the group ring $\mathbb{Z}G$ is called a Schur ring (S-ring, for short) over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathfrak{A})$ of G such that*

- (SR1) $\{\alpha\} \in \mathcal{S}$;
- (SR2) $X \in \mathcal{S} \ \Rightarrow \ X^{-1} \in \mathcal{S}$;
- (SR3) $\mathfrak{A} = \mathrm{Span}\,\underline{\mathcal{S}}$, *where* $\underline{\mathcal{S}} = \{\underline{X} : \ X \in \mathcal{S}\}$.

The elements of $\mathcal{S}$ and the number $\mathrm{rk}(\mathfrak{A}) = |\mathcal{S}|$ are called respectively the *basic sets* and the *rank* of the S-ring $\mathfrak{A}$. Any union of basic sets is called an $\mathfrak{A}$-*set*. Each subgroup of G that is at the same time $\mathfrak{A}$-set, is called $\mathfrak{A}$-*group*; the set of all of them is denoted by $\mathcal{E}(\mathfrak{A})$. The condition (SR3) implies that for any $X, Y \in \mathcal{S}$,

$$\underline{X}\,\underline{Y} = \sum_{Z \in \mathcal{S}} c_{X,Y}^{Z} \underline{Z},$$

where $c_{X,Y}^{Z}$ are nonnegative integers.

Example 2.4.15. *Let G be the additive group of the ring $\mathbb{Z}_5$; here $\alpha = 0$ and the multiplication in G is written in the additive form. The sets*

$$X = \{0\}, \quad Y = \{1, -1\}, \quad Z = \{2, -2\}$$

form a partition $\mathcal{S}$ of G. This partition obviously satisfies the condition (SR1)*, and satisfies* (SR2)*, because $X^{-1} = X$, $Y^{-1} = Y$, and $Z^{-1} = Z$.*

Since

$$\underline{Y}^2 = 2\underline{X} + \underline{Z} \quad \text{and} \quad \underline{Z}^2 = 2\underline{X} + \underline{Y}$$

and

$$\underline{Y}\,\underline{Z} = \underline{Z}\,\underline{Y} = \underline{Y} + \underline{Z}$$

the linear space $\mathfrak{A}$ *defined by the condition* (SR3) *is an S-ring over* G*.*

The set of all S-rings over G is partially ordered under inclusion. The smallest and largest elements in this order are the trivial S-ring corresponding to the partition of G into the singleton $\{\alpha\}$ and (if $|G| \geq 2$) its complement in G, and the group ring corresponding to the partition of G into singletons.

Let $\mathfrak{A}$ be an S-ring over a group G; in what follows we extend $\mathfrak{A}$ linearly to a subalgebra of $\mathbb{C}G$. Then the image of $\mathfrak{A}$ with respect to the monomorphism τ defined in (1.4.6) is a subalgebra of Mat_G. By Exercise 1.4.14, this subalgebra satisfies the conditions (A1), (A2), and (A3) and hence is coherent. It follows that $\mathfrak{A}^\tau$ is the adjacency algebra of a coherent configuration $\mathcal{X} = \mathcal{X}(G, \mathfrak{A})$ (Theorem 2.3.7), and the standard basis of it coincides with $\underline{\mathcal{S}}(\mathfrak{A})^\tau$ (see the condition (SR3)). Thus,

$$\mathfrak{A}^\tau = \mathrm{Adj}(\mathcal{X}) \quad \text{and} \quad \underline{\mathcal{S}}(\mathfrak{A})^\tau = \mathcal{M}(\mathcal{X}). \tag{2.4.9}$$

The monomorphism τ preserves inclusion between the subalgebras of the group algebra $\mathbb{C}G$. Since the latter is the largest S-ring over G, it immediately implies that

$$(\mathbb{C}G)^\tau \geq \mathfrak{A}^\tau.$$

In accordance with Example 2.3.4 and the first equality in (2.4.9), this means that inclusion (2.4.2) holds.

Finally in view of the Galois correspondence, we have

$$G_{right} \leq \mathrm{Aut}(\mathcal{X}),$$

i.e., $\mathcal{X}$ is a Cayley scheme over G. The construction of this scheme from the S-ring $\mathfrak{A}$ is obviously reversible. Thus we arrive to the following statement.

Theorem 2.4.16. *For any group* G*, the mapping* $\mathfrak{A} \mapsto \mathcal{X}(G, \mathfrak{A})$ *defined by formulas* (2.4.9) *is a partial order isomorphism between the S-rings over* G *and Cayley schemes over* G*.*

Let us clarify the relationship between an S-ring $\mathfrak{A}$ over a group G and the corresponding Cayley scheme $\mathcal{X}$. For any basic set $X \in \mathcal{S}(\mathfrak{A})$, the matrix $\tau(\underline{X})$ belongs to the set $\mathcal{M}(\mathcal{X})$ (see the second equality in (2.4.9)). In terms of the mapping ρ defined in Exercise 1.4.15, we have

$$A_{\rho(X)} = \tau(\underline{X}^{-1}) \tag{2.4.10}$$

and so $\rho(X) \in S(\mathcal{X})$. This defines a one-to-one correspondence $X \mapsto \rho(X)$ between the basic sets of $\mathfrak{A}$ (respectively, the $\mathfrak{A}$-sets) and the basis relations (respectively, the relations) of $\mathcal{X}$.

The monomorphism τ takes the structure constants with respect to the basis $\underline{\mathcal{S}}(\mathfrak{A})$ to those with respect to the basis $\mathcal{M}(\mathcal{X})$. Therefore,

$$c_{\rho(X),\rho(Y)}^{\rho(Z)} = c_{X^{-1},Y^{-1}}^{Z^{-1}}$$

for all $X, Y, Z \in \mathcal{S}(\mathfrak{A})$.

Finally, by statement (6) of Exercise 1.4.16, a set $X \subseteq G$ is an $\mathfrak{A}$-group if and only if $\rho(X)$ is a parabolic of $\mathcal{X}$.

We summarize what we said above in the following statement.

Theorem 2.4.17. *Let G be a group, $\mathfrak{A}$ an S-ring over G, and let $\mathcal{X} = \mathcal{X}(G, \mathfrak{A})$. Then*

$$\mathcal{S}(\mathfrak{A})^\rho = S(\mathcal{X}), \quad (\mathcal{S}(\mathfrak{A})^\cup)^\rho = S(\mathcal{X})^\cup, \quad \mathcal{E}(\mathfrak{A})^\rho = E(\mathcal{X}).$$

Theorems 2.4.16 and 2.4.17 explains why the Cayley schemes and S-rings are the same up to language. The isomorphisms, algebraic isomorphisms, and Cayley isomorphisms for S-rings are defined via the corresponding Cayley schemes. The same can be said on the concepts of schurity, separability, cyclotomicity, and normality.

The following statement is an immediate consequence of Theorem 2.4.17 and Exercise 1.4.15. It suggests a way to construct the S-ring corresponding to a Cayley scheme.

Corollary 2.4.18. *Let $\mathcal{X}$ be a Cayley scheme and $\mathfrak{A}$ the corresponding S-ring. Then*

$$\mathcal{S}(\mathfrak{A}) = \{\alpha s : \ s \in S(\mathcal{X})\}.$$

2.5 Finite geometries

A huge source of coherent configurations comes from finite geometries: two of the oldest constructions are the schemes of generalized polygons introduced by D. Higman in [**64**] and the rank 3 schemes of partial geometries studied by R. Bose in [**16**]. It does not look amazing, because many nice geometries correspond to invariant relations of permutation groups.

On the other hand, not all geometries have rich automorphism group and the corresponding coherent configurations are often neither schurian nor separable. In this section, we deal with coherent configurations arising from projective and affine planes, and from a special type of designs which are called here coherent. More detail on finite geometries can be found in [**19**].

2.5.1 Coherent configurations of a projective plane

A *projective plane* is a triple consisting of a set of points, a set of lines, and an incidence relation between the points and lines, and satisfying the following axioms:

(P1) any two different points are incident to a unique line;
(P2) any two different lines are incident to a unique point;
(P3) there exist four different points any three of which are incident to no line.

An isomorphism of projective planes is a bijection taking the points (respectively, lines) of one plane to the points (respectively, lines) of the other, and preserving the incidence relation.

An important example of a projective plane is given by the *Galois plane*, the points and lines of which are the lines and planes of a 3-dimensional linear space over a (finite) field and the incidence is defined by inclusion. The smallest Galois plane is obtained for a field $\mathbb{F}_2$ and is known as the *Fano plane*: it has seven points and seven lines, each of the points is incident to exactly three lines and each line contains exactly three points. In Fig. 2.3, the points and lines of the Fano plane are depicted by the seven small black circles and six segments together with the big circle, respectively.

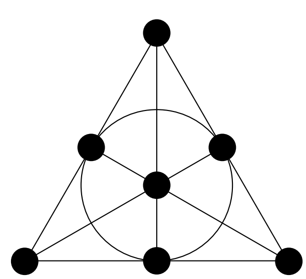


FIGURE 2.3 The Fano plane.

Let $\mathcal{P}$ be a projective plane, and let P and L be the point and line sets of $\mathcal{P}$, respectively. In the finite case (which we are only interested in), there exists an integer $q \geq 2$, called the *order* of $\mathcal{P}$, such that

$$|P| = |L| = q^2 + q + 1,$$

each line is incident to exactly $q + 1$ points, and each point is incident to exactly $q+1$ lines. The order of the Galois plane is equal to the order of the corresponding finite field; in particular, the order of the Fano plane equals 2. Note that the Galois planes are precisely those finite projective planes for which the well-known Desargues theorem is true (see [**20**]). Though there is a lot of non-Galois finite projective planes, it is not known if there exists such a plane, the order of which is not a prime power.

In order to construct a coherent configuration associated with the projective plane $\mathcal{P}$, denote by $\mathcal{I} \subseteq P \times L$ the incidence relation of $\mathcal{P}$: the pair (p, l) belongs to $\mathcal{I}$ if and only if the point p is incident to the line l. We consider $\mathcal{I}$ as a relation on the set Ω of all elements of $\mathcal{P}$ (points and lines).

The relations

$$s_1 = 1_P, \qquad s_2 = 1_L, \qquad s_3 = P^2 \setminus 1_P, \qquad s_4 = L^2 \setminus 1_L, \tag{2.5.1}$$

$$s_5 = \mathcal{I}, \qquad s_6 = s_5^*, \qquad s_7 = (P \times L) \setminus \mathcal{I}, \qquad s_8 = s_7^*, \tag{2.5.2}$$

obviously form a partition of the set Ω^2. This partition satisfies the condition (CC1), because $1_\Omega = s_1 \cup s_2$, and the condition (CC2), because

$$s_i = s_i^* \ \ (i \leq 4), \qquad s_5^* = s_6, \qquad s_7^* = s_8.$$

Thus the pair $\mathcal{X} = (\Omega, S)$ with $S = \{s_1, \ldots, s_8\}$, is a rainbow.

Assume that the projective plane $\mathcal{P}$ is of order q. Then for each i, any point in $\Omega_-(s_i)$ has exactly $n_i := n_{s_i}$ neighbors in s_i, where

$$n_i = \begin{cases} 1, & \text{if } i = 1, 2, \\ q^2 + q, & \text{if } i = 3, 4, \\ q + 1, & \text{if } i = 5, 6, \\ q^2, & \text{if } i = 7, 8. \end{cases} \tag{2.5.3}$$

More generally, the following statement holds.

Lemma 2.5.1. *For any $i, j, k \in \{1, \ldots, 8\}$, the number $|\alpha s_i \cap \beta s_j^*|$ is a polynomial in q that does not depend on the pair $(\alpha, \beta) \in s_k$.*

Proof. Let $(\alpha, \beta) \in s_k$. Without loss of generality, we assume that $s_i \cdot s_j \neq \varnothing$, otherwise $|\alpha s_i \cap \beta s_j^*| = 0$. Next, if $i = 1$ or $j = 1$, then obviously,

$$|\alpha s_i \cap \beta s_j^*| = \begin{cases} \delta_{k,j}, & \text{if } i = 1, \\ \delta_{k,i}, & \text{if } j = 1. \end{cases}$$

In the case $k = 1$, we are done with $|\alpha s_i \cap \beta s_j^*| = n_k$, see formula (2.5.3). Similarly, one can prove the required statement if $i = 2$, $j = 2$, or $k = 2$. The nontrivial remaining cases are

$$(i,j,k) = (3,5,5) \text{ or } (4,6,6) \quad \text{and} \quad (i,j,k) = (5,6,3) \text{ or } (6,5,4).$$

By the duality between the points and lines, we consider the cases $(3,5,5)$ and $(5,6,3)$ only.

In the first case, α is a point and β is a line incident to α. Since the number of points other than α and incident to β equal the cardinality of the line β reduced by one,

$$|\alpha s_3 \cap \beta s_5^*| = q.$$

In the second case, α and β are different points, and the required number equals the number of lines incident both α and β. Thus in view of the conditions (P1) and (P2),

$$|\alpha s_5 \cap \beta s_6^*| = 1,$$

which completes the proof. □

Lemma 2.5.1 shows that the rainbow $\mathcal{X}$ satisfies the condition (CC3). Together with the definition of the basis relations of $\mathcal{X}$, this immediately implies the following theorem.

Theorem 2.5.2. *For any projective plane $\mathcal{P}$ of order q,*

(1) *the rainbow $\mathcal{X}$ is a coherent configuration of degree $2(q^2+q+1)$ and rank 8;*
(2) *for any indices i, j, k, there exists a polynomial $p_{ijk}(x)$ such that $c_{s_i s_j}^{s_k} = p_{ijk}(q)$;*
(3) *$F(\mathcal{X}) = \{P, L\}$ and the homogeneous components of $\mathcal{X}$ are trivial schemes.*

We say that the coherent configuration $\mathcal{X}$ from statement (1) of Theorem 2.5.2 is associated with the projective plane $\mathcal{P}$. Any coherent configuration algebraically isomorphic to $\mathcal{X}$ is the coherent configuration associated with a certain projective plane. Indeed, given $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, we have

$$c_{s_5' s_6'}^{s_3'} = 1 \quad \text{and} \quad c_{s_6' s_5'}^{s_4'} = 1,$$

where $s_i' = \varphi(s_i)$ for all i. This implies that the conditions (P1) and (P2) are satisfied for the incidence structure $\mathcal{P}'$ consisting of the point set $P' = P^\varphi$, the line set $L' = L^\varphi$, and the incidence relation $\mathcal{I}' = s_5'$. If, in addition, $q \geq 2$, then the condition (P3) is also satisfied. Thus, $\mathcal{P}'$ is a projective plane and the coherent configuration $\mathcal{X}'$ is associated with $\mathcal{P}'$.

Theorem 2.5.3. *The coherent configuration associated with a finite projective plane $\mathcal{P}$ is schurian if and only if $\mathcal{P}$ is a Galois plane.*

Proof. By the main theorem of projective geometry, the automorphism group $\mathrm{Aut}(\mathcal{P})$ of the Galois plane $\mathcal{P}$ of order q is equal to the full projective semilinear group $\mathrm{P\Gamma L}(3,q)$. This group acts 2-transitively on the points and on the lines of $\mathcal{P}$; moreover, the induced actions of $\mathrm{Aut}(\mathcal{P})$ on the flags and on the antiflags are transitive (here, the flags and antiflags are the pairs belonging to the relations s_5 and s_7, respectively). It follows that

$$\mathcal{X} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{P}), \Omega),$$

where $\mathcal{X}$ is coherent configuration associated with the plane $\mathcal{P}$ and Ω is the union of the points and lines of $\mathcal{P}$. The sufficiency is proved.

Conversely, we assume that the coherent configuration $\mathcal{X}$ associated with a projective plane $\mathcal{P}$ is schurian. Then by statement (3) of Theorem 2.5.2 and statement (1) of Corollary 2.2.6, the group $\mathrm{Aut}(\mathcal{X})$ acts 2-transitively on the point set of $\mathcal{P}$. Since

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathcal{P}),$$

$\mathcal{P}$ is a Galois plane by the Ostrom–Wagner theorem stating that a projective plane $\mathcal{P}$ is isomorphic to a Galois plane whenever the group $\mathrm{Aut}(\mathcal{P})$ acts 2-transitively on the points of $\mathcal{P}$ [**14**, p. 114]. □

Let $\mathcal{X}$ and $\mathcal{X}'$ be the coherent configurations associated with projective planes $\mathcal{P}$ and $\mathcal{P}'$, respectively. It is easily seen that $\mathcal{X}$ and $\mathcal{X}'$ are not algebraically isomorphic unless the order of $\mathcal{P}$ equals the order of $\mathcal{P}'$ (Corollary 2.3.24 and statement (1) of Theorem 2.5.2).

Assume that $\mathcal{P}$ and $\mathcal{P}'$ have the same order, say q. Then by statement (2) of Theorem 2.5.2, the bijection

$$\varphi : S \to S',\ s_i \mapsto s_i',$$

is an algebraic isomorphism, where $S = S(\mathcal{X})$ and $S' = S(\mathcal{X}')$. Thus we come to the following statement.

Theorem 2.5.4. *The coherent configurations associated with projective planes $\mathcal{P}$ and $\mathcal{P}'$ are algebraically isomorphic if and only if the orders of $\mathcal{P}$ and $\mathcal{P}'$ are equal.*

The duality principle for the projective planes expresses the symmetry of the roles played by points and lines in the definitions and theorems of the theory of projective planes. One of the consequences of this principle is that the coherent configurations associated with projective planes $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic if and only if $\mathcal{P}$ is isomorphic to $\mathcal{P}'$ or to the dual of $\mathcal{P}'$ (which is obtained by interchanging points and lines and reversing the incidence relation).

Another consequence of the duality is that the mapping

$$\varphi : S \to S,\ s_i \ \leftrightarrow \ s_{i+1} \quad (i = 1, 3, 5, 7), \tag{2.5.4}$$

is an algebraic automorphism of the coherent configuration $\mathcal{X}$ associated with a projective plane $\mathcal{P}$. If q is the order of $\mathcal{P}$, then by Theorem 2.5.4 this implies that $\mathcal{X}$ is separable if and only if any projective plane of order q is isomorphic to $\mathcal{P}$.

For each $q \leq 8$, there is a unique projective plane of order q. Therefore the coherent configuration $\mathcal{X}$ is separable for $q \leq 8$. However, for $q = 9$, there are two non-isomorphic projective planes: the Galois plane and the Hall plane (see [**53**, Chap. 20]). Thus none of the corresponding coherent configurations is separable. Such a situation occurs for infinitely many prime powers q, see, e.g., [**14**].

The coherent configuration $\mathcal{X}$ associated with a projective plane $\mathcal{P}$ has a homogeneous fusion. Namely, denote by Φ the subgroup of $\mathrm{Aut}_{alg}(\mathcal{X})$ generated by the algebraic automorphism of $\mathcal{X}$ defined in (2.5.4). Then $|\Phi| = 2$ and the algebraic fusion

$$\mathcal{X}^{\Phi} = (\Omega, S^{\Phi})$$

is a homogeneous scheme of rank 4. The valencies of the basis relations of this scheme are $1, q^2 + q, q + 1, q^2$. The graph associated with the relation $s_5 \cup s_6$ is distance-regular and has diameter 3 (see Subsection 2.6.2).

2.5.2 Affine schemes

An affine plane is an incidence structure consisting of a set Ω of points and a set $\mathcal{L}$ of lines being subsets of Ω and such that

(AP1) any two different points α and β belong to a unique line $\alpha\beta$;[5]
(AP2) for any point α and any line l not containing α, there exists a unique line containing α and parallel to l (i.e., the line does not intersect l);
(AP3) there exist three different points not belonging to the same line.

An isomorphism of two affine planes is defined to be a line-preserving bijection between their point sets. In what follows, the set Ω is assumed to be finite. In this case, there exists an integer $q \geq 2$, called the *order* of the plane, such that

$$|\Omega| = q^2 \quad \text{and} \quad |\mathcal{L}| = q^2 + q,$$

and each line consists of q points. There are exactly $q+1$ classes $\mathcal{L}_1, \ldots, \mathcal{L}_{q+1}$ of pairwise parallel (i.e., equal or disjoint) lines so that

$$|\mathcal{L}_i| = q, \qquad i = 1, \ldots, q+1.$$

The *affine Galois plane* of order q is obtained from a two-dimensional space over the field $\mathbb{F}_q$ by taking as the points and lines, the 0- and 1-dimensional affine subspaces of this space. The minimal example is obtained for $q = 2$, see below.

Example 2.5.5. *Let $\mathfrak{X}$ be an undirected complete graph with four vertices. Then the vertex and edge sets of $\mathfrak{X}$ can be considered, respectively, as the points and lines of an affine plane of order* 2. *It has* 4 *points and* 6 *lines partitioned in three parallel classes each of which consists of two disjoint edges.*

Let $\mathcal{A}$ be an affine plane with point set Ω and line set $\mathcal{L}$. Denote by S the partition of Ω^2 into $q+2$ classes: 1_Ω and $q+1$ classes of the form

$$\{(\alpha, \beta) \in \Omega^2 : \ \alpha \neq \beta \ \text{ and } \ \alpha\beta \in \mathcal{L}_i\}, \qquad i = 1, \ldots, q+1.$$

Thus the irreflexive relations in S are in one-to-one correspondence with the parallel classes of $\mathcal{A}$, and each such relation is the disjoint union of q undirected loopless complete graphs of order q whose vertex sets are the lines belonging to the corresponding class. In Example (2.5.5), the partition S consists of three relations, each of which is the graph with 4 vertices and two disjoint undirected edges.

Theorem 2.5.6. *The pair $\mathcal{X} = (\Omega, S)$ is a symmetric scheme whose nonzero intersection numbers c_{rs}^t with $1 \notin \{r, s\}$, are as follows:*

[5] The line $\alpha\beta$ is not uniquely determined if $\alpha = \beta$.

$$c_{rs}^t = \begin{cases} q-1, & \textit{if } r = s \textit{ and } t = 1, \\ q-2, & \textit{if } r = s = t, \\ 1, & \textit{if } r \ne s \ne t \ne r. \end{cases} \tag{2.5.5}$$

Proof. The definition of S implies that $\mathcal{X}$ is a symmetric rainbow. We have to prove that given relations $r, s, t \in S$, the number $|\alpha r \cap \beta s|$ does not depend on the choice of the pair $(\alpha, \beta) \in t$.

Without loss of generality, we may assume that $1 \notin \{r, s\}$ and t intersects $r \cdot s$. In particular,

$$\alpha \ne \beta \quad \text{and} \quad |\{r, s, t\}| = 1 \text{ or } 3.$$

The first statement is clear. To prove the second one, we assume that two relations of r, s, t, say r and s, are equal. Taking into account that

$$t \cap (r \cdot s) \ne \varnothing,$$

we see that the lines $\alpha\gamma$ and $\gamma\beta$ are also equal. Therefore, $\alpha\beta = \alpha\gamma = \gamma\beta$. Thus either $r = s = t$ or $r \ne s \ne t \ne r$.

Let $r = s = t$. Recall that $r = e \setminus 1$, where e is an equivalence relation on Ω with q classes each of cardinality q. It follows that

$$A_r A_s = (A_e - I)(A_e - I) = qA_e - 2A_e + I = (q-2)A_t + (q-1)I,$$

where $I = I_\Omega$. Taking the (α, β)-entry of the matrices on the left-hand and right-hand sides, we conclude that

$$|\alpha r \cap \beta s| = \begin{cases} q-2, & \text{if } t \ne 1, \\ q-1, & \text{otherwise.} \end{cases}$$

This proves the first two equalities in formula (2.5.5) (see statement (4) of Exercise 1.4.8).

Now let $r \ne s \ne t \ne r$. Denote by $\alpha_1, \ldots, \alpha_{q-1}$ the points other than α that belong the line $\alpha r \in \mathcal{L}$. Then no two distinct lines among $\beta\alpha_1, \ldots, \beta\alpha_{q-1}$ are parallel (they have a common point β). It follows that the basis relations $s_i = r(\beta, \alpha_i)$ are pairwise distinct. Thus,

$$\alpha r \cap \beta s = \begin{cases} \{\alpha_i\}, & \text{if } s = s_i \text{ for some } i \in \{1, \ldots, q-1\}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

which proves the third equality in formula (2.5.5). □

The scheme $\mathcal{X}$ from Theorem 2.5.6 is said to be *affine* or the scheme of the affine plane $\mathcal{A}$. Formula (2.5.5) implies that $n_s = q - 1$ for all irreflexive $s \in S$. Consequently, $\mathcal{X}$ is an equivalenced scheme of valency $q - 1$.

Theorem 2.5.7. *The scheme of a finite affine plane $\mathcal{A}$ is schurian if and only if $\mathcal{A}$ is an affine Galois plane.*

Proof. By the well-known result of finite geometries (see, e.g., [**31**]), a finite affine plane $\mathcal{A}$ is either non-Desarguesian or an affine Galois plane. In the latter case, the scheme of $\mathcal{A}$ coincides with the coherent configuration associated with the group

$$A\,G \le \mathrm{AGL}(2,q),$$

where A is the translation group of the underlying linear space and G is the center of the group $\mathrm{GL}(2,q)$. This proves the sufficiency.

To prove the necessity, let $\mathcal{X}$ be the scheme of an affine plane $\mathcal{A}$. Assume that $\mathcal{X}$ is schurian. We have to verify that $\mathcal{A}$ is Desarguesian, i.e., given three lines containing a common point δ and given points α, α' lying on the first line, β, β' lying on the second line, and γ, γ' lying on the third line,

$$\alpha\gamma \parallel \alpha'\gamma' \quad \text{and} \quad \beta\gamma \parallel \beta'\gamma' \qquad \Rightarrow \qquad \alpha\beta \parallel \alpha'\beta',$$

where $\parallel$ denotes the relation of parallelism, see Fig 2.4.

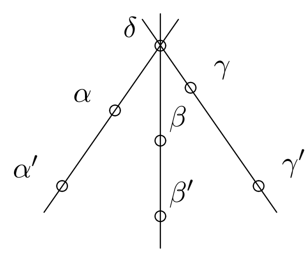


FIGURE 2.4 The points and lines of a Desarguesian configuration.

To this end, we note that $r(\delta,\gamma) = r(\delta,\gamma')$, because $\delta\gamma = \delta\gamma'$. The schurity assumption implies that there exists an automorphism $k \in \mathrm{Aut}(\mathcal{X})$ such that

$$\delta^k = \delta \quad \text{and} \quad \gamma^k = \gamma'.$$

Since k preserves the basis relations of $\mathcal{X}$, we have

$$r(\alpha,\delta) = r(\alpha^k,\delta) \quad \text{and} \quad r(\alpha,\gamma) = r(\alpha^k,\gamma').$$

By the first equality, $\alpha^k \in \alpha\delta$, and by the second equality, $\alpha\gamma \parallel \alpha^k\gamma'$. In view of $\alpha\gamma \parallel \alpha'\gamma'$, the axiom (AP2) for $l = \alpha\gamma$ and $\alpha = \gamma'$ shows that

$$\alpha^k\gamma' = \alpha'\gamma'.$$

This line intersects $\alpha\delta$ in the point α^k (see above) and in the point α' (by the assumption). By the axiom (AP1), this proves the first of the following equalities

$$\alpha^k = \alpha' \quad \text{and} \quad \beta^k = \beta',$$

the second one is proved similarly. On the other hand, $r(\alpha,\beta) = r(\alpha^k,\beta^k)$ or equivalently, $\alpha\beta \parallel \alpha^k\beta^k$. Since $\alpha^k\beta^k = \alpha'\beta'$, we conclude that $\alpha\beta \parallel \alpha'\beta'$ as required. □

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be affine schemes. If the underlying affine planes have different orders, then the degrees of $\mathcal{X}$ and $\mathcal{X}'$ are different. Therefore these schemes cannot be algebraically isomorphic (Corollary 2.3.24).

Suppose that the affine planes have the same order q. Then

$$|S| = q + 2 = |S'|,$$

From formula (2.5.5), it follows that for any bijection $\varphi : S \to S'$, $s \mapsto s'$ taking 1_Ω to $1_{\Omega'}$, we have

$$c_{rs}^{t} = c_{r's'}^{t'}, \qquad r, s, t \in S.$$

Therefore, φ is an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

Theorem 2.5.8. *Two affine schemes are algebraically isomorphic if and only if the underlying affine planes have the same order. Moreover, for any affine scheme* $\mathcal{X}$*,*

$$\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}) = \mathrm{Sym}(S(\mathcal{X}))_{1_\Omega}.$$

As in the case of coherent configurations, one can easily prove that a bijection f between the point sets of two affine planes is an isomorphism if and only if f is an isomorphism of the schemes of the corresponding planes. By Theorem 2.5.8, this implies that the scheme of an affine plane $\mathcal{A}$ of order q is separable if and only if $\mathcal{A}$ is a unique (up to isomorphism) affine plane of order q. However, for infinitely many prime powers q, there exist at least two non-isomorphic affine planes of order q. In each of these cases, the scheme of any affine plane of order q is not separable.

The affine schemes were introduced and studied in [**52**]. A stronger version of Theorem 2.5.7 can be found in [**102**].

2.5.3 Designs

Let Ω be a set of cardinality n, and let $\mathfrak{B}$ be a collection of subsets of Ω, all of cardinality $k \geq 1$; the elements of Ω and $\mathfrak{B}$ are called points and blocks, respectively. A pair $(\Omega, \mathfrak{B})$ is called a *design* with parameters (n, k, λ), or (n, k, λ)-design, if any two distinct points belong to exactly λ blocks. Note that if $\lambda = 0$, then $k = 1$ and $\mathfrak{B}$ consists of singletons. A huge literature on designs can be found in [**13**] (see also [**18**] and references therein).

Example 2.5.9. *Every projective plane $\mathcal{P}$ of order q produces a natural $(q^2+q+1, q+1, 1)$-design, in which Ω is the point set of $\mathcal{P}$ and $\mathfrak{B}$ consists of all sets of points incident to a line of $\mathcal{P}$.*

Example 2.5.10. *Any affine plane $\mathcal{A}$ of order q is a $(q^2, q, 1)$-design, in which Ω is the point set of $\mathcal{A}$ and $\mathfrak{B}$ consists of the lines of $\mathcal{A}$.*

In Example 2.5.9, the points consist a fiber of the coherent configuration associated with projective plane and the homogeneous component of this fiber is trivial (statement (3) of Theorem 2.5.2). The following observation shows that inside any coherent configuration $\mathcal{X}$ having trivial homogeneous component, one can find several designs whose parameters are among the intersection numbers of $\mathcal{X}$.

Proposition 2.5.11. *Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. Suppose that $\mathcal{X}_\Delta = \mathcal{T}_\Delta$ for some fiber Δ of $\mathcal{X}$. Then for each irreflexive $s \in S_{\Omega,\Delta}$, the collection*

$$\mathfrak{B}_s = \{\alpha s : \ \alpha \in \Omega_-(s)\}$$

is the set of blocks of an (n, k, λ)-design, where

$$n = |\Delta|, \quad k = n_s, \quad \lambda = c_{s^*s}^t$$

with $t = \Delta^2 \setminus 1_\Delta$.

Proof. Let $s \in S_{\Omega,\Delta}$ be a irreflexive relation, and let $\delta, \gamma \in \Delta$ be distinct points. Then $(\delta, \gamma) \in t$, and for any $\alpha \in \Omega_-(s)$,

$$\delta, \gamma \in \alpha s \quad \Leftrightarrow \quad \alpha \in \delta s^* \cap \gamma s^*.$$

Therefore the number of sets αs containing δ and γ equals the intersection number $|\delta s^* \cap \gamma s^*| = c_{s^*s}^t$. □

The designs from Proposition 2.5.11 can naturally be called *coherent*. More precisely, a design $\mathfrak{D}$ is said to be coherent if there exists a coherent configuration $\mathcal{X}$ which has a trivial homogeneous component $\mathcal{X}_\Delta$ and a basis relation s such that $\mathfrak{D} = (\Delta, \mathfrak{B}_s)$. In this case,

$$A_s A_{s^*} = \sum_{i=0}^{k} i A_i, \tag{2.5.6}$$

where $k = n_s$ and A_i is the adjacency matrix of the relation t_i consisting of all pairs of blocks having exactly i common points; in particular, $t_k = 1_\Gamma$, where $\Gamma = \Omega_-(s)$. The set

$$T = \{t_0, \ldots, t_k\}$$

forms a partition of Γ^2 that defines a (symmetric) rainbow (Γ, T). By the Wielandt principle, formula (2.5.6) implies that $T \subseteq S(\mathcal{X})^\cup$. It follows that the rainbow has a homogeneous fission, namely, $\mathcal{X}_\Gamma$.

In a similar way for any (n, k, λ)-design $\mathfrak{D}$, one can define a rainbow $\mathcal{X}(\mathfrak{D})$ of rank at most $k+1$ on the block set: the basis relations consist of all pairs of blocks with fixed cardinality of their intersection. The following statement proved in the previous paragraph gives a strong necessary condition for a design to be coherent.

Proposition 2.5.12. *A design $\mathfrak{D}$ is coherent only if the rainbow $\mathcal{X}(\mathfrak{D})$ is a fusion of a certain scheme.*

In the extremal case, the scheme from Proposition 2.5.12 coincides with the rainbow $\mathcal{X}(\mathfrak{D})$, i.e., this rainbow is a symmetric scheme. As we will see below, this is always true if the rank of the rainbow $\mathcal{X}(\mathfrak{D})$ is at most 3; in this case, the design $\mathfrak{D}$ is said to be *quasisymmetric*. Thus, $\mathfrak{D}$ is a quasisymmetric design if and only if there exists constants x and y such that any two distinct blocks of $\mathfrak{D}$ intersect at either x or y points.

The designs associated with projective and affine planes are examples of quasisymmetric designs (in the former case, $x = y = 1$, and in the latter case, $x = 0$ and $y = 1$). In the proof of the theorem below, we follow that of [**79**, Theorem 8.2.14].

Theorem 2.5.13. *Every quasisymmetric design is coherent.*

Proof. Let $\mathfrak{D} = (\Delta, \mathfrak{B})$ be a quasisymmetric (n, k, λ)-design. Assume that any two distinct blocks intersect in either x or y points. If $x = y$, then the required statement follows from Exercise 2.7.47. Let $x > y$.

Denote by r the number of blocks containing a point of Δ (Exercise 2.7.46), and set A to be rectangular $\{0,1\}$-matrix with rows and columns indexed by Δ and $\mathfrak{B}$, respectively, and such that $A_{\alpha,B} = 1$ if and only if $\alpha \in B$. By the definition of design, we have

$$J_\Delta A = kJ_{\Delta,\mathfrak{B}} \quad \text{and} \quad AJ_\mathfrak{B} = rJ_{\Delta,\mathfrak{B}}, \tag{2.5.7}$$

and

$$A^T J_\Delta = kJ_{\mathfrak{B},\Delta} \quad \text{and} \quad J_\mathfrak{B} A^T = rJ_{\mathfrak{B},\Delta}. \tag{2.5.8}$$

Lemma 2.5.14.

$$AA^T = \lambda(J_\Delta - I_\Delta) + rI_\Delta. \tag{2.5.9}$$

In particular, the matrix AA^T has exactly two eigenvalues, both nonzero.

Proof. The first statement follows from the definition of λ and r. Formula (2.5.9) implies that $\underline{\Delta}$ and each column of the matrix $I_\Delta - \frac{1}{n}J_\Delta$ are eigenvectors of the matrix AA^T that correspond to the eigenvalues

$$r + \lambda(n-1) \quad \text{and} \quad r - \lambda,$$

respectively. Since the multiplicities of the them are 1 and $n-1$, we are done. □

Denote by A_z the adjacency matrix of the relation $t_z \subseteq \mathfrak{B}^2$, which consists of all pairs of blocks intersecting in exactly z points. Since the design $\mathfrak{D}$ is quasisymmetric, this matrix is nonzero only if $z = k$, x, or y. It follows that

$$J_{\mathfrak{B}} = I_{\mathfrak{B}} + A_x + A_y \quad \text{and} \quad A^T A = kI_{\mathfrak{B}} + xA_x + yA_y. \tag{2.5.10}$$

Lemma 2.5.15. *The linear space* $\mathcal{A}_{\mathfrak{B}} = \operatorname{Span}\{I_{\mathfrak{B}}, A_x, A_y\}$ *is a* 3*-dimensional coherent algebra.*

Proof. From formulas (2.5.7) and (2.5.8), it follows that

$$A^T A J_{\mathfrak{B}} = rA^T J_{\Delta,\mathfrak{B}} = krJ_{\mathfrak{B}} = rJ_{\mathfrak{B},\Delta}A = J_{\mathfrak{B}}A^T A.$$

It follows that $\mathcal{A}_{\mathfrak{B}}$ consists of pairwise commuting matrices; in particular, the symmetric matrices $I_{\mathfrak{B}}$, A_x, and $J_{\mathfrak{B}}$ are pairwise commute.

By a well-known result in linear algebra, one can find an orthogonal matrix C such that the matrices $C^T A_x C$ and $C^T J_{\mathfrak{B}} C$ are diagonal. In view of (2.5.10), the matrix

$$\begin{aligned} C^T A^T AC &= C^T(kI_{\mathfrak{B}} + xA_x + yA_y)C \\ &= C^T((k-y)I_{\mathfrak{B}} + (x-y)A_x + yJ_{\mathfrak{B}})C \\ &= (k-y)I_{\mathfrak{B}} + (x-y)C^T A_x C + yC^T J_{\mathfrak{B}} C \end{aligned} \tag{2.5.11}$$

is also a diagonal one. The matrices AA^T and $A^T A$ have the same nonzero eigenvalues (including multiplicities). By Lemma 2.5.14, this implies that $A^T A$ has exactly two nonzero eigenvalues: one of multiplicity 1 and the other one of multiplicity $n-1$.

One can see that $\underline{\mathfrak{B}}$ is a common eigenvector of the largest eigenvalues of the matrices $A^T A$ and $J_{\mathfrak{B}}$. Thus up to permutation of indices,

$$C^T A^T AC = \operatorname{diag}(\lambda, \mu, \ldots, \mu, 0, \ldots, 0) \quad \text{and} \quad C^T J_{\mathfrak{B}} C = \operatorname{diag}(\nu, 0, \ldots, 0)$$

for some reals λ, μ, and ν. Consequently, the linear space

$$\operatorname{Span}\{I_{\mathfrak{B}}, C^T A_x C, C^T J_{\mathfrak{B}} C\} = \operatorname{Span}\{I_{\mathfrak{B}}, C^T A_x C, C^T A_y C\} = C^T \mathcal{A}_{\mathfrak{B}} C$$

and hence the space $\mathcal{A}_{\mathfrak{B}}$, is a 3-dimensional commutative algebra. It has a linear basis $\{I_{\mathfrak{B}}, A_x, A_y\}$ consisting of symmetric $\{0,1\}$-matrices. Thus it is coherent by Corollary 2.3.8. □

Let $\Omega = \Delta \cup \mathfrak{B}$. From now on, it is convenient to consider the linear spaces Mat_Δ, $\mathrm{Mat}_\mathfrak{B}$, $\mathrm{Mat}_{\Delta,\mathfrak{B}}$, and $\mathrm{Mat}_{\mathfrak{B},\Delta}$ as subspaces of Mat_Ω via the natural injections $\Delta \to \Omega$ and $\mathfrak{B} \to \Omega$. In particular, the matrices A, A^T, I_Δ, $I_\mathfrak{B}$, ... are treated as the matrices of Mat_Ω.

Let $\mathcal{X}$ be a rainbow on Ω corresponding to the partition of Ω^2 into the relations belonging to the union of the sets

$$S_\Delta = S(\mathcal{T}_\Delta), \qquad S_\mathfrak{B} = S(\mathcal{X}_\mathfrak{B}), \qquad S_{\Delta,\mathfrak{B}} = \{s, s'\}, \qquad S_{\mathfrak{B},\Delta} = S^*_{\Delta,\mathfrak{B}},$$

where $\mathcal{X}_\mathfrak{B}$ is the coherent configuration corresponding to the algebra $\mathcal{A}_\mathfrak{B}$ (Lemma 2.5.15 and Theorem 2.3.7), and s and s' are relations on Ω such that

$$A_s = A \quad \text{and} \quad A_{s'} = J_{\Delta,\mathfrak{B}} - A.$$

In these notation, $\mathfrak{B} = \mathfrak{B}_s$. Thus it suffices to verify that $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ is a coherent algebra, or even that $\mathcal{A}$ is an algebra with respect to the matrix multiplication (Corollary 2.3.8).

In view of formulas (2.5.7) and (2.5.8), and Lemmas 2.5.14 and 2.5.15, the only non-obvious case is to verify that $A_u A_v \in \mathcal{A}$ for $u \in S_{\Delta,\mathfrak{B}}$ and $v \in S_\mathfrak{B}$. Assume that $u = s$ and $v = t_x$ (the statement for the other possibilities follows from this one). Then by formulas (2.5.7) and (2.5.9),

$$(AA^T)A = (\lambda(J_\Delta - I_\Delta) + rI_\Delta)A = \lambda k J_{\Delta,\mathfrak{B}} + (r - \lambda)A,$$

whereas by formulas (2.5.10) and (2.5.7),

$$\begin{aligned} A(A^T A) &= A((k-y)I_\mathfrak{B} + (x-y)A_x + yJ_\mathfrak{B}) \\ &= (k-y)A + (x-y)A_u A_v + yJ_{\Delta,\mathfrak{B}}. \end{aligned}$$

This shows that the product $A_u A_v$ is the linear combination of the matrices $A = A_s$ and $J_{\Delta,\mathfrak{B}} = A_s + A_{s'}$, as required. □

The type of a coherent configuration (Ω, S) with two fibers Ω_1 and Ω_2 is defined to be a 2×2 matrix, the (i,j)-entry of which equals the number $|S_{\Omega_i,\Omega_j}|$. By Proposition 2.5.11, any coherent configuration of type

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \tag{2.5.12}$$

produces two complementary designs, which are obviously quasisymmetric; the points and blocks of the design *complementary* to a design $\mathfrak{D}$ are the points of $\mathfrak{D}$ and the complements of the blocks of $\mathfrak{D}$, respectively. Conversely, the coherent configuration constructed in the proof of Theorem 2.5.13 for a quasisymmetric design, has one of the two above types.

Corollary 2.5.16. *The pairs of complementary quasisymmetric designs are in 1-1 correspondence with the coherent configurations of types* (2.5.12).

Numerous examples of quasisymmetric designs can be found in monograph [**116**]. Here, we mention only the *Steiner designs*, i.e., those with

$\lambda = 1$. In such a design, no two distinct blocks have more than one common point. Therefore, all the Steiner designs are quasisymmetric (with $x \le 1$ and $y = 1$) and hence coherent (Theorem 2.5.13). The examples of Steiner designs include the designs of projective and affine planes, and the designs whose blocks form a Steiner triple system ($k = 3$).

The schurity problem for the coherent configurations corresponding to the coherent designs is reduced to determining the subgroups of 2-transitive groups, which are known modulo the Classification of Finite Simple Groups (see [**23**, Sec. 7.3, 7.4]). Indeed, statement (1) of Corollary 2.2.6 implies the following proposition.

Proposition 2.5.17. *Let $\mathcal{X}$ be a schurian coherent configuration for which the hypothesis of Proposition 2.5.11 is satisfied. Then the group* $\mathrm{Aut}(\mathcal{X})^{\Delta}$ *is 2-transitive.*

The separability problem in the class $\mathcal{K}$ seems to be hopeless, because it includes identifying all Steiner designs determined by its parameters up to isomorphism. Concerning this problem, we refer the reader to [**116**].

2.6 Graphs

Every graph can be considered as a relation of a coherent configuration, for instance, the discrete one. Among such coherent configurations, there is the smallest one, the *coherent closure*, and it is exactly one constructed by the Weisfeiler–Leman algorithm applied to the original graph.

The coherent closure controls the automorphisms and isomorphisms of the graph in question. In this section, we will see in which sense this is true and what happens in the most interesting case of distance-regular graphs. These graphs are quite well studied (see e.g. [**17**]) and lead to various nice coherent configurations.

2.6.1 Coherent closure

Let T be a collection of relations on Ω. Denote by $\mathfrak{T}$ the set of all coherent configurations on Ω that contain T as a set of relations,

$$\mathfrak{T} = \mathfrak{T}(\Omega, T) = \{\mathcal{X} \leq \mathcal{D}_\Omega : \ T \subseteq S(\mathcal{X})^\cup\}.$$

In particular, $\mathfrak{T}$ includes the discrete configuration and hence is nonempty. On the other hand, $\mathfrak{T}$ is closed under intersections, see Subsection 2.3.1. Consequently, $\mathfrak{T}$ has the smallest element, namely, the intersection of all coherent configurations in $\mathfrak{T}$,

$$\mathrm{WL}(T) = \bigcap_{\mathcal{X} \in \mathfrak{T}(\Omega, T)} \mathcal{X}.$$

Definition 2.6.1. *The coherent configuration* $\mathrm{WL}(T)$ *is called the coherent closure of the collection* T*.*

The notation WL is explained by the fact that the coherent closure can explicitly be found with the help of the Weisfeiler–Leman algorithm (the WL-algorithm), see below. The mapping

$$\mathcal{X} \mapsto \mathrm{WL}(\mathcal{X}), \qquad (2.6.1)$$

defines a closure operator on the rainbows on Ω (Exercise 2.7.49), where $\mathrm{WL}(\mathcal{X})$ is defined to be the coherent closure of $T = S(\mathcal{X})$.

Example 2.6.2. *Let us find the coherent closure* $\mathcal{X}$ *of* $T = \{s\}$*, where* s *is the arc set of an undirected* 6*-cycle on* $\Omega = \{1, \ldots, 6\}$*, see the first graph in Fig.* 2.5*.*

Among the relations of $\mathcal{X}$*, there are obviously* $s_0 = 1$ *and* $s_1 = s$*. By Proposition* 2.1.4*,*

$$s_2 = (s \cdot s) \setminus s_0 \quad \text{and} \quad s_3 = \Omega^2 \setminus (s_0 \cup s_1 \cup s_2)$$

are also relations of $\mathcal{X}$*, see the second and third graph in Fig.*2.5*. It is easily seen that*

$$\{s_0, s_1, s_2, s_3\} = \mathrm{Orb}(K, \Omega^2),$$

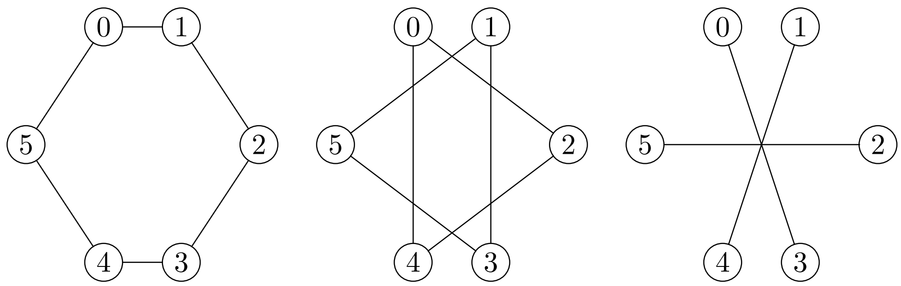


FIGURE 2.5 The coherent configuration of an undirected 6-cycle.

where $K = D_{12}$ *is the dihedral group of degree* 6*. The scheme* $\mathrm{Inv}(K, \Omega)$ *belongs to* $\mathfrak{T}$*, and is a fusion of* $\mathrm{WL}(T) = \mathcal{X}$*. By the minimality of the coherent closure, this shows that* $\mathcal{X} = \mathrm{Inv}(K, \Omega)$*.*

In general, the technique based on Proposition 2.1.4 only does not help to find the coherent closure (like as in Example 2.6.2) and a reason is that this technique does not take into account the intersection numbers of the resulting coherent configuration. A solution is to replace Proposition 2.1.4 by the Wielandt principle (Theorem 2.3.10). To explain this in more detail, we need some preparation.

In finding the coherent closure of a collection T of relations, we may assume (without loss of generality) that T equals the set S of basis relations of a rainbow $\mathcal{X}$ on Ω (Exercise 2.7.51). For any integer $k \geq 0$ and any $r, s, t \in S$, define the relation

$$w_k(r, s; t) = \{(\alpha, \beta) \in t : \ |\alpha r \cap \beta s^*| = k\}.$$

Certainly, this relation is empty if $k > n$, and equals t if $\mathcal{X}$ is a coherent configuration and $k = c_{rs}^t$. Furthermore, it is easily seen that

$$w_k(r, s; t)^* = w_k(s^*, r^*; t^*). \tag{2.6.2}$$

Denote by S' the partition of Ω^2 such that (α, β) and (α', β') belong to the same class if and only if for every integer $k \geq 0$ and any $r, s, t \in S$,

$$(\alpha, \beta) \in w_k(r, s; t) \ \Leftrightarrow \ (\alpha', \beta') \in w_k(r, s; t).$$

Lemma 2.6.3. *Let* $\mathcal{X} = (\Omega, S)$ *be a rainbow and* S' *as above. Then*

(1) $\mathcal{X}' = (\Omega, S')$ *is a rainbow;*
(2) $\mathcal{X} \leq \mathcal{X}'$*, and* $\mathcal{X} = \mathcal{X}'$ *if and only if* $\mathcal{X}$ *is a coherent configuration;*
(3) $\mathrm{WL}(S) = \mathrm{WL}(S')$.

Proof. Statement (1) follows from Exercise 2.7.51. The obvious inclusion $w_k(r, s; t) \subseteq t$ that holds for all k and r, s, t, proves the inclusion in statement (2). Note that

$$w_k(r, s; t) = t \quad \Leftrightarrow \quad |\alpha r \cap \beta s^*| = k \ \text{ for all } \ (\alpha, \beta) \in t.$$

Therefore, $\mathcal{X} = \mathcal{X}'$ if and only if the rainbow $\mathcal{X}$ satisfies the condition (CC3), i.e. is a coherent configuration.

To prove statement (3), we note that the relations of S' are intersections of the relations $w_k(r,s;t)$. Thus it suffices to verify that $w_k(r,s;t)$ is a relation of the coherent configuration $\mathrm{WL}(S)$ for all k and r,s,t. By statement (4) of Exercise 1.4.8, we have

$$w_k(r,s;t) = t \cap s_f(A_rA_s),$$

where $s_f(\cdot)$ is as in formula (2.3.4) with $f(x) = \delta_{x,k}$ (the Kronecker delta). Since t is a relation of $\mathrm{WL}(S)$, it remains to prove that so is $s_f(A_rA_s)$. However, this is a consequence of the Wielandt principle, because r and s are relations of $\mathrm{WL}(S)$ and so the matrix A_rA_s belongs to the adjacency algebra of $\mathrm{WL}(S)$. □

Lemma 2.6.3 suggests the following algorithm for finding the coherent closure of a rainbow.

The Weisfeiler–Leman algorithm

Input a rainbow $\mathcal{X} = (\Omega, S)$.
Output the coherent closure $\mathrm{WL}(\mathcal{X})$.

Step 1 Find the set S' defined in Lemma 2.6.3.

Step 2 If $|S| < |S'|$, then $S := S'$ and go to Step 1.

Step 3 Output $\mathrm{WL}(\mathcal{X}) = (\Omega, S')$.

At each iteration except for the last one, the number $|S| \le n^2$ strictly increases. Therefore, the algorithm stops after at most $n^2 - |S|$ iterations. By statements (2) and (3) of Lemma 2.6.3, the resulting rainbow is equal to the coherent closure of $\mathcal{X}$.

To make the algorithm more clear, let us take noncommuting variables x_s, $s \in S$. Then at each iteration, one needs to calculate a polynomials

$$\Big(\sum_{s\in S} x_sA_s\Big)^2_{\alpha,\beta} = \sum_{\gamma\in\Omega} x_{r(\alpha,\gamma)}x_{r(\gamma,\beta)} = \sum_{r,s\in S} a_{rs}x_rx_s$$

for each pair $(\alpha,\beta) \in \Omega^2$, where a_{rs} are suitable integers. Using the set of all these linear combinations, the set S' can easily be constructed by comparing the corresponding multivariable polynomials, because

$$(\alpha,\beta) \in w_k(r,s;t) \quad \Leftrightarrow \quad r(\alpha,\beta) = t \quad \text{and} \quad a_{rs} = k.$$

This modification of Step 1 leads to the standard Weisfeiler–Leman algorithm or the 2-dim WL; for details, see Section 4.6.

The most important property of the coherent closure is that it preserves the automorphism group of the initial collection of relations. In this sense, taking the coherent closure reduces the problem of finding the automorphism group of a collection of relations on Ω to that of a coherent configuration on Ω.

Theorem 2.6.4. *Let T and T' be sets of relations on Ω and Ω', respectively. Denote by* $\mathrm{Iso}(T, T')$ *the set of all bijections $\Omega \to \Omega'$ taking T to T'. Then*

$$\mathrm{Iso}(T, T') \subseteq \mathrm{Iso}(\mathcal{X}, \mathcal{X}') \quad \text{and} \quad \mathrm{Aut}(T) = \mathrm{Aut}(\mathcal{X}), \tag{2.6.3}$$

where $\mathcal{X} = \mathrm{WL}(T)$ *and* $\mathcal{X}' = \mathrm{WL}(T')$.

Proof. Let $f \in \mathrm{Iso}(T, T')$. Then obviously,

$$\mathcal{X} \in \mathfrak{T}(\Omega, T) \quad \Leftrightarrow \quad \mathcal{X}^f \in \mathfrak{T}(\Omega', T').$$

Consequently, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$, which proves the first inclusion in (2.6.3).

To prove the equality, we note that every automorphism of the coherent configuration $\mathcal{X}$ leaves any of its relation, in particular, any relation of T, fixed. Thus,

$$\mathrm{Aut}(\mathcal{X}) \le \mathrm{Aut}(T).$$

Conversely, each relation of T is obviously $\mathrm{Aut}(T)$-invariant. By statement (1) of Exercise 2.7.17, this implies that

$$\mathrm{Inv}(\mathrm{Aut}(T)) \in \mathfrak{T}(\Omega, T).$$

It follows that

$$\mathrm{Inv}(\mathrm{Aut}(T)) \ge \mathcal{X}$$

and hence by the Galois correspondence (Theorem 2.2.8),

$$\mathrm{Aut}(\mathrm{Inv}(\mathrm{Aut}(T))) \le \mathrm{Aut}(\mathcal{X}).$$

The left-hand side contains $\mathrm{Aut}(T)$ (see the second inclusion in (2.2.7)). Thus, $\mathrm{Aut}(T) \le \mathrm{Aut}(\mathcal{X})$, as required. □

Definition 2.6.5. *The coherent configuration of a graph $\mathfrak{X}$ on Ω with arc set D is defined to be the coherent closure*

$$\mathrm{WL}(\mathfrak{X}) := \mathrm{WL}(\{D\}),$$

i.e., the smallest coherent configuration on Ω that contains D among of its relations.

Clearly, the coherent configuration of a complete or empty graph on Ω is equal to $\mathcal{T}_\Omega$. This shows that the coherent configurations of non-isomorphic graphs can be equal. However, the following corollary of Theorem 2.6.4 shows that such graphs always have the same automorphism groups.

Corollary 2.6.6. *Let $\mathfrak{X}$ be a graph and $\mathcal{X} = \mathrm{WL}(\mathfrak{X})$. Then*

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathfrak{X}). \tag{2.6.4}$$

In particular,

(1) *if $\mathfrak{X}$ is vertex-transitive, then $\mathcal{X}$ is homogeneous;*
(2) *if $\mathfrak{X}$ is arc-transitive, then the arc set of $\mathfrak{X}$ is a basis relation of $\mathcal{X}$;*
(3) *$\mathfrak{X}$ is a Cayley graph if and only if $\mathcal{X}$ is a Cayley scheme.*

By definition, the arc set of a graph $\mathfrak{X}$ is a relation of its coherent configuration $\mathcal{X}$. It follows that the adjacency matrix of $\mathfrak{X}$ belongs to the adjacency algebra of $\mathcal{X}$. The use of the Wielandt principle enables us to get some information on the relations of $\mathcal{X}$.

Theorem 2.6.7. *Let $\mathfrak{X}$ be a graph, $\mathcal{X} = \mathrm{WL}(\mathfrak{X})$, and $d \geq 0$ an integer. Then*

(1) *the vertices of $\mathfrak{X}$ of valency d form a homogeneity set of $\mathcal{X}$;*
(2) *the pairs of vertices of $\mathfrak{X}$ at distance d form a relation of $\mathcal{X}$.*

Proof. Denote by Ω_d the set of all vertices of $\mathfrak{X}$ that have valency d. Clearly,

$$\alpha \in \Omega_d \quad \Leftrightarrow \quad (AA^T)_{\alpha,\alpha} = d,$$

where A is the adjacency matrix of $\mathfrak{X}$. It follows that

$$1_{\Omega_d} = 1_\Omega \cap s_f(AA^T),$$

where $s_f(\cdot)$ is as in formula (2.3.4) with $f(x) = \delta_{d,x}$ (the Kronecker delta). Note that $A \in \mathrm{Adj}(\mathcal{X})$. By the Wielandt principle, this implies that 1_{Ω_d} is a relation of $\mathcal{X}$, which proves statement (1).

To prove statement (2), let $i \geq 0$ be an integer and

$$A_i = \sum_{j=0}^{i} A^j.$$

It is easily seen that for any vertices α and β of the graph $\mathfrak{X}$,

$$d(\alpha,\beta) \leq i \quad \Leftrightarrow \quad (A_i)_{\alpha,\beta} > 0. \tag{2.6.5}$$

Denote by f the function such that $f(x) = 1$ or 0 depending on whether or not x is a positive integer. By the Wielandt principle,

$$s_i := s_f(A_i)$$

is a relation of $\mathcal{X}$ for all i. It follows that so is the relation $s_d \setminus s_{d-1}$. In view of (2.6.5), it coincides with the relation "to be at distance d", which proves statement (2). □

The following useful property of the coherent configuration of a graph $\mathfrak{X}$ concerns the equivalence relation $e_{con}(\mathfrak{X})$ corresponding to the partition of the vertices of $\mathfrak{X}$ into the vertex sets of connected components. It immediately follows from Proposition 2.1.18.

Proposition 2.6.8. *For any graph $\mathfrak{X}$, $e_{con}(\mathfrak{X})$ is a parabolic of the coherent configuration $\mathrm{WL}(\mathfrak{X})$.*

It was proved in [7], that almost all graphs have pairwise distinct valencies. By statement (1) of Theorem 2.6.7, the coherent configuration of any of these graphs is discrete and hence does not contain any specific information about the initial graph. In fact, the theory of coherent configurations can help in the study of only those graphs that have a sufficiently large combinatorial symmetry and the measure of this symmetry is precisely the coherent closure: the smaller a configuration is, the more information it gives about the graph in question.

2.6.2 Distance-regular graphs

Throughout this subsection, $\mathfrak{X}$ is a connected undirected loopless graph on a set Ω. Denote by d the diameter of $\mathfrak{X}$. By statement (2) of Theorem 2.6.7, the coherent configuration $\mathcal{X}$ of this graph contains $d+1$ pairwise disjoint relations

(2.6.6) $$s_i = \{(\alpha,\beta) \in \Omega^2 : \ d(\alpha,\beta) = i\},$$

where $i = 0, 1, \ldots, d$.

It may happen that all of these relation are basis ones; in this case the Weisfeiler–Leman algorithm (applied to the rainbow formed by the s_i) stops after the first iteration. Then $\mathcal{X}$ is a symmetric scheme of rank $d+1$. Moreover, the intersection numbers

$$b_i = c_{s_{i+1},s_1}^{s_i} \quad \text{and} \quad c_i = c_{s_{i-1},s_1}^{s_i}$$

satisfy the condition

(2.6.7) $$|\alpha s_i \cap \beta s_1| = \begin{cases} b_{i-1}, & \text{if } d(\alpha,\beta) = i-1, \\ c_{i+1}, & \text{if } d(\alpha,\beta) = i+1 \end{cases}$$

for any two vertices $\alpha, \beta \in \Omega$ and all i.

Example 2.6.9. *The undirected* 6*-cycle in Example* 2.6.2 *has diameter* 3*. The four basis relations of its coherent configuration are exactly the relations defined by formula* (2.6.6) (*see Fig.* 2.5)*. A direct calculation shows that*

$$(b_0, b_1, b_2) = (2, 1, 1) \quad \text{and} \quad (c_1, c_2, c_3) = (1, 1, 2).$$

Definition 2.6.10. *A graph $\mathfrak{X}$ is said to be distance-regular if there exist integers $b_0, \ldots, b_{d-1}$ and $c_1, \ldots, c_d$ for which relations* (2.6.7) *hold for all vertices α and β and all suitable i. The sequence*

$$\mathrm{IA}(\mathfrak{X}) = \{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$$

is called the intersection array of $\mathfrak{X}$.

Note that $c_1 = 1$ and $|\alpha s_1| = b_0$ for all vertices α; in particular, $\mathfrak{X}$ is a regular graph of valency b_0. If the group $\mathrm{Aut}(\mathfrak{X})$ acts transitively on each relation s_i, then the graph $\mathfrak{X}$ is called *distance-transitive*.

Theorem 2.6.11. *Let $\mathfrak{X}$ be a distance-regular graph of diameter d and let S be the partition of Ω^2 into the relations s_i defined by formula* (2.6.6)*, $i = 0, \ldots, d$. Then*

(1) *$\mathcal{X} = (\Omega, S)$ is a symmetric scheme of rank $d+1$, and $\mathrm{WL}(\mathfrak{X}) = \mathcal{X}$;*
(2) *$b_i = c_{s_{i+1},s_1}^{s_i}$ and $c_i = c_{s_{i-1},s_1}^{s_i}$ for all suitable i;*
(3) *the intersection numbers of $\mathcal{X}$ are uniquely determined by $\mathrm{IA}(\mathfrak{X})$;*
(4) *$\mathcal{X}$ is schurian if and only if $\mathfrak{X}$ is distance-transitive;*
(5) *$\mathcal{X}$ is separable if and only if $\mathfrak{X}$ is uniquely determined by $\mathrm{IA}(\mathfrak{X})$.*

Proof. The arc set of $\mathfrak{X}$ is equal to s_1. Therefore if α and β are vertices of $\mathfrak{X}$ at distance $i \in \{1, \ldots, d-1\}$, then the set $\beta s_j \cap \alpha s_1$ is obviously empty for all j other than $i-1$, i, and $i+1$. Thus,

$$\alpha s_1 \subseteq \beta s_{i-1} \cup \beta s_i \cup \beta s_{i+1}.$$

In particular, the number $a_i = |\alpha s_1 \cap \beta s_i|$ is equal to $b_0 - c_i - b_i$ (recall that $b_0 = |\alpha s_1|$). Since the graph $\mathfrak{X}$ is distance-regular, formula (2.6.7) implies that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}, \tag{2.6.8}$$

where A_i is the adjacency matrix of s_i for all i. Furthermore, $A_0 = I$, and also $c_{i+1} \neq 0$ by the connectedness of $\mathfrak{X}$. Therefore using induction on i, one can see that A_i is a polynomial in A_1 of degree i.

Now the linear space

$$\mathcal{A} = \mathrm{Span}\{A_0, A_1, \ldots, A_d\}$$

is closed under multiplication by A_1. Therefore, $\mathcal{A}$ is a commutative subalgebra in Mat_Ω. Moreover, the matrices $A_0, A_1, \ldots, A_d$ are $\{0,1\}$, symmetric, and form a linear basis of $\mathcal{A}$. Consequently, $\mathcal{A}$ is a coherent algebra (Corollary 2.3.8) and $\mathcal{X}$ is a symmetric scheme of rank $d+1$. Since $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ is a minimal coherent algebra containing A_{s_1}, we conclude that

$$\mathcal{X} = \mathrm{WL}(\mathfrak{X}), \tag{2.6.9}$$

which completes the proof of statement (1).

Statement (2) follows from formula (2.6.8). This formula allows to express the coefficients of the polynomial in A_1 defining A_i via the intersection array of $\mathfrak{X}$. Therefore, the intersection numbers of $\mathcal{X}$ are uniquely determined by $\mathrm{IA}(\mathfrak{X})$. This proves statement (3) and hence statement (5). Finally, formulas (2.6.4) and (2.6.9) imply that $\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathfrak{X})$, and hence the schurity of $\mathcal{X}$ exactly means that the graph $\mathfrak{X}$ is distance-transitive. This proves statement (4) and completes the proof of the theorem. □

We will not go into the theory of distance-regular graphs. Instead, we present three infinite classical families of distance-transitive graphs (all the details can be found in [**17**]). By statement (4) of Theorem 2.6.11, the corresponding schemes are schurian. In Subsection 4.2.3, we return to the question of their separability.

The Hamming graph. Let $d \geq 0$ and $q \geq 2$ be integers. The *Hamming graph* $H(d,q)$ has vertex set $\Omega = \{1, \ldots, q\}^d$ and two vertices are adjacent if and only if the corresponding d-tuples differ in exactly one coordinate. Thus, $H(d,2)$ is the graph of a d-dimensional cube. For all d and q, the Hamming graph is distance-transitive, has diameter d, and

$$\mathrm{Aut}(H_{d,q}) = \mathrm{Sym}(q) \wr \mathrm{Sym}(d)$$

(the wreath product is taken in the primitive action).

The coherent configuration of the graph $H(d,q)$, the *Hamming scheme*, is a symmetric scheme of degree q^d and rank $d+1$; the ith basis relation of this scheme is of the form

$$s_i = \{(\alpha,\beta)\in\Omega^2 :\ |\{j:\ \alpha_j\neq\beta_j\}| = i\},$$

and has valency $\binom{d}{i}(q-1)^i$, where α_j and β_j are the jth entries of the d-tuples α and β, respectively.

The Johnson graph. Let n,k be nonnegative integers, $k\leq n$. The *Johnson graph* $J(n,k)$ has vertex set $\binom{n}{k}$ (the k-subsets of $\{1,\dots,n\}$) and two vertices are adjacent if the corresponding k-subsets have exactly $k-1$ common elements. Thus, $J(n,1)$ is a complete graph with n vertices. For all possible n and k, the Johnson graph $J(n,k)$ is distance-transitive, has diameter $d=\min\{k,n-k\}$, and

$$\mathrm{Aut}(J_{n,k}) = \mathrm{Sym}(n)^{\binom{n}{k}},\qquad n\neq 2k,$$

where the group in the right-hand side is induced by the action of $\mathrm{Sym}(n)$ on the set $\binom{n}{k}$.

The coherent configuration of the graph $J(n,k)$, the *Johnson scheme*, is a symmetric scheme of degree $\binom{n}{k}$ and rank $d+1$; the ith basis relation of this scheme is of the form

$$s_i = \{(\alpha,\beta)\in\Omega^2 :\ |\alpha\cap\beta| = k-i\},$$

and has valency $\binom{k}{k-i}\binom{n-k}{i}$.

The Grassmann graph. Let n and k be integers, $1\leq k\leq n-1$. The *Grassmann graph* $J_q(n,k)$ has as the vertex set Ω the k-subspaces of an n-dimensional linear space $V_{n,q}$ over the field $\mathbb{F}_q$, and two vertices α and β are adjacent if and only if the subspace $\alpha\cap\beta$ has dimension $k-1$. Thus, $J_q(n,1)$ and $J_q(n,n-1)$ are complete graphs. For all q,n,k, the Grassmann graph is distance-transitive, has diameter $d=\min(k,n-k)$, and

$$\mathrm{Aut}(J_q(n,k)) = \mathrm{P\Gamma L}(V_{n,q}),\qquad 1<n<k \text{ and } 2k\neq n.$$

The coherent configuration of the graph $J_q(n,k)$, the *Grassmann scheme*, is a symmetric scheme of degree

$$\begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{(q^n-1)\cdots(q^n-q^{k-1})}{(q^k-1)\cdots(q-1)}$$

and rank $1+\min\{k,n-k\}$; the ith basis relation of this scheme is of the form

$$s_i = \{(\alpha,\beta)\in\Omega^2 :\ \dim(\alpha\cap\beta) = k-i\}$$

and

$$n_{s_i} = \begin{bmatrix} k \\ k-i \end{bmatrix}_q \begin{bmatrix} n-k \\ i \end{bmatrix}_q .$$

2.6.3 Strongly regular graphs

An undirected graph $\mathfrak{X}$ with arc set s is said to be *strongly regular* if there exist nonnegative integers k, λ, μ such that for any two vertices α and β,

$$|\alpha s \cap \beta s| = \begin{cases} k, & \text{if } \alpha = \beta, \\ \lambda, & \text{if } (\alpha,\beta) \in s, \\ \mu, & \text{if } \alpha \ne \beta \text{ and } (\alpha,\beta) \notin s. \end{cases} \tag{2.6.10}$$

The number n of vertices together with the numbers k, λ, μ are called the *parameters* of $\mathfrak{X}$. In what follows, we exclude the trivial cases $k = 0$ and $k = n - 1$, in which $\mathfrak{X}$ is the empty and complete graph, respectively.

Example 2.6.12. *Any strongly regular graph with parameters* $(5, 2, 0, 1)$ *is isomorphic to a pentagon. Any strongly regular graph with parameters* $(10, 3, 0, 1)$ *is isomorphic to the Petersen graph, i.e., the complement of the Johnson graph* $J(5, 2)$ (*see Fig.* 2.6).

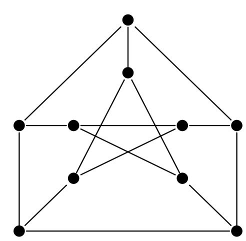


FIGURE 2.6 The Petersen graph.

Let $\mathfrak{X}$ be a connected strongly regular graph. Then there are at least two vertices at distance 2 unless $\mathfrak{X}$ is a complete graph. It follows that $\mu > 0$ and hence the diameter of $\mathfrak{X}$ is equal to 2. Define the relations s_0, s_1, and s_2 by formula (2.6.6). Then from the definition of λ and μ, it easily follows that condition (2.6.7) is satisfied for $i = 1$, $b_0 = k$, $c_2 = \mu$, and for $i = 2$, $b_1 = k - 1 - \lambda$; for $i = 0$, the condition is satisfied trivially. This proves the following statement.

Proposition 2.6.13. *A connected strongly regular graph with parameters* (n, k, λ, μ) *is either complete or a distance-regular graph of diameter* 2 *with the intersection array* $(k, k - 1 - \lambda; 1, \mu)$.

Proposition (2.6.13) and statement (1) of Theorem 2.6.11 imply that the coherent configuration $\mathcal{X}$ of a connected non-complete strongly regular graph $\mathfrak{X}$ is a symmetric scheme of rank 3. The irreflexive basis relations $s = s_1$ and $s' := s_2$ have valencies k and $n - k - 1$, respectively, and

$$c_{ss}^{s} = \lambda \quad \text{and} \quad c_{ss}^{s'} = \mu.$$

By formula (2.1.8), this shows that

$$k^2 = n_s n_s = n_{s_0} c_{ss}^{s_0} + n_s c_{ss}^{s} + n_{s'} c_{ss}^{s'} = k + k\lambda + (n - k - 1)\mu,$$

which shows that the parameters of a strongly regular graph are not independent and satisfy the relation

$$k(k - 1 - \lambda) = (n - k - 1)\mu.$$

In contrast to general distance-regular graphs, the distance-transitive strongly regular graphs, called the *rank* 3 graphs, are in principle known. Indeed, for any rank 3 graph $\mathfrak{X}$, there exists a group $K \le \mathrm{Aut}(\mathfrak{X})$ acting transitively on the set of vertices, on the set of ordered pairs of adjacent distinct vertices, and on the set of ordered pairs of nonadjacent vertices. Such a group K is called a rank 3 group and all such groups are known, see [**25**]. Using this result, a solution to the schurity problem for the coherent configurations of rank at most 3 is given by the following theorem.

Theorem 2.6.14. *Let $\mathcal{X}$ be a coherent configuration of rank 3 and $\mathfrak{X}$ a basis graph of an irreflexive basis relation of $\mathcal{X}$. Then one of the following two statements holds:*

(1) *$\mathcal{X}$ is symmetric and $\mathfrak{X}$ is strongly regular;*
(2) *$\mathcal{X}$ is antisymmetric and $\mathfrak{X}$ is a doubly regular tournament.*

Moreover, $\mathcal{X}$ is schurian if and only if either $\mathfrak{X}$ is a graph of rank 3 or isomorphic to a Paley tournament.

Proof. Denote by s and s' the two irreflexive basis relations of $\mathcal{X}$, and let the arc set of $\mathfrak{X}$ equal s. If the coherent configuration $\mathcal{X}$ is symmetric, then $\mathfrak{X}$ is an undirected loopless graph satisfying condition (2.6.10) with

$$k = n_s, \quad \lambda = c_{ss}^{s}, \quad \mu = c_{ss}^{s'}.$$

Thus in this case the graph $\mathfrak{X}$ is strongly regular and the schurity criterion for the scheme $\mathcal{X}$ immediately follows from the definition of a rank 3 graph.

To complete the proof, we assume that $\mathcal{X}$ is not symmetric. Then $s' = s^*$ and $\mathcal{X}$ is an antisymmetric scheme of rank 3. Thus, the required statements follow from Exercises 2.7.57 and 2.7.58. □

The separability question for the schemes of rank 3 seems to be hopeless. There are strongly regular graphs determined up to isomorphisms by its parameters, e.g., the pentagon, the Petersen graph, etc. In all these cases, the scheme of such a graph is separable (statement (5) of Theorem 2.6.11). On the other hand, a lot of non-separable schemes of rank 3 can be constructed as fusions of the affine schemes.

Example 2.6.15. *Let $\mathcal{X}$ be the scheme of the affine Galois plane of prime order $p > 2$. From Theorem 2.5.8, it follows that for each positive integer $m \le p + 1$, there are*

$$N = \binom{p+1}{m} \approx (p+1)^m$$

distinct equivalenced schemes of rank 3: each of them is the algebraic fusion $\mathcal{X}^\Phi$, where $\Phi \le \mathrm{Sym}(p+1)$ is the setwise stabilizer of an m-subset of $S(\mathcal{X})^\#$.

For a fixed m, all the above schemes are algebraically isomorphic and can be treated as Cayley schemes over the same group $C_p \times C_p$. Among them, there are at most

$$N_0 = \frac{\mathrm{Aut}(C_p \times C_p)}{|\mathbb{F}^\times|} = \frac{|\,\mathrm{GL}(2,p)|}{|p^2-1|} = \frac{(p^2-1)(p^2-p)}{(p^2-1)}$$

cyclotomic schemes of rank 3 over a field $\mathbb{F} = \mathbb{F}_{p^2}$: these schemes are in a one-to-one correspondence with groups

$$C_{p^2-1} \le \mathrm{GL}(2,p)$$

considered as the multiplicative groups of the fields $\mathbb{F} \subset \mathrm{Mat}_2(\mathbb{F}_p)$. It follows that each of these schemes is not separable whenever $N > N_0$.

The latter inequality is obviously true if $p \ge 5$ and $m = (p+1)/2$. Under the latter condition, $N > |\,\mathrm{GL}(2,p)|$ which implies that there are exponentially many schemes $\mathcal{X}^\Phi$ that are not schurian.

The construction in Example 2.6.15 can be generalized by replacing the affine scheme with any amorphic scheme defined as follows. For any coherent configuration $\mathcal{X}$ on Ω, and any partition Π of the set $S = S(\mathcal{X})$, put

$$S_\Pi = \big\{\bigcup_{s\in T} s : \ T \in \Pi\big\}.$$

Following [**52**], a scheme $\mathcal{X}$ is said to be *amorphic* if the pair

$$\mathcal{X}^\Pi = (\Omega, S_\Pi)$$

is a coherent configuration for any partition Π such that $\{1\} \in \Pi$. The examples of amorphic schemes include all schemes of rank 3, and all affine schemes (Exercise 2.7.59).

One can easily see that any amorphic scheme of rank at least 4 is symmetric. It follows that if $s \in S$ is irreflexive and

$$\Pi = \{\{1\}, \{s\}, S^\# \setminus \{s\}\},$$

then $\mathcal{X}^\Pi$ is a symmetric scheme of rank 3, i.e., the basis graph s is strongly regular (Theorem 2.6.14). Thus the basis graphs of an amorphic scheme are strongly regular. A survey on amorphic schemes can be found in [**120**].

Let us mention one more general construction of strongly regular graphs that can be treated as a faithful functor from the category of finite groups to the category of symmetric Cayley schemes of rank 3. Namely, given a group G, denote by $\mathfrak{X}_G$ a graph on $\Omega = G \times G$ with the arc set

$$s_G = \{(\alpha, \beta) \in \Omega^2 : \ \alpha_1 = \beta_1 \ \text{ or } \ \alpha_2 = \beta_2 \ \text{ or } \ \alpha_1^{-1}\alpha_2 \ = \beta_1^{-1}\beta_2 \ \}, \tag{2.6.11}$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are distinct.

The following statement is straightforward; some other properties of the graph $\mathfrak{X}_G$ can be found in Exercise 2.7.61.

Proposition 2.6.16. *Let G be a group of order $n > 1$. Then $\mathfrak{X}_G$ is a Cayley graph over the group $G \times G$, and strongly regular with parameters $(n^2, 3n - 3, n, 6)$.*

Example 2.6.17. *Let $G = C_4$ and $G' = C_2 \times C_2$. Then the graphs $\mathfrak{X}_G$ and $\mathfrak{X}_{G'}$ are strongly regular with parameters $(16, 9, 4, 6)$. The complement of the first graph is known as the Shrikhande graph, whereas the complement of the second one is isomorphic to the Hamming graph $H(2, 4)$.*

The schemes of $\mathfrak{X}_G$ and $\mathfrak{X}_{G'}$ are, respectively, non-schurian and schurian, both of degree 16, rank 3, and with valencies $\{1, 6, 9\}$. These schemes are algebraically isomorphic but not isomorphic (Exercise 2.7.61). In particular, none of them is separable.

We complete the section by mentioning a famous conjecture suggesting a combinatorial solution to the schurity problem for the class of symmetric schemes of rank 3. Before proceeding further, we need some preparation.

Let $\mathfrak{X}$ be an undirected graph. Following [**60**], two induced subgraphs of $\mathfrak{X}$ are said to be of the *same type with respect to a pair* (α, β) of vertices if both contain α and β and there exists an isomorphism of one onto the other mapping α to α and β to β.

Definition 2.6.18. *The graph $\mathfrak{X}$ satisfies the t-vertex condition for $t \geq 2$ if the number of t-vertex subgraphs of a given type with respect to a given pair (α, β) of vertices depends only on whether α and β are equal, adjacent, or non-adjacent.*

The following statement is obvious.

Proposition 2.6.19. *An undirected graph is regular (respectively, strongly regular, of rank 3) if and only if it satisfies the t-vertex condition for $t = 2$ (respectively, $t = 3$, $t = n$).*

The smallest examples of non-rank 3 strongly regular graphs satisfying the 4-vertex condition can be found in [**83**]. The largest known t for which there exists a non-rank 3 strongly regular graph satisfying the t-condition, is equal to 7; the corresponding example was found by S. Reichard in [**111**].

On the other hand, any strongly regular graph of the form $\mathfrak{X}_G$, where G is a group, is of rank 3 if and only if it satisfies the 4-condition [**34**]. A similar result is true for another large class of strongly regular graphs [**82**]. All of the above supports the following conjecture.

Conjecture 2.6.20. (M.Klin, see [**49**, p.74]) *There exists an integer* t_0 *such that any strongly regular graph satisfying the* t_0*-vertex condition is of rank* 3*.*

The concept of the t-condition can naturally be extended to colored graphs. Namely, let $\mathfrak{X}$ be a colored graph on Ω. Following [**49**, p.71], for any sets $\Delta, \Gamma \subseteq \Omega$, the induced colored graphs $\mathfrak{X}_\Delta$ and $\mathfrak{X}_\Gamma$ are said to be of the same type with respect to a pair $(\alpha, \beta) \in (\Delta \cap \Gamma)^2$ if there exists an isomorphism of one onto the other mapping α to α and β to β.

Now a colored graph satisfies the t-condition for $t \geq 2$ if the number of the induced t-vertex colored subgraphs of a given type with respect to a given pair (α, β) of vertices depends only on the color class containing (α, β).

The t-condition for a rainbow $\mathcal{X}$ is defined via associated colored graph; of course, the fulfillment of this condition does not depend on the chosen colors of standard coloring of $\mathcal{X}$. We will return to this concept in Subsection 4.2.2.

2.7 Exercises

In what follows, unless otherwise stated, $\mathcal{X}$ is a coherent configuration on Ω and $S = S(\mathcal{X})$, $F = F(\mathcal{X})$, and $E = E(\mathcal{X})$. The notations $\mathcal{X}'$ and Ω', S', F', and E' have the same meaning.

2.7.1 [86] The conditions (CC1), (CC2), and (CC3) are independent.

2.7.2 Find all coherent configurations of degree at most 4.

2.7.3 Denote by s_i the relation on the vertex set Ω of a three-dimensional cube that is defined by the property "to be at distance i", $i = 0, 1, 2, 3$. Then the pair (Ω, S) with $S = \{s_0, s_1, s_2, s_3\}$, is a coherent configuration.

2.7.4 Let $\Delta, \Gamma \in F$ and $s \in S_{\Delta,\Gamma}$. Then $\Omega_-(s) = \Delta$ and $\Omega_+(s) = \Gamma$. In particular, $\Omega_-(r)$, $\Omega_+(r)$, and $\Omega(r)$ are homogeneity sets of $\mathcal{X}$ for all $r \in S^\cup$.

2.7.5 Let $M \subset \mathbb{N}$ and $T \subseteq S^\cup$. Then $\{\alpha \in \Omega : \ |\alpha s| \in M \text{ for all } s \in T\}$ is a homogeneity set of $\mathcal{X}$.

2.7.6 Let $r, s, t \in S$ and $\Delta \in F$. Then

(1) $c_{rs}^{1_\Delta} \neq 0$ if and only if $s = r^*$ and $\Omega_-(r) = \Delta$;
(2) $c_{rs}^t \leq \min\{n_r, n_{s^*}\}$;
(3) $\sum_{s \in S_{\Gamma,\Delta}} n_s = |\Delta|$ for all $\Gamma \in F$;
(4) $\sum_{w \in S} c_{rs}^w c_{wu}^v = \sum_{w \in S} c_{rw}^v c_{su}^w$ for all $u, v \in S$.

2.7.7 [123, p.28] Let $\mathcal{X}$ be a scheme and $r, s, t \in S$. Then

(1) c_{rs}^t is a multiple of $\frac{n_s n_t \operatorname{GCD}(n_r, n_s, n_t)}{\operatorname{GCD}(n_r, n_s) \operatorname{GCD}(n_s, n_t) \operatorname{GCD}(n_t, n_r)}$;
(2) $n_t c_{rs}^t = 0 \pmod{m}$, where $m = \operatorname{LCM}(n_r, n_s)$.

2.7.8 Let $s \in S^\cup$. Then

(1) $e(s) = \{(\alpha, \beta) \in \Omega^2 : \ \alpha s = \beta s\}$ belongs to E;
(2) $s \cdot s^* \in E$ if $s \in S$ and $n_s = 1$.

2.7.9 Let $e \in E$. For $\alpha \in \Omega$ and $\Delta \in \Omega/e$, set

$$S(\alpha, \Delta) = \{s \in S : \ \alpha s \cap \Delta \neq \varnothing\}.$$

Then

(1) for any $\alpha' \in \Omega$, the sets $S(\alpha, \Delta)$ and $S(\alpha', \Delta)$ are equal or disjoint;
(2) for any $\Delta' \in \Omega/e$, the sets $S(\alpha, \Delta)$ and $S(\alpha, \Delta')$ are equal or disjoint.

2.7.10 Let $e \in E$ and $\Delta \in F$ be such that $e_\Delta \neq \varnothing$. Then $e \cdot 1_\Delta \cdot e$ is an indecomposable component of e.

2.7.11 Let $s \in S$ and $e \in E$. Then

(1) the number $|\alpha s \cap \Delta|$ does not depend on $\alpha \in \Omega$ and $\Delta \in \Omega/e$ for which $\alpha s \cap \Delta \neq \varnothing$;
(2) if $\Omega(s) \subseteq \Omega(e)$ and $e \cdot s = s \cdot e$, then $n_{s_{\Omega/e}}$ divides n_s.

2.7.12 Let $\mathcal{X}$ be a regular scheme. Then

(1) the closed subsets of S and the subgroups of S_1 are in a one-to-one correspondence;
(2) any fission of $\mathcal{X}$ is semiregular.

2.7.13 Let $\mathcal{X}$ be a semiregular coherent configuration. Then

(1) $|\Omega| = |\Delta| \cdot |F|$ and $|S| = |F|^2 \cdot |\Delta|$ for all $\Delta \in F$;
(2) if $\Delta, \Gamma \in F$ and $s \in S_{\Delta,\Gamma}$, then $f_s \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_\Gamma)$;
(3) there exists a full system T of distinct representatives of the family $\{S_{\Delta,\Gamma}\}_{\Delta,\Gamma \in F}$ such that such that every nonempty composition $r \cdot s$ with $r, s \in T$, belongs to T.

2.7.14 Let $s \in S$ be such that ss^* consists of thin relations. Then

$$ss^*s = \{s\}.$$

2.7.15 Let e be the equivalence relation on Ω such that $\Omega/e = F$. Then $e \in E$ and $e \cdot s = s \cdot e$ for all $s \in S$.

2.7.16 Let $\mathcal{X}$ be a cyclotomic scheme over a field $\mathbb{F}$. Then

$$\mathrm{A\Gamma L}(1, \mathbb{F}) \le \mathrm{Iso}(\mathcal{X}).$$

2.7.17 Let $K \le \mathrm{Sym}(\Omega)$ and $\mathcal{X} = \mathrm{Inv}(K, \Omega)$. Then

(1) $S^\cup$ equals the set of all K-invariant relations on Ω;
(2) if $e \in E$ and $\Delta \in \Omega/e$, then $\mathcal{X}_\Delta = \mathrm{Inv}(K^\Delta, \Delta)$;
(3) K is of odd oder if and only if $\mathcal{X}$ is antisymmetric;
(4) K is a p-group if and only if $|s|$ is a p-power for each $s \in S$.[6]

2.7.18 Let $\mathcal{X}$ be a schurian coherent configuration. Then the group $\mathrm{Iso}(\mathcal{X})$ equals the normalizer of $\mathrm{Aut}(\mathcal{X})$ in $\mathrm{Sym}(\Omega)$.

2.7.19 Let $\mathcal{X}$ be a *quasiregular* coherent configuration, i.e., every its homogeneous component is regular. Then the group $\mathrm{Aut}(\mathcal{X})$ is abelian if each homogeneous component of $\mathcal{X}$ is commutative. The converse is true if $\mathcal{X}$ is schurian.

2.7.20 [**86**] In the notation of Theorem 2.2.7, suppose that the group K is transitive and H is a point stabilizer of K. Then for any $r, s, t \in S$, the number $|H|c_{rs}^t$ is equal to the number of distinct decomposition $z = xy$ with a fixed $z \in D_{t^*}$ and all suitable $x \in D_{r^*}$ and $y \in D_{s^*}$.

2.7.21 Let $e \in E$ and $\Delta \in \Omega/e$. Then

(1) the mapping $S_\Delta \to S$, $s_\Delta \mapsto s$ is an injection; it induces injections from $F(\mathcal{X}_\Delta)$ and $E(\mathcal{X}_\Delta)$ to F and E, respectively;
(2) the coherent configuration $\mathcal{X}_\Delta$ is schurian whenever so is $\mathcal{X}$;
(3) the restriction of a schurian coherent configuration to any homogeneity set is schurian.

2.7.22 For any 2-orbit s of the group $\mathrm{Sym}(\Omega)$ acting on Ω^m ($m \ge 1$), there exists an equivalence relation e on $\{1, \ldots, 2m\}$ such that

$$s = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : \ (\alpha \cdot \beta)_i = (\alpha \cdot \beta)_j \ \Leftrightarrow \ (i, j) \in e\}.$$

Conversely, any such s is a 2-orbit of $\mathrm{Sym}(\Omega)$ acting on Ω^m.

---

[6] To prove the sufficiency, one can use a result [**110**, Theorem 1.1] stating that if $\mathcal{X}$ is a coherent configuration such that $|s|$ is a p-power for each $s \in S$, then any fission of $\mathcal{X}$ has this property.

2.7.23 Let $K \le \mathrm{Sym}(\Omega)$. Then

(1) $K^{(1)}$ equals the direct product of $\mathrm{Sym}(\Delta)$, $\Delta \in \mathrm{Orb}(K)$;
(2) K is 2-transitive if and only if $K^{(2)} = \mathrm{Sym}(\Omega)$;
(3) $(K^{(a)})^{(b)} = K^{(m)}$, where $m = \min\{a, b\}$;
(4) if $L \le K$, then $L^{(m)} \le K^{(m)}$.

2.7.24 [39] Given a matrix $A \in \mathrm{Mat}_\Omega$, set

$$e(A) = \{(\alpha,\beta) \in \Omega^2 : \; A\alpha = A\beta \neq 0\}. \tag{2.7.1}$$

Then $e(A) \in E$ whenever $A \in \mathrm{Adj}(\mathcal{X})$.

2.7.25 Let $m \ge 2$ be an integer, $r \in S$, and $s_1, \ldots, s_{m-1} \in S^\cup$. Then the number $p_r(\alpha, \beta; s_1, \ldots, s_{m-1})$ of all tuples $(\alpha_1, \ldots, \alpha_m) \in \Omega^m$ such that

$$(\alpha_1, \alpha_m) = (\alpha, \beta) \quad \text{and} \quad r(\alpha_i, \alpha_{i+1}) = s_i, \; i = 1, \ldots, m-1,$$

does not depend on the choice of $(\alpha, \beta) \in r$.

2.7.26 The scalar product on the adjacency algebra $\mathrm{Adj}(\mathcal{X})$ defined by the formula

$$\langle \sum_{s \in S} c_s A_s, \sum_{s \in S} b_s A_s \rangle = \frac{1}{|\Omega|} \sum_{s \in S} c_s \bar{b}_s |s|$$

is associative, i.e., $\langle AB, C\rangle = \langle B, A^*C\rangle$ for all $A, B, C \in \mathrm{Adj}(\mathcal{X})$.

2.7.27 [102, Lemma 2.3] If $\mathcal{X}$ is a scheme and $r, s \in S^\#$, then

$$rr^* \cap ss^* = \{1_\Omega\} \quad \Leftrightarrow \quad c_{r^*s}^t \le 1 \text{ for all } t \in S.$$

2.7.28 Let s be a relation of $\mathcal{X}$. Then so is $\{(\alpha, \beta) \in \Omega^2 : \; \alpha \xrightarrow{s} \beta\}$. Moreover, if $\mathcal{X}$ is a scheme, then this relation is symmetric.

2.7.29 Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ and $r, s \in S^\cup$. Then

(1) $\varphi(r \cup s) = \varphi(r) \cup \varphi(s)$ and $\varphi(r \cap s) = \varphi(r) \cap \varphi(s)$;
(2) $\varphi(\langle s \rangle) = \langle \varphi(s) \rangle$ and $\varphi(\mathrm{rad}(s)) = \mathrm{rad}(\varphi(s))$.

2.7.30 Every algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ induces a lattice isomorphism from E to E'.

2.7.31 Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, e an indecomposable partial parabolic of $\mathcal{X}$, $\Delta \in \Omega/e$, and $e' = \varphi(e)$. Then for any $\Delta' \in \Omega'/e'$, the bijection

$$\varphi_{\Delta, \Delta'} : S_\Delta \to S'_{\Delta'}, \; s_\Delta \mapsto \varphi(s)_{\Delta'}$$

is an algebraic isomorphism from $\mathcal{X}_\Delta$ to $\mathcal{X}'_{\Delta'}$.

2.7.32 If one of two algebraically isomorphic coherent configurations is half-homogeneous (respectively, homogeneous, equivalenced, regular, semiregular, quasiregular), then so is the other.

2.7.33 The coherent configuration of a dihedral group D_{2n} of degree n is separable for all $n \ge 1$.

2.7.34 [72] Every quasiregular coherent configuration with at most three fibers is schurian and separable.

2.7.35 Every semiregular coherent configuration is schurian and separable.

2.7.36 Let $K \le \mathrm{Iso}(\mathcal{X})$. Then $K \le \mathrm{Aut}(\mathcal{X}^K)$.

2.7.37 Let $\Psi \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$, $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, and $\Psi' = \varphi\Psi\varphi^{-1}$. Then

(1) $\Psi' \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}')$;

(2) $\varphi_\Psi : S(\mathcal{X}^\Psi) \to S(\mathcal{X}'^{\Psi'})$, $s^\Psi \mapsto \varphi(s)^{\Psi'}$ is a well-defined bijection;

(3) $\varphi_\Psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}^\Psi, \mathcal{X}'^{\Psi'})$.

2.7.38 Find an example of schurian scheme which is the algebraic fusion of a non-schurian scheme.

2.7.39 Let G be a group, $K = \langle G_{right}, G_{left}\rangle$, and $\mathcal{X} = \mathrm{Inv}(K, G)$. Then

(1) the stabilizer K_1 of the identity of G in K equals $\mathrm{Inn}(G)$;

(2) $\mathrm{Orb}(K_1, G) = \{x^G :\ x \in G\}$;

(3) $\mathrm{Adj}(\mathcal{X})$ is isomorphic to the center of $\mathbb{C}G$;

(4) the scheme $\mathcal{X}$ is commutative.

2.7.40 Let $\mathcal{X}$ be a Cayley scheme and $\mathcal{X} \ge \mathcal{X}'$. Then $\mathcal{X}$ is normal whenever so is $\mathcal{X}'$.

2.7.41 Let $\mathcal{X}$ be a cyclotomic scheme over a group G, H a characteristic subgroup of G, and ρ the mapping in Exercise 1.4.15. Then $H^\rho \in E$.

2.7.42 Let $\mathfrak{A}$ and $\mathfrak{A}'$ be S-rings over groups G and G', respectively. Then

(1) a ring isomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is an algebraic isomorphism if and only if $\underline{X}^\varphi \in \underline{\mathcal{S}}(\mathfrak{A}')$ for all $X \in \mathcal{S}(\mathfrak{A})$;

(2) a bijection $f : G \to G'$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$ if and only if there exists an algebraic isomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}'$ such that

$$f(Xy) = X^\varphi y^f \quad \text{for all } \ X \in \mathcal{S}(\mathfrak{A}),\ y \in G.$$

2.7.43 Let Ω be the set of flags of a projective plane of order q, where the flag is a pair of a point and a line incident to it. Every two flags (p, l) and (p', l') belongs to one of the relations in the set $S = \{s_0, \dots, s_5\}$ that are defined as in Fig. 2.7, where the double line and arrow denote

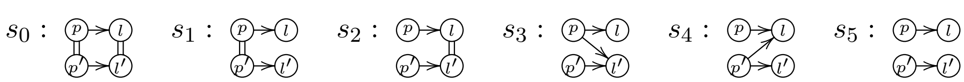


FIGURE 2.7 The scheme on flags of a projective plane: basis relations.

the equality and incidence, respectively, and the absence of any line means general position. For example, $s_0 = 1$ and s_1 consists the pairs of flags having common point. Then

(1) $s_i = s_i^*$ if and only if $i \ne 3, 4$, and $s_3^* = s_4$;

(2) $(s_3, s_4) = (s_1 \cdot s_2,\ s_2 \cdot s_1)$ and $s_5 = s_1 \cdot s_2 \cdot s_1 = s_2 \cdot s_1 \cdot s_2$;

(3) the rainbow (Ω, S) is a scheme of degree $(q^2 + q + 1)(q + 1)$ and rank 6.

2.7.44 Any scheme algebraically isomorphic to the scheme associated with a projective (respectively, affine) plane, is associated with a projective (respectively, affine) plane of the same order.

2.7.45 Among the affine schemes, there exist

(1) schurian schemes, which are not separable;
(2) normal Cayley schemes, which are not schurian.

2.7.46 In any (n, k, λ)-design, the number r of blocks containing a point does not depend on the choice of this point. Moreover,

$$nr = bk \quad \text{and} \quad \lambda(n-1) = r(k-1),$$

where b is the number of blocks.

2.7.47 A design $\mathfrak{D}$ is said to be *symmetric* if the number of blocks is equal to the number of points. The following three statements are equivalent:

(1) $\mathfrak{D}$ is symmetric;
(2) any two distinct blocks of $\mathfrak{D}$ have the same number of common points;
(3) $\mathfrak{D}$ is a coherent design and the corresponding coherent configuration of $\mathfrak{D}$ has type $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$.

2.7.48 [**22**] Define a system of linked designs to be a set $\{\Omega_1, \ldots, \Omega_m\}$ of sets ($m \geq 3$) and an incidence relation $I_{ij} \subseteq \Omega_i \times \Omega_j$ for all distinct i and j, such that for all distinct i, j, and k,

(LD1) the pair $(\Omega_i, \{\alpha I_{ji} : \ \alpha \in \Omega_j\})$ is a symmetric design;
(LD2) the number of elements in Ω_k incident with both $\alpha \in \Omega_i$ and $\beta \in \Omega_j$ depends only on whether or not $(\alpha, \beta) \in I_{ij}$.

Then every such system defines a coherent configuration $\mathcal{X}$ on the union of the Ω_i, such that

(1) $F = \{\Omega_1, \ldots, \Omega_m\}$;
(2) $\mathcal{X}_{\Omega_i} = \mathcal{T}_{\Omega_i}$ for all i;
(3) $S_{\Omega_i, \Omega_j} = \{I_{ij}, I'_{ij}\}$ for all $i \neq j$, where $I'_{ij} = (\Omega_i \times \Omega_j) \setminus I_{ij}$.

2.7.49 The mapping (2.6.1) is a closure operator in the class of all rainbows $\mathcal{X}$ on Ω, i.e., the following statements hold:

(1) $\mathcal{X} \leq \mathrm{WL}(\mathcal{X})$;
(2) if $X \leq \mathcal{X}'$, then $\mathrm{WL}(\mathcal{X}) \leq \mathrm{WL}(\mathcal{X}')$;
(3) $\mathrm{WL}(\mathrm{WL}(\mathcal{X})) = \mathrm{WL}(\mathcal{X})$.

2.7.50 Let S and T be sets of relations on Ω. Assume that $S^{\cup} \subseteq T^{\cup}$. Then $\mathrm{WL}(S) \leq \mathrm{WL}(T)$.

2.7.51 Let T be a collection of relations on Ω. Denote by S the partition of Ω^2 such that (α, β) and (α', β') belong to the same class if and only if

$$\forall t \in \{1_\Omega\} \cup T \cup T^* : \qquad (\alpha, \beta) \in t \ \Leftrightarrow \ (\alpha', \beta') \in t.$$

Then (Ω, S) is a rainbow and $\mathrm{WL}(T) = \mathrm{WL}(S)$.

2.7.52 Let $\mathfrak{X} = (\Omega, D)$ be a colored graph, and let φ be an algebraic isomorphism from $\mathcal{X} = \mathrm{WL}(\mathcal{P}_{c_\mathfrak{X}})$ to another coherent configuration. Define a graph $\mathfrak{X}' = \mathfrak{X}^\varphi$ by

$$\Omega(\mathfrak{X}') = \Omega^\varphi \quad \text{and} \quad D(\mathfrak{X}') = D^\varphi$$

with a coloring $c_{\mathfrak{X}'}$ each color class of which is of the form $(c_{\mathfrak{X}}^{-1}(i))^\varphi$ for some color i of $c_{\mathfrak{X}}$. Then the colored graphs $\mathfrak{X}$ and $\mathfrak{X}^\varphi$ are isomorphic if and only if φ is induced by an isomorphism.

2.7.53 Let $\mathfrak{X}$ be an undirected cycle on n vertices. Then $\mathrm{WL}(\mathfrak{X}) = \mathrm{Inv}(D_{2n})$.

2.7.54 Let $\mathfrak{X}$ be a vertex-disjoint union of two connected graphs $\mathfrak{X}_1$ and $\mathfrak{X}_2$ on Ω_1 and Ω_2, respectively. Assume that $\Delta \in F(\mathrm{WL}(\mathfrak{X}))$ is such that

$$|\Delta \cap \Omega_1| \neq |\Delta \cap \Omega_2|.$$

Then the graphs $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are not isomorphic.

2.7.55 Let $\mathfrak{X}$ be a graph and φ an algebraic isomorphism from $\mathrm{WL}(\mathfrak{X})$ to another coherent configuration. Then

(1) if $s_d(\mathfrak{X})$ is the relation on $\Omega(\mathfrak{X})$ consisting of all pairs of vertices at distance d in $\mathfrak{X}$, then $s_d(\mathfrak{X})^\varphi = s_d(\mathfrak{X}^\varphi)$;

(2) if the graph $\mathfrak{X}$ is distance-regular, then the graph $\mathfrak{X}^\varphi$ is also distance-regular and $\mathrm{IA}(\mathfrak{X}) = \mathrm{IA}(\mathfrak{X}^\varphi)$.

2.7.56 Let $\mathfrak{X}$ be a connected but not 2-connected undirected graph[7] with at least 3 vertices. Then the coherent configuration of $\mathfrak{X}$ is not homogeneous.

2.7.57 Let $\mathcal{X}$ be an antisymmetric scheme of rank 3, and $S = \{s_0, s_1, s_2\}$, where $s_0 = 1$. Then the basis graphs of s_1 and s_2 are doubly regular tournaments, $n_{s_1} = n_{s_2} = (n-1)/2$, and the intersection numbers of $\mathcal{X}$ are determined from the formulas

$$c_{11}^0 = 0, \quad c_{12}^0 = \frac{n-1}{2}, \quad c_{11}^1 = c_{12}^1 = c_{12}^2 = \frac{n-3}{4}, \quad c_{11}^2 = \frac{n+1}{4}, \tag{2.7.2}$$

where $c_{ij}^k = c_{s_i s_j}^{s_k}$ for all i, j, k. In particular, $n = 3 \pmod 4$.

2.7.58 [12] An antisymmetric scheme of rank 3 is schurian if and only if each irreflexive basis graph is isomorphic to a Paley tournament.

2.7.59 [52] The following statements hold:

(1) any affine scheme is amorphic;

(2) the degree of any amorphic scheme of rank at least 4 is a square.

2.7.60 [102] A finite affine plane is Desarguesian if and only if the corresponding scheme satisfies the 4-condition.

2.7.61 [58, 34] For a group G, denote by $\mathcal{X}_G$ the scheme of the strongly regular graph $\mathfrak{X}_G$ defined by formula (2.6.11). Then

(1) $\mathrm{Aut}(\mathcal{X}_G) \cong ((G \times G)\,\mathrm{Aut}(G))\,\mathrm{Sym}(3)$ whenever $|G| \geq 5$;

(2) $\mathcal{X}_G$ is schurian if and only if it satisfies the 4-condition;

[7] An undirected graph is said to be k-connected if no two of its vertices are separated by fewer than k other vertices.

(3) $\mathcal{X}_G$ and $\mathcal{X}_{G'}$ are algebraically isomorphic if and only if $|G| = |G'|$;
(4) $\mathcal{X}_G$ and $\mathcal{X}_{G'}$ are isomorphic if and only if G and G' are isomorphic.

2.7.62 A complete colored n-vertex graph $\mathfrak{X}$ satisfies the t-vertex condition for $t = 3$ (respectively, for $t = n$) if and only if the color classes of $\mathfrak{X}$ form a coherent configuration (respectively, a schurian coherent configuration).

CHAPTER 3

Machinery and constructions

This chapter fully deserves the title "group theory without groups". Indeed, all the concepts and constructions studied here have direct analogs in the permutation group theory. Based on the Galois correspondence between the coherent configurations and permutation groups, one can, more or less in a natural way, translate the imprimitivity, direct sum and product, as well as wreath product into the language of coherent configurations.

In order to do the same for the stabilizers and actions on tuples, one needs the concept of coherent closure (Subsection 2.6.1). The translation of the representation theory goes much more smoothly due to the fact that the representations of permutation groups and coherent configurations can be considered in the framework of the representation theory of coherent algebras.

3.1 Primitivity and quotients

Let e be an equivalence relation on a set Ω. It is easily seen that the permutation groups on Ω that leave e fixed, form a lattice. Analogously, one can consider the lattice of coherent configurations on Ω that contain e as a partial parabolic. In this subsection, we will see that the Galois correspondence between the coherent configurations and permutation groups on Ω induces the Galois correspondence between these two lattices associated with e.

The results from Subsections 3.1.1 and 3.1.2 are essentially contained in Chapters H, I, and K of book [**123**]. The concept of thin residue defined in Subsection 3.1.3 was introduced and studied for homogeneous coherent configurations in [**128**]. In our presentation, we follow papers [**45, 103**].

3.1.1 Primitive and imprimitive schemes

Let $\mathcal{X}$ be a scheme on Ω.

Definition 3.1.1. *We say that $\mathcal{X}$ is primitive if every parabolic of $\mathcal{X}$ equals 1_Ω or Ω^2; otherwise $\mathcal{X}$ is said to be imprimitive.*

Every scheme of prime degree is primitive (Corollary 2.1.23), whereas a regular scheme of composite degree is always imprimitive (statement (4) of Theorem 2.1.25).

Example 3.1.2. *Let $\mathcal{X}$ be an imprimitive scheme of rank 3. Then there is a nontrivial parabolic $e \in E(\mathcal{X})$. It is the union of at least two basis* *relations, $s_0 = 1_\Omega$ and s_1, and the relation $s_2 = \Omega^2 \setminus e$ is nonempty. Since* $\mathrm{rk}(\mathcal{X}) = 3$*, this implies that*

$$S(\mathcal{X}) = \{s_0, s_1, s_2\}.$$

It follows that $e = s_0 \cup s_1$ and the graph of s_1 is the disjoint union of cliques of the same order.

A statement below, which is an immediate consequence of statement (1) of Exercise 2.7.21, indicates that every imprimitive scheme has nontrivial "primitive parts".

Proposition 3.1.3. *Let $\mathcal{X}$ be a scheme and $e \ne 1$ a minimal parabolic of $\mathcal{X}$. Then the restriction of $\mathcal{X}$ to any class of e is primitive.*

The primitivity of a schurian scheme can easily be determined with the help of the corresponding permutation group. In order to see this, we recall that a transitive group $K \le \mathrm{Sym}(\Omega)$ is primitive if and only if no nontrivial equivalence relation on Ω is K-invariant. However, all K-invariant equivalence relations on Ω are exactly the parabolics of the scheme $\mathrm{Inv}(K, \Omega)$ (statement (1) of Exercise 2.7.17). Thus the following statement holds.

Proposition 3.1.4. *Let K be a transitive group. Then the scheme* $\mathrm{Inv}(K)$ *is primitive (respectively, imprimitive) if and only if the group K is primitive (respectively, imprimitive).*

Proposition 3.1.4 shows that one of the two mapping in the Galois correspondence (namely, that from permutation groups to coherent configurations) respects the property "to be primitive". However, the other mapping does not. First, because the automorphism group $\mathrm{Aut}(\mathcal{X})$ of a primitive scheme $\mathcal{X}$ is not necessarily transitive (for example, if $\mathcal{X}$ is the antisymmetric scheme of degree 15 and rank 3, then a computer computation shows that the group $\mathrm{Aut}(\mathcal{X})$ has three orbits of lengths 1, 7, and 7). Second, even if $\mathrm{Aut}(\mathcal{X})$ is transitive, then it is not necessarily primitive (for example, take any scheme $\mathcal{X}_G$ in Exercise 2.7.61 for which the group G has a nontrivial characteristic subgroup).

Example 3.1.2 shows that a scheme of rank 3 is imprimitive if and only if one of its irreflexive basis graphs (namely, s_1) is the union of at least two complete loopless graphs of the same order. In particular, this graph is disconnected. In fact, the absence of such graphs in a scheme is necessary and sufficient for the scheme to be primitive.

Theorem 3.1.5. *A scheme is primitive if and only if each irreflexive basis relation of it is strongly connected.*

Proof. It suffices to verify that a scheme $\mathcal{X}$ is imprimitive if and only if at least one of its irreflexive basis relations is not strongly connected. First, we assume that $\mathcal{X}$ is imprimitive. Then there is a nontrivial parabolic

$e \in E(\mathcal{X})$. Take two distinct classes Δ and Γ, and an irreflexive basis relation $s \subseteq e$. We have

$$\alpha \in \Delta \text{ and } \alpha \xrightarrow{s} \alpha' \quad \Rightarrow \quad \alpha' \in \Delta.$$

Thus there is no s-path connecting α with a point belonging to Γ, i.e., s is not strongly connected.

Conversely, let s be an irreflexive basis relation of $\mathcal{X}$ that is not strongly connected. By Exercise 2.7.28, the relation $e = \{(\alpha, \beta) : \alpha \xrightarrow{s} \beta\}$ is a parabolic of $\mathcal{X}$. Moreover, e is not trivial, because s is neither reflexive nor strongly connected. Thus the scheme $\mathcal{X}$ is imprimitive. □

Example 2.6.15 shows that there are many primitive schemes of rank 3 which are neither schurian nor separable. In all these schemes, the minimal valency of basis relation is sufficiently large in comparison with the degree of a scheme in question. On the other hand, if this valency is at most 2, then the scheme is schurian and separable. This follows from the theorem below and Exercises 2.7.35 and 2.7.33.

Theorem 3.1.6. *Let $\mathcal{X}$ be a primitive scheme on Ω. Assume that the minimum valency n_{min} of an irreflexive basis relation of $\mathcal{X}$ is at most 2. Then $p := |\Omega|$ is a prime and*

(1) *if $n_{min} = 1$, then $\mathcal{X} = \mathrm{Inv}(C_p, \Omega)$;*
(2) *if $n_{min} = 2$, then $\mathcal{X} = \mathrm{Inv}(D_{2p}, \Omega)$.*

Proof. Assume first that $n_{min} = 1$, i.e., there exists an irreflexive thin basis relation. Then the thin radical parabolic of $\mathcal{X}$ is different from 1, and hence is equal to Ω^2 by the primitivity assumption. This means that all basis relations of $\mathcal{X}$ are thin, i.e., the scheme $\mathcal{X}$ is regular (Theorem 2.1.29). By Theorem 2.2.11, this implies that

$$\mathcal{X} = \mathrm{Inv}(K, \Omega),$$

where K is a regular group. This group must be primitive by Proposition 3.1.4. Since the only regular group which is primitive is a cyclic group C_p for a prime p, we conclude that $K = C_p$ and statement (1) holds.

Let $n_{min} = 2$, i.e., $n_s = 2$ for some $s \in S$. Using formulas (2.1.12) and (2.1.8), we obtain

$$4 = n_s n_{s^*} = \sum_{t \in S} n_t c_{ss^*}^t = n_s + \sum_{t \in S^\#} n_t c_{ss^*}^t \geq 2 + 2 \sum_{t \in S^\#} c_{ss^*}^t.$$

Thus, $c_{ss^*}^t \neq 0$ for exactly one irreflexive basis relation t. Moreover, $n_t = 2$, and t is symmetric, because $t^* \in (ss^*)^* = ss^*$.

Now the basis graph $\mathfrak{X}_t$ of the relation t is the vertex-disjoint union of undirected cycles. By Theorem 3.1.5, this graph is connected and hence is an undirected cycle. Thus in accordance with Exercise 2.7.53,

$$\mathcal{X} \geq \mathrm{WL}(\mathfrak{X}_t) = \mathrm{Inv}(D_{2n}, \Omega).$$

It is not hard to see that any proper fission of the scheme on the right-hand side has an irreflexive thin basis relation. Therefore, $\mathcal{X}$ cannot be proper fission of $\mathrm{WL}(\mathfrak{X}_t)$. Consequently,

$$\mathcal{X} = \mathrm{Inv}(D_{2n}, \Omega).$$

The primitivity of $\mathcal{X}$ implies that the dihedral group D_{2n} is primitive (Proposition 3.1.4). Since this is true only if n is prime, we are done. □

By the Sims conjecture proved in [**26**] with the help of the Classification of Finite Simple Groups, the maximal subdegree of a primitive group is bounded from above by a function of its minimal subdegree. The subdegrees of a transitive group are equal to the valencies of the corresponding scheme (statement (3) of Proposition 2.2.5). So the following conjecture is true in the schurian case (if $n_{min} \le 2$, then this follows from Theorem 3.1.6).

Conjecture 3.1.7. (L. Babai) *The maximal valency n_{max} of a primitive scheme is bounded from above by a function of its minimal valency n_{min}.*

In the non-schurian case, the conjecture is true if the degree of a primitive scheme is prime (this immediately follows from Theorem 4.5.6 to be proved in Section 4.5). In general case, even for $n_{min} = 3$, the validity of the conjecture is not known (some partial results on primitive schemes with $n_{min} = 3$ can be found in [**15**] and [**69**]).

The valencies of a primitive scheme satisfy additional conditions. Some of them are collected in the following statement.

Theorem 3.1.8. *Let $\mathcal{X}$ be a primitive scheme and*

$$\{n_s : \ s \in S^{\#}\} = \{n_1, \ldots, n_k\},$$

where $S = S(\mathcal{X})$ and

$$1 = n_0 < n_1 < n_2 < \cdots < n_k.$$

Then for $i = 1, \ldots, k-1$,

(1) *$n_{i+1} \le n_1 n_i$; in particular, $n_k \le n_1^{d-1}$, where $d = \mathrm{rk}(\mathcal{X})$;*
(2) $\mathrm{GCD}(n_i, n_k) \ge \frac{n_k}{n_{k-1}} > 1$;
(3) *if p is a prime divisor of n_i, then $p \le n_1$* [**99**].

Proof. For any nonempty set $I \subseteq \{1, \ldots, k\}$, denote by s_I the union of all $s \in S$ such that $n_s = n_i$ with $i \in I$. The relation s_I is symmetric (formula (2.1.12)) and connected (Theorem 3.1.5). Consequently, $\langle s_I \rangle = \Omega^2$. This implies that if $J \subsetneq \{1, \ldots, k\}$, then

$$s_I \cdot s_J \not\subseteq s_J,$$

for otherwise $\langle s_I \rangle \cdot s_J \subseteq s_J$ and hence $\Omega^2 \subseteq s_J$, a contradiction. This formula shows that $s_I \cdot s_J$ contains a basis relation belonging to $s_{J'}$ for some nonempty set $J' \subseteq \{1,\dots,k\}$ not intersecting J. Thus there exist $r,s,t \in S$ such that

$$r \subseteq s_I,\ s \subseteq s_J,\ t \subseteq s_{J'},\quad \text{and}\quad c_{rs}^t \neq 0. \tag{3.1.1}$$

To prove statement (1), take $I=\{1\}$ and $J=\{1,\dots,i\}$. Then $n_i < n_t$. By formula (2.1.8), this implies that

$$n_{i+1} \le n_t \le n_r n_s \le n_1 n_i.$$

To prove statement (2), take $I=\{i\}$ and $J=\{k\}$. Then

$$n_r = n_i,\quad n_s = n_k,\quad n_t \le n_{k-1}. \tag{3.1.2}$$

By formulas (2.1.14) and (2.1.12), we have $n_r c_{st^*}^{r^*} = n_s c_{t^*r}^{s^*}$ and this number equals $n_t c_{rs}^t \neq 0$. It follows that the number $c_{st^*}^{r^*}$ is a multiple of the quotient of n_s by $\mathrm{GCD}(n_r,n_s)$. Taking into account that $c_{st^*}^{r^*} \le n_t$, we obtain

$$n_t \ge c_{st^*}^{r^*} \ge \frac{n_s}{\mathrm{GCD}(n_r,n_s)}.$$

In view of relations (3.1.2), this immediately implies that

$$\mathrm{GCD}(n_i,n_k) = \mathrm{GCD}(n_s,n_r) \ge \frac{n_s}{n_t} \ge \frac{n_k}{n_{k-1}}.$$

To prove statement (3), we assume on the contrary that $p > n_1$. Take $I=\{1\}$ and

$$J = \{j \in \{1,\dots,k\} :\ n_j = 0 \ (\mathrm{mod}\, p)\}.$$

Note that $i \in J$ and $1 \notin J$. Since $c_{rs}^t \le n_r = n_1$ and $t \nsubseteq s_J$, we have

$$\mathrm{GCD}(c_{rs}^t, p) = 1 = \mathrm{GCD}(n_t, p).$$

It follows that $\mathrm{GCD}(n_t c_{rs}^t, p) = 1$. On the other hand,

$$n_t c_{rs}^t = n_s c_{t^*r}^{s^*} = 0 \ (\mathrm{mod}\, p)$$

by the choice of s, a contradiction. □

3.1.2 Quotients

For any permutation group K and any K-invariant equivalence relation e, there is a permutation group induced by the action of K on the classes of e. A similar picture arises for schemes. However, the quotient schemes which correspond to the induced actions are defined in a more complicated way. In this subsection, we introduce quotient coherent configurations and trace how the Galois correspondence respects the operation of taking quotient.

Throughout this subsection, $\mathcal{X} = (\Omega, S)$ is a coherent configuration, $E = E(\mathcal{X})$, and e a parabolic of $\mathcal{X}$. Our first goal is to show that the relations $s_{\Omega/e}$, $s \in S$, defined by formula (1.1.3), form a coherent configuration on Ω/e.

In view of Proposition 2.1.4, for any $s \in S$ the composition

$$s^e = e \cdot s \cdot e = \bigcup_{(\Delta,\Gamma)\in s_{\Omega/e}} \Delta \times \Gamma. \tag{3.1.3}$$

is a relation of $\mathcal{X}$. Set

$$S^e = \{s^e : \; s \in S\}$$

Lemma 3.1.9. *S^e is a partition of Ω^2 satisfying the condition* (CC2).

Proof. Assume that the intersection $t = r^e \cap s^e$ is not empty for some $r \in S$. From the right-hand side equality of formula (3.1.3), it follows that in this case $t \cap r \neq \varnothing$. Since $r \in S$ and $t \in S^{\cup}$, this implies that $r \subseteq t$. Therefore,

$$r^e = e \cdot r \cdot e \subseteq e \cdot t \cdot e = t \subseteq r^e.$$

Consequently, $t = r^e$. Similarly, one can check that $t = s^e$. Thus any two relations of S^e are disjoint or equal. Since obviously $(s^e)^* = (s^*)^e$, we are done. □

In fact, the partition S^e satisfies the coherence condition (CC3). In order to see this, we need some preparation. Let $r, s \in S$ be such that $r^e = s^e$. It immediately follows that for any Δ and Γ in Ω/e,

$$r_{\Delta,\Gamma} \neq \varnothing \quad \Leftrightarrow \quad s_{\Delta,\Gamma} \neq \varnothing.$$

The number $n_e(r) = |r_{\Delta,\Gamma}|$ does not depend on the classes Δ and Γ such that $r_{\Delta,\Gamma} \neq \varnothing$ (Proposition 2.1.17). Therefore the number

$$n(s^e) := |(s^e)_{\Delta,\Gamma}| = \sum_{\substack{r\in S \\ r\subseteq s^e}} n_e(r) \tag{3.1.4}$$

also does not depend on such Γ and Δ. By formula (3.1.3), this implies that

$$|s^e| = n(s^e)\,|s_{\Omega/e}|.$$

Note that if $s = 1_\Delta$ for some $\Delta \in F$, then s^e is a partial equivalence relation and, in fact, an indecomposable component of e (Exercise 2.7.10).

Therefore, in this case each class of s^e is of cardinality $\sqrt{n(s^e)}$ (statement (2) of Theorem 2.1.22).

Lemma 3.1.10. *Let $r, s, t \in S$ and $(\alpha, \beta) \in t$. Then the set $\alpha r^e \cap \beta s^e$ is a disjoint union of some classes of the parabolic e. Moreover, if $n_e(r, s, t)$ and $c_e(r, s, t)$ denote the cardinality of this set and the number of these classes, respectively, then*

$$n_e(r,s,t) = \sum_{\substack{u\in S\\ u\subseteq r^e}} \sum_{\substack{v\in S\\ v\subseteq s^e}} c_{uv^*}^t, \qquad c_e(r,s,t) = n_e(r,s,t)\sqrt{\frac{n(t^e)}{n(r^e)n(s^e)}}. \tag{3.1.5}$$

Proof. The first statement follows from decomposition (3.1.3). The left-hand side equality in (3.1.5) is true, because

$$\gamma \in \alpha r^e \cap \beta s^e \quad \Leftrightarrow \quad \gamma \in \alpha u \cap \beta v$$

for some $u, v \in S$ such that $u \subseteq r^e$ and $v \subseteq s^e$.

To prove the right-hand side equality, we make use of decomposition (3.1.3) again to conclude that

$$\sum_{\delta\in\Delta} |\delta r^e \cap \beta s^e| = \sum_{\substack{r_{\Delta,\Lambda}\neq\varnothing,\\ s_{\Gamma,\Lambda}\neq\varnothing}} |\Delta \times \Lambda| \quad \text{and} \quad \sum_{\gamma\in\Gamma} |\alpha r^e \cap \gamma s^e| = \sum_{\substack{r_{\Delta,\Lambda}\neq\varnothing,\\ s_{\Gamma,\Lambda}\neq\varnothing}} |\Gamma \times \Lambda|,$$

where Λ runs over the classes of e, $\Delta = \alpha e$, and $\Gamma = \beta e$. The cardinalities of the sets $\Delta \times \Lambda$ and $\Gamma \times \Lambda$ in the above formulas are equal to $n(r^e)$ and $n(s^e)$, respectively. Therefore the first equality in (3.1.5) implies that

$$|\Delta|\, n_e(r,s,t) = n(r^e)c_e(r,s,t) \quad \text{and} \quad |\Gamma|\, n_e(r,s,t) = n(s^e)c_e(r,s,t).$$

Thus,

$$|\Delta \times \Gamma|\, n_e(r,s,t)^2 = n(r^e)\, n(s^e)\, c_e(r,s,t)^2,$$

which proves the required equality, because $|\Delta \times \Gamma| = n(t^e)$. □

From Lemmas 3.1.9 and 3.1.10, it follows that the pair (Ω, S^e) satisfies the conditions (CC2) and (CC3). However, the condition (CC1) may not hold and hence the pair is not necessarily a rainbow. On the other hand, Lemma 3.1.9 and formula (3.1.3) show that for all $r, s \in S$,

$$r_{\Omega/e} = s_{\Omega/e} \quad \Leftrightarrow \quad r^e = s^e.$$

This implies that the set

$$S_{\Omega/e} = \{s_{\Omega/e} : \ s \in S\}$$

forms a partition of $(\Omega/e)^2$. It immediately follows that

$$\mathcal{X}_{\Omega/e} = (\Omega/e, S_{\Omega/e})$$

is a rainbow.

Furthermore, by the right-hand side equality in (3.1.5), for any $r, s, t \in S$ and any $(\Delta, \Gamma) \in t_{\Omega/e}$,

$$|\Delta r_{\Omega/e} \cap \Gamma s_{\Omega/e}| = c_e(r, s, t). \tag{3.1.6}$$

By Lemma 3.1.10, the number $c_e(r, s, t)$ does not depend on the choice of Δ and Γ. Consequently, the rainbow $\mathcal{X}_{\Omega/e}$ satisfies the condition (CC3). This proves the first part of the theorem below; the second part is obvious.

Theorem 3.1.11. *The rainbow* $\pi_e(\mathcal{X}) := \mathcal{X}_{\Omega/e}$ *is a coherent configuration with intersection numbers*

$$c_{r_{\Omega/e} s_{\Omega/e}}^{t_{\Omega/e}} = c_e(r, s^*, t), \qquad r, s, t \in S. \tag{3.1.7}$$

The mapping π_e *defined by formula* (1.1.4) *induces surjections from* $S(\mathcal{X})$, $F(\mathcal{X})$, *and* $E(\mathcal{X})$ *onto* $S(\mathcal{X}_{\Omega/e})$, $F(\mathcal{X}_{\Omega/e})$, *and* $E(\mathcal{X}_{\Omega/e})$, *respectively.*

Definition 3.1.12. *The coherent configuration* $\mathcal{X}_{\Omega/e}$ *from Theorem* 3.1.11 *is called the quotient of* $\mathcal{X}$ *modulo the parabolic* e.

Clearly, the quotient is canonically isomorphic to $\mathcal{X}$ whenever $e = 1_\Omega$. It should be mentioned that $\mathcal{X}_{\Omega/e}$ can be homogeneous even when $\mathcal{X}$ does not; this occurs if the parabolic e is indecomposable (Exercise 3.7.6).

Example 3.1.13. *Let* $\mathcal{X}$ *be the coherent configuration of an undirected* 6*-cycle* (*Example* 2.6.2) *and* $e = s_0 \cup s_3$. *Then*

$$(s_0)^e = (s_3)^e \quad \text{and} \quad (s_1)^e = (s_2)^e.$$

In particular,

$$|\Omega/e| = 3 \quad \text{and} \quad |S_{\Omega/e}| = 2.$$

Thus the quotient scheme $\mathcal{X}_{\Omega/e}$ *equals* $\mathcal{T}_{\Omega/e}$.

From Theorem 3.1.11, it follows that the parabolics of the quotient coherent configuration $\mathcal{X}_{\Omega/e}$ are in a one-to-one correspondence with the parabolics of $\mathcal{X}$ that contain e. This observation proves the following statement providing a reduction of arbitrary schemes to primitive ones.

Proposition 3.1.14. *Let* $\mathcal{X}$ *be a scheme on* Ω *and* $e \neq \Omega^2$ *a maximal parabolic of* $\mathcal{X}$. *Then the quotient scheme* $\mathcal{X}_{\Omega/e}$ *is primitive.*

Now assume that $\mathcal{X}$ is a Cayley scheme over a group G. Then in view of statement (6) of Exercise 1.4.16, there exists a group $H \leq G$ such that $e = \rho(H)$, where ρ is the mapping (1.4.8). In this case,

$$\Omega/e = G/H,$$

i.e., the classes of e are the right H-cosets of G (Proposition 2.4.9).

If the group H is normal in G, then the mapping π_e is the canonical epimorphism from G to $\overline{G} = G/H$ such that

$$g_{right}\pi_e = \pi_e \overline{g}_{right}, \qquad g \in G,$$

where $\overline{g} = Hg = \pi_e(g)$. It follows that for any $s \in S$,

$$\pi_e(s)^{\overline{g}_{right}} = (s^e)^{\pi_e \overline{g}_{right}} = (s^e)^{g_{right}\pi_e} = (s^e)^{\pi_e} = \pi_e(s).$$

Thus,

$$\overline{G}_{right} \leq \mathrm{Aut}(\mathcal{X}_{\Omega/e}),$$

i.e., the following statement holds.

Proposition 3.1.15. *Let $\mathcal{X}$ be a Cayley scheme over a group G, and let e be a parabolic of $\mathcal{X}$. Assume that the group $H = \rho^{-1}(e)$ is normal in G. Then $\mathcal{X}_{G/H} := \mathcal{X}_{\Omega/e}$ is a Cayley scheme over the group G/H.*

For an arbitrary partial parabolic $e \in E$, the support $\Lambda = \Omega(e)$ is a homogeneity set of $\mathcal{X}$ (Exercise 2.7.4) and e_Λ is a parabolic of $\mathcal{X}_\Lambda$. The quotient of $\mathcal{X}$ modulo e is defined to be

$$\mathcal{X}_{\Omega/e} := (\mathcal{X}_\Lambda)_{\Lambda/e_\Lambda}.$$

Thus the restriction of a coherent configuration to a homogeneity set Δ is canonically isomorphic to the quotient of $\mathcal{X}$ modulo 1_Δ.

A natural analog of quotient in permutation group theory is obtained by taking the group induced by the action of a permutation group on the classes of an invariant equivalence relation. The following statement shows to what extent the operation of taking a quotient respects the Galois correspondence between the coherent configurations and permutation groups.

Theorem 3.1.16. *Let e be a partial equivalence relation on Ω. Then*

$$\mathrm{Inv}(K^{\Omega/e}) = \mathrm{Inv}(K)_{\Omega/e} \tag{3.1.8}$$

for any group $K \leq \mathrm{Sym}(\Omega)$ such that e is K-invariant, and

$$\mathrm{Aut}(\mathcal{X}_{\Omega/e}) \geq \mathrm{Aut}(\mathcal{X})^{\Omega/e} \tag{3.1.9}$$

for any coherent configuration $\mathcal{X}$ on Ω such that e is a partial parabolic of $\mathcal{X}$.

Proof. Assume that e is K-invariant. Then K acts on Ω/e. Therefore statement (1) follows from the obvious equality

$$\mathrm{Orb}(K, \Omega \times \Omega)_{\Omega/e} = \mathrm{Orb}(K^{\Omega/e}, \Omega/e \times \Omega/e).$$

Next, let $s \in S$. Then any $k \in \mathrm{Aut}(\mathcal{X})$ leaves s fixed and permutes the classes of e. It follows that k also permutes the nonempty sets $s_{\Delta,\Gamma}$ with $\Delta, \Gamma \in \Omega/e$. Consequently, the permutation $k^{\Omega/e}$ leaves the relation $s_{\Omega/e}$ fixed. Thus this permutation is an automorphism of the quotient $\mathcal{X}_{\Omega/e}$. This proves statement (2). □

Corollary 3.1.17. *Any quotient of a schurian coherent configuration is schurian.*

It should be noted that inclusion (3.1.9) may be strict (even if the coherent configuration $\mathcal{X}$ is schurian). For instance, let p be a prime,

$$K = C_p \rtimes C_{p-1} \le \mathrm{Sym}(\Omega)$$

a regular group, and $\mathcal{X} = \mathrm{Inv}(K, \Omega)$. Then

$$e = (C_{p-1})^\rho$$

is a parabolic of $\mathcal{X}$ with p classes, each of cardinality $p-1$.

One can see that $K^{\Omega/e}$ is a transitive group of degree p and order $p(p-1)$. In particular, it is 2-transitive. By statement (1) of Proposition 3.1.16, this implies that

$$\mathcal{X}_{\Omega/e} = \mathcal{T}_{\Omega/e}.$$

Therefore if $p \ge 5$, then

$$\mathrm{Aut}(\mathcal{X}_{\Omega/e}) = \mathrm{Aut}(\mathcal{T}_{\Omega/e}) \cong \mathrm{Sym}(p) \not\cong K \cong K^{\Omega/e}.$$

In contrast to Corollary 3.1.17, the quotient (and, in particular, the restriction) of a separable coherent configuration is not necessary separable (a lot of examples can be constructed with the help of the standard wreath product defined in Section 3.4.1). A reason for this is that some algebraic isomorphisms of the quotient are not induced, in the sense explained below, by those of the coherent configuration in question.

Let e be a partial parabolic of $\mathcal{X}$, $\mathcal{X}' = (\Omega', S')$ a coherent configuration, and $\varphi : S \to S'$, $s \mapsto s'$, an algebraic isomorphism. In view of Proposition 2.3.25, $e' = \varphi(e)$ is a partial parabolic of $\mathcal{X}'$. By statement (2) of Proposition 2.3.18, we also have

$$\varphi(s^e) = \varphi(e \cdot s \cdot e) = \varphi(e) \cdot \varphi(s) \cdot \varphi(e) = e' \cdot s' \cdot e' = (s')^{e'},$$

for all $s \in S$. It follows that φ induces a bijection from S^e onto $(S')^{e'}$ and hence a bijection

$$\varphi_{\Omega/e} : S_{\Omega/e} \to S'_{\Omega'/e'},\ s_{\Omega/e} \mapsto s'_{\Omega'/e'}. \tag{3.1.10}$$

This bijection preserves the intersection numbers of the quotient coherent configuration, because of formula (3.1.7) and equalities

$$n_e(r,s,t) = n_{e'}(r',s',t') \quad \text{and} \quad c_e(r,s,t) = c_{e'}(r',s',t')$$

that hold for all $r, s, t \in S$: the first of them is obvious, whereas the second one follows from Proposition 2.1.17. Thus,

$$\varphi_{\Omega/e} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{\Omega/e}, \mathcal{X}'_{\Omega'/e'}).$$

We say that $\varphi_{\Omega/e}$ is the algebraic isomorphism induced by φ. In the special case, when $e = 1_\Delta$ for a homogeneity set Δ of $\mathcal{X}$, the algebraic isomorphism $\varphi_{\Omega/e}$ is denoted by φ_Δ and is called the restriction of φ to Δ. It should also be mentioned that if $\mathcal{X} = \mathcal{X}'$ and $e = e'$, then the restriction mapping

$\varphi \mapsto \varphi_{\Omega/e}$ is a group homomorphism from $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$ to $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_{\Omega/e})$; in other words,

$$\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})^{\Omega/e} \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_{\Omega/e}). \tag{3.1.11}$$

We complete this subsection by finding the adjacency algebra of the quotient $\mathcal{X}_{\Omega/e}$. To this end, in what follows for any $\Delta, \Gamma \subseteq \Omega$, we set $J_{\Delta,\Gamma}$ to be the adjacency matrix of the relation $\Delta \times \Gamma \subseteq \Omega^2$, and put $J_\Delta = J_{\Delta,\Delta}$.

Let us define a matrix

$$P = P_e = \sum_{\Delta \in \Omega/e} \frac{1}{|\Delta|} J_\Delta. \tag{3.1.12}$$

It is easily seen that $P = P^2$, and that P is the identity element of the algebra

$$P \,\mathrm{Adj}(\mathcal{X}) P \subseteq \mathrm{Mat}_\Omega\,.$$

Lemma 3.1.18. *For any $s \in S$, the matrix PA_sP is a multiple of A_{s^e}. In particular, $\{A_{s^e} :\ s \in S\}$ is a linear basis of the algebra $P\,\mathrm{Adj}(\mathcal{X})P$.*

Proof. The second statement follows from the first one and Lemma 3.1.9. To prove the first one, let $s \in S$. Then

$$\begin{aligned} PA_sP &= \sum_{\Delta,\Gamma \in \Omega/e} \frac{1}{|\Delta|\,|\Gamma|} J_\Delta A_s J_\Gamma \\ &= \sum_{\substack{\Delta,\Gamma \in \Omega/e \\ s_{\Delta,\Gamma} \ne \varnothing}} \frac{n_e(s)}{|\Delta|\,|\Gamma|} J_{\Delta,\Gamma}, \end{aligned}$$

where $n_e(s)$ is the number defined in Proposition 2.1.17.

Denote by e' the indecomposable component of e, the support of which contains the fiber $\Omega_-(s)$ (Lemma 2.1.21). Then for each $\Delta \in \Omega/e$,

$$s_{\Delta,\Omega} \ne \varnothing \quad \Rightarrow \quad \Delta \in \Omega/e'.$$

By statement (2) of Theorem 2.1.22, this implies that the cardinality of Δ with $s_{\Delta,\Omega} \ne \varnothing$, is constant. Denote it by a_s. Similarly, one can check that the cardinality of each $\Gamma \in \Omega/e$ with $s_{\Omega,\Gamma} \ne \varnothing$ is constant, say b_s.

Thus the above formula takes the form

$$PA_sP = \frac{n_e(s)}{a_s b_s} \sum_{\substack{\Delta,\Gamma \in \Omega/e \\ s_{\Delta,\Gamma} \ne \varnothing}} J_{\Delta,\Gamma} = \frac{n_e(s)}{a_s b_s} A_{s^e},$$

as required. □

By Lemma 3.1.18, the linear spaces $P\operatorname{Adj}(\mathcal{X})P$ and $\operatorname{Adj}(\mathcal{X}_{\Omega/e})$ are isomorphic. In fact, they are isomorphic as algebras and an explicit isomorphism is given in the following statement.

Theorem 3.1.19. *Let $\mathcal{X}$ be a coherent configuration, e a partial parabolic of $\mathcal{X}$, and $P = P_e$ the matrix defined by (3.1.12). Then the linear mapping*

$$f : P\operatorname{Adj}(\mathcal{X})P \to \operatorname{Adj}(\mathcal{X}_{\Omega/e}),\ A_{s^e} \mapsto \sqrt{n(s^e)}A_{s_{\Omega/e}} \tag{3.1.13}$$

is an algebra isomorphism.

Proof. The fact that f is a linear isomorphism is a consequence of Lemma 3.1.18. Next, by Lemma 3.1.10, for all $r, s, t \in S$ we have

$$A_{r^e}A_{s^e} = \sum_{t^e \in S^e} n_e(r, s^*, t)A_{t^e}.$$

By formulas (3.1.5) and (3.1.7), this implies

$$\begin{aligned}
(A_{r^e}A_{s^e})^f &= (\sum_{t^e \in S^e} n_e(r, s^*, t)A_{t^e})^f \\
&= \sum_{t_{\Omega/e} \in S_{\Omega/e}} n_e(r, s^*, t)\sqrt{n(t^e)}A_{t_{\Omega/e}} \\
&= \sqrt{n(r^e)}A_{r_{\Omega/e}}\sqrt{n(s^e)}A_{s_{\Omega/e}} \\
&= (A_{r^e})^f(A_{s^e})^f.
\end{aligned}$$

Thus, f is an algebra isomorphism. □

3.1.3 Residually thin extension

The quotient scheme considered in the previous subsection is an analog (in the sense of the Galois correspondence) of the permutation group induced by the action of a transitive group on the blocks of imprimitivity system. The kernel of this action also corresponds to a certain coherent configuration.

Let $K \leq \mathrm{Sym}(\Omega)$ and e a partial K-invariant equivalence relation. Then the kernel of the action of K on Ω/e is equal to the group

$$K_e = \{k \in K : \ \Delta^k = \Delta \ \text{ for all } \ \Delta \in \Omega/e\},$$

see formula (1.4.11). This group can equivalently be defined as the largest subgroup of K with respect to which the relation 1_Δ is invariant for each $\Delta \in \Omega/e$. Thus if $\mathcal{X}$ is a coherent configuration on Ω and e is a partial parabolic of $\mathcal{X}$, then the coherent configuration

$$\mathcal{X}_e = \mathrm{WL}(\mathcal{X}, \{1_\Delta : \ \Delta \in \Omega/e\}),$$

can be considered as a natural analog of K_e, where here and below for a set T of relations on Ω, we denote by $\mathrm{WL}(\mathcal{X}, T)$ the coherent closure of $S \cup T$ with $S = S(\mathcal{X})$.

Definition 3.1.20. *The coherent configuration $\mathcal{X}_e$ is called the extension of $\mathcal{X}$ with respect to e.*

Certainly, $\mathcal{X}_e = \mathcal{X}$ if $e = \Omega^2$, and $\mathcal{X}_e = \mathcal{D}_\Omega$ if $e = 1_\Omega$.

Theorem 3.1.21. *Let e be a partial equivalence relation on Ω. Then*

$$\mathrm{Inv}(K_e) \geq \mathrm{Inv}(K)_e \tag{3.1.14}$$

for any group $K \leq \mathrm{Sym}(\Omega)$ such that e is K-invariant, and

$$\mathrm{Aut}(\mathcal{X}_e) = \mathrm{Aut}(\mathcal{X})_e \tag{3.1.15}$$

for any coherent configuration $\mathcal{X}$ on Ω such that e is a partial parabolic of $\mathcal{X}$.

Proof. Let $K \leq \mathrm{Sym}(\Omega)$. Then obviously $K_e \leq K$, and in view of the Galois correspondence,

$$\mathrm{Inv}(K_e) \geq \mathrm{Inv}(K).$$

By the monotonicity of the coherent closure (Exercise 2.7.50), this implies that

$$\mathrm{Inv}(K_e)_e \geq \mathrm{Inv}(K)_e.$$

Since the relation 1_Δ is K_e-invariant for all $\Delta \in \Omega/e$, the left-hand side coherent configuration is equal to $\mathrm{Inv}(K_e)$, which proves inclusion (3.1.14). To prove formula (3.1.15), set $T := \{1_\Delta : \Delta \in \Omega/e\}$. Then

$$\mathrm{Aut}(\mathcal{X}_e) = \mathrm{Aut}(S(\mathcal{X}) \cup T) = \mathrm{Aut}(\mathcal{X}) \cap \mathrm{Aut}(T) = \mathrm{Aut}(\mathcal{X})_e,$$

as required. □

It seems that inclusion (3.1.14) can be strict, but no example is known. In fact, the explicit calculation of the extension of a coherent configuration with respect to a partial parabolic is not easy, except for some special cases. One of them is considered in the rest of the subsection.

In what follows, $\mathcal{X} = (\Omega, S)$ is a coherent configuration, $E = E(\mathcal{X})$, and $e \in E$ is a parabolic of $\mathcal{X}$. It is assumed that e is *proper*, i.e., each class of e is contained in a fiber of $\mathcal{X}$.

Definition 3.1.22. *The parabolic e is said to be residually thin if the coherent configuration $\mathcal{X}_{\Omega/e}$ is semiregular.*

Any coherent configuration has at least one residually thin parabolic: the classes of it are the fibers. In any scheme, each parabolic with at most two classes is obviously residually thin.

Example 3.1.23. *Let $\mathcal{X} = \mathrm{Inv}(K, \Omega)$, where $K \le \mathrm{Sym}(\Omega)$ is a transitive group. Let e be a parabolic of $\mathcal{X}$ such that $K_{\{\Delta\}} \trianglelefteq K$ for some $\Delta \in \Omega/e$. Then*

$$K_{\{\Delta\}} = K_e. \tag{3.1.16}$$

Indeed, the inclusion $K_{\{\Delta\}} \ge K_e$ is obvious, whereas the reverse inclusion is true, because $K_{\{\Delta\}} \trianglelefteq K$ and

$$K_{\{\Delta^k\}} = k^{-1} K_{\{\Delta\}} k$$

for all $k \in K$.

Now equality (3.1.16) implies that the group $K^{\Omega/e}$ and hence the scheme $\mathrm{Inv}(K^{\Omega/e})$ is regular. However in view of formula (3.1.8),

$$\mathcal{X}_{\Omega/e} = \mathrm{Inv}(K)_{\Omega/e} = \mathrm{Inv}(K^{\Omega/e}).$$

Thus the scheme $\mathcal{X}_{\Omega/e}$ is regular and the parabolic e is residually thin.

The concept of residually thin parabolic is, in a sense, dual to that of thin parabolic (the exact meaning of this statement can be seen in the class of Cayley schemes over an abelian group G, where the duality comes from the group dual to G, see Exercises 3.7.52–3.7.54). Therefore, the fact that every coherent configuration has the smallest (with respect to inclusion) residually thin parabolic does not seem too surprising.

Lemma 3.1.24. *The intersection of any set of residually thin parabolics is residually thin.*

Proof. Let e_1 and e_2 be residually thin parabolics of $\mathcal{X}$. Without loss of generality (see statement (1) of Exercise 3.7.9), we may assume that

$$e_1 \cap e_2 = 1.$$

We need to verify that $\mathcal{X} = \mathcal{X}_{\Omega/1_\Omega}$ is semiregular, i.e., $|\alpha s| = 1$ for all $\alpha \in \Omega$ and $s \in S$ such that $\alpha s \ne \varnothing$.

Note that the relations s_{Ω/e_1} and s_{Ω/e_2} being the basis relations of the semiregular quotients $\mathcal{X}_{\Omega/e_1}$ and $\mathcal{X}_{\Omega/e_2}$ are thin. Therefore, there exist classes $\Delta_1 \in \Omega/e_1$ and $\Delta_2 \in \Omega/e_2$ such that

$$(\alpha e_1)\, s_{\Omega/e_1} = \{\Delta_1\} \quad \text{and} \quad (\alpha e_2)\, s_{\Omega/e_2} = \{\Delta_2\}.$$

This implies that αs is contained in the singleton $\Delta_1 \cap \Delta_2$, as required. □

By Lemma 3.1.24, there exists the smallest residually thin parabolic of a coherent configuration $\mathcal{X}$ that is the intersection of all residually thin parabolics. It is called the *thin residue parabolic* of $\mathcal{X}$. Clearly, $\mathcal{X}$ is semiregular if and only if the thin residue parabolic of $\mathcal{X}$ is equal to 1_Ω.

Definition 3.1.25. *The set of basis relations contained in the thin residue parabolic of $\mathcal{X}$ is called the thin residue of $\mathcal{X}$.*

An extremely useful property of a residually thin parabolics e as stated in the theorem below, is that the extension of $\mathcal{X}$ with respect to e is completely under control. This means not only that the basis relations of $\mathcal{X}_e$ admit an explicit description but also that any algebraic isomorphism from $\mathcal{X}$ to another coherent configuration can be extended to an algebraic isomorphism of $\mathcal{X}_e$.

Theorem 3.1.26. [45, Theorem 2.1] *Let $\mathcal{X}$ be a coherent configuration on Ω, e a residually thin parabolic of $\mathcal{X}$, $S = S(\mathcal{X})$, and $S_e = S(\mathcal{X}_e)$. Then*

(1) $S_e = \{s_{\Delta,\Gamma} :\ s \in S,\ \Delta, \Gamma \in \Omega/e\}^\natural$;
(2) *given $k \in \mathrm{Aut}(\mathcal{X}_{\Omega/e})$, the mapping*

$$\psi_k : S_e \to S_e,\ s_{\Delta,\Gamma} \mapsto s_{\Delta^k,\Gamma^k}$$

is an algebraic automorphism of $\mathcal{X}_e$;
(3) $\mathcal{X} = (\mathcal{X}_e)^\Psi$, *where* $\Psi = \{\psi_k :\ k \in \mathrm{Aut}(\mathcal{X}_{\Omega/e})\}$;
(4) *each $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ can be extended to $\varphi_e \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_e, \mathcal{X}'_{e'})$, where $e' = \varphi(e)$.*

Proof. Let us prove that given $r, s, t \in S$ and $\Delta, \Delta', \Delta'', \Gamma, \Gamma', \Gamma'' \in \Omega/e$ such that $r_{\Delta,\Gamma} \neq \varnothing$ and $s_{\Delta',\Gamma'} \neq \varnothing$, we have

$$|\alpha\, r_{\Delta,\Gamma}\ \cap\ \beta\, s_{\Delta',\Gamma'}| = \delta_{\Delta'',\Delta}\delta_{\Gamma,\Gamma'}\delta_{\Delta',\Gamma''}c^t_{rs^*}, \quad (\alpha,\beta) \in t_{\Delta'',\Gamma''}, \tag{3.1.17}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta. Without loss of generality we may assume that

$$\Delta'' = \Delta, \quad \Gamma = \Gamma', \quad \Delta' = \Gamma''.$$

Since $r_{\Delta,\Gamma} \neq \varnothing$ and $s_{\Delta',\Gamma'} \neq \varnothing$, the semiregularity of $\mathcal{X}_{\Omega/e}$ implies that

$$\Delta\, r_{\Omega/e} = \{\Gamma\} \quad \text{and} \quad \Delta'\, s_{\Omega/e} = \{\Gamma'\}.$$

Therefore,

$$|\alpha r \cap \beta s| = |\alpha\, r_{\Delta,\Gamma}\ \cap\ \beta\, s_{\Delta',\Gamma'}|, \quad (\alpha,\beta) \in \Delta \times \Delta'.$$

If $(\alpha,\beta)\in t$, then the left-hand side of the above equality equals $c_{rs^*}^t$ and (3.1.17) follows.

Denote by S' the set on the right-hand side of the equality in statement (1). Then obviously the pair $\mathcal{X}'=(\Omega,S')$ is a rainbow. By formula (3.1.17), this implies that $\mathcal{X}'$ is a coherent configuration with the intersection numbers

$$c_{r_{\Delta,\Gamma}\,s_{\Gamma',\Delta'}}^{t_{\Delta'',\Gamma''}}=\delta_{\Delta'',\Delta}\delta_{\Gamma,\Gamma'}\delta_{\Delta',\Gamma''}c_{rs}^t. \tag{3.1.18}$$

Next, by the definition of the coherent configuration $\mathcal{X}_e$, each class of e is a homogeneity set of $\mathcal{X}_e$. It follows (Proposition 2.1.4) that

$$s_{\Delta,\Gamma}=1_\Delta\cdot s\cdot 1_\Gamma\in(S_e)^\cup\qquad\text{for all }\ \Delta,\Gamma\in\Omega/e,\ s\in S.$$

Consequently, $S'\subseteq(S_e)^\cup$ and hence $\mathcal{X}'\le\mathcal{X}_e$. The reverse inclusion follows from the minimality of the coherent closure $\mathcal{X}_e$. Thus, $\mathcal{X}'=\mathcal{X}_e$ and $S'=S_e$. Statement (1) is proved.

Statement (2) immediately follows from (3.1.18) and the definition of ψ_k. Furthermore, the coherent configuration $\mathcal{X}_{\Omega/e}$ is semiregular and so schurian (Exercise 2.7.35). Consequently, for any $s\in S$, the group $\mathrm{Aut}(\mathcal{X}_{\Omega/e})$ acts regularly on $s_{\Omega/e}$. Consequently, the relation s has the form

$$s=\bigcup_{(\Delta,\Gamma)\in s_{\Omega/e}}s_{\Delta,\Gamma}=\bigcup_{k\in\mathrm{Aut}(\mathcal{X}_{\Omega/e})}s_{\Delta_0^k,\Gamma_0^k}=\bigcup_{\psi\in\Psi}(s_{\Delta_0,\Gamma_0})^\psi=(s_{\Delta_0,\Gamma_0})^\Psi,$$

where Δ_0 and Γ_0 are classes of e such that the relation s_{Δ_0,Γ_0} is not empty. This shows that $S=(S_e)^\Psi$, which proves statement (3).

To prove statement (4), let $\varphi\in\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{X}')$. By formula (3.1.10), this algebraic isomorphism induces the algebraic isomorphism

$$\varphi_{\Omega/e}\in\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{\Omega/e},\mathcal{X}'_{\Omega'/e'}).$$

The coherent configuration $\mathcal{X}_{\Omega/e}$ is semiregular, and hence separable (Exercise 2.7.35). Therefore there exists an isomorphism

$$f_e\in\mathrm{Iso}(\mathcal{X}_{\Omega/e},\mathcal{X}'_{\Omega'/e'},\varphi_{\Omega/e})$$

inducing $\varphi_{\Omega/e}$. Since parabolic e' is residually thin (Exercise 3.7.15), formula (3.1.18) holds also for all basis relations of $\mathcal{X}'$ and all classes of e'. It follows that the mapping

$$\varphi_e:S(\mathcal{X}_e)\to S(\mathcal{X}'_{e'}),\ s_{\Delta,\Gamma}\mapsto s'_{\Delta^{f_e},\Gamma^{f_e}}$$

is an algebraic isomorphism from $\mathcal{X}_e$ to $\mathcal{X}'_{e'}$, where $s'=\varphi(s)$. Consequently,

$$\varphi_e(s)=\bigcup_{(\Delta,\Gamma)\in s_{\Omega/e}}\varphi_e(s_{\Delta,\Gamma})=\bigcup_{(\Delta',\Gamma')\in s'_{\Omega'/e'}}s'_{\Delta',\Gamma'}=s'=\varphi(s),$$

where $\Delta'=\Delta^{f_e}$ and $\Gamma'=\Gamma^{f_e}$. This means that φ_e extends φ. $\square$

Remark 3.1.27. *The algebraic isomorphism* φ_e *in statement* (4) *of Theorem* 3.1.26 *is obviously unique in the following sense: for any algebraic isomorphism* $\psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_e, \mathcal{X}'_{e'})$ *extending* φ,

$$\psi_{\Omega/e} = (\varphi_e)_{\Omega/e} \qquad \Rightarrow \qquad \psi = \varphi_e.$$

3.1.4 Schurity and separability of residually thin extension

The close connection between a coherent configuration $\mathcal{X}$ and its extension $\mathcal{X}_e$ with respect to a residually thin parabolic e that was established in Theorem 3.1.26 suggests that the schurity and separability of $\mathcal{X}$ is somehow related to those of $\mathcal{X}_e$. However, a possible reduction cannot be direct. For example, an algebraic isomorphism from $\mathcal{X}_e$ to another coherent configuration does not necessarily induce an algebraic isomorphism of $\mathcal{X}$, and hence it is difficult to expect that the separability of $\mathcal{X}$ implies the separability of $\mathcal{X}_e$. The following two theorems were proved in [**103**].

Theorem 3.1.28. *Let $\mathcal{X}$ be a coherent configuration and e a residually thin parabolic of $\mathcal{X}$. Then*

(1) *if $\mathcal{X}_e$ is separable, then $\mathcal{X}$ is separable;*
(2) *if $\mathcal{X}$ is schurian, then $\mathcal{X}_e$ is schurian.*

Proof. To prove statement (1), let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. By statement (4) of Theorem 3.1.26, there exists an algebraic isomorphism

$$\varphi_e \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_e, \mathcal{X}'_{e'})$$

extending φ, where $e' = \varphi(e)$. Assume that the coherent configuration $\mathcal{X}_e$ is separable. Then φ_e is induced by an isomorphism $f \in \mathrm{Iso}(\mathcal{X}_e, \mathcal{X}'_{e'})$. Now if $s \in S(\mathcal{X})$, then $s \in S(\mathcal{X}_e)^{\cup}$, and

$$\varphi(s) = \varphi_e(s) = s^f.$$

Thus, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$, as required.

To prove statement (2), let us verify that each $s_0 \in S(\mathcal{X}_e)$ is a 2-orbit of the group $\mathrm{Aut}(\mathcal{X}_e)$. By formula (3.1.15),

$$(\alpha_0, \beta_0)^{\mathrm{Aut}(\mathcal{X})_e} \subseteq s_0 \tag{3.1.19}$$

for any $(\alpha_0, \beta_0) \in s_0$. Denote by Δ and Γ the classes of e that contain α_0 and β_0, respectively. By statement (1) of Theorem 3.1.26, there exists $s \in S(\mathcal{X})$ such that

$$s_0 = s_{\Delta,\Gamma}.$$

Assume that the coherent configuration $\mathcal{X}$ is schurian. Then s is a 2-orbit of the group $\mathrm{Aut}(\mathcal{X})$. It follows that there exists a set $K_0 \subseteq \mathrm{Aut}(\mathcal{X})$ such that

$$s_0 = (\alpha_0, \beta_0)^{K_0}. \tag{3.1.20}$$

Clearly, $\Delta^{K_0} = \Delta$. On the other hand, the group $\mathrm{Aut}(\mathcal{X})^{\Omega/e} \le \mathrm{Aut}(\mathcal{X}_{\Omega/e})$ is semiregular, because the coherent configuration $\mathcal{X}_{\Omega/e}$ is semiregular. Thus,

$$(K_0)^{\Omega/e} \subseteq (\mathrm{Aut}(\mathcal{X})_{\{\Delta\}})^{\Omega/e} = \{\mathrm{id}_{\Omega/e}\},$$

i.e., $K_0 \subseteq \mathrm{Aut}(\mathcal{X})_e$. Now by formulas (3.1.19) and (3.1.20), we have

$$s_0 = (\alpha_0, \beta_0)^{K_0} \subseteq (\alpha_0, \beta_0)^{\mathrm{Aut}(\mathcal{X})_e} \subseteq s_0,$$

which shows that s_0 is a 2-orbit of $\mathrm{Aut}(\mathcal{X}_e)$. □

There are examples showing that none of statements of Theorem 3.1.28 can be reversed. However, the situation is changed if the property of a coherent configuration to be schurian or separable is replaced by the property to be schurian and separable simultaneously.

Theorem 3.1.29. *Let $\mathcal{X}$ be a coherent configuration and e a residually thin parabolic of $\mathcal{X}$. Then $\mathcal{X}$ is schurian and separable if and only if $\mathcal{X}_e$ is schurian and separable.*

Proof. First, we assume that $\mathcal{X}$ is schurian and separable. Then, the coherent configuration $\mathcal{Y} = \mathcal{X}_e$ is schurian by statement (2) of Theorem 3.1.28. To verify that $\mathcal{Y}$ is separable, let $\mathcal{Y}'$ be a coherent configuration on Ω' and

$$\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}').$$

Denote by $\overline{\varphi}$ the restriction of φ to $S = S(\mathcal{X})$, and set $\mathcal{X}' = \mathcal{X}^{\overline{\varphi}}$. Then by Corollary 2.3.21 applied for $\mathcal{Y} = \mathcal{X}$,

$$\overline{\varphi} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}').$$

In accordance with Exercise 3.7.15, the parabolic $e' = e^{\overline{\varphi}}$ is residually thin in $\mathcal{X}'$. Furthermore, the definition of $\overline{\varphi}$ implies that

$$\varphi(s_{\Delta,\Gamma}) = \overline{\varphi}(s)_{\Delta^{\varphi},\Gamma^{\varphi}}, \qquad s \in S,\ \Delta, \Gamma \in \Omega/e.$$

Thus by statement (1) of Theorem 3.1.26 we have

$$\mathcal{Y}' = (\mathcal{X}')_{e'} \quad \text{and} \quad F(\mathcal{Y}') = \Omega'/e'.$$

In particular, the algebraic isomorphism φ induces a bijection

$$g : \Omega/e \to \Omega'/e',\ \Delta \mapsto \Delta^{\varphi}.$$

This bijection induces the algebraic isomorphism $\varphi_{\Omega/e}$, which extends the algebraic isomorphism $\overline{\varphi}_{\Omega/e}$. Consequently,

$$g \in \mathrm{Iso}(\mathcal{X}_{\Omega/e}, \mathcal{X}'_{\Omega'/e'}).$$

By the separability of the coherent configuration $\mathcal{X}$, there exists an isomorphism $\bar{f} \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \overline{\varphi})$. Because $\bar{f}^{\Omega/e}$ and g induce the same algebraic isomorphism $\overline{\varphi}_{\Omega/e}$,

$$\bar{f}^{\Omega/e} g^{-1} \in \mathrm{Aut}(\mathcal{X}_{\Omega/e}).$$

The schurity of $\mathcal{X}$ and the fact that $\mathcal{X}_{\Omega/e}$ is semiregular, implies that this automorphism can be extended to an automorphism $\widetilde{f}$ of $\mathcal{X}$. After replacing

$\bar{f}$ by $\widetilde{f}^{-1}\bar{f}$, we get $\bar{f}^{\Omega/e} = g$ and hence

$$(\overline{\varphi}_e)_{\Omega/e} = \varphi_{\Omega/e}.$$

Thus, $\varphi = \overline{\varphi}_e$ by Remark 3.1.27. Since the isomorphism $\bar{f}$ induces the algebraic isomorphism $\overline{\varphi}_e$, we obtain

$$\bar{f} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}') = \mathrm{Iso}(\mathcal{X}_e, \mathcal{X}'_{e'}, \varphi).$$

This completes the proof that the coherent configuration $\mathcal{X}_e$ is separable.

Now assume that $\mathcal{X}_e$ is schurian and separable. Then $\mathcal{X}$ is separable by statement (1) of Theorem 3.1.28. Let us verify that any $s \in S$ is a 2-orbit of the group $\mathrm{Aut}(\mathcal{X})$. To this end, let (α_1, β_1) and (α_2, β_2) belong to s. Set

$$\Delta_i = \alpha_i e \quad \text{and} \quad \Gamma_i = \beta_i e, \qquad i = 1, 2.$$

Then (Δ_1, Γ_1) and (Δ_2, Γ_2) belong to $s_{\Omega/e}$. Since the quotient $\mathcal{X}_{\Omega/e}$ is semiregular and hence schurian, there exists $k \in \mathrm{Aut}(\mathcal{X}_{\Omega/e})$ such that

$$(\Delta_1, \Gamma_1)^k = (\Delta_2, \Gamma_2).$$

By statement (2) of Theorem 3.1.26, the automorphism k induces an algebraic automorphism $\psi_k \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_e)$ such that

$$(\Delta_1)^{\psi_k} = \Delta_2 \quad \text{and} \quad (\Gamma_1)^{\psi_k} = \Gamma_2. \tag{3.1.21}$$

By the separability of $\mathcal{X}_e$, there exists $f \in \mathrm{Iso}(\mathcal{X}_e, \mathcal{X}_e, \psi_k)$. From the definition of ψ_k, it follows that $f \in \mathrm{Aut}(\mathcal{X})$. Furthermore by (3.1.21),

$$(\Delta_1)^f = \Delta_2 \quad \text{and} \quad (\Gamma_1)^f = \Gamma_2.$$

Thus without loss of generality, we may assume that $\Delta_1 = \Delta_2$ and $\Gamma_1 = \Gamma_2$. Denote these sets by Δ and Γ. Then

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in s_{\Delta,\Gamma}.$$

The schurity of the coherent configuration $\mathcal{X}_e$ implies that $s_{\Delta,\Gamma}$ is a 2-orbit of $\mathrm{Aut}(\mathcal{X}_e) \le \mathrm{Aut}(\mathcal{X})$. Thus the pairs (α_1, β_1) and (α_2, β_2) belong to a 2-orbit of $\mathrm{Aut}(\mathcal{X})$, as required. □

A quite general situation, where a residually thin parabolic arises in a natural way, was considered in [**41**, Theorem 2.2]. Namely, let $\mathcal{X}$ be a coherent configuration on Ω. Assume that it admits an isomorphism group G acting regularly on the fibers of $\mathcal{X}$. Then

$$e = \bigcup_{\Delta \in F(\mathcal{X})} \Delta^2, \tag{3.1.22}$$

is a residually thin parabolic of $\mathcal{X}^G$. If, in addition, the extension of $\mathcal{X}^G$ with respect to e is equal to $\mathcal{X}$, then statement (2) of Theorem 3.1.28 can be reversed. Furthermore, the automorphism group of $\mathcal{X}^G$ is explicitly calculated modulo the group $\mathrm{Aut}(\mathcal{X})$.

Theorem 3.1.30. *Let $\mathcal{X}$ be a coherent configuration, $G \le \mathrm{Iso}(\mathcal{X})$, and*

$$f : G \to \mathrm{Sym}(F)$$

the homomorphism induced by the action of G on $F = F(\mathcal{X})$. Assume that the following conditions are satisfied:

(1) $\mathrm{Im}(f)$ *is a regular subgroup of* $\mathrm{Sym}(F)$*;*
(2) $\ker(f) \le \mathrm{Aut}(\mathcal{X})$.

Then

$$\mathrm{Aut}(\mathcal{X}^G) = G\,\mathrm{Aut}(\mathcal{X}). \tag{3.1.23}$$

Furthermore, the coherent configurations $\mathcal{X}$ and $\mathcal{X}^G$ are schurian or not simultaneously.

Proof. Equality (3.1.23) is a direct consequence of Lemma 3.1.31 below: conditions (1) and (2) of this lemma are satisfied by Corollary 2.2.23 and obvious equality

$$\ker(f) = \bigcap_{\Delta \in F} G_{\{\Delta\}},$$

respectively. Furthermore, if $\mathcal{X}$ is schurian, then so is $\mathcal{X}^G$ (Proposition 2.3.28). The reverse statement immediately follows from statement (2) of Theorem 3.1.28 for $\mathcal{X}$ replaced by $\mathcal{X}^G$ and residually thin parabolic e defined by formula (3.1.22).

Lemma 3.1.31. *In the notation of Theorem 3.1.30, equality (3.1.23) holds whenever the following conditions are satisfied:*

(1) *the group* $\mathrm{Im}(f)$ *is* 2*-closed;*
(2) *for any $\Delta_1, \Delta_2 \in F$, the induced action of the group $G_{\{\Delta_1\}} \cap G_{\{\Delta_2\}}$ on the set S_{Δ_1,Δ_2} is trivial, where $S = S(\mathcal{X})$.*

Proof. In accordance with Exercise 2.7.36, we have $G \le \mathrm{Aut}(\mathcal{X}^G)$. Furthermore, $\mathrm{Aut}(\mathcal{X}) \le \mathrm{Aut}(\mathcal{X}^G)$, because $\mathcal{X} \ge \mathcal{X}^G$. Thus,

$$\mathrm{Aut}(\mathcal{X}^G) \ge G\,\mathrm{Aut}(\mathcal{X}).$$

To prove the reverse inclusion let $r \in S(\mathcal{X}^G)$. Then $r = s^G$ for some $s \in S$. Note that if Δ and Γ are the fibers of $\mathcal{X}$ such that $s \in S_{\Delta,\Gamma}$, then $s^g \in S_{\Delta^g,\Gamma^g}$ for all $g \in G$. When the element g runs over G, the pair (Δ^g, Γ^g) runs over a certain 2-orbit t of the group $\mathrm{Im}(f)$. Therefore,

$$r = \bigcup_{(\Delta_1,\Delta_2) \in t} r_{\Delta_1,\Delta_2}. \tag{3.1.24}$$

Obviously, r_{Δ_1,Δ_2} is the union of all s^g for which $(\Delta_1, \Delta_2)^g = (\Delta_1, \Delta_2)$, or equivalently $g \in G_{\{\Delta_1\}} \cap G_{\{\Delta_2\}}$. By condition (2), this implies that

$$r_{\Delta_1,\Delta_2} \in S_{\Delta_1,\Delta_2} \quad \text{for all} \quad (\Delta_1, \Delta_2) \in t. \tag{3.1.25}$$

The parabolic e defined in (3.1.22), being G-invariant, is a parabolic of the coherent configuration $\mathcal{X}^G$. Since F is the set of classes of e, there is a homomorphism

$$f' : \mathrm{Aut}(\mathcal{X}^G) \to \mathrm{Sym}(F)$$

induced by the action of the group $\mathrm{Aut}(\mathcal{X}^G)$ on the set F. Clearly, $f'|_G = f$.

By formula (3.1.24) the groups $f(G)$ and $\mathrm{Im}(f')$ have the same 2-orbits. Therefore, condition (1) implies that

$$f(G) = \mathrm{Im}(f').$$

On the other hand, formula (3.1.25) shows that $\ker(f') \leq \mathrm{Aut}(\mathcal{X})$. Thus,

$$\mathrm{Aut}(\mathcal{X}^G) \leq G\,\mathrm{Aut}(\mathcal{X})$$

and we are done. □

3.2 Direct sum and tensor product

The direct product of two permutation groups has two natural actions: on the disjoint union and on the Cartesian product of the underlying sets. For coherent configurations these actions correspond to a direct sum and a tensor product, respectively. In this section, we define them explicitly and show that both respect the Galois correspondence between the coherent configurations and permutation groups. The most part of the results goes back to monograph [**123**].

Throughout the section, $\mathcal{X}_i = (\Omega_i, S_i)$ is a coherent configuration, $F_i = F(\mathcal{X}_i)$, and $E_i = E(\mathcal{X}_i)$, $i = 1, 2$.

3.2.1 The direct sum

Let Ω be the disjoint union of Ω_1 and Ω_2. Set

$$S_{1,2} = \{\Delta_1 \times \Delta_2 : \ (\Delta_1, \Delta_2) \in F_1 \times F_2\} \quad \text{and} \quad S_{2,1} = S_{1,2}^*.$$

Any pair contained in no relation belonging to the union $S_{1,2} \cup S_{2,1}$ lies in a basis relation of either $\mathcal{X}_1$ or $\mathcal{X}_2$. Therefore, the set

$$S_1 \boxplus S_2 = S_1 \, \cup \, S_2 \, \cup \, S_{1,2} \, \cup \, S_{2,1}, \tag{3.2.1}$$

forms a partition of Ω^2. It is easily seen that this partition satisfies the conditions (CC1) and (CC2). Thus the pair

$$\mathcal{X}_1 \boxplus \mathcal{X}_2 := (\Omega_1 \cup \Omega_2, S_1 \boxplus S_2)$$

is a rainbow on Ω.

Theorem 3.2.1. *The rainbow $\mathcal{X} = \mathcal{X}_1 \boxplus \mathcal{X}_2$ is a coherent configuration of degree $|\Omega_1|+|\Omega_2|$ and rank $|S_1|+|S_2|+2|F_1|\,|F_2|$. Moreover, $F(\mathcal{X}) = F_1 \cup F_2$, Ω_1 and Ω_2 are homogeneity sets of $\mathcal{X}$, and also $\mathcal{X}_{\Omega_1} = \mathcal{X}_1$ and $\mathcal{X}_{\Omega_2} = \mathcal{X}_2$.*

Proof. It suffices to verify that $\mathcal{X}$ satisfies the condition (CC3). Let $r, s \in S$, where $S = S_1 \boxplus S_2$. Without loss of generality, we may assume that $r \cdot s \neq \varnothing$. Then one of the following statements holds for $i = 1$ or 2:

(a) $r, s \in S_i$;
(b) $r \in (S_i)_{\Delta_i, \Gamma_i}$ and $s = \Gamma_i \times \Gamma_{3-i}$;
(c) $r = \Delta_{3-i} \times \Delta_i$ and $s \in (S_i)_{\Delta_i, \Gamma_i}$;
(d) $r = \Delta_{3-i} \times \Delta_i$ and $s = \Delta_i \times \Gamma_{3-i}$,

where $\Delta_i, \Gamma_i \in F_i$, and $\Delta_{3-i}, \Gamma_{3-i} \in F_{3-i}$.

Let us prove that given $t \in S$, the number $|\alpha r \cap \beta s^*|$ does not depend on the choice of $(\alpha, \beta) \in t$. This is true in the case (a), because

$$|\alpha r \cap \beta s^*| = \begin{cases} c_{rs}^t, & \text{if } t \in S_i, \\ 0, & \text{if } t \notin S_i. \end{cases}$$

In the cases (b) and (c), the assumption implies that t is equal to, respectively, $\Delta_i \times \Gamma_{3-i}$ and $\Delta_{3-i} \times \Gamma_i$, and

$$\alpha r \cap \beta s^* = \begin{cases} \alpha r \cap \Gamma_i, & \text{if } t = \Delta_i \times \Gamma_{3-i}, \\ \Delta_i \cap \beta s^*, & \text{if } t = \Delta_{3-i} \times \Gamma_i. \end{cases}$$

By formula (2.1.4), this implies that $|\alpha r \cap \beta s^*|$ equals, respectively, n_r and n_s.

In the remaining case (d), the assumption yields $t \subseteq \Delta_{3-i} \times \Gamma_{3-i}$. It follows that

$$|\alpha r \cap \beta s^*| = |\Delta_i|.$$

Thus the condition (CC3) is satisfied in each case and we are done. □

Definition 3.2.2. *The coherent configuration $\mathcal{X}_1 \boxplus \mathcal{X}_2$ is called the direct sum of the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$.*

It is straightforward to define the direct sum of more than two coherent configurations; it is obviously monotonic with respect to each summand, commutative, and associative. The uniqueness of the decomposition into the direct sum of indecomposable summands is established in the following statement.

Theorem 3.2.3. *Let $\mathcal{X}$ be a coherent configuration. Denote by $\mathfrak{X} = \mathfrak{X}(\mathcal{X})$ a loopless undirected graph the vertices of which are the fibers of $\mathcal{X}$, and two fibers Δ and Γ are adjacent if and only if $|S_{\Delta,\Gamma}| > 1$. Then*

$$\mathcal{X} = \boxplus_{i=1}^{m} \mathcal{X}_{\Omega_i},$$

where m is the number of connected components of $\mathfrak{X}$ and Ω_i is the union of fibers belonging to the ith component. Moreover, the restriction $\mathcal{X}_{\Omega_i}$ is indecomposable with respect to the direct sum.

Proof. The first statement is true, because if $i \neq j$ and $\Delta_i \subseteq \Omega_i$ and $\Delta_j \subseteq \Omega_j$ are fibers of $\mathcal{X}$, then Δ_i and Δ_j are not adjacent in the graph $\mathfrak{X}$, i.e., $|S_{\Delta_i,\Delta_j}| = 1$ or equivalently, $\Delta_i \times \Delta_j \in S$. To prove the second statement, we assume on the contrary that

$$\mathcal{X}_{\Omega_i} = \mathcal{X}_\Lambda \boxplus \mathcal{X}_{\Lambda'},$$

where Λ and Λ' are disjoint homogeneity sets of $\mathcal{X}$ contained in Ω_i. Then no fiber contained in Λ is adjacent to a fiber contained in Λ'. Therefore Λ and Λ' lie in different components of the graph $\mathfrak{X}(\mathcal{X}_{\Omega_i})$. Consequently, this graph is not connected, a contradiction. □

In the lattice of all coherent configuration on Ω, the direct sum $\mathcal{X}_1 \boxplus \mathcal{X}_2$ is distinguished by the minimality condition established in the following statement.

Theorem 3.2.4. *The direct sum $\mathcal{X}_1 \boxplus \mathcal{X}_2$ is the smallest coherent configuration among all coherent configurations $\mathcal{X}$ on Ω such that*

(3.2.2) $$\Omega_i \in F(\mathcal{X})^{\cup} \quad \text{and} \quad \mathcal{X}_{\Omega_i} = \mathcal{X}_i, \qquad i = 1, 2.$$

Proof. By Theorem 3.2.1, the coherent configuration $\mathcal{X}_1 \boxplus \mathcal{X}_2$ satisfies conditions (3.2.2). Let $\mathcal{X}$ be an arbitrary coherent configuration on Ω satisfying these conditions. For any its basis relation s,

$$s \in S(\mathcal{X})_{\Omega_i} \quad \text{or} \quad s \in S(\mathcal{X})_{\Omega_i, \Omega_{3-i}},$$

where $i = 1$ or 2. This implies that in any case s is contained in a basis relation of $\mathcal{X}_1 \boxplus \mathcal{X}_2$. Thus, $\mathcal{X} \geq \mathcal{X}_1 \boxplus \mathcal{X}_2$. □

Obviously, every discrete coherent configuration $\mathcal{X}$ is the direct sum of its homogeneous components. Furthermore in this case, $\mathcal{X} = \mathcal{X}_{\Omega_1} \boxplus \mathcal{X}_{\Omega_2}$, where Ω_1 is an arbitrary nonempty subset of Ω, and Ω_2 is the complement of Ω_1 in Ω.

The direct sum of coherent configurations is a combinatorial analog of the direct product of permutation groups in the intransitive action. The following statement shows that it is invariant with respect to the Galois correspondence between the coherent configurations and permutation groups.

Theorem 3.2.5. *Assume that Ω_1 and Ω_2 are disjoint sets. Then*

(3.2.3) $$\mathrm{Inv}(K_1 \times K_2, \Omega_1 \cup \Omega_2) = \mathrm{Inv}(K_1, \Omega_1) \boxplus \mathrm{Inv}(K_2, \Omega_2)$$

for all groups $K_1 \leq \mathrm{Sym}(\Omega_1)$ and $K_2 \leq \mathrm{Sym}(\Omega_2)$, and

(3.2.4) $$\mathrm{Aut}(\mathcal{X}_1 \boxplus \mathcal{X}_2) = \mathrm{Aut}(\mathcal{X}_1) \times \mathrm{Aut}(\mathcal{X}_2)$$

for all coherent configurations $\mathcal{X}_1$ on Ω_1 and $\mathcal{X}_2$ on Ω_2.

Proof. Denote by $\mathcal{X}$ the coherent configuration on the left-hand side of formula (3.2.3). By the definition of the intransitive direct product (see (1.3.1)),

$$\mathcal{X}_{\Omega_1} = \mathrm{Inv}(K_1, \Omega_1) \quad \text{and} \quad \mathcal{X}_{\Omega_2} = \mathrm{Inv}(K_2, \Omega_2),$$

and

$$\Delta_1 \times \Delta_2 \in \mathrm{Orb}(K_1 \times K_2, (\Omega_1 \cup \Omega_2)^2)$$

for any fibers Δ_1 and Δ_2 of the coherent configurations $\mathrm{Inv}(K_1, \Omega_1)$ and $\mathrm{Inv}(K_2, \Omega_2)$, respectively. Thus any basis relation of the direct sum on the right-hand side of formula (3.2.3) is a 2-orbit of the group

$$K_1 \times K_2 \leq \mathrm{Sym}(\Omega_1 \cup \Omega_2).$$

This proves formula (3.2.3).

To prove formula (3.2.4), set $\mathcal{X} := \mathcal{X}_1 \boxplus \mathcal{X}_2$. Clearly, Ω_1 and Ω_2 are $\mathrm{Aut}(\mathcal{X})$-invariant sets and

$$\mathrm{Aut}(\mathcal{X})^{\Omega_1} \leq \mathrm{Aut}(\mathcal{X}_1) \quad \text{and} \quad \mathrm{Aut}(\mathcal{X})^{\Omega_2} \leq \mathrm{Aut}(\mathcal{X}_2).$$

Therefore, the group on the left-hand side of (3.2.4) is contained in the group on the right-hand side.

Conversely, let $k \in \mathrm{Aut}(\mathcal{X}_1) \times \mathrm{Aut}(\mathcal{X}_2)$, and let $s \in S$. Now if $s \in S_i$ for $i = 1$ or 2, then
$$s^k = s^{k_i} = s,$$
where k_i is the ith component of k. Assume that $s = \Delta_i \times \Delta_{3-i}$ for some $\Delta_i \in F_i$ and $\Delta_{3-i} \in F_{3-i}$. Then
$$s^k = (\Delta_i)^{k_i} \times (\Delta_{3-i})^{k_{3-i}} = \Delta_i \times \Delta_{3-i} = s.$$
Thus in any case, $s^k = s$. This is true for all $s \in S$, and hence $k \in \mathrm{Aut}(\mathcal{X})$, as required. □

Corollary 3.2.6. *$\mathcal{X}_1 \boxplus \mathcal{X}_2$ is schurian if and only if so are $\mathcal{X}_1$ and $\mathcal{X}_2$.*

Proof. The necessity follows from the last statement of Theorem 3.2.1 and the fact that the schurity is preserved under restrictions (Exercise 2.7.21). To prove the sufficiency, we assume that the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$ are schurian, i.e.,
$$\mathcal{X}_1 = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1)) \quad \text{and} \quad \mathcal{X}_2 = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_2)).$$
Then by Theorem 3.2.5,
$$\begin{aligned}\mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1 \boxplus \mathcal{X}_2)) &= \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1) \times \mathrm{Aut}(\mathcal{X}_2))\\ &= \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1)) \boxplus \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_2))\\ &= \mathcal{X}_1 \boxplus \mathcal{X}_2,\end{aligned}$$
i.e., the coherent configuration $\mathcal{X}_1 \boxplus \mathcal{X}_2$ is schurian. □

Certainly, the direct sum of coherent configurations respects the isomorphisms. The same is true for the algebraic isomorphisms. The following statement together with Exercise 3.7.33 completely describes algebraic isomorphisms of direct sum. In particular, it shows that any coherent configuration algebraically isomorphic to direct sum, is also direct sum.

Theorem 3.2.7. *Let $\mathcal{X} = \mathcal{X}_1 \boxplus \mathcal{X}_2$ and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. Then*

(1) *$\mathcal{X}' = \mathcal{X}'_1 \boxplus \mathcal{X}'_2$, where $\mathcal{X}'_1 = \mathcal{X}'_{\Omega_1^\varphi}$ and $\mathcal{X}'_2 = \mathcal{X}'_{\Omega_2^\varphi}$;*
(2) *$\varphi_{\Omega_1} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}'_1)$ and $\varphi_{\Omega_2} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}'_2)$.*

Proof. From statement (1) of Proposition 2.3.22, it follows that the algebraic isomorphism φ induces an isomorphism of the graph $\mathfrak{X} = \mathfrak{X}(\mathcal{X})$ to the graph $\mathfrak{X}' = \mathfrak{X}(\mathcal{X}')$ defined in Theorem 3.2.3. Since no edge of $\mathfrak{X}$ join a vertex of Ω_1 with vertex of Ω_2, the same statement holds for $\mathfrak{X}'$ and the sets Ω_1^φ and Ω_2^φ. This immediately implies statement (1), and also statement (2) by the definition of the restriction of algebraic isomorphism. □

Under the condition of Theorem 3.2.7, the algebraic isomorphism φ is uniquely determined by its restrictions $\varphi_1 = \varphi_{\Omega_1}$ and $\varphi_2 = \varphi_{\Omega_2}$ (see Exercise 3.7.33). It easily follows that
$$\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) \neq \varnothing \quad \Leftrightarrow \quad \mathrm{Iso}(\mathcal{X}_1, \mathcal{X}'_1, \varphi_1) \neq \varnothing \text{ and } \mathrm{Iso}(\mathcal{X}_2, \mathcal{X}'_2, \varphi_2) \neq \varnothing.$$

This gives the following criterion for the direct sum of coherent configurations to be separable.

Corollary 3.2.8. $\mathcal{X}_1 \boxplus \mathcal{X}_2$ *is separable if and only if so are* $\mathcal{X}_1$ *and* $\mathcal{X}_2$.

3.2.2 Tensor product

Let $\Omega = \Omega_1 \times \Omega_2$; the first and second coordinates of $\alpha \in \Omega$ are denoted by α_1 and α_2, respectively. There are two natural equivalence relations on Ω defined by the equalities of the first and second coordinates, namely,

$$e_1 = \{(\alpha,\beta) \in \Omega^2 :\ \alpha_1 = \beta_1\}, \qquad e_2 = \{(\alpha,\beta) \in \Omega^2 :\ \alpha_2 = \beta_2\}. \tag{3.2.5}$$

The classes of e_1 and e_2 are of the form $\{\alpha_1\}\times\Omega_2$ and $\Omega_1\times\{\alpha_2\}$, respectively, and the associated canonical mappings

$$f_1 : \Omega_1 \to \Omega/e_1, \alpha_1 \mapsto \{\alpha_1\}\times\Omega_2, \quad f_2 : \Omega_2 \to \Omega/e_2, \alpha_2 \mapsto \Omega_1\times\{\alpha_2\} \tag{3.2.6}$$

are bijections.

The tensor product of any two relations, one on Ω_1 and the other on Ω_2, defined by formula (1.1.1), is a relation on Ω. This allows us to define a rainbow

$$\mathcal{X}_1 \otimes \mathcal{X}_2 = (\Omega_1 \times \Omega_2, S_1 \otimes S_2)$$

for any rainbows $\mathcal{X}_1 = (\Omega_1, S_1)$ and $\mathcal{X}_2 = (\Omega_2, S_2)$. In what follows, any $s \in S_1 \otimes S_2$ is written in the form $s = s_1 \otimes s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$.

Theorem 3.2.9. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be coherent configurations. Then the rainbow $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$ is a coherent configuration of degree $|\Omega_1|\cdot|\Omega_2|$, rank $|S_1|\cdot|S_2|$, and intersection numbers*

$$c_{rs}^t = c_{r_1 s_1}^{t_1} \cdot c_{r_2 s_2}^{t_2}, \qquad r,s,t \in S_1 \otimes S_2. \tag{3.2.7}$$

In particular, $n_s = n_{s_1} n_{s_2}$ for all s. Moreover,

(1) $F(\mathcal{X}) = \{\Delta_1 \times \Delta_2 :\ \Delta_1 \in F_1,\ \Delta_2 \in F_2\}$;
(2) *e_1 and e_2 are parabolics of $\mathcal{X}$;*
(3) *$f_1 \in \mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_{\Omega/e_1})$ and $f_2 \in \mathrm{Iso}(\mathcal{X}_2, \mathcal{X}_{\Omega/e_2})$.*

Proof. We have $\alpha r = \alpha_1 r_1 \times \alpha_2 r_2$ for all $\alpha \in \Omega$ and $r = r_1 \otimes r_2$. It follows that for all $\beta \in \Omega$ and $s \subseteq \Omega^2$,

$$\alpha r \cap \beta s^* = (\alpha_1 r_1 \cap \beta_1 s_1^*) \times (\alpha_2 r_2 \cap \beta_2 s_2^*).$$

Consequently, given $r,s,t \in S$ the cardinality of the set $\alpha r \cap \beta s^*$ is equal to the number on the right-hand side of (3.2.7) and does not depend on the pair $(\alpha,\beta) \in t$. Thus, $\mathcal{X}$ satisfies the condition (CC3) and hence is a coherent configuration with intersection numbers defined by formula (3.2.7).

Statements (1) and (2) immediately follow from the obvious equalities

$$1_{\Delta_1} \otimes 1_{\Delta_2} = 1_{\Delta_1 \times \Delta_2},$$

where $\Delta_1 \subseteq \Omega_1$ and $\Delta_2 \subseteq \Omega_2$, and

$$e_1 = \bigcup_{s_2 \in S_2} 1_{\Omega_1} \otimes s_2 \quad \text{and} \quad e_2 = \bigcup_{s_1 \in S_1} s_1 \otimes 1_{\Omega_2}.$$

The definition of the bijection f_1 implies that for all $s_1 \in S_1$ and $s_2 \in S_2$,

$$(s_1)^{f_1} = s_1 \otimes \Omega_2^2 = (s_1 \otimes s_2)_{\Omega/e_1}.$$

Thus, $f_1 \in \mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_{\Omega/e_1})$. The second part of statement (3) is proved similarly. □

Definition 3.2.10. *The coherent configuration $\mathcal{X}_1 \otimes \mathcal{X}_2$ is called the tensor product of the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$. It is said to be nontrivial if the degrees of both $\mathcal{X}_1$ and $\mathcal{X}_2$ are greater than* 1.

The following statement is an immediate consequence of Theorem 3.2.9.

Corollary 3.2.11. *The coherent configuration $\mathcal{X}_1 \otimes \mathcal{X}_2$ is homogeneous (respectively, symmetric, commutative, semiregular) if and only if so are $\mathcal{X}_1$ and $\mathcal{X}_2$.*

Example 3.2.12. *For integers $n, m \geq 2$, the tensor product $\mathcal{X} = \mathcal{T}_n \otimes \mathcal{T}_m$ is a scheme of degree nm, rank 4, and valencies*

$$\{1, n-1, m-1, (n-1)(m-1)\}.$$

The irreflexive basis relations of valencies $n-1$ and $m-1$ are

$$s_n \otimes 1_m = e_2 \setminus 1_{mn} \quad \text{and} \quad 1_n \otimes s_m = e_1 \setminus 1_{mn},$$

where s_n and 1_n (respectively, s_m and 1_m) are the irreflexive and reflexive basis relations of $\mathcal{T}_n$ (respectively, $\mathcal{T}_m$).

The union of $s_n \otimes 1_m$ and $1_n \otimes s_m$ equals the edge set of so-called $n \times m$-grid graph; the edge set of the complement graph gives the basis relation of $\mathcal{X}$ that is of valency $(n-1)(m-1)$.

Let $\mathcal{Y}$ be a fission of the tensor product $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$. Then e_1 and e_2 are also parabolics of $\mathcal{Y}$. The quotients $\mathcal{Y}_{\Omega/e_1}$ and $\mathcal{Y}_{\Omega/e_2}$ are definitely fissions of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively.

Definition 3.2.13. *The coherent configuration $\mathcal{Y} \geq \mathcal{X}$ is called a subtensor product of $\mathcal{X}_1$ and $\mathcal{X}_2$ if*

$$\mathcal{Y}_{\Omega/e_1} = \mathcal{X}_1 \quad \text{and} \quad \mathcal{Y}_{\Omega/e_2} = \mathcal{X}_2. \tag{3.2.8}$$

Thus the tensor product of coherent configurations is a special case of the subtensor product, and exactly in the same sense as the direct product in group theory is a special case of the subdirect product.

Example 3.2.14. *Let $\mathcal{Y}$ be the scheme of the transitive permutation group K of degree 9 defined as follows:*

$$K = \{(k_1, k_2) \in \mathrm{Sym}(3) \times \mathrm{Sym}(3) : \ f(k_1) = f(k_2)\},$$

where $f : \mathrm{Sym}(3) \to \mathrm{Sym}(2)$ is the natural epimorphism. Then $\mathcal{Y}$ is a fission of the tensor product $\mathcal{X} = \mathcal{T}_3 \times \mathcal{T}_3$, and conditions (3.2.8) are satisfied for

$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{T}_3$. *Thus,* $\mathcal{Y}$ *is a subtensor product of* $\mathcal{X}_1$ *and* $\mathcal{X}_2$*, but not a tensor product, because* $\mathrm{rk}(\mathcal{Y}) = 5$*, whereas* $\mathrm{rk}(\mathcal{X}) = 4$.

As in the case of the direct sum, the tensor product is distinguished by the minimality condition in the sense that the rank of the subtensor product $\mathcal{Y}$ of the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$ is greater than or equal to $\mathrm{rk}(\mathcal{X}_1 \otimes \mathcal{X}_2)$ with equality if and only if $\mathcal{Y} = \mathcal{X}_1 \otimes \mathcal{X}_2$.

Theorem 3.2.15. *The tensor product* $\mathcal{X}_1 \otimes \mathcal{X}_2$ *is the smallest coherent configuration among the subtensor products of* $\mathcal{X}_1$ *and* $\mathcal{X}_2$.

In order to characterize the subtensor product internally, let e_1 and e_2 be *orthogonal* equivalence relations on Ω, i.e.,

$$e_1 \cap e_2 = 1_\Omega \quad \text{and} \quad e_1 \cdot e_2 = e_2 \cdot e_1 = \Omega^2. \tag{3.2.9}$$

Clearly, if the underlying set is the Cartesian product $\Omega = \Omega_1 \times \Omega_2$, then the equivalence relations e_1 and e_2 defined by formulas (3.2.5) are orthogonal.

Thus a necessary condition for a coherent configuration to be isomorphic to a subtensor product, is the existence of a pair of orthogonal parabolics. Before proving that this condition is also sufficient (see Theorem 3.2.17 below), we give an explicit example of orthogonal parabolics.

Example 3.2.16. *Let* $\mathcal{X}$ *be a semiregular coherent configuration, and let* $T \subseteq S$ *be as in statement* (3) *of Exercise* 2.7.13*. Then*

$$e_1 = \bigcup_{t \in T} t \quad \textit{and} \quad e_2 = \bigcup_{\Delta \in F} \Delta^2$$

are orthogonal parabolics of $\mathcal{X}$.

Let e_1 and e_2 be orthogonal equivalence relations on Ω. Set

$$\Omega_1 = \Omega/e_1 \quad \text{and} \quad \Omega_2 = \Omega/e_2.$$

Any point $\alpha \in \Omega$ lies in a unique class $\alpha_1 \in \Omega/e_1$ and in a unique class $\alpha_2 \in \Omega/e_2$. In view of the first condition in (3.2.9), these classes have at most one common point, and in our case such a point does exist, namely, α. So the mapping

$$f : \Omega \to \Omega_1 \times \Omega_2, \quad \alpha \mapsto (\alpha_1, \alpha_2), \tag{3.2.10}$$

is well-defined and injective.

Furthermore, let $\beta_1 \in \Omega_1$ and $\gamma_2 \in \Omega_2$. By the second condition in (3.2.9), for any points $\beta \in \beta_1$ and $\gamma \in \gamma_2$, there exist a point $\alpha \in \Omega$ such that

$$(\beta, \alpha) \in e_1 \quad \text{and} \quad (\alpha, \gamma) \in e_2.$$

Therefore, $\alpha \in \beta_1 \cap \gamma_2$ and hence $(\beta_1, \gamma_2) = (\alpha_1, \alpha_2)$, i.e., $f(\alpha) = (\beta_1, \gamma_2)$.

Thus the mapping f is surjective and hence a bijection. It is called the standard bijection associated with orthogonal equivalence relations e_1 and e_2.

Theorem 3.2.17. *Let $\mathcal{X}$ be a coherent configuration, e_1 and e_2 orthogonal parabolics of $\mathcal{X}$, and f the standard bijection associated with them. Then the coherent configuration $\mathcal{X}^f$ is the subtensor product of $\mathcal{X}_{\Omega/e_1}$ and $\mathcal{X}_{\Omega/e_2}$.*

Proof. From the definition of f, it follows that $\mathcal{X}' = \mathcal{X}^f$ is a coherent configuration on $\Omega' = \Omega_1 \times \Omega_2$, where $\Omega_1 = \Omega/e_1$ and $\Omega_2 = \Omega/e_2$, and the parabolics $e_1' = (e_1)^f$ and $e_2' = (e_2)^f$ are of the form (3.2.5).

Furthermore, for all basis relations s of $\mathcal{X}$,

$$f(s^{e_1}) = s_{\Omega_1} \otimes \Omega_2^2 \quad \text{and} \quad f(s^{e_2}) = \Omega_1^2 \otimes s_{\Omega_2}, \tag{3.2.11}$$

where s^{e_1} and s^{e_2} are defined by formula (3.1.3). It follows that $f\pi_{e_1'} = \pi_{e_1}$ and $f\pi_{e_2'} = \pi_{e_2}$, see (1.1.4). Thus,

$$\mathcal{X}'_{\Omega'/e_1'} = \mathcal{X}_{\Omega/e_1} \quad \text{and} \quad \mathcal{X}'_{\Omega'/e_2'} = \mathcal{X}_{\Omega/e_2}. \tag{3.2.12}$$

By formula (3.2.11), for all basis relations r and s of $\mathcal{X}$, the product

$$r_{\Omega_1} \otimes s_{\Omega_2} = (r_{\Omega_1} \otimes \Omega_2^2) \cap (\Omega_1^2 \otimes s_{\Omega_2})$$

is a relation of $\mathcal{X}'$. Thus, $\mathcal{X}'$ is a fission of the tensor product $\mathcal{X}_{\Omega_1} \otimes \mathcal{X}_{\Omega_2}$. Together with (3.2.12), this shows that $\mathcal{X}'$ is the subtensor product of $\mathcal{X}_{\Omega/e_1}$ and $\mathcal{X}_{\Omega/e_2}$. □

Let $\mathcal{X} = (\Omega, S)$ be a semiregular coherent configuration and e_1 and e_2 are orthogonal parabolics from Example 3.2.16. In accordance with Exercises 3.7.7 and 3.7.10,

$$\mathcal{X}_{\Omega/e_1} \cong \mathcal{X}_\Delta \quad \text{and} \quad \mathcal{X}_{\Omega/e_2} = \mathcal{D}_F,$$

where $F = \mathcal{F}(\mathcal{X})$ and $\Delta \in F$. By Theorem 3.2.17, the coherent configuration $\mathcal{X}$ is isomorphic to the subtensor product of $\mathcal{D}_F$ and $\mathcal{X}_\Delta$. On the other hand,

$$\mathrm{rk}(\mathcal{X}) = |F|^2 \cdot |\Delta| = \mathrm{rk}(\mathcal{D}_F) \cdot \mathrm{rk}(\mathcal{X}_\Delta),$$

see statement (1) of Exercise 2.7.13. Thus $\mathcal{X}$ is isomorphic to $\mathcal{D}_F \otimes \mathcal{X}_\Delta$. This shows that semiregular coherent configurations can be characterized as follows.

Corollary 3.2.18. *Any semiregular coherent configuration $\mathcal{X}$ is isomorphic to the tensor product of the discrete coherent configuration $\mathcal{D}_{F(\mathcal{X})}$ and a homogeneous component of $\mathcal{X}$.*

The following result shows that under a special commutativity condition, the subtensor product equals tensor product if one of the factors is semiregular. In commutative case, it was proved in [**43**].

Theorem 3.2.19. *Let $\mathcal{X}$ be the subtensor product of the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$. Suppose that $\mathcal{X}_1$ is semiregular, $\mathcal{X}_2$ is homogeneous, and*

$$s \cdot e_2 = e_2 \cdot s \quad \textit{for all } s \in S,\ s \subseteq e_1 \tag{3.2.13}$$

where e_1 *and* e_2 *are as in* (3.2.5), *and* $S = S(\mathcal{X})$. *Then* $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$.

Proof. Fix any $\Delta \in \Omega/e_1$. By the orthogonality of the parabolics e_1 and e_2, we have $s^{e_2} \cap \Delta^2 \ne \varnothing$ for every $s \in S$. Therefore the mapping

$$S_\Delta \to S^{e_2},\ s_\Delta \mapsto s^{e_2}$$

is surjective. Moreover, if $r^{e_2} = s^{e_2}$ for some relations $r, s \in S$ contained in e_1, then by condition (3.2.13) we have

$$r \cdot e_2 = e_2 \cdot r \cdot e_2 = r^{e_2} = s^{e_2} = e_2 \cdot s \cdot e_2 = s \cdot e_2.$$

Now if $r_\Delta \ne \varnothing \ne s_\Delta$, then $r_\Delta = (r \cdot e_2)_\Delta = (s \cdot e_2)_\Delta = s_\Delta$, because $(e_2)_\Delta = 1_\Delta$, and hence $r = s$. Thus the above mapping is injective and

$$|S_\Delta| = |S_{\Omega/e_2}| = \operatorname{rk}(\mathcal{X}_2). \tag{3.2.14}$$

The parabolic e_1 is proper by the assumption on $\mathcal{X}_2$ and condition (3.2.13). So it is residually thin by the assumption on $\mathcal{X}_1$. Every class of e_1 is a homogeneity set of the extension $\mathcal{X}_{e_1}$ of $\mathcal{X}$ with respect to e_1 (statement (1) of Theorem 3.1.26). Let us verify that

$$|(S_{e_1})_\Delta| = |(S_{e_1})_\Gamma| = |(S_{e_1})_{\Delta,\Gamma}| \tag{3.2.15}$$

for all classes Δ and Γ of the parabolic e_1, where $S_{e_1} = S(\mathcal{X}_{e_1})$.

Indeed, the orthogonality of e_1 and e_2 implies that

$$s := (e_2)_{\Delta,\Gamma}$$

is a thin relation of the restriction $(\mathcal{X}_{e_1})_{\Delta\cup\Gamma}$. The bijection f_s defined by formula (1.1.5) is an isomorphism from $(\mathcal{X}_{e_1})_{\Delta\cup\Gamma}$ to itself (Example (2.2.2)) such that $\Delta^{f_s} = \Gamma$. This proves the first equality in (3.2.15), whereas the second follows from the fact that

$$(S_{e_1})_{\Delta,\Gamma} = (S_{e_1})_\Delta \cdot s.$$

By (3.2.15), the number $|(S_{e_1})_{\Delta,\Gamma}|$ does not depend on the classes Δ and Γ. Consequently, the rank of $\mathcal{X}_{e_1}$ is equal to this number multiplied by $|\Omega/e_1|^2$. Taking into account equalities (3.2.14) and $(S_{e_1})_\Delta = S_\Delta$ (statement (1) of Theorem 3.1.26), we obtain

$$\operatorname{rk}(\mathcal{X}_{e_1}) = \operatorname{rk}(\mathcal{X}_2) \cdot |\Omega/e_1|^2.$$

Note that $|\Omega/e_1| = |\Omega_1|$. Besides, from statement (3) of Theorem 3.1.26 and the semiregularity of the action of $\operatorname{Aut}(\mathcal{X}_1)$ on Ω_1, it follows that $\operatorname{rk}(\mathcal{X})$ is equal to $\operatorname{rk}(\mathcal{X}_{e_1})$ divided by $|\operatorname{Aut}(\mathcal{X}_1)|$. Thus,

$$\operatorname{rk}(\mathcal{X}) = \frac{\operatorname{rk}(\mathcal{X}_2) \cdot |\Omega_1|^2}{|\operatorname{Aut}(\mathcal{X}_1)|}. \tag{3.2.16}$$

Since the coherent configuration $\mathcal{X}_1$ is semiregular, the group $\operatorname{Aut}(\mathcal{X}_1)$ acts regularly on each fiber Δ and hence its order equals $|\Delta|$. By statement (1) of Exercise 2.7.13, we obtain

$$|\Omega_1|^2 = |\Delta|^2 \cdot |F(\mathcal{X}_1)|^2 = |\Delta| \cdot (|\Delta| \cdot |F(\mathcal{X}_1)|^2) = |\operatorname{Aut}(\mathcal{X}_1)| \cdot \operatorname{rk}(\mathcal{X}_1).$$

Together with formula (3.2.16), this implies that

$$\operatorname{rk}(\mathcal{X}) = \operatorname{rk}(\mathcal{X}_2) \cdot \operatorname{rk}(\mathcal{X}_1) = \operatorname{rk}(\mathcal{X}_1 \otimes \mathcal{X}_2),$$

and we are done because $\mathcal{X} \geq \mathcal{X}_1 \otimes \mathcal{X}_2$. □

Corollary 3.2.20. *A commutative subtensor product of a regular scheme with any other scheme equals the tensor product of these schemes.*

Condition (3.2.13) in Theorem 3.2.19 is essential. Indeed, take $\mathcal{X}$ to be the regular scheme on Ω associated with a Frobenius group with kernel G and complement H. Then the groups G and H are semiregular, the equivalence relations e_1 and e_2 defined by

$$\Omega/e_1 = \operatorname{Orb}(G, \Omega) \quad \text{and} \quad \Omega/e_2 = \operatorname{Orb}(H, \Omega)$$

are orthogonal parabolics of $\mathcal{X}$, and the quotient scheme $\mathcal{X}_1 = \mathcal{X}_{\Omega/e_1}$ is regular. By Theorem 3.2.17, $\mathcal{X}$ is isomorphic to the subtensor product of the schemes $\mathcal{X}_1$ and $\mathcal{X}_2 = \mathcal{X}_{\Omega/e_2}$. On the other hand, $\operatorname{rk}(\mathcal{X}) > \operatorname{rk}(\mathcal{X}_1)\operatorname{rk}(\mathcal{X}_2)$, and hence $\mathcal{X}$ is not isomorphic to the tensor product of these schemes.

The tensor product of coherent configurations is a combinatorial analog of the direct product of permutation groups in the transitive action. The following statement shows that it is invariant with respect to the Galois correspondence between the coherent configurations and permutation groups.

Theorem 3.2.21. *For arbitrary sets Ω_1 and Ω_2,*

$$\operatorname{Inv}(K_1 \times K_2, \Omega_1 \times \Omega_2) = \operatorname{Inv}(K_1, \Omega_1) \otimes \operatorname{Inv}(K_2, \Omega_2) \tag{3.2.17}$$

for all groups $K_1 \leq \operatorname{Sym}(\Omega_1)$ and $K_2 \leq \operatorname{Sym}(\Omega_2)$, and

$$\operatorname{Aut}(\mathcal{X}_1 \otimes \mathcal{X}_2) = \operatorname{Aut}(\mathcal{X}_1) \times \operatorname{Aut}(\mathcal{X}_2) \tag{3.2.18}$$

for all coherent configurations $\mathcal{X}_1$ on Ω_1 and $\mathcal{X}_2$ on Ω_2.

Proof. A straightforward check shows that given $(\alpha, \beta) \in (\Omega_1 \times \Omega_2)^2$,

$$(\alpha, \beta)^{K_1 \times K_2} = (\alpha_1, \beta_1)^{K_1} \otimes (\alpha_2, \beta_2)^{K_2}.$$

This shows that the basis relations of the coherent configuration on the left-hand side of (3.2.17) coincide with those on the right-hand side, and the first statement follows.

To prove the second statement, set $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$. Let e_1 and e_2 be the equivalence relations on $\Omega = \Omega_1 \times \Omega_2$ defined in (3.2.5). Without loss of generality, we may assume that $\Omega_1 = \Omega/e_1$ and $\Omega_2 = \Omega/e_2$. Then by formula (3.1.9),

$$\operatorname{Aut}(\mathcal{X})^{\Omega_1} \leq \operatorname{Aut}(\mathcal{X}_1) \quad \text{and} \quad \operatorname{Aut}(\mathcal{X})^{\Omega_2} \leq \operatorname{Aut}(\mathcal{X}_2).$$

Therefore, the group on the left-hand side of (3.2.18) is contained in the group on the right-hand side.

Conversely, let $(k_1, k_2) \in \mathrm{Aut}(\mathcal{X}_1) \times \mathrm{Aut}(\mathcal{X}_2)$. Given $s_1 \otimes s_2 \in S(\mathcal{X})$,

$$(s_1 \otimes s_2)^{(k_1,k_2)} = (s_1)^{k_1} \otimes (s_2)^{k_2} = s_1 \otimes s_2 = s.$$

Thus, $(k_1, k_2) \in \mathrm{Aut}(\mathcal{X})$ and we are done. □

Corollary 3.2.22. *$\mathcal{X}_1 \otimes \mathcal{X}_2$ is schurian if and only if so are $\mathcal{X}_1$ and $\mathcal{X}_2$.*

Proof. The necessity follows from Corollary 3.1.17. To prove the sufficiency, we assume that the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$ are schurian. Then by Theorem 3.2.21,

$$\begin{aligned}\mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1 \otimes \mathcal{X}_2)) &= \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1) \times \mathrm{Aut}(\mathcal{X}_2)) \\ &= \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1)) \otimes \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_2)) \\ &= \mathcal{X}_1 \otimes \mathcal{X}_2,\end{aligned}$$

i.e., the coherent configuration $\mathcal{X}_1 \otimes \mathcal{X}_2$ is schurian. □

Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. Assume that e_1 and e_2 are orthogonal parabolics of $\mathcal{X}$. Then $e_1' = \varphi(e_1)$ and $e_2' = \varphi(e_2)$ are orthogonal parabolics of $\mathcal{X}'$, and

$$\varphi_{\Omega_1} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{\Omega_1}, \mathcal{X}'_{\Omega_1'}) \quad \text{and} \quad \varphi_{\Omega_2} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{\Omega_2}, \mathcal{X}'_{\Omega_2'}),$$

where $\Omega_i = \Omega/e_i$ and $\Omega_i' = \Omega'/e_i'$ for $i = 1, 2$. Thus the following statement is a consequence of Theorem 3.2.17 and the minimality of the tensor product.

Theorem 3.2.23. *Let $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$, e_1 and e_2 the parabolics (3.2.5), and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. Then in the above notations,*

1. *$(\mathcal{X}')^{f'} = \mathcal{X}_1' \otimes \mathcal{X}_2'$, where $\mathcal{X}_1' = \mathcal{X}'_{\Omega_1'}$, $\mathcal{X}_2' = \mathcal{X}'_{\Omega_2'}$, and f' is the standard bijection associated with e_1' and e_2';*
2. *$\varphi_{\Omega_1} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}_1')$ and $\varphi_{\Omega_2} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}_2')$.*

Under the condition of Theorem 3.2.23, the algebraic isomorphism φ is uniquely determined by its restrictions $\varphi_1 = \varphi_{\Omega_1}$ and $\varphi_2 = \varphi_{\Omega_2}$ (see statement (1) of Exercise 3.7.33). It easily follows that

$$\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) \ne \varnothing \quad \Leftrightarrow \quad \mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1) \ne \varnothing \text{ and } \mathrm{Iso}(\mathcal{X}_2, \mathcal{X}_2', \varphi_2) \ne \varnothing.$$

This gives the following criterion for the tensor product of coherent configurations to be separable.

Corollary 3.2.24. *$\mathcal{X}_1 \otimes \mathcal{X}_2$ is separable if and only if so are $\mathcal{X}_1$ and $\mathcal{X}_2$.*

The adjacency matrix of a relation $s_1 \otimes s_2$ is equal to the Kronecker product $A_{s_1} \otimes A_{s_2}$ (Exercise 1.4.9). When s_1 and s_2 run over basis relations of the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$, the matrix $A_{s_1} \otimes A_{s_2}$ runs over the standard basis of the tensor product $\mathrm{Adj}(\mathcal{X}_1) \otimes \mathrm{Adj}(\mathcal{X}_2)$. This proves the following statement.

Theorem 3.2.25. $\mathrm{Adj}(\mathcal{X}_1 \otimes \mathcal{X}_2) = \mathrm{Adj}(\mathcal{X}_1) \otimes \mathrm{Adj}(\mathcal{X}_2)$.

It is straightforward to define the tensor product of more than two coherent configurations; it is obviously noncommutative but associative and monotonic with respect to each factor.

In general, the decomposition of any coherent configuration into the tensor product of indecomposable factors is not unique. For example, if $\mathcal{X}$ is the scheme of regular elementary abelian p-group G, then each decomposition of G into the direct product of subgroups of order p induces a decomposition of $\mathcal{X}$ into the tensor product of regular schemes of degree p. The uniqueness of subtensor product decomposition was proved for commutative schemes in [**50**].

3.3 Point extensions

Use of pointwise stabilizers in permutation group theory seems to be quite natural. In terms of them, one can express various properties of a permutation group (e.g., primitivity) and define important invariants (e.g., subdegree or base number). A combinatorial analog of the pointwise stabilizer was introduced in [**123**, p. 111] in algorithmic way. Systematically, this concept, in the form of a special coherent closure and under the name point extension, began to be studied since from the late 1990s, see, e.g., [**36, 40**].

In this section, the point extensions of coherent configurations are considered in the framework of the Galois correspondence between the coherent configurations and permutation groups. The three main results proved here are a combinatorial analog of the Wielandt theorem on imprimitive Frobenius group [**125**, Theorem 10.4], the Babai theorem on the base number of a primitive scheme [**5**], and the schurity and separability of partly regular coherent configurations [**41**, Theorem 9.3].

3.3.1 One-point extension

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration and T a set of relations on Ω. Recall that $\mathrm{WL}(\mathcal{X}, T)$, the *extension of* $\mathcal{X}$ *by* T, is the smallest coherent configuration on Ω that is a fission of $\mathcal{X}$ and contains T as a set of relations. In other words,

$$\mathrm{WL}(\mathcal{X}, T) = \mathrm{WL}(S \cup T).$$

In this subsection, we are interested in a special extension, namely, for the set $T = \{1_\alpha\}$ with $\alpha \in \Omega$.

Definition 3.3.1. *The coherent configuration*

$$\mathcal{X}_\alpha = \mathrm{WL}(\mathcal{X}, \{1_\alpha\})$$

is called an α*-extension or one-point extension of the coherent configuration* $\mathcal{X}$*.*

Clearly,

$$\mathcal{X} \le \mathcal{X}_\alpha \le \mathcal{D}_\Omega.$$

Both the equalities are attained: the first if and only if $\{\alpha\}$ is a fiber of $\mathcal{X}$ (see Corollary 3.3.6 below), and the second if, for example, $\mathcal{X}$ is a semiregular coherent configuration.

Example 3.3.2. *Let* $\mathcal{X} = \mathcal{T}_\Omega$ *and* $\alpha \in \Omega$*. Then the minimality of the direct sum* (*Theorem* 3.2.4) *implies that*

$$\mathcal{X}_\alpha = \mathcal{D}_{\{\alpha\}} \boxplus \mathcal{T}_{\Omega \setminus \{\alpha\}}.$$

In view of the monotonicity of the coherent closure operator (Exercise 2.7.49),

$$\mathcal{X} \le \mathcal{X}' \quad \Rightarrow \quad \mathcal{X}_\alpha \le \mathcal{X}'_\alpha \tag{3.3.1}$$

for all coherent configurations $\mathcal{X}'$ on Ω. Moreover, the one-point extension respects the Galois correspondence between the coherent configurations and permutation groups in the following sense.

Proposition 3.3.3. *For any $\alpha \in \Omega$,*

(1) *if $\mathcal{X}$ is a coherent configuration on Ω, then $\mathrm{Aut}(\mathcal{X}_\alpha) = \mathrm{Aut}(\mathcal{X})_\alpha$;*
(2) *if K is a permutation group on Ω, then $\mathrm{Inv}(K_\alpha) \ge \mathrm{Inv}(K)_\alpha$.*

Proof. Statement (1) immediately follows from Theorem 2.6.4. To prove statement (2), we observe that the inclusion $K_\alpha \le K$ and formula (2.2.6) imply that

$$\mathrm{Inv}(K_\alpha) \ge \mathrm{Inv}(K).$$

Since $\{\alpha\}$ is a fiber of the coherent configuration $\mathrm{Inv}(K_\alpha)$, this implies that

$$\mathrm{Inv}(K_\alpha) = \mathrm{Inv}(K_\alpha)_\alpha \ge \mathrm{Inv}(K)_\alpha,$$

as required. □

The inclusion in statement (2) is attained, see, e.g., Example 3.3.2 or statement (1) of Theorem 4.4.14. In general, the inclusion is not strict; in other words, a one-point extension of a schurian coherent configuration is not necessarily schurian.[1] On the other hand, there are non-schurian coherent configurations for which all one-point extensions are schurian.

Example 3.3.4. *Let $\mathcal{X}$ be a unique antisymmetric coherent configuration of degree 15 and rank 3. Although it is not schurian, all one-point extensions of $\mathcal{X}$ are schurian. Indeed, if this is not true, then there exists a point α such that*

$$\mathcal{Y} = (\mathcal{X}_\alpha)_{\Omega \setminus \{\alpha\}}$$

is a non-schurian coherent configuration of degree 14. A straightforward computation shows that $\mathcal{Y}$ has fibers of cardinality 3 or 7. Therefore, $\mathcal{Y}$ cannot be the unique non-schurian coherent configuration described in Subsection 2.2.3. Thus, $\mathcal{Y}$ is schurian, a contradiction.

There is also no explicit relationship between the separability of a coherent configuration and the separability of its one-point extension. For example, the coherent configuration $\mathcal{X}$ from Example 3.3.4 is not separable but one can show (and this is a good exercise) that all its one-point extensions are separable. On the other hand, the coherent configurations in the proof of Theorem 4.2.4 for $cn \ge 2$, are non-separable as well as all their one-point extensions.

[1]The smallest example we know is obtained for the coherent configuration of the Galois plane of order 3; the non-schurity of the point extension can be derived from results in [**45**].

In general, finding the one-point extension of a coherent configuration in an explicit form is a quite difficult problem. The following statement can sometimes simplify it. For example, it immediately implies that any one-point extension of a semiregular coherent configuration is discrete.

Lemma 3.3.5. *Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration, $\alpha \in \Omega$. Then for all $r, s, t \in S^{\cup}$,*

(1) *$\alpha r \in (F_\alpha)^{\cup}$, where $F_\alpha = F(\mathcal{X}_\alpha)$;*
(2) *$r_{s,t} \in (S_\alpha)^{\cup}$, where $r_{s,t} = r \cap (\alpha s \times \alpha t)$ and $S_\alpha = S(\mathcal{X}_\alpha)$;*
(3) *if $s, t \in S$, then $|\beta r_{s,t}| = c_{tr^*}^s$ for all $\beta \in \alpha s$.*

Proof. By the definition of the α-extension,

$$\{\alpha\} \in F_\alpha \quad \text{and} \quad r \in S^{\cup} \subseteq (S_\alpha)^{\cup}.$$

This implies that $1_\alpha \cdot r$ is a relation of $\mathcal{X}_\alpha$ (Proposition 2.1.4). In accordance with Exercise 2.7.4,

$$\alpha r = \Omega_+(1_\alpha \cdot r) \in (F_\alpha)^{\cup}.$$

This proves statement (1).

From statement (1), it follows that $\alpha t \in (F_\alpha)^{\cup}$. Therefore, $\alpha s \times \alpha t$ is a relation of $\mathcal{X}_\alpha$ (Proposition 2.1.6). Consequently, $r_{s,t} \in (S_\alpha)^{\cup}$. Statement (3) follows from the definition of the intersection numbers. □

Corollary 3.3.6. *In the notation of Lemma 3.3.5, $\alpha s \in F_\alpha$ for any thin relation $s \in S$. In particular, if $\mathcal{X}$ is semiregular, then $\mathcal{X}_\alpha = \mathcal{D}_\Omega$.*

In the schurian case, a little bit more can be said about the structure of a point extension. In particular, the following statement gives a necessary condition for a coherent configuration to be schurian in terms of the point extensions.

Theorem 3.3.7. *Let $\mathcal{X} = (\Omega, S)$ be a schurian coherent configuration. Then given $\alpha \in \Omega$ and $\Delta \in F(\mathcal{X})$ containing α,*

(1) *$F(\mathcal{X}_\alpha) = \{\alpha s : \ s \in S_{\Delta,\Omega}\}$;*
(2) *for any $\beta \in \Delta$, there is $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ extending id_S.*

Proof. Let $\mathcal{X} = \mathrm{Inv}(K)$, where $K \leq \mathrm{Sym}(\Omega)$. Then by statement (3) of Proposition 2.2.5,

$$\alpha s \in \mathrm{Orb}(K_\alpha) \ \text{ for all } \ s \in S_{\Delta,\Omega}.$$

By statement (2) of Proposition 3.3.3, this implies that αs is contained in a fiber of the coherent configuration $\mathcal{X}_\alpha$. However, αs is a homogeneity set of $\mathcal{X}_\alpha$ (statement (1) of Lemma 3.3.5). Thus,

$$\alpha s \in F(\mathcal{X}_\alpha).$$

This proves statement (1).

Next, the schurity of $\mathcal{X}$ implies that Δ is an orbit of $\mathrm{Aut}(\mathcal{X})$ (statement (1) of Proposition 2.2.5). It follows that there exists an automorphism $f \in \mathrm{Aut}(\mathcal{X})$ taking α to β. By Theorem 2.6.4,

$$f \in \mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}_\beta).$$

Thus statement (2) holds for the algebraic isomorphism $\varphi_f \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ induced by f. □

We complete the subsection by an example, in which each one-point extension of a coherent configuration can be found more or less explicitly. To explain the example, we recall that any 3/2-transitive permutation group K is either primitive or a Frobenius group [**125**, Theorem 10.4]. The latter means that a point stabilizer of K acts semiregularly on the underlying set with the fixed point removed. Thus in view of statements (1) and (2) of Theorem 2.2.6, the following statement proved in [**39**, Lemma 5.13] is a generalization of this result to coherent configurations.

Theorem 3.3.8. *Let $\mathcal{X}$ be an equivalenced imprimitive scheme on Ω. Then for any $\alpha \in \Omega$, the restriction of $\mathcal{X}_\alpha$ to $\Omega \setminus \{\alpha\}$ is semiregular.*

Proof. Let k be the valency of an irreflexive basis relation of $\mathcal{X}$, and let e be a nontrivial parabolic of $\mathcal{X}$. Then for any $\Delta \in \Omega/e$,

$$|\Delta| = \sum_{s \in S,\ s \subseteq e} n_s = k(a-1)+1$$

where $S = S(\mathcal{X})$ and a is the number of all $s \in S$ contained in e. It follows that

$$\mathrm{GCD}(k, |\Delta|) = 1. \tag{3.3.2}$$

For a point $\alpha \in \Omega$, set

$$T = S(\alpha, \Delta) = \{s \in S : \ \alpha s \cap \Delta \neq \varnothing\}.$$

Then for different Δ, the sets $S(\alpha, \Delta)$ either coincide or disjoint (statement (2) of Exercise 2.7.9). Consequently, αT is the union of some classes of e.

Assume that $\alpha \notin \Delta$. Then

$$k|T| = \sum_{s \in T} n_s = m|\Delta|$$

for some positive integer m. From formula (3.3.2), it follows that $|\Delta|$ divides $|T|$. Since also $|T| \le |\Delta|$, we conclude that $|T| = |\Delta|$. Thus,

$$|\alpha s \cap \Delta| \le 1 \tag{3.3.3}$$

for all $\Delta \in \Omega/e$, $\alpha \in \Omega \setminus \Delta$, and $s \in S$.

Let $\alpha \in \Omega$, $S_\alpha = S(\mathcal{X}_\alpha)$, and $\Omega^\# = \Omega \setminus \{\alpha\}$. To prove that the restriction of the coherent configuration $\mathcal{X}_\alpha$ to $\Omega^\#$ is semiregular, it suffices to verify that

$$n_r = n_{r^*} = 1 \quad \text{for all } \ r \in (S_\alpha)_{\Omega^\# \times \Omega^\#}.$$

To this end, set $\Delta = \alpha e$. It is a homogeneity set of $\mathcal{X}_\alpha$ by statement (1) of Lemma 3.3.5. Therefore any fiber of $\mathcal{X}_\alpha$ is either contained in or disjoint with Δ. Let Γ and Γ' be fibers of $\mathcal{X}_\alpha$ such that

$$\Gamma \subseteq \Delta^\# \quad \text{and} \quad \Gamma' \cap \Delta = \varnothing,$$

where $\Delta^\# = \Delta \setminus \{\alpha\}$. Then by inequality (3.3.3),

$$r(\gamma, \delta) \neq r(\gamma, \delta') \quad \text{for all } \ \gamma \in \Gamma' \ \text{ and distinct } \ \delta, \delta' \in \Gamma.$$

It follows that

$$|(S_\alpha)_{\Gamma', \Gamma}| = |\Gamma| = |\Gamma'|.$$

Consequently,

$$r \in (S_\alpha)_{\Delta^\#, \Omega \setminus \Delta} \quad \Rightarrow \quad n_r = n_{r^*} = 1.$$

To complete the proof, it remains to note that each relation of S_α contained in

$$\Delta^\# \times \Delta^\# \quad \text{or} \quad (\Omega \setminus \Delta) \times (\Omega \setminus \Delta)$$

can be written as the composition of two relations, one in $(S_\alpha)_{\Delta^\#, \Omega \setminus \Delta}$ and the other in $(S_\alpha)_{\Omega \setminus \Delta, \Delta^\#}$. □

Under a *Frobenius scheme* we mean the scheme of a Frobenius group in its standard permutation representation, in which a point stabilizer coincides with the Frobenius complement acting semiregularly on the points other than the fixed one. It should be noted that the automorphism group of a Frobenius scheme $\mathcal{X}$ is not necessarily a Frobenius group, for example, if $\mathcal{X}$ is the scheme of a Paley graph and the ground field is of composite order.

Corollary 3.3.9. *An imprimitive equivalenced scheme $\mathcal{X}$ is schurian if and only if $\mathcal{X}$ is a Frobenius scheme.*

Proof. The sufficiency is trivial. To prove the necessity, let $\mathcal{X} = \mathrm{Inv}(K)$, where $K = \mathrm{Aut}(\mathcal{X})$. Then the group K is imprimitive (Proposition 3.1.4). In view of statement (1) of Proposition 3.3.3, we have $K_\alpha = \mathrm{Aut}(\mathcal{X}_\alpha)$ for any point α. By Theorem 3.3.8, this implies that K_α acts semiregularly on $\Omega \setminus \{\alpha\}$. Thus, K is a Frobenius group. □

3.3.2 The base number

Let $\mathcal{X}$ be a coherent configuration on Ω. A base of the group $\mathrm{Aut}(\mathcal{X})$ as defined in Subsection 2.2.4 is a subset of Ω that controls the automorphisms of $\mathcal{X}$. In this sense, a combinatorial analog of the base can be defined in different ways, see survey [**9**]. The concept introduced below, corresponds to what was called the EP-base there. We start with a combinatorial analog of the pointwise stabilizer for a coherent configuration $\mathcal{X}$.

Definition 3.3.10. *The extension of $\mathcal{X}$ with respect to $m \geq 1$ points $\alpha, \beta, \ldots$, or briefly an m-point extension of $\mathcal{X}$, is defined to be the coherent closure*

$$\mathcal{X}_{\alpha,\beta,\ldots} = \mathrm{WL}(\mathcal{X}, \{1_\alpha, 1_\beta, \ldots\}).$$

For $m = 1$, this is the α-extension of $\mathcal{X}$. Note that the ordering of the points is unessential. From Proposition 3.3.3, one can deduce by induction on m, that

$$\text{(3.3.4)} \quad \mathrm{Aut}(\mathcal{X}_{\alpha,\beta,\ldots}) = \mathrm{Aut}(\mathcal{X})_{\alpha,\beta,\ldots} \quad \text{and} \quad \mathrm{Inv}(K_{\alpha,\beta,\ldots}) \geq \mathrm{Inv}(K)_{\alpha,\beta,\ldots}$$

for all points $\alpha, \beta, \ldots$ and all groups $K \leq \mathrm{Sym}(\Omega)$.

Definition 3.3.11. *A set $\{\alpha, \beta, \ldots\} \subseteq \Omega$ is called a base of $\mathcal{X}$ if the corresponding point extension is equal to the discrete configuration,*

$$\mathcal{X}_{\alpha,\beta,\ldots} = \mathcal{D}_\Omega.$$

The minimum cardinality $b(\mathcal{X})$ of the base is called the base number of $\mathcal{X}$.

In the definition of the base number, one can assume without loss of generality that the minimum is taken over the *irredundant* bases, i.e., those containing no proper base. It is also clear that

$$0 \leq b(\mathcal{X}) \leq n - 1$$

with equalities if and only if $\mathcal{X} = \mathcal{D}_n$ and $\mathcal{X} = \mathcal{T}_n$, respectively. From Corollary 3.3.6, it immediately follows that $b(\mathcal{X}) = 1$ for any non-discrete semiregular coherent configuration $\mathcal{X}$.

By the left-hand side equality in (3.3.4), every base of the coherent configuration $\mathcal{X}$ is also a base of the group $\mathrm{Aut}(\mathcal{X})$. Consequently,

$$\text{(3.3.5)} \qquad b(K) \leq b(\mathrm{Inv}(K))$$

for any permutation group K. The equality is attained, for instance, for a semiregular group K: in this case both numbers in (3.3.5) are equal to 1 (or to 0 if $K = 1$).

On the other hand, if $K = C_p \rtimes \mathrm{Aut}(C_p)$ acts naturally on the elements of C_p, then

$$b(K) = 2 \quad \text{and} \quad b(\mathrm{Inv}(K)) = b(\mathcal{T}_p) = p - 1.$$

We believe that there are 2-closed groups K for which inequality (3.3.5) is strict, but at present no such example is known.

From formula (3.3.1), it follows that every base of a coherent configuration is also a base of any of its fissions,

$$\mathcal{X} \le \mathcal{X}' \quad \Rightarrow \quad b(\mathcal{X}) \ge b(\mathcal{X}').$$

It seems that there is no explicit relationship between the base number of a coherent configuration and that of its quotient. For example, the direct sum $\mathcal{X}$ of trivial coherent configurations of degrees at least 2 has enough large base number (Exercise 3.7.34), whereas the quotient of $\mathcal{X}$ modulo the parabolic the classes of which are fibers, has the base number 0.

Example 3.3.12. *Let $M \le \mathrm{Aut}(C_{p^2})$ be the group of order $p-1$. Then the scheme*

$$\mathcal{X} = \mathrm{Cyc}(M, C_{p^2})$$

defined in (2.4.3), has a parabolic e with p classes of cardinality p. In particular, $\mathcal{X}$ is imprimitive. Since it is also equivalenced of valency $p-1$, Theorem 3.3.8 implies that

$$b(\mathcal{X}) \le 2.$$

On the other hand, the quotient of $\mathcal{X}$ modulo e is isomorphic to $\mathcal{T}_p$. Thus the base number of the quotient equals $p-1$.

In the rest of this section, we prove the Babai theorem on the base number of a primitive scheme, which implies, as a direct consequence, an upper bound for the order of a uniprimitive permutation group.

Theorem 3.3.13. *The base number of a nontrivial primitive scheme of degree n is at most $4\sqrt{n}\log n$.*

The estimate given in the theorem is close to the exact one. Indeed, let $\mathcal{X}$ be the scheme of the Johnson graph $J(m,2)$. Then $\mathcal{X}$ is primitive, has degree $n = \binom{m}{2}$, and is nontrivial for $m \ge 4$. Moreover,

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Sym}(m).$$

By inequalities (2.2.13) and (3.3.5), this implies that

$$\binom{m}{2}^b = n^b \ge n^{b(K)} \ge |K| = m!,$$

where $b = b(\mathcal{X})$ and $K = \mathrm{Aut}(\mathcal{X})$. It follows that $b \ge c\sqrt{n}$ for a positive constant c.

Proof of Theorem 3.3.13. Let $\mathcal{X} = (\Omega, S)$ be a scheme of degree n. For any two points $\alpha, \beta \in \Omega$, set

$$\overline{\Omega}_{\alpha,\beta} = \{\gamma \in \Omega :\ r(\alpha,\gamma) \ne r(\beta,\gamma)\}.$$

Lemma 3.3.14. *Let $\alpha, \beta \in \Omega$. Then*

(1) $\overline{\Omega}_{\alpha,\beta} = \overline{\Omega}_{\beta,\alpha}$;
(2) $\overline{\Omega}_{\alpha,\alpha} = \varnothing$;
(3) *if* $\alpha \neq \beta$, *then* $\alpha, \beta \in \overline{\Omega}_{\alpha,\beta}$;
(4) *for* $s \in S$, *the number* $\overline{c}(s) := |\overline{\Omega}_{\alpha,\beta}|$ *does not depend on* $(\alpha,\beta) \in s$.

Proof. The first three statements are obvious, whereas the fourth one follows from the equality $\overline{c}(s) = n - c(s)$, see (2.1.15). □

We say that $\Delta \subseteq \Omega$ is a *test set* of the scheme $\mathcal{X}$ if for all distinct $\alpha, \beta \in \Omega$,

$$\Delta \cap \overline{\Omega}_{\alpha,\beta} \neq \varnothing.$$

In view of statement (1) of Lemma 3.3.5, no two distinct points belong to the same fiber of extension of $\mathcal{X}$ with respect to the points of a test set. Therefore, every test set is a base of $\mathcal{X}$.

Thus if the scheme $\mathcal{X}$ is nontrivial and primitive, then the required statement immediately follows from Lemmas 3.3.15 and 3.3.16 below, where we set

$$d = \min_{\alpha \neq \beta} |\overline{\Omega}_{\alpha,\beta}|. \tag{3.3.6}$$

Lemma 3.3.15. *The minimum cardinality of a test set of* $\mathcal{X}$ *is less than or equal to* $2n \log n/d$.

Proof. Let $m+1$ be the minimum cardinality of a test set. For each pair of points α and β, we define a function

$$\chi_{\alpha,\beta} : \Omega^{\{m\}} \to \{0,1\},\ \Delta \mapsto \begin{cases} 1, & \text{if } \overline{\Omega}_{\alpha,\beta} \cap \Delta = \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

where $\Omega^{\{m\}}$ is the set of all m-subsets of Ω. Let us calculate the number

$$a = \sum_{\Delta \in \Omega^{\{m\}}} \sum_{\alpha \neq \beta} \chi_{\alpha,\beta}(\Delta)$$

in two ways.

First, for fixed distinct α and β, the sum $\sum_{\Delta} \chi_{\alpha,\beta}(\Delta)$ is equal to the number of ways to choose the m-subset Δ in the complement of $\overline{\Omega}_{\alpha,\beta}$. In view of (3.3.6), the last set has at least d elements. Therefore,

$$a \leq n(n-1)\binom{n-d}{m}. \tag{3.3.7}$$

On the other hand, by the definition of m no $\Delta \in \Omega^{\{m\}}$ is a test set of $\mathcal{X}$. Consequently, for each Δ there exists a pair (α,β) such that $\overline{\Omega}_{\alpha,\beta} \cap \Delta = \varnothing$. This implies that $\chi_{\alpha,\beta}(\Delta) = 1$, and by statement (1) of Lemma 3.3.14 also $\chi_{\beta,\alpha}(\Delta) = 1$. Summing up over all $\Delta \in \Omega^{\{m\}}$, we obtain

$$a \geq 2|\Omega^{\{m\}}| = 2\binom{n}{m}.$$

By inequality (3.3.7), this shows that

$$2 \le \frac{n(n-1)\binom{n-d}{m}}{\binom{n}{m}} = n(n-1)\prod_{k=n-m+1}^{n}\left(1-\frac{d}{k}\right) < n^2\left(1-\frac{d}{n}\right)^m.$$

Taking the logarithm of both parts of the last expression, we derive

$$1 < 2\log n + m\log(1-d/n) < 2\log n - md/n.$$

Thus, $m < n(2\log n - 1)/d$. □

Lemma 3.3.16. $d > \sqrt{n}/2$.

Proof. The proof goes in three auxiliary steps.

Claim 1. *For each* $s \in S^{\#}$,

$$n_s\,\overline{c}(s) \ge n, \tag{3.3.8}$$

where the number $\overline{c}(s)$ *as in statement* (4) *of Lemma* 3.3.14.

Proof. By formula (2.1.5) and statement (4) of Lemma 3.3.14, the number of all triples (α,β,γ) such that $r(\beta,\gamma)=s$ and $\alpha \in \overline{\Omega}_{\beta,\gamma}$, is equal to

$$\sum_{(\beta,\gamma)\in s} |\overline{\Omega}_{\beta,\gamma}| = |s|\,\overline{c}(s) = nn_s\,\overline{c}(s). \tag{3.3.9}$$

It suffices to verify that for any distinct α and β, there exists $\gamma \in \beta s$ such that $r(\alpha,\beta) \ne r(\alpha,\gamma)$. Indeed, then the number on the left-hand side of (3.3.9) is greater than or equal to n^2, which proves inequality (3.3.8).

Assume on the contrary that $\beta s \subseteq \alpha r$ for some points α and β, where $r = r(\alpha,\beta)$. Now if $\beta' \in \alpha r$, then $|\alpha r \cap \beta' s| = c_{rs^*}^r = |\alpha r \cap \beta s| = |\beta s| = |\beta' s|$. It follows that $\beta' s \subseteq \alpha r$. Consequently, no point in αr is connected by a directed s-path with a point α, contrary to the primitivity of $\mathcal{X}$ (Theorem 3.1.5). □

For any $r \in S$ and any nonnegative integer i, denote by r_i the set of all pairs of points at distance i in the graph $\mathfrak{X}_r$ associated with the relation $r \cup r^*$.

Claim 2. *For all* $r, s \in S$,

$$r_i \cap s \ne \varnothing \quad \Rightarrow \quad \overline{c}(s) \le i\,\overline{c}(r\cup r^*).^2 \tag{3.3.10}$$

Proof. Without loss of generality, we may assume that $r, s \in S^{\#}$. Take any pair $(\alpha,\beta) \in r_i \cap s$. By the primitivity of the scheme $\mathcal{X}$, the graph $\mathfrak{X}_r$ (Theorem 3.1.5) is strongly connected. Therefore, there is an $(r\cup r^*)$-path

$$\alpha = \alpha_0,\ \alpha_1,\ \ldots,\ \alpha_i = \beta$$

[2] From statements (1) and (4) of Lemma 3.3.14, it follows that $\overline{c}(r) = \overline{c}(r^*)$, and hence the number $\overline{c}(r \cup r^*)$ is well-defined and is equal to $\overline{c}(r)$.

(Theorem 3.1.5). Now let $\gamma \in \overline{\Omega}_{\alpha,\beta}$. Then the relations $r(\alpha,\gamma)$ and $r(\beta,\gamma)$ are different. Therefore, there exists an index $0 \le j \le i-1$ such that the relations $r(\alpha_j,\gamma)$ and $r(\alpha_{j+1},\gamma)$ are also different. Hence, $\gamma \in \overline{\Omega}_{\alpha_j,\alpha_{j+1}}$. Thus,

$$\overline{\Omega}_{\alpha,\beta} \subseteq \bigcup_{j=0}^{i-1} \overline{\Omega}_{\alpha_j,\alpha_{j+1}}. \tag{3.3.11}$$

Now inequality (3.3.10) holds, because the cardinalities of the sets on the left- and right-hand sides are equal to $\overline{c}(s)$ and $i\,\overline{c}(r \cup r^*)$, respectively. □

Claim 3. *There exists $s_0 \in S^\#$ such that*

$$\overline{c}(s_0) > \sqrt{n}. \tag{3.3.12}$$

Proof. Assume the contrary. Then inequality (3.3.8) shows that $n_s \ge \sqrt{n}$ for all $s \in S^\#$. By formula (2.1.13) with taking into account that $|S^\#| \ge 2$, this implies that

$$\begin{aligned} n^2 - \sum_{s\in S} n_s^2 &= \big(\sum_{s\in S} n_s\big)^2 - \sum_{s\in S} n_s^2 \\ &= \sum_{s\in S} n_s(n-n_s) \\ &\ge \sum_{s\in S^\#} \sqrt{n}(n-n_s) \\ &= \sqrt{n}\big(\sum_{s\in S^\#} n - \sum_{s\in S^\#} n_s\big) \\ &\ge \sqrt{n}(2n-n+1) \\ &> n\sqrt{n}. \end{aligned}$$

On the other hand, for any point γ denote by k_γ the number of all pairs (α,β) such that $\gamma \in \overline{\Omega}_{\alpha,\beta}$. Then obviously,

$$k_\gamma = \sum_{r\ne s} |\gamma r| \cdot |\gamma s| = \sum_{r\ne s} n_r\, n_s = n^2 - \sum_{s\in S} n_s^2.$$

In particular, the number $k := k_\gamma$ does not depend on the choice of γ.

Now comparing the left-hand side of the above calculations with the right-hand side of the last equality, we conclude that every point γ belongs to at least $n\sqrt{n}$ sets $\overline{\Omega}_{\alpha,\beta}$. Since there are exactly $n(n-1)$ such sets (and n choices for γ), at least one of them, say $\overline{\Omega}_{\alpha,\beta}$, contains at least

$$\frac{nk}{n(n-1)} \ge \frac{n\sqrt{n}}{n-1} > \sqrt{n}$$

points. This implies that $\overline{c}(s) > \sqrt{n}$ for $s = r(\alpha,\beta)$, a contradiction. □

To complete the proof of Lemma 3.3.16, let $s \in S^{\#}$. We have to verify that

$$\overline{c}(s) > \sqrt{n}/2.$$

Without loss of generality, we may assume that s does not equal s_0 in formula (3.3.12). Now if the basis graph of $s \cup s^*$ has diameter 2, then relation (3.3.10) for $r = s$, $i = 2$, and $s = s_0$, yields

$$\overline{c}(s_0) \leq 2\overline{c}(s),$$

and we are done by (3.3.12).

Let the above graph contains two vertices α and β at distance 3. Then

$$\overline{c}(r) = |\overline{\Omega}_{\alpha,\beta}| \geq |\alpha s' \cup \beta s'| \geq 2n_s,$$

where $r = r(\alpha, \beta)$ and $s' = s \cup s^*$. On the other hand, from relation (3.3.10) for $i = 3$, $r = s$, and $s = r$, it follows that $\overline{c}(r) \leq 3\overline{c}(s')$. Thus using inequality (3.3.8), we obtain

$$\overline{c}(s)^2 = \overline{c}(s')^2 \geq \frac{\overline{c}(s')\overline{c}(r)}{3} \geq \frac{2\overline{c}(s')n_s}{3} > \frac{2n}{3},$$

whence $\overline{c}(s) > \sqrt{n}/2$. □

Recently, the upper bound in Theorem 3.3.13 has been reduced to a function of order $cn^{1/3}\log n^{4/3}$ in the case when the primitive scheme is nontrivial and other than the schemes of the Johnson graph $J(m, 2)$ and the Hamming graph $H(2, m)$ for the suitable m [**118**].

3.3.3 Partly regular coherent configurations

The class of partly regular coherent configurations studied in this subsection, occurs in many different situations. A group theoretical counterpart of it is formed by permutation groups with base number at most 1. Our main goal is to show that the partly regular coherent configurations are schurian and separable.

Definition 3.3.17. *A coherent configuration $\mathcal{X} = (\Omega, S)$ is said to be partly regular*[3] *if there exists $\alpha \in \Omega$ such that*

$$|\alpha s| \le 1 \quad \textit{for all } s \in S. \tag{3.3.13}$$

Any point α satisfying condition (3.3.13) is called a *regular* point of $\mathcal{X}$. By Exercise 2.7.5 for $M = \{0, 1\}$, all regular points form a homogeneity set. The restriction of $\mathcal{X}$ to this set is obviously a semiregular coherent configuration. Of course, every semiregular coherent configuration is partly regular, and a scheme is partly regular if and only if it is regular.

Theorem 3.3.18. *Let K be a permutation group. Then the coherent configuration $\mathrm{Inv}(K)$ is partly regular if and only if $b(K) \le 1$.*

Proof. Assume that $\mathcal{X} := \mathrm{Inv}(K)$ is a partly regular coherent configuration. Denote by α any of its regular points. Then each fiber of the α-extension $\mathcal{X}_\alpha$ is a singleton (statement (1) of Theorem 3.3.7) and hence the coherent configuration $\mathcal{X}_\alpha$ is a discrete one. It follows that $b(\mathcal{X}) \le 1$. Thus, $b(K) \le 1$ by inequality (3.3.5).

Conversely, we assume that the group K_α is trivial for a certain point α. Take any 2-orbit $s \subseteq \Delta \times \Omega$, where Δ is the K-orbit containing α. Then by statement (3) of Proposition 2.2.5,

$$\alpha s = \beta^{K_\alpha} = \{\beta\} \quad \text{for all } \beta \in \alpha s.$$

This implies that $|\alpha s| = 1$. Since $\alpha s = \varnothing$ for all 2-orbits $s \not\subseteq \Delta \times \Omega$ of the group K, we conclude that α is a regular point of $\mathrm{Inv}(K)$, i.e., this coherent configuration is partly regular. □

Theorem 3.3.18 and the following statement show that the Galois correspondence between coherent configurations and permutation groups defines a one-to-one correspondence between the partly regular coherent configurations and the permutation groups with base number at most 1.

[3] This concept was introduced in [41] in the form of 1-regular cellular algebras.

Theorem 3.3.19. *Any partly regular coherent configuration is schurian and separable.*

Proof. Let $\mathcal{X}$ be a partly regular coherent configuration. Denote by Δ the set of all regular points of $\mathcal{X}$. Then the restriction $\mathcal{X}_\Delta$ being semiregular is schurian and separable (Exercise 2.7.35). Furthermore, by the definition of regular point,

$$\Gamma \in F \text{ and } \Gamma \cap \Delta = \varnothing \quad \Rightarrow \quad n_s = 1 \text{ for some } s \in S_{\Delta,\Gamma}, \tag{3.3.14}$$

where $S = S(\mathcal{X})$ and $F = F(\mathcal{X})$. Thus the required statement follows from the lemma below.

Lemma 3.3.20. *Let $\mathcal{X}$ be a coherent configuration satisfying condition (3.3.14) for some homogeneity set Δ. Then*

(1) *for any $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, the restriction map*

$$\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) \to \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}'_{\Delta^\varphi}, \varphi_\Delta), \quad f \mapsto f^\Delta \tag{3.3.15}$$

is a bijection; in particular, the restriction map $\mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\mathcal{X}_\Delta)$ is a group isomorphism;

(2) *$\mathcal{X}$ is schurian (respectively, separable) whenever $\mathcal{X}_\Delta$ is schurian (respectively, separable).*

Proof. For any fiber $\Gamma \in F$, set

$$s_\Gamma = \begin{cases} s, & \text{if } \Gamma \cap \Delta = \varnothing, \\ 1_\Gamma, & \text{otherwise,} \end{cases}$$

where s is as in condition (3.3.14). By statement (2) of Exercise 2.7.8, the composition $s_\Gamma \cdot s_\Gamma^*$ is a partial parabolic of the coherent configuration $\mathcal{X}$. Any class of this partial parabolic is of the form γs_Γ^*, where $\gamma \in \Gamma$. Therefore,

$$s_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma s_\Gamma^* \times \{\gamma\} \tag{3.3.16}$$

and the union is disjoint.

Next, for any fibers $\Gamma, \Lambda \in F$ and any $s \in S_{\Gamma,\Lambda}$, choose a basis relation

$$r \in s_\Gamma \, s \, s_\Lambda^*.$$

Certainly, $r \in S_\Delta$, see Fig. 3.1. Since the relations s_Γ and s_Λ have valency 1, we have $s_\Gamma^* \cdot s_\Gamma = 1_\Gamma$ and $s_\Lambda^* \cdot s_\Lambda = 1_\Lambda$. Therefore,

$$s_\Gamma^* \cdot r \cdot s_\Lambda \subseteq s_\Gamma^* \cdot (s_\Gamma \cdot s \cdot s_\Lambda^*) \cdot s_\Lambda = (s_\Gamma^* \cdot s_\Gamma) \cdot s \cdot (s_\Lambda^* \cdot s_\Lambda) = s.$$

Taking into account that the relation s is a basis one, we conclude that

$$s_\Gamma^* \cdot r \cdot s_\Lambda = s. \tag{3.3.17}$$

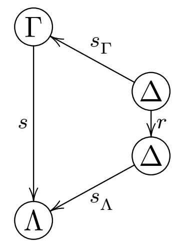


FIGURE 3.1 The relations $s \in S_{\Gamma,\Lambda}$ and $r \in S_\Delta$.

To prove statement (1), let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, where $\mathcal{X}'$ is a coherent configuration on Ω'. From Corollary 2.3.20, it follows that $\mathcal{X}'$ satisfies condition (3.3.14) for Δ and the s_Γ replaced by, respectively, $\Delta' = \Delta^\varphi$ and $s_{\Gamma'} = \varphi(s_\Gamma)$ with $\Gamma' = \Gamma^\varphi$. Since $s_{\Gamma'}$ is of valency 1 for all $\Gamma \in F$,

$$s_{\Gamma'} = \bigcup_{\gamma' \in \Gamma'} \gamma' s^*_{\Gamma'} \times \{\gamma'\} \tag{3.3.18}$$

and the union is disjoint.

It suffices to verify that mapping (3.3.15) is surjective. To this end, let

$$f_0 \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}'_{\Delta'}, \varphi_\Delta). \tag{3.3.19}$$

For each $\Gamma \in F$, it takes the partial parabolic $s_\Gamma \cdot s^*_\Gamma$ to the partial parabolic $s_{\Gamma'} \cdot s^*_{\Gamma'}$. Therefore, f_0 induces a bijection f_Γ between the classes of these two partial parabolics.

For any $\gamma \in \Gamma$, denote by γ' the unique point of Γ' such that

$$\gamma'\, s^*_{\Gamma'} = (\gamma s^*_\Gamma)^{f_\Gamma}.$$

In view of formulas (3.3.16) and (3.3.18), the mapping

$$f : \Omega \to \Omega',\ \ \gamma \mapsto \gamma' \tag{3.3.20}$$

is a bijection, and also

$$(s_\Gamma)^f = s_{\Gamma'}.$$

This is true for all $\Gamma \in F$. Therefore equality (3.3.17) with taking into account that $r^f = r^{f_0} = r^\varphi$ shows that

$$s^f = s^\varphi, \quad s \in S.$$

Thus,

$$f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi).$$

Since $s_\Gamma = 1_\Gamma$ for all fibers $\Gamma \subseteq \Delta$, we have

$$f^\Delta = f_0.$$

Consequently, the mapping (3.3.15) is surjective. On the other hand, for an arbitrary isomorphism $f' \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$, the bijection (3.3.20) constructed as above for $f_0 = f'_\Delta$ coincides with f'. Thus the mapping (3.3.15) is also injective.

To prove statement (2) suppose first that the restriction $\mathcal{X}_\Delta$ is separable. Again let $\mathcal{X}'$ be a coherent configuration on Ω' and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. By the separability of $\mathcal{X}_\Delta$, one can find an isomorphism of the form (3.3.19). By statement (1), there exists $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ such that $f_0 = f_\Delta$. Thus, the coherent configuration $\mathcal{X}$ is separable.

Now assume that the restriction $\mathcal{X}_\Delta$ is schurian. Let $s \in S_{\Gamma,\Lambda}$ be an arbitrary basis relation of $\mathcal{X}$, where $\Gamma, \Lambda \in F$. Take the relation $r \in S_\Delta$ as in formula (3.3.17) and set

$$\overline{r} = \{(\gamma s_\Gamma^*, \lambda s_\Lambda^*) :\ (\gamma, \lambda) \in \Gamma \times \Lambda,\ \ (\gamma s_\Gamma^*) r \,\cap\, \lambda s_\Lambda^* \neq \varnothing\}.$$

Since $n_{s_\Gamma} = n_{s_\Lambda}$, formula (3.3.17) implies that the mapping

$$f : \overline{r} \to s,\ (\gamma s_\Gamma^*, \lambda s_\Lambda^*) \mapsto (\gamma, \lambda)$$

is a bijection.

By formula (3.3.16) and the invariance of the relations s_Γ and s_Λ with respect to the group $\mathrm{Aut}(\mathcal{X})$, this bijection defines an isomorphism between the actions of $\mathrm{Aut}(\mathcal{X})$ on the sets $\overline{r}$ and s. By the schurity of $\mathcal{X}_\Delta$ the group $\mathrm{Aut}(\mathcal{X}_\Delta)$ acts transitively on r and hence on $\overline{r}$. Therefore, $\mathrm{Aut}(\mathcal{X})$ acts transitively on s by statement (1). It follows that s is a 2-orbit of the group $\mathrm{Aut}(\mathcal{X})$. Since this is true for all $s \in S$, the coherent configuration $\mathcal{X}$ is schurian. □

3.4 Wreath products

It is more or less clear how to define an imprimitive wreath product of two coherent configurations that respects the Galois correspondence between the coherent configurations and permutation groups. This is done in the first part of this section. However, in general, the resulted configuration can be algebraically isomorphic to a coherent configuration which is not the wreath product. This explains the fact that the universal embedding theorem [**33**, Theorem 2.6A] for the wreath product of permutation groups does not hold for coherent configurations.

There is no obvious way how to define a combinatorial analog of the primitive wreath product of permutation groups. In the second part of this section, a weaker analog, the exponentiation of a coherent configuration by a permutation group, is introduced and studied.

The third part of the section is devoted to a concept of a generalized wreath product introduced by D. K. Faddeev for abstract groups in the early 1950s (see [**80**, p. 46]). The corresponding construction proved to be extremely useful in the theory of S-rings over a cyclic group [**42, 89, 90**]. In our exposition we deal with the generalized wreath of Cayley schemes only, a more general construction can be found in [**101**].

3.4.1 Canonical wreath product

Let $\mathcal{X}_1 = (\Omega_1, S_1)$ and $\mathcal{X}_2 = (\Omega_2, S_2)$ be schemes. Define a rainbow $\mathcal{X} = (\Omega, S)$ with

$$\Omega = \Omega_1 \times \Omega_2 \quad \text{and} \quad S = S^{(1)} \cup S^{(2)},$$

where

$$S^{(1)} = \{s_1 \otimes 1_{\Omega_2} : \ s_1 \in S_1\} \quad \text{and} \quad S^{(2)} = \{\Omega_1^2 \otimes s_2 : \ s_2 \in S_2^{\#}\}. \tag{3.4.1}$$

The union of the relations in $S^{(1)}$ coincides with the equivalence relation e defined by the equalities of the second coordinates, i.e., one whose classes are $\Delta = \Omega_1 \times \{\delta\}$, $\delta \in \Omega_2$. One can identify the class Δ with Ω_1, and the set Ω/e with Ω_2 via the bijections

$$\pi_1 : \Delta \to \Omega_1, \alpha \mapsto \alpha_1 \quad \text{and} \quad \pi_2 : \Omega/e \to \Omega_2, \ \Omega_1 \times \{\delta\} \mapsto \delta, \tag{3.4.2}$$

respectively.

Theorem 3.4.1. *The rainbow $\mathcal{X}$ is a homogeneous coherent configuration of degree $|\Omega_1| \cdot |\Omega_2|$ and rank $|S_1| + |S_2| - 1$. Moreover, e is a parabolic of $\mathcal{X}$, and in the above notation,*

$$\mathcal{X}_1 = (\mathcal{X}_\Delta)^{\pi_1} \quad \text{and} \quad \mathcal{X}_2 = (\mathcal{X}_{\Omega/e})^{\pi_2}. \tag{3.4.3}$$

Proof. It suffices to prove that $\mathcal{X}$ satisfies the condition (CC3), i.e., that given $r, s, t \in S$, the number $|\alpha r \cap \beta s^*|$ does not depend on the choice of the pair $(\alpha,\beta)\in t$. Without loss of generality, we may assume that $r\cdot s$ intersects t.

Let $r,s\in S^{(1)}$. Then obviously, $r\cdot s\subseteq e$. The assumption implies that $t\subseteq e$, i.e., that $t\in S^{(1)}$. Thus the points α and β belong to a certain class $\Delta\in\Omega/e$. It follows that

$$|\alpha r\cap\beta s^*|=|\alpha_1 r_1\cap\beta_1 s_1^*|=c_{r_1 s_1}^{t_1}, \tag{3.4.4}$$

where α_1 and β_1 are the first coordinates of α and β, and r_1, s_1, and t_1 are the π_1-images of respectively, r, s, and t.

Let $r,s\in S^{(2)}$. Then $\alpha r\cap\beta s^*$ is the union of $\Delta\in\Omega/e$ such that

$$(\overline{\alpha},\Delta)\in r_{\Omega/e}\quad\text{and}\quad(\Delta,\overline{\beta})\in s_{\Omega/e},$$

where $\overline{\alpha}=\pi_e(\alpha)$ and $\overline{\beta}=\pi_e(\beta)$. It follows that

$$|\alpha r\cap\beta s^*|=|\Delta|\cdot|\overline{\alpha}r_{\Omega/e}\cap\overline{\beta}s^*_{\Omega/e}|=|\Omega_1|c_{r_2 s_2}^{t_2}, \tag{3.4.5}$$

where r_2 and s_2 are the π_2-images of $r_{\Omega/e}$ and $s_{\Omega/e}$, respectively.

Let $r\in S^{(1)}$ and $s\in S^{(2)}$ (the case $r\in S^{(2)}$ and $s\in S^{(1)}$ is considered similarly). Every point in $\alpha r\cap\beta s^*$ lies in the class $\Delta=\alpha e$. Moreover, the assumption implies that $t_{\Omega/e}=s_{\Omega/e}$ and hence $\Delta\times\{\beta\}\subseteq s$. Thus,

$$\alpha r\cap\beta s^*=\alpha r, \tag{3.4.6}$$

whence $|\alpha r\cap\beta s^*|=n_r$. □

Definition 3.4.2. *The coherent configuration $\mathcal{X}$ from Theorem 3.4.1 is called the canonical wreath product or, briefly, the wreath product of the coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$, and is denoted by $\mathcal{X}_1\wr\mathcal{X}_2$.*

The wreath product is said to be *nontrivial* if the parabolic e is not trivial. In particular, a nontrivial wreath product must be imprimitive. It is also easily seen that the wreath product is monotonic with respect to each factor.

Example 3.4.3. *The scheme of an undirected cycle with four vertices is isomorphic to the wreath product $\mathcal{T}_2\wr\mathcal{T}_2$.*

One can also define the canonical wreath product for non-homogeneous coherent configurations. However, the explicit formulas for basis relations become a little bit more complicated. An alternative way is indicated in the following statement.

Proposition 3.4.4. *In the above notations, $\mathcal{X}_1\wr\mathcal{X}_2=\mathcal{Y}\cap\mathcal{Z}$, where*

$$\mathcal{Y}=\boxplus_{\Delta\in\Omega/e}\mathcal{X}_\Delta\quad\text{and}\quad\mathcal{Z}=\mathcal{X}_1\otimes\mathcal{X}_2. \tag{3.4.7}$$

Remark 3.4.5. *The coherent configuration $\mathcal{Y}$ is isomorphic to the direct sum of $|\Omega_2|$ copies of the scheme $\mathcal{X}_1$.*

Proof. Set $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$. In accordance with formulas (3.4.1), we have $S(\mathcal{X}) \subseteq S(\mathcal{Y})^{\cup}$ and $S(\mathcal{X}) \subseteq S(\mathcal{Z})^{\cup}$. Thus,

$$S(\mathcal{X}) \subseteq S(\mathcal{Y})^{\cup} \cap S(\mathcal{Z})^{\cup},$$

which proves the inclusion

$$\mathcal{X} \leq \mathcal{Y} \cap \mathcal{Z}. \tag{3.4.8}$$

To prove the reverse inclusion, let us verify that each $s \in S(\mathcal{Y} \cap \mathcal{Z})$ belongs to $S(\mathcal{X})^{\cup}$. Depending on whether or not s is contained in the parabolic e we consider two cases.

Let $s \subseteq e$. Then in view of (3.4.8), there exists $r \in S^{(1)}$ such that

$$s \subseteq r = \bigcup_{\Delta \in \Omega/e} r_{\Delta}.$$

Since also $\mathcal{Y} \cap \mathcal{Z} \leq \mathcal{Y}$ and $r_{\Delta} \in S(\mathcal{Y})$, it follows that for all $\Delta \in \Omega/e$,

$$s_{\Delta} \neq \varnothing \quad \Rightarrow \quad s_{\Delta} = r_{\Delta}.$$

Now if $s_{\Delta} \neq \varnothing$ for all Δ, then $s = r$ as required. Assume that $s_{\Delta} = \varnothing$ for some Δ. Then the parabolic

$$e \in E(\mathcal{Y}) \cap E(\mathcal{Z}) \subseteq E(\mathcal{Y} \cap \mathcal{Z})$$

is not indecomposable (statement (1) of Theorem 2.1.22). However, this contradicts Exercise 3.7.6.

Let $s \cap e = \varnothing$. Then in view of (3.4.8), there exists $r \in S^{(2)}$ such that

$$s \subseteq r = \bigcup_{(\Delta,\Gamma) \in r_{\Omega/e}} r_{\Delta,\Gamma}.$$

Since also $\mathcal{Y} \cap \mathcal{Z} \leq \mathcal{Y}$ and $r_{\Delta,\Gamma} = \Delta \times \Gamma$ belongs to $S(\mathcal{Y})$ for all $(\Delta, \Gamma) \in r_{\Omega/e}$, it follows that

$$s_{\Delta,\Gamma} \neq \varnothing \quad \Rightarrow \quad s_{\Delta,\Gamma} = r_{\Delta,\Gamma}.$$

Now if $s_{\Delta,\Gamma} \neq \varnothing$ for all suitable pairs (Δ, Γ), then $s = r$, and we are done. Assume that $s_{\Delta,\Gamma} = \varnothing$ and $r_{\Delta,\Gamma} \neq \varnothing$ for some Δ and Γ. Then

$$s_{\Omega/e} \subsetneq r_{\Omega/e}. \tag{3.4.9}$$

On the other hand, $\mathcal{Y} \cap \mathcal{Z} \leq \mathcal{Z}$ implies that

$$(\mathcal{Y} \cap \mathcal{Z})_{\Omega/e} \leq \mathcal{Z}_{\Omega/e} = \mathcal{X}_2,$$

in particular, $s_{\Omega/e} \in (S_2)^{\cup}$. Besides, $r_{\Omega/e} \in S_2$ by the assumption on r. Together with (3.4.9), this shows that $s_{\Omega/e} = \varnothing$. Therefore, $s \subseteq e$, a contradiction. □

The construction of the wreath product of $\mathcal{X}_1$ by $\mathcal{X}_2$ with respect to a family of algebraic isomorphisms (see Exercise 3.7.31), shows that, in general, the canonical wreath product $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$ is not a unique scheme on Ω that has a parabolic e such that

$$\mathcal{X}_\Delta = \mathcal{X}_1 \quad \text{and} \quad \mathcal{X}_{\Omega/e} = \mathcal{X}_2$$

for some $\Delta \in \Omega/e$. A weaker minimality condition of the wreath product established in Exercise 3.7.32 implies that the uniqueness of the canonical wreath product holds up to algebraic isomorphisms.

The canonical wreath product of schemes is a combinatorial analog of the wreath product of transitive permutation groups in the imprimitive action. The following statement shows that it respects the Galois correspondence between the coherent configurations and permutation groups.

Theorem 3.4.6. *For arbitrary sets Ω_1 and Ω_2,*

$$\operatorname{Inv}(K_1 \wr K_2, \Omega_1 \times \Omega_2) = \operatorname{Inv}(K_1, \Omega_1) \wr \operatorname{Inv}(K_2, \Omega_2) \tag{3.4.10}$$

for all transitive groups $K_1 \le \operatorname{Sym}(\Omega_1)$ and $K_2 \le \operatorname{Sym}(\Omega_2)$, and

$$\operatorname{Aut}(\mathcal{X}_1 \wr \mathcal{X}_2) = \operatorname{Aut}(\mathcal{X}_1) \wr \operatorname{Aut}(\mathcal{X}_2) \tag{3.4.11}$$

for all schemes $\mathcal{X}_1$ on Ω_1 and $\mathcal{X}_2$ on Ω_2.

Proof. Set $K = K_1 \wr K_2$ and $\Omega = \Omega_1 \times \Omega_2$. One can easily see that

$$K = \langle L, M \rangle,$$

where $L \le \operatorname{Sym}(\Omega)$ is the base group of K and $M = K_1 \times K_2 \le \operatorname{Sym}(\Omega)$. By formula (2.3.7), this implies that

$$\operatorname{Inv}(K, \Omega) = \operatorname{Inv}(\langle L, M \rangle) = \operatorname{Inv}(L) \cap \operatorname{Inv}(M). \tag{3.4.12}$$

The groups L and M are equal to the intransitive direct product of $|\Omega_2|$ copies of K_1, and the transitive direct product of K_1 and K_2, respectively. Thus by formulas (3.2.3) and (3.2.17),

$$\operatorname{Inv}(L) = \boxplus_{\Delta \in \Omega/e} \operatorname{Inv}(K)_\Delta \quad \text{and} \quad \operatorname{Inv}(M) = \operatorname{Inv}(K_1) \otimes \operatorname{Inv}(K_2).$$

By Proposition 3.4.4 this implies that

$$\operatorname{Inv}(K_1, \Omega_1) \wr \operatorname{Inv}(K_2, \Omega_2) = \operatorname{Inv}(L) \cap \operatorname{Inv}(M),$$

which proves equality (3.4.10) by formula (3.4.12).

Let us prove equality (3.4.11). The group $\operatorname{Aut}(\mathcal{X}_1) \wr \operatorname{Aut}(\mathcal{X}_2)$ is generated by the base group $\operatorname{Aut}(\mathcal{X}_1)^{\Omega_2}$ of this wreath product and the transitive direct product $\operatorname{Aut}(\mathcal{X}_1) \times \operatorname{Aut}(\mathcal{X}_2)$. By equalities (3.2.4) and (3.2.18), these groups are equal to $\operatorname{Aut}(\mathcal{Y})$ and $\operatorname{Aut}(\mathcal{Z})$, respectively, where $\mathcal{Y}$ and $\mathcal{Z}$ are the schemes defined in (3.4.7). Thus,

$$\operatorname{Aut}(\mathcal{X}_1) \wr \operatorname{Aut}(\mathcal{X}_2) = \langle \operatorname{Aut}(\mathcal{Y}), \operatorname{Aut}(\mathcal{Z}) \rangle.$$

By Proposition 3.4.4 and formula (2.3.8), this implies that

$$\mathrm{Aut}(\mathcal{X}_1 \wr \mathcal{X}_2) = \mathrm{Aut}(\mathcal{Y} \cap \mathcal{Z}) \geq \langle \mathrm{Aut}(\mathcal{Y}), \mathrm{Aut}(\mathcal{Z}) \rangle = \mathrm{Aut}(\mathcal{X}_1) \wr \mathrm{Aut}(\mathcal{X}_2).$$

To prove the reverse inclusion, let $f \in \mathrm{Aut}(\mathcal{X}_1 \wr \mathcal{X}_2)$. Denote by e the parabolic of $\mathcal{X}_1 \wr \mathcal{X}_2$ defined by the equalities of the second coordinates. Then obviously, $e^f = e$. It follows that for any $\alpha_2 \in \Omega_2$, there exists a uniquely determined $\alpha_2' \in \Omega_2$ such that

$$(\Omega_1 \times \{\alpha_2\})^f = \Omega_1 \times \{\alpha_2'\}. \tag{3.4.13}$$

This defines a permutation g of $\Omega_1 \times \Omega_2$ taking a pair (α_1, α_2) to (α_1, α_2').

We claim that

$$g \in \mathrm{Aut}(\mathcal{X}_1) \wr \mathrm{Aut}(\mathcal{X}_2). \tag{3.4.14}$$

Indeed, g is equal to the permutation $(h, f^{\Omega/e}) \in \mathrm{Sym}(\Omega_1) \wr \mathrm{Sym}(\Omega_2)$, where the function $h : \Omega_2 \to \mathrm{Aut}(\mathcal{X}_1)$ takes any point of Ω_2 to the identity of $\mathrm{Aut}(\mathcal{X}_1)$. Thus the claim follows, because $f^{\Omega/e} \in \mathrm{Aut}(\mathcal{X}_2)$.

One can easily see that $s^g = s$ for all $s \in S^{(1)} \cup S^{(2)}$. Consequently, g is an automorphism of $\mathcal{X}_1 \wr \mathcal{X}_2$ and hence

$$f' := g^{-1} f \in \mathrm{Aut}(\mathcal{X}_1 \wr \mathcal{X}_2).$$

From (3.4.13) and the definition of g, it follows that the permutation f' leaves each class of e fixed (as a set). Therefore,

$$(f')^{\Delta} \in \mathrm{Aut}(\mathcal{X}_1) \quad \text{for all } \ \Delta \in \Omega/e.$$

This shows that the permutation f' lies in the base of the wreath product $\mathrm{Aut}(\mathcal{X}_1) \wr \mathrm{Aut}(\mathcal{X}_2)$. Together with formula (3.4.14), this implies that

$$f = g(g^{-1} f) = g f' \in \mathrm{Aut}(\mathcal{X}_1) \wr \mathrm{Aut}(\mathcal{X}_2).$$

Thus,

$$\mathrm{Aut}(\mathcal{X}_1 \wr \mathcal{X}_2) \leq \mathrm{Aut}(\mathcal{X}_1) \wr \mathrm{Aut}(\mathcal{X}_2),$$

and we are done. □

Corollary 3.4.7. *$\mathcal{X}_1 \wr \mathcal{X}_2$ is schurian if and only if so are $\mathcal{X}_1$ and $\mathcal{X}_2$.*

Proof. The necessity follows from formula (3.4.3) and the fact that the schurity is preserved under restrictions and quotients (statement (2) of Exercise 2.7.21 and Corollary 3.1.17).

To prove the sufficiency, we assume that the schemes $\mathcal{X}_1$ and $\mathcal{X}_2$ are schurian. Then by Theorem 3.4.6,

$$\begin{aligned}\mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1 \wr \mathcal{X}_2)) &= \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1) \wr \mathrm{Aut}(\mathcal{X}_2)) \\ &= \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_1)) \wr \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_2)) \\ &= \mathcal{X}_1 \wr \mathcal{X}_2,\end{aligned}$$

i.e., the scheme $\mathcal{X}_1 \wr \mathcal{X}_2$ is schurian. □

The property of a scheme to be isomorphic to the canonical wreath product is not preserved by algebraic isomorphisms. The reason is that the restrictions $\mathcal{X}_\Delta$ in the canonical wreath product are pairwise isomorphic and this is not true for every scheme algebraically isomorphic to $\mathcal{X}$, see the construction in Exercise 3.7.31.

In order to describe the algebraic isomorphisms φ between two canonical wreath products $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$ and $\mathcal{X}' = \mathcal{X}'_1 \wr \mathcal{X}'_2$, denote by e and e' the parabolics of $\mathcal{X}$ and $\mathcal{X}'$ that are defined by the equalities of the second coordinates. Suppose that $e' = \varphi(e)$ and set

$$\varphi_{\Omega_1} = \pi_1^{-1}\,\varphi_{\Delta,\Delta'}\,\pi'_1 \quad \text{and} \quad \varphi_{\Omega_2} = \pi_2^{-1}\,\varphi_{\Omega/e}\,\pi'_2,$$

where Δ and Δ' are classes of e and e', respectively, $\varphi_{\Delta,\Delta'}$ is the algebraic isomorphism from Exercise 2.7.31, and π'_1 and π'_2 are the bijections defined by formulas (3.4.2). Note that the mapping $\varphi_{\Omega_1} : S(\mathcal{X}_1) \to S(\mathcal{X}_2)$ does not depend on the choice of the classes Δ and Δ'.

Theorem 3.4.8. *In the above notation,*

(1) *for any* $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1 \wr \mathcal{X}_2, \mathcal{X}'_1 \wr \mathcal{X}'_2)$ *such that* $e' = \varphi(e)$,

$$\varphi_{\Omega_1} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}'_1) \quad \text{and} \quad \varphi_{\Omega_2} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}'_2);$$

(2) *for any* $\varphi_1 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}'_1)$ *and* $\varphi_2 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}'_2)$, *there exists a unique algebraic isomorphism*

$$\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1 \wr \mathcal{X}_2, \mathcal{X}'_1 \wr \mathcal{X}'_2)$$

such that $e' = \varphi(e)$, $\varphi_1 = \varphi_{\Omega_1}$, *and* $\varphi_2 = \varphi_{\Omega_2}$.

Proof. Statement (1) immediately follows from the definitions of algebraic isomorphisms φ_{Ω_1} and φ_{Ω_2}, whereas statement (2) is easily derived from the explicit formulas (3.4.4), (3.4.5), and (3.4.6) for the intersection numbers of the canonical wreath product. □

We complete the subsection by the following result showing that the canonical wreath product respects the separability.

Theorem 3.4.9. $\mathcal{X}_1 \wr \mathcal{X}_2$ *is separable if and only if so are* $\mathcal{X}_1$ *and* $\mathcal{X}_2$.

Proof. To prove the necessity, let

$$\varphi_1 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}'_1) \quad \text{and} \quad \varphi_2 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}'_2)$$

for some schemes $\mathcal{X}'_1$ and $\mathcal{X}'_2$. Let $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$ and $\mathcal{X}' = \mathcal{X}'_1 \wr \mathcal{X}'_2$. By statement (2) of Theorem 3.4.8, there exists $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ such that $\varphi(e) = e'$,

$$\varphi_{\Omega_1} = \varphi_1 \quad \text{and} \quad \varphi_{\Omega_2} = \varphi_2.$$

Assuming $\mathcal{X}$ to be separable, one can find an isomorphism

$$f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi).$$

Since $e^f = \varphi(e) = e'$, any $\Delta \in \Omega/e$ goes under f to a certain $\Delta' \in \Omega'/e'$.

Define the composition

$$f_1 = \pi_1^{-1}\, f_{\Delta,\Delta'}\, \pi_1',$$

where $f_{\Delta,\Delta'} : \Delta \to \Delta'$ is the bijection induced by f. Note that $f_{\Delta,\Delta'}$ induces $\varphi_{\Delta,\Delta'}$, because f induces φ.

Thus for any $s_1 \in S(\mathcal{X}_1)$,

$$(s_1)^{f_1} = (s_1)^{\pi_1^{-1} f_{\Delta,\Delta'} \pi_1'} = \varphi_{\Delta,\Delta'}(s_1^{\pi_1^{-1}})^{\pi_1'} = \varphi_1(s_1).$$

Therefore, f_1 induces φ_1. This proves that $\mathcal{X}_1$ is separable. Similarly, one can verify that the composition

$$f_2 = \pi_2^{-1}\, f^{\Omega/e} \pi_2'$$

induces φ_2 and the scheme $\mathcal{X}_2$ is also separable.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ for some scheme $\mathcal{X}'$. Then $e' = \varphi(e)$ is a parabolic of $\mathcal{X}'$. The scheme $\mathcal{X}_{\Omega/e}$ is isomorphic to $\mathcal{X}_2$ and hence is separable. Therefore, there exists an isomorphism

$$\overline{f} \in \mathrm{Iso}(\mathcal{X}_{\Omega/e}, \mathcal{X}'_{\Omega'/e'}, \varphi_{\Omega/e}).$$

For each $\Delta \in \Omega/e$, set $\Delta' = \Delta^{\overline{f}}$. The scheme $\mathcal{X}_\Delta$ is isomorphic to $\mathcal{X}_1$ and hence is separable. Therefore, there exists an isomorphism

$$f_\Delta \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}'_{\Delta'}, \varphi_{\Delta,\Delta'}),$$

where $\varphi_{\Delta,\Delta'}$ is the algebraic isomorphism from Exercise 2.7.31.

At this point, the mapping

$$f : \Omega \to \Omega',\ \alpha \mapsto \alpha^{f_\Delta}$$

is a bijection, where Δ is the class of e that contains α.

Now if $s \in S^{(1)}$, then $s \subseteq e$ and

$$s^f = \big(\bigcup_{\Delta} s_\Delta\big)^f = \bigcup_{\Delta'} (s^f)_{\Delta'} = \bigcup_{\Delta'} \varphi_{\Delta,\Delta'}(s) = \bigcup_{\Delta'} s'_{\Delta'} = \varphi(s).$$

Similarly, if $s \in S^{(2)}$ and $\overline{s} = s_{\Omega/e}$, then

$$s^f = \big(\bigcup_{(\Delta,\Gamma)\in\overline{s}} \Delta \times \Gamma\big)^f = \bigcup_{(\Delta',\Gamma')\in \overline{s}^{f_{\Omega/e}}} \Delta' \times \Gamma' = \bigcup_{(\Delta',\Gamma')\in\varphi_{\Omega/e}(\overline{s})} \Delta' \times \Gamma' = \varphi(s).$$

Thus, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ and the scheme $\mathcal{X}$ is separable. □

3.4.2 Exponentiation

It is not quite clear how to define a pure combinatorial analog of the primitive wreath product of permutation groups. Instead, a wreath product of a coherent configuration and a permutation group is introduced below; in contrast to the canonical wreath product, the resulting coherent configuration can be primitive. The most part of the material of this subsection is taken from [**39**, Section 3].

Let $\mathcal{X}$ be a coherent configuration on Ω, and K a permutation group on a set Δ. Define an action of K on the Cartesian product Ω^Δ by

$$\lambda^k(\delta) = \lambda(\delta^{k^{-1}}), \qquad \delta \in \Delta,\ k \in K,$$

where the element $\lambda \in \Omega^\Delta$ is considered as a function $\lambda : \Delta \to \Omega$. The induced permutation group acts on the relations on Ω^Δ. This action preserves tensor products,

$$\Big(\bigotimes_{\delta\in\Delta} s_\delta\Big)^k = \bigotimes_{\delta\in\Delta} s_{\delta^{k^{-1}}}, \tag{3.4.15}$$

where s_δ is a relation on Ω for all $\delta \in \Delta$. If each s_δ is a basis relation of $\mathcal{X}$, then the left-hand and right-hand sides of the equality are the basis relations of the coherent configuration

$$\mathcal{X}^\Delta = \underbrace{\mathcal{X} \otimes \cdots \otimes \mathcal{X}}_{\Delta}.$$

It follows that the group K can naturally be embedded into the group $\mathrm{Iso}(\mathcal{X}^\Delta)$.

Definition 3.4.10. *The algebraic fusion of $\mathcal{X}^\Delta$ with respect to K is denoted by $\mathcal{X} \uparrow K$ and called the exponentiation of the coherent configuration $\mathcal{X}$ by the group K.*

Certainly, the exponentiation of $\mathcal{X}$ with trivial group is nothing else than the tensor power of $\mathcal{X}$. The scheme of the Hamming graph $H(d, q)$ is isomorphic to the exponentiation $\mathcal{T}_q \uparrow \mathrm{Sym}(d)$, see Exercise 3.7.37.

Proposition 3.4.11. *Let $\mathcal{X}$ be a coherent configuration and K, L permutation groups. Then there are the following canonical isomorphisms:*

(1) $\mathcal{X} \uparrow (K \times L) = (\mathcal{X} \uparrow K) \otimes (\mathcal{X} \uparrow L)$, *where the direct product $K \times L$ acts on the disjoint union of underlying sets;*
(2) $(\mathcal{X} \uparrow K) \uparrow L = \mathcal{X} \uparrow (K \wr L)$, *where $K \wr L$ is the wreath product in the imprimitive action.*

Proof. Both statements are proved by a straightforward computation. Let us do this for the second statement. Assume that $K \le \mathrm{Sym}(\Delta)$ and

$L \le \mathrm{Sym}(\Gamma)$. For any family $\{s_{\delta\gamma}\}_{\delta\in\Delta,\gamma\in\Gamma}$ of basis relations of $\mathcal{X}$, we have

$$\begin{aligned}\bigcup_{l\in L}\bigotimes_{\gamma\in\Gamma}\Big(\bigcup_{k\in K}\bigotimes_{\delta\in\Delta}s_{\delta^{k^{-1}}\gamma^{l^{-1}}}\Big) &= \bigcup_{l\in L}\Big(\bigcup_{f\in K^\Gamma}\bigotimes_{(\delta,\gamma)\in\Delta\times\Gamma}s_{\delta^{f(\gamma)^{-1}}\gamma^{l^{-1}}}\Big)\\ &= \bigcup_{(f,l)\in K\wr L}\bigotimes_{(\delta,\gamma)\in\Delta\times\Gamma}s_{\delta^{f(\gamma)^{-1}}\gamma^{l^{-1}}}\\ &= \bigcup_{g\in K\wr L}\bigotimes_{\alpha\in\Delta\times\Gamma}s_{\alpha^{g^{-1}}}.\end{aligned}$$

Since any basis relation of the coherent configuration on the left- (respectively, right-) hand side in the required equality is of the form on the left- (respectively, right-) hand side of the last formula for a suitable family $\{s_{\delta\gamma}\}$, we are done. □

The exponentiation of a coherent configuration by a permutation group is an almost combinatorial analog of the primitive wreath product of permutation groups. The following statement shows that it respects the Galois correspondence between the coherent configurations and permutation groups.

Theorem 3.4.12. *For any permutation group K,*

$$\mathrm{Inv}(L\uparrow K) = \mathrm{Inv}(L)\uparrow K \tag{3.4.16}$$

for all permutation groups L, and

$$\mathrm{Aut}(\mathcal{X}\uparrow K) \ge \mathrm{Aut}(\mathcal{X})\uparrow K \tag{3.4.17}$$

for all coherent configurations $\mathcal{X}$.

Proof. Let $L \le \mathrm{Sym}(\Omega)$ and $K \le \mathrm{Sym}(\Delta)$. A straightforward check shows that

$$\mathrm{Inv}(L^\Delta)\cap \mathrm{Inv}(K,\Omega^\Delta) = \mathrm{Inv}(L)\uparrow K.$$

By formula (2.3.7), this implies that

$$\mathrm{Inv}(L\uparrow K) = \mathrm{Inv}(\langle L^\Delta, K\rangle) = \mathrm{Inv}(L^\Delta)\cap\mathrm{Inv}(K,\Omega^\Delta) = \mathrm{Inv}(L)\uparrow K,$$

which proves equality (3.4.16). Inclusion (3.4.17) follows from Exercise 2.7.36 and the inclusion

$$\mathrm{Aut}(\mathcal{X}\uparrow K)\ge \mathrm{Aut}(\mathcal{X})^\Delta,$$

which is a consequence of the Galois correspondence. □

The equality in inclusion (3.4.17) can be attained, for example, if one takes $\mathcal{X} = \mathcal{T}_2$ and $K = \mathrm{Alt}(5) \le \mathrm{Sym}(6)$.

We have no explicit formula for the automorphism group of the exponentiation. However, one can see that

$$\mathrm{Aut}(\mathcal{X}\uparrow K) \le \mathrm{Aut}(\mathcal{X})\uparrow K^{(1)}. \tag{3.4.18}$$

Indeed by Exercise 3.7.37, we have

$$\operatorname{Aut}(\mathcal{T}_\Omega \uparrow \operatorname{Sym}(\Delta)) = \operatorname{Sym}(\Omega) \uparrow \operatorname{Sym}(\Delta).$$

Then each permutation from $\operatorname{Aut}(\mathcal{X} \uparrow K)$ is of the form $\sigma = (\lambda, k)$ with $\lambda \in \Omega^\Delta$ and $k \in K$. Moreover, $\lambda(\delta) \in \operatorname{Aut}(\mathcal{X})$ for all $\delta \in \Delta$, and $k \in K^{(1)}$, which follows from considering the action of the permutation σ on the basis relations of $\mathcal{X} \uparrow K$ having the form r^Δ and $r^\Lambda \times s^{\Delta \setminus \Lambda}$, where r, s are different basis relations of $\mathcal{X}$ and Λ is an orbit of the group K.

Theorem 3.4.13. *For a nontrivial coherent configuration $\mathcal{X}$,*

$$\operatorname{Aut}(\mathcal{X} \uparrow K) \le \operatorname{Aut}(\mathcal{X}) \uparrow K^{(2)}.$$

Proof. Let $K \le \operatorname{Sym}(\Delta)$. By the hypothesis, there exist two distinct irreflexive basis relations x and y of $\mathcal{X}$. For any two points $\delta_1, \delta_2 \in \Delta$, we define a relation $t(\delta_1, \delta_2)$ on Ω^Δ consisting of all pairs (λ, μ) such that

$$(\lambda(\delta_1), \mu(\delta_1)) \in x, \qquad (\lambda(\delta_2), \mu(\delta_2)) \in y,$$

and

$$\lambda(\delta) = \mu(\delta) \text{ for all } \delta \notin \{\delta_1, \delta_2\}.$$

One can see that

$$t(\delta_1, \delta_2)^{(f,k)} = t(\delta_1^k, \delta_2^k), \quad f \in \operatorname{Sym}(\Omega)^\Delta,\ k \in \operatorname{Sym}(\Delta). \tag{3.4.19}$$

Since $t(\delta_1, \delta_2)$ is a relation of the coherent configuration $\mathcal{X}^\Delta$, this implies that for any 2-orbit u of the group K,

$$t(u) = \bigcup_{(\delta_1, \delta_2) \in u} t(\delta_1, \delta_2)$$

is a relation of the exponentiation $\mathcal{X} \uparrow K$. Furthermore, in view of the fact that $t(\delta_1, \delta_2)$ and $t(\delta_1', \delta_2')$ are disjoint for $(\delta_1, \delta_2) \ne (\delta_1', \delta_2')$, the mapping $u \mapsto t(u)$ is injective.

Now let $(f, k) \in \operatorname{Aut}(\mathcal{X} \uparrow K)$. By formula (3.4.18), we may assume that

$$(f, k) \in \operatorname{Aut}(\mathcal{X}) \uparrow K^{(1)}.$$

Thus it suffices to verify that $k \in K^{(2)}$. To this end, we note that the relation $t(u)$ is $\operatorname{Aut}(\mathcal{X} \uparrow K)$-invariant for any 2-orbit u of the group K. By formula (3.4.19), this implies that

$$t(u) = t(u)^{(f,k)} = t(u^k).$$

In view of the injectivity of the mapping $u \mapsto t(u)$, this shows that $u^k = u$ for all u. Thus, $k \in K^{(2)}$. □

The following theorem gives a necessary and sufficient condition for exponentiation to be schurian. In particular, it shows one more way how given a non-schurian coherent configuration one can construct an infinite family of non-schurian coherent configurations.

Theorem 3.4.14. *$\mathcal{X} \uparrow K$ is schurian if and only if so is $\mathcal{X}$.*

Proof. The sufficiency immediately follows from Corollary 3.2.22 and Proposition 2.3.28. Conversely, we assume that $\mathcal{X} \uparrow K$ is schurian. Then the group $\mathrm{Aut}(\mathcal{X} \uparrow K)$ acts transitively on each basis relation of the exponentiation $\mathcal{X} \uparrow K$, in particular, on the relation

$$s^{\Delta} \supseteq \{(\lambda_\alpha, \lambda_\beta) \in (\Omega^{\Delta})^2 : (\alpha, \beta) \in s\}$$

for each $s \in S(\mathcal{X})$, where $\lambda_\alpha(\delta) = \alpha$ and $\lambda_\beta(\delta) = \beta$ for all $\delta \in \Delta$. Therefore for any two pairs (α, β) and (α', β') lying in s, there exists an automorphism $\sigma \in \mathrm{Aut}(\mathcal{X} \uparrow K)$ such that

$$(\lambda_\alpha)^\sigma = \lambda_{\alpha'} \quad \text{and} \quad (\lambda_\beta)^\sigma = \lambda_{\beta'}.$$

From Theorem 3.4.13, it follows that $\sigma = (f, k)$, where $f \in \mathrm{Aut}(\mathcal{X})^{\Delta}$ and $k \in K^{(2)}$. By formula (1.3.3), for any $\delta \in \Delta$ we have

$$\alpha' = (\lambda_\alpha)^\sigma(\delta) = \alpha^{f(\delta')} \quad \text{and} \quad \beta' = (\lambda_\beta)^\sigma(\delta) = \beta^{f(\delta')}$$

where $\delta' = \delta^{k^{-1}}$. Thus the element $f(\delta') \in \mathrm{Aut}(\mathcal{X})$ takes (α, β) to (α', β'). Consequently, s is a 2-orbit of $\mathrm{Aut}(\mathcal{X})$ as required. □

From Exercise 2.7.37, it follows that every $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{Y})$ induces an algebraic isomorphism from $\mathcal{X} \uparrow K$ to $\mathcal{Y} \uparrow K$. In particular,

$$\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{Y}) \neq \varnothing \quad \Rightarrow \quad \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X} \uparrow K, \mathcal{Y} \uparrow K) \neq \varnothing.$$

However, it is not clear whether or not the reverse statement is true. In fact, we do not also know whether the structure of exponentiation is preserved by algebraic isomorphisms. The following example shows that the exponentiation of a separable coherent configuration is not necessarily separable.

Example 3.4.15. *Let $\mathcal{X}$ and $\mathcal{Y}$ be the schemes of the Hamming graph $H(2,4)$ and the Shrikhande graph, respectively. In accordance with Example 2.6.17, they are algebraically isomorphic. This implies that for all integers $m \geq 1$, so are the schemes*

$$\mathcal{X}' = \mathcal{X} \uparrow \mathrm{Sym}(m) \quad \text{and} \quad \mathcal{Y}' = \mathcal{Y} \uparrow \mathrm{Sym}(m).$$

One can see that $\mathcal{X}' = \mathcal{T}_4 \uparrow \mathrm{Sym}(2m)$ is the scheme of the Hamming graph $H(2m, 4)$ (Exercise 3.7.37); in particular, $\mathcal{X}'$ is the exponentiation of a separable scheme.

On the other hand, $\mathcal{Y}'$ is the scheme of so called Doob graph, which is distance-regular, has the same intersection array as $H(2m, 4)$, but is not distance-transitive. It follows that $\mathcal{X}'$ is schurian, whereas $\mathcal{Y}'$ is not (statement (4) of Theorem 2.6.11) and hence $\mathcal{X}'$ and $\mathcal{Y}'$ are not isomorphic. Thus none of $\mathcal{X}'$ and $\mathcal{Y}'$ is separable.

It is easily seen that the coherent configuration $\mathcal{X} \uparrow K$ is homogeneous if and only if so is $\mathcal{X}$. The following statement gives a necessary and sufficient condition for the exponentiation to be primitive. In view of formula (3.4.16), it generalizes a well-known criterion for the primitivity of the exponentiation of permutation groups.

Theorem 3.4.16. *Let $\mathcal{X}$ be a coherent configuration on Ω, and let $K \le \mathrm{Sym}(\Delta)$. Assume that $|\Omega| > 1$ and $|\Delta| > 1$. Then $\mathcal{X} \uparrow K$ is primitive if and only if $\mathcal{X}$ is primitive and nonregular and K is transitive.*

Remark 3.4.17. *If $|\Omega| = 1$, then $\mathcal{X} \uparrow K$ is of degree one and hence is primitive, whereas if $|\Delta| = 1$, then $\mathcal{X} \uparrow K = \mathcal{X}$ is primitive if and only if so is $\mathcal{X}$.*

Proof. Assume that $\mathcal{X} \uparrow K$ is primitive. Then $\mathcal{X}$ is homogeneous and each its parabolic e defines a parabolic e^{Δ} of $\mathcal{X} \uparrow K$, which is trivial only if e is trivial. Thus, $\mathcal{X}$ is primitive.

Furthermore, if $\mathcal{X}$ is regular, then each relation $s \in S(\mathcal{X})$ is thin and hence $s^{\Delta} \in S_1(\mathcal{X}^{\Delta})$. Note that the relation s^{Δ} is K-invariant. Therefore,

$$s^{\Delta} \in S_1(\mathcal{X} \uparrow K).$$

Since $|\Omega| > 1$, statement (1) of Theorem 3.1.6 implies that $|\Omega^{\Delta}|$ is prime. Consequently, $|\Delta| = 1$, a contradiction. Thus, $\mathcal{X}$ is not regular.

Finally if $\Gamma \subsetneq \Delta$ is an orbit of K, then

$$\{(\lambda, \mu) \in (\Omega^{\Delta})^2 : \ \lambda(\delta) = \mu(\delta) \text{ for all } \delta \in \Delta \setminus \Gamma\}$$

is a nontrivial partial parabolic of $\mathcal{X} \uparrow K$. Thus, K is transitive.

Conversely, we assume that $\mathcal{X}$ is primitive and nonregular and K is transitive. Then the exponentiation $\mathcal{X} \uparrow K$ is homogeneous. Let $e \ne 1_{\Omega^{\Delta}}$ be a parabolic of $\mathcal{X} \uparrow K$.

Denote by T the set of all basis relations of $\mathcal{X}^{\Delta}$ contained in e. Each of them can be written in the form

$$s = \bigotimes_{\delta \in \Delta} s_{\delta},$$

where $s_{\delta} \in S(\mathcal{X})$ for all δ. Choose a relation $s \in T$ for which the number

$$m_s = |\{\delta \in \Delta : \ s_{\delta} = 1_{\Omega}\}|$$

is as small as possible. Since $s \ne 1_{\Omega^{\Delta}}$ (the choice of e), we have $m_s < |\Delta|$.

We claim that $m_s = 0$. Indeed, let

$$s_{\delta} \ne 1_{\Omega} \quad \text{and} \quad s_{\gamma} = 1_{\Omega}$$

for some $\delta, \gamma \in \Delta$. By the transitivity of K, there exists $k \in K$ such that $\delta^k = \gamma$. Note that $s^k \in T$, because $e^k = e$. It follows that

$$ss^k \subseteq T.$$

On the other hand, since $\mathcal{X}$ is primitive and nonregular, the complex product of any two basis relations belonging to ss^k is equal to $\{1_\Omega\}$ if and only if both of them equal 1_Ω. Since the γ position of ss^k is equal to

$$s_\gamma s_{\gamma^{k-1}} = 1_\Omega s_\delta = s_\delta \neq 1_\Omega,$$

this implies that there exists $t \in ss^k \subseteq T$ such that $m_t < m_s$, contrary to the choice of s.

By the above claim $s_\delta \neq 1_\Omega$ for all $\delta \in \Delta$. Since $\mathcal{X}$ is primitive and nonregular, this implies that there exists a positive integer m_δ such that

$$(s_\delta)^{m_\delta} = S(\mathcal{X}),$$

see Exercise 3.7.3. It follows that any basis relation of $\mathcal{X}^\Delta$ belongs to s^m for a suitable m (for instance, one can take m to be the product of m_δ, $\delta \in \Delta$). Since $s^m \subseteq e$, this shows that

$$e = \Omega^\Delta \times \Omega^\Delta.$$

Thus the scheme $\mathcal{X} \uparrow K$ has only trivial parabolics and hence is primitive.□

3.4.3 Generalized wreath product of Cayley schemes

Saying on the imprimitive wreath product $K \wr L$ of the permutation groups K and L, we do not assume that these groups are connected in some way. Passing to the generalized wreath product, one needs to require that K and L have sections of a special form that are permutation isomorphic.

The exact definition of the generalized wreath product of permutation groups introduced in [**46**] is too cumbersome to be discussed here. A combinatorial analog of this operation for homogeneous coherent configurations had been studied in [**101**] under the name "wedge product". In studying generalized wreath product we restrict ourselves to Cayley schemes.

Throughout this section, $\mathcal{X}$ is a Cayley scheme over a group G. By Proposition 2.4.9, any parabolic of $\mathcal{X}$ is of the form $e_H = \rho(H)$ (see also statement (6) of Exercise 1.4.16) for a uniquely determined subgroup H of G. Any such subgroup is called an $\mathcal{X}$-subgroup of G; in particular, the parabolics of $\mathcal{X}$ are in one-to-one correspondence with the $\mathcal{X}$-subgroups of G. In what follows, we also use the following abbreviations:

$$\mathcal{X}_H := \mathcal{X}_{e_H} \quad \text{and} \quad \mathcal{X}_{G/H} := \mathcal{X}_{G/e_H}.$$

Let L and U be $\mathcal{X}$-subgroups of G such that $L \le U$ and $L \trianglelefteq G$. Assume that for each $s \in S$ with $S = S(\mathcal{X})$,

$$s \not\subseteq e_U \quad \Rightarrow \quad e_L \subseteq \operatorname{rad}(s), \tag{3.4.20}$$

where $\operatorname{rad}(s)$ is the radical of s, see (1.1.2). In other words, the parabolic e_L is contained in the radical of each basis relation outside the parabolic e_U.

Definition 3.4.18. *The scheme $\mathcal{X}$ is called an U/L-wreath product of the Cayley schemes $\mathcal{X}_U$ and $\mathcal{X}_{G/L}$.*[4]

When the explicit reference to the groups L and U is not important, we use the term *generalized wreath product*. The U/L-wreath product is *nontrivial* or *proper* if $1 \ne L$ and $U \ne G$.

Example 3.4.19. *Let $p > 2$ be a prime, $G = C_{p^3}$, and $M \le \operatorname{Aut}(G)$ is the group of order p. Let L and U be the subgroups of G that are isomorphic to C_p and C_{p^2}, respectively. Then the cyclotomic scheme $\operatorname{Cyc}(M, G)$ is the U/L-wreath product of two regular Cayley schemes over C_{p^2}.*

The basis relations of the U/L-wreath product $\mathcal{X}$ are uniquely determined by the basis relations of the factors $\mathcal{X}_U$ and $\mathcal{X}_{G/L}$. Indeed, condition (3.4.20) implies that

$$S = \pi_U(S_U) \cup \pi_L^{-1}(S_{G/L}), \tag{3.4.21}$$

where $S_U = S(\mathcal{X}_U)$, $S_{G/L} = S(\mathcal{X}_{G/L})$ and

$$\pi_U : S_U \to S,\ s_U \mapsto s \quad \text{and} \quad \pi_L : S \to S_{G/L},\ s \mapsto s_{G/e_L}. \tag{3.4.22}$$

[4]The normality of L in G provides that $\mathcal{X}_{G/L}$ is a Cayley scheme over G/L.

This interpretation of the basis relations of $\mathcal{X}$ indicates a way how to construct the generalized wreath product from the factors. The following statement shows that the canonical wreath product of Cayley schemes is a special case of the generalized wreath product.

Proposition 3.4.20. *Let $\mathcal{X}$ be the U/L-wreath product with $U = L$. Then $\mathcal{X}$ is isomorphic to the canonical wreath product $\mathcal{X}_U \wr \mathcal{X}_{G/L}$.*

Proof. Let us choose representatives of the right cosets of U in G. For an element $g \in G$, the representative of Ug is denoted by v_g; in particular, $g = u_g v_g$ for a uniquely determined $u_g \in U$. Now setting $V = G/U$, we get a bijection

$$f : G \to U \times V,\ g \mapsto (u_g, Uv_g).$$

Let $s \in S_U$ and $(g,h) \in s$. Then $v_g = v_h$ and hence $gh^{-1} = u_g u_h^{-1}$. It immediately implies that

$$s^f = s_U \otimes 1_V.$$

Now let $s \notin S_U$. Then by formula (3.4.20) with taking into account that $L = U$, we have

$$s^f = U^2 \otimes s_{G/e_U}.$$

Thus $S^f = S^{(1)} \cup S^{(2)}$, where the summands are defined by (3.4.1) with $\Omega_1 = U$, $S_1 = S_U$ and $\Omega_2 = V$, $S_2 = S_{G/U}$. Thus,

$$\mathcal{X}^f = \mathcal{X}_U \wr \mathcal{X}_{G/U},$$

as required. □

In contrast to the canonical wreath product, the factors $\mathcal{X}_U$ and $\mathcal{X}_{G/L}$ of the U/L-wreath product $\mathcal{X}$ are not independent. In fact, a certain quotient of the former is equal to a certain restriction of the latter,

$$(\mathcal{X}_U)_{U/L} = (\mathcal{X}_{G/L})_{U/L},$$

which follows from the fact that the operations of taking factor and taking quotient are permutable (statement (2) of Exercise 3.7.9). Thus the generalized wreath product of two Cayley schemes can be considered as "gluing" them along a common section of the underlying groups.

Let us establish a necessary and sufficient condition for a Cayley scheme $\mathcal{X}$ over a group G to be a generalized wreath product in terms of certain automorphisms of $\mathcal{X}$. This condition shows, in particular, that the automorphism group of a proper generalized wreath product is always greater that G_{right}.

Theorem 3.4.21. *Let $\mathcal{X}$ be a Cayley scheme over a group G, and let L and U be $\mathcal{X}$-subgroups of G such that $L \le U$ and $L \trianglelefteq G$. Then $\mathcal{X}$ is the U/L-wreath product if and only if*

$$\mathrm{Aut}(\mathcal{X}) \ge \prod_{\Lambda \in G/U} (L_{right})^{\Lambda}, \qquad (3.4.23)$$

where the group L_{right} is treated as a permutation group on G induced by right multiplications by the elements of L.

Proof. To prove the necessity, we assume that $\mathcal{X}$ is the U/L-wreath product. We claim that

$$\mathcal{X} \le \boxplus_{\Lambda\in G/U} (\mathcal{X}_{e_L})_\Lambda, \tag{3.4.24}$$

where $\mathcal{X}_{e_L}$ is the extension of $\mathcal{X}$ with respect to the parabolic e_L. It suffices to verify that each $s \in S(\mathcal{X})$ is a relation of the direct sum on the right-hand side.

Now if $s \subseteq e_U$, then we are done, because s is the disjoint union of the s_Λ with $\Lambda \in G/U$, and

$$s_\Lambda \in S(\mathcal{X})_\Lambda \subseteq (S(\mathcal{X}_{e_L}))^\cup_\Lambda.$$

Let $s \not\subseteq e_U$. Then by the definition of generalized wreath product and formula (1.4.2), we have

$$s = \bigcup_{(\Delta,\Gamma)\in s_{G/e_L}} \Delta\times\Gamma. \tag{3.4.25}$$

However, $(\Delta,\Gamma) \in s_{G/e_L}$ only if Δ and Γ are contained in distinct classes of e_U (for otherwise $s \cap e_U \ne \varnothing$, contrary to the assumption). Therefore, Δ and Γ are homogeneity sets of different summands of the direct sum on the right-hand side of (3.4.24). Thus, $\Delta\times\Gamma$ is a relation of this direct sum, which implies that so is s.

Taking the automorphism groups of the coherent configurations on both sides of inclusion (3.4.24) and using formula (3.2.4), we obtain

$$\mathrm{Aut}(\mathcal{X}) \ge \prod_{\Lambda\in G/U} \mathrm{Aut}((\mathcal{X}_{e_L})_\Lambda) \ge \prod_{\Lambda\in G/U} \mathrm{Aut}(\mathcal{X}_{e_L})^\Lambda. \tag{3.4.26}$$

In view of formula (3.1.15), we have

$$\mathrm{Aut}(\mathcal{X}_{e_L}) = \mathrm{Aut}(\mathcal{X})_{e_L} \ge (G_{right})_{e_L} = L_{right}.$$

Together with (3.4.26), this completes the proof of the necessity.

To prove the sufficiency, let s be a basis relation of $\mathcal{X}$ not contained in the parabolic e_U. We have to verify that s is of the form (3.4.25), or equivalently, that

$$s_{\Delta,\Gamma} = \Delta\times\Gamma$$

for all $\Delta,\Gamma \in G/L$ such that $s_{\Delta,\Gamma} \ne \varnothing$.

To this end, denote by Λ_Δ and Λ_Γ the right cosets of U in G that contain Δ and Γ, respectively. Then $\Lambda_\Delta \ne \Lambda_\Gamma$, because $s \not\subseteq e_U$. By the assumption,

$$(L_{right})^{\Lambda_\Delta} \times (L_{right})^{\Lambda_\Gamma} \le \mathrm{Aut}(\mathcal{X})^{\Lambda_\Delta\cup\Lambda_\Gamma}$$

and hence the group

$$K := (L_{right})^{\Delta} \times (L_{right})^{\Gamma}$$

is contained in $\mathrm{Aut}(\mathcal{X})^{\Delta\cup\Gamma}$. Recall that Δ and Γ are cosets of L in G. Therefore for any $(\alpha,\beta)\in s_{\Delta,\Gamma}$, we have

$$\Delta\times\Gamma=(\alpha,\beta)^K\subseteq s_{\Delta,\Gamma}\subseteq\Delta\times\Gamma$$

as required. □

Corollary 3.4.22. *Let $\mathcal{X}$ be a Cayley scheme. Assume that $\mathcal{X}$ is a proper generalized wreath product. Then a point stabilizer of* $\mathrm{Aut}(\mathcal{X})$ *is nontrivial.*

It should be remarked that the generalized wreath product of schurian (respectively, separable) Cayley schemes is not necessarily schurian (respectively, separable). The first examples of non-schurian and non-separable Cayley schemes over a cyclic group was constructed in [**42**] just by using the generalized wreath product, see Exercise 3.7.40. Furthermore, as was proved later in [**41**], every non-schurian or non-separable Cayley scheme over a cyclic group is a proper generalized wreath product.

On the other hand, there are some criteria for the schurity and separability of generalized wreath product. Most of them are applicable only for Cayley schemes over abelian groups [**47, 113, 114**]. A general result of such a type is given below (the schurity part of it was proved in [**101**]).

Theorem 3.4.23. *The generalized wreath product of two regular Cayley schemes is schurian and separable.*

Proof. Let $\mathcal{X}$ be a Cayley scheme over a group G. Assume that $\mathcal{X}$ is the U/L-wreath product such that the schemes $\mathcal{X}_U$ and $\mathcal{X}_{G/L}$ are regular. In this case, e_L is obviously a residually thin parabolic of $\mathcal{X}$.

By statement (1) of Theorem 3.1.26,

$$S(\mathcal{X}_{e_L})=\{s_{\Delta,\Gamma}:\ s\in S,\ (\Delta,\Gamma)\in s_{G/e_L}\},$$

where $\mathcal{X}_{e_L}$ is the extension of $\mathcal{X}$ with respect to e_L. In particular, the fibers of $\mathcal{X}_{e_L}$ are the cosets of L. This and formula (3.4.20) imply that given two such cosets Δ and Γ, and a relation s$\not\subseteq e_U$, we have

$$\Lambda_\Delta\neq\Lambda_\Gamma\quad\Rightarrow\quad s_{\Delta,\Gamma}=\Delta\times\Gamma,$$

where Λ_Δ and Λ_Γ are the right cosets of U that contain Δ and Γ, respectively. It follows that

$$\mathcal{X}_{e_L}=\boxplus_{\Lambda\in G/U}(\mathcal{X}_{e_L})_\Lambda.$$

For each coset Λ, the scheme $\mathcal{X}_\Lambda$ is algebraically isomorphic to the regular scheme $\mathcal{X}_U$ (Example 2.3.16). Therefore, $\mathcal{X}_\Lambda$ is regular. Consequently, the coherent configuration

$$(\mathcal{X}_{e_L})_\Lambda\geq\mathcal{X}_\Lambda$$

is semiregular.

Thus, $\mathcal{X}_{e_L}$ is the direct sum of semiregular coherent configurations. Since each of them is schurian and separable (Exercise 2.7.35), so is the coherent configuration $\mathcal{X}_{e_L}$ (Corollaries 3.2.6 and 3.2.8). Now the required statement follows from Theorem 3.1.29. □

The following special result generalizes Example 3.4.19 and will be used in Section 4.4. It gives a characterization of generalized wreath products in the class of cyclotomic schemes over a cyclic p-group.

Proposition 3.4.24. *Let $G = C_n$, where $n = p^k$ with prime p and $k \geq 1$, and let $M \leq \mathrm{Aut}(G)$. Then the cyclotomic scheme $\mathrm{Cyc}(M, G)$ is a proper generalized wreath product if and only if one of the following statements holds:*

(1) *p is odd and divides $|M|$;*
(2) *$p = 2$, and $|M| \geq 4$ or $M = \{1, \sigma_{1+n/2}\}$ and $k \geq 2$, see (1.4.10).*

Proof. Using the one-to-one correspondence (1.4.8) and Exercise 1.4.16, the fact that $\mathrm{Cyc}(M, G)$ is the U/L-wreath product can be rewritten as follows:

$$L \leq \mathrm{rad}(X) \ \text{ for all } \ X \in \mathrm{Orb}(M, G \setminus U). \tag{3.4.27}$$

Furthermore, any subgroup of G is characteristic and hence an $\mathcal{X}$-group (Exercise 2.7.41). Therefore, under condition (3.4.27), the scheme $\mathrm{Cyc}(M, G)$ is the U'/L'-wreath product for any groups $L' \leq L$ and $U' \geq U$ (statement (2) of Exercise 3.7.38). Thus without loss of generality, we may assume that

$$|L| = p \quad \text{and} \quad |U| = p^{k-1}.$$

In the rest of the proof, X denotes the orbit of M that contains a generator g of G; in particular,

$$X \subseteq G \setminus U \quad \text{and} \quad |X| = |g^M| = |M|.$$

To prove the necessity, we assume that $\mathrm{Cyc}(M, G)$ is a proper generalized wreath product. Then $L \leq \mathrm{rad}(X)$. It follows that X is the union of some cosets of L. In particular,

$$p = |L| \ \text{ divides } \ |X| = |M|.$$

This leaves us with the case $p = 2$ and $|M| < 4$. But then $M \leq \mathrm{Aut}(G)$ must be one of the following groups:

$$\{\sigma_1\}, \quad \{\sigma_1, \sigma_{-1}\}, \quad \{\sigma_1, \sigma_{-1+n/2}\}, \quad \{\sigma_1, \sigma_{1+n/2}\}. \tag{3.4.28}$$

A straightforward computation shows that if $n > 4$ and M is one of the first three groups, then X is not the union of some L-cosets of G. Thus, $M = \{\sigma_1, \sigma_{1+n/2}\}$ as required.

Let us prove the necessity. We have

$$\mathrm{Orb}(M, G \setminus U) = \{X^{\sigma_m} : \ \mathrm{GCD}(m, n) = 1\}.$$

Since the group $\mathrm{rad}(X^{\sigma_m})$ does not depend on the automorphism σ_m, it suffices to verify that

$$L \le \mathrm{rad}(X).$$

Without loss of generality, we may assume that either condition (1) is satisfied or $p = 2$ and $|M| \ge 4$ (for otherwise, the required statement can be checked directly). Then M contains a subgroup $M_0 \le \mathrm{Aut}(G)$ of order p,

$$M_0 = \{\sigma_{1+m_i} : \ i = 0, \ldots, p-1\},$$

where $m_i = ip^{k-1}$. Since g is a generator of G and $g \in X$, we have

$$gL = \{g^{1+m_0}, \ldots, g^{1+m_{p-1}}\} = g^{M_0} \subseteq X^{M_0} = X.$$

This implies that

$$X \subseteq XL = g^M L = (gL)^M \subseteq X^M = X,$$

whence $X = XL = LX$. Thus, $L \le \mathrm{rad}(X)$, and we are done. □

We complete the subsection by studying algebraic isomorphisms of generalized wreath products. Let $\mathcal{X}$ and $\mathcal{X}'$ be Cayley schemes over groups G and G', respectively, and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$.

For an $\mathcal{X}$-group H denote by $\varphi(H)$ the $\mathcal{X}'$-subgroup of G' such that

$$\varphi(e_H) = e_{\varphi(H)}.$$

Since H and $H' = \varphi(H)$ are classes of the (indecomposable) parabolics e_H and $e_{H'}$, the algebraic isomorphism φ induces the algebraic isomorphism

$$\varphi_H \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_H, \mathcal{X}'_{H'}),$$

which coincides with $\varphi_{H,H'}$ defined in Exercise 2.7.31.

If the subgroups H and H' are normal, then $\mathcal{X}_{G/H}$ and $\mathcal{X}'_{G'/H'}$ are Cayley schemes; in this case the quotient algebraic isomorphism φ_{G/e_H} is denoted by $\varphi_{G/H}$,

$$\varphi_{G/H} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{G/H}, \mathcal{X}'_{G'/H'}).$$

Theorem 3.4.25. *In the above notation, suppose that $\mathcal{X}$ and $\mathcal{X}'$ are the U/L- and U'/L'-wreath products, respectively. Then*

(1) *for any $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ such that $L' = \varphi(L)$ and $U' = \varphi(U)$,*

$$\varphi_U \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_U, \mathcal{X}'_{U'}) \quad \text{and} \quad \varphi_{G/L} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{G/L}, \mathcal{X}'_{G'/L'});$$

(2) *for any $\varphi_1 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_U, \mathcal{X}'_{U'})$, $\varphi_2 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_{G/L}, \mathcal{X}'_{G'/L'})$ such that*

$$(\varphi_1)_{U/L} = (\varphi_2)_{U/L}, \tag{3.4.29}$$

there exists a unique $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ for which

$$L' = \varphi(L), \ U' = \varphi(U) \quad \textit{and} \quad \varphi_1 = \varphi_U, \ \varphi_2 = \varphi_{G/L}.$$

Proof. Statement (1) immediately follows from the definitions of the algebraic isomorphisms φ_U and $\varphi_{G/L}$. In order to prove statement (2), we make use of formula (3.4.21) to define a bijection

$$\varphi : S \to S',\ s \mapsto s' = \begin{cases} \pi_{U'}(\varphi_1(\pi_U^{-1}(s))), & \text{if } s \subseteq e_U, \\ \pi_{L'}^{-1}(\varphi_2(\pi_L(s))), & \text{if } s \not\subseteq e_U, \end{cases}$$

where $S = S(\mathcal{X})$, $S' = S(\mathcal{X}')$, and the mappings π_U, π_L and $\pi_{U'}$, $\pi_{L'}$ are defined as in formula (3.4.22). By condition (3.4.29), we have $\varphi_1 = \varphi_U$ and $\varphi_2 = \varphi_{G/L}$. To prove that φ is an algebraic isomorphism, let $r, s, t \in S$.

Let $r, s \subseteq e_U$. Then the π_U-preimages of r and s are equal to r_U and s_U, respectively. Therefore, $r', s' \subseteq e_{U'}$ and the $\pi'_{U'}$-preimages of r' and s' are equal to $r'_{U'}$ and $s'_{U'}$, respectively. If, in addition, $t \not\subseteq e_U$, then $t' \not\subseteq e_{U'}$ and

$$c_{rs}^t = 0 = c_{r's'}^{t'},$$

whereas if $t \subseteq e_U$, then $t' \subseteq e_{U'}$, and

$$c_{rs}^t = c_{r_U s_U}^{t_U} = c_{r'_{U'} s'_{U'}}^{t'_{U'}} = c_{r's'}^{t'}.$$

Let $r \subseteq e_U$ and $s \not\subseteq e_U$; the case $r \not\subseteq e_U$ and $s \subseteq e_U$ is considered similarly. Then $s = s^{e_L}$ and a straightforward check shows that

$$c_{rs}^t = \begin{cases} m_r c_{\bar{r}\,\bar{s}}^{\bar{t}}, & \text{if } t \not\subseteq e_U, \\ 0, & \text{if } t \subseteq e_U, \end{cases} \tag{3.4.30}$$

where m_r is the number defined in statement (1) of Exercise 2.7.11 for $s = r$ and $e = e_L$, and $\bar{r} = r_{G/L}$, $\bar{s} = s_{G/L}$, and $\bar{t} = t_{G/L}$. Thus assuming $t \not\subseteq e_U$, we obtain

$$c_{rs}^t = m_r c_{\bar{r}\,\bar{s}}^{\bar{t}} = m_{r'} c_{\bar{r}'\,\bar{s}'}^{\bar{t}'} = c_{r's'}^{t'},$$

where $\bar{r}' = r'_{G'/L'}$, $\bar{s}' = s'_{G'/L'}$, and $\bar{t}' = t'_{G'/L'}$.

Now let $r \not\subseteq e_U$ and $s \not\subseteq e_U$. Then for any $t \in S$,

$$c_{rs}^t = |L| c_{\bar{r}\,\bar{s}}^{\bar{t}}. \tag{3.4.31}$$

Thus,

$$c_{rs}^t = |L| c_{\bar{r}\,\bar{s}}^{\bar{t}} = |L'| c_{\bar{r}'\,\bar{s}'}^{\bar{t}'} = c_{r's'}^{t'},$$

and we are done. □

3.5 Multidimensional extensions

The method of invariant relations was mentioned by H. Wielandt in [**127**] among the three major tools for studying the actions of a group on a set. An illustration of this method is given in Corollary 2.2.24, where the group $K \le \mathrm{Sym}(\Omega)$ is naturally approximated by a series of permutation groups induced by the action of K on the Cartesian power Ω^m, $m \ge 1$.

The aim of this section is to define a similar series for an arbitrary coherent configuration. To this end a combinatorial analog of the group

$$\widehat{K}^{(m)} \le \mathrm{Sym}(\Omega^m)$$

induced by the action of K on Ω^m is introduced. The key point here is that $\widehat{K}^{(m)}$ coincides with the setwise stabilizer of the diagonal of Ω^m in the direct product $K^m \le \mathrm{Sym}(\Omega^m)$,

$$\widehat{K}^{(m)} = (K^m)_{\{\mathrm{Diag}(\Omega^m)\}}.$$

In the case of coherent configurations, the Galois correspondence suggests to replace the direct product and setwise stabilizer in the above formula by the tensor product and coherent closure. In this way, we arrive at the concept of the m-dimensional extension of a coherent configuration. This leads naturally to the m-dimensional intersection numbers and algebraic isomorphisms.

As in the case of permutation groups, such an approach allows to define the operator of the m-dimensional closure and the concept of m-closed coherent configuration. Any coherent configuration is 1-closed; some group-like properties of the 2-closed coherent configurations are studied at the end of the section. The most part of this theory was developed by S. Evdokimov and I. Ponomarenko in the early 2000s, see [**44**] and references therein.

3.5.1 The m-dimensional extension

Let $\mathcal{X}$ be a coherent configuration on Ω and m a positive integer.

Definition 3.5.1. *The m-dimensional extension of $\mathcal{X}$ is defined to be the coherent closure*

$$\widehat{\mathcal{X}}^{(m)} = \mathrm{WL}(\mathcal{X}^m, 1_{\mathrm{Diag}(\Omega^m)}), \tag{3.5.1}$$

i.e., the smallest fission of $\mathcal{X}^m$ containing $\mathrm{Diag}(\Omega^m)$ *as a homogeneity set.*

The intersection numbers of this coherent configuration are called the m-dimensional intersection numbers of $\mathcal{X}$. Obviously, the 1-dimensional extension of $\mathcal{X}$ coincides with $\mathcal{X}$, and 1-dimensional intersection numbers are usual intersection numbers of $\mathcal{X}$.

From Theorem 2.6.4, it follows that any isomorphism from $\mathcal{X}$ to another coherent configuration induces an isomorphism between their m-dimensional extensions. In this sense, the m-extension and the m-dimensional intersection numbers can be treated as m-dimensional invariants of $\mathcal{X}$.

Example 3.5.2. *Let $\mathcal{X} = \mathcal{T}_3$. Then the scheme $\mathcal{X}^2 = \mathcal{X} \otimes \mathcal{X}$ is of degree* 9 *and rank* 4. *The action of* $\mathrm{Aut}(\mathcal{X}) = \mathrm{Sym}(3)$ *on* Ω^2 *induces an automorphism group of the coherent closure* (3.5.1) (*Theorem* 2.6.4). *Therefore,*

$$\widehat{\mathcal{X}}^{(2)} \le \mathrm{Inv}(\mathrm{Sym}(3), \Omega^2). \tag{3.5.2}$$

The coherent configuration on the right-hand side has two homogeneous components: trivial scheme on $\mathrm{Diag}(\Omega^2)$ *and a regular scheme associated with* $\mathrm{Sym}(3)$. *A straightforward computation shows that in this case, inclusion* (3.5.2) *is an equality.*

The following statement shows that in the framework of the Galois correspondence between the coherent configurations and permutation groups, the m-dimensional extension can be considered as a combinatorial analog of the permutation group induced by the action of a group on the m-tuples.

Theorem 3.5.3. *For any positive integer m,*

$$\widehat{\mathrm{Inv}(K)}^{(m)} \le \mathrm{Inv}(\widehat{K}^{(m)}) \tag{3.5.3}$$

for all groups $K \le \mathrm{Sym}(\Omega)$, and

$$\mathrm{Aut}(\widehat{\mathcal{X}}^{(m)}) = \widehat{\mathrm{Aut}(\mathcal{X})}^{(m)} \tag{3.5.4}$$

for all coherent configurations $\mathcal{X}$ on Ω.

Proof. Inclusion (3.5.3) follows from formula (3.2.17) and statement (1) of Exercise 3.7.12:

$$\begin{aligned}\widehat{\mathrm{Inv}(K)}^{(m)} &= \mathrm{WL}(\mathrm{Inv}(K)^m, 1_{\Delta_m})\\ &= \mathrm{WL}(\mathrm{Inv}(K^m), 1_{\Delta_m})\\ &\le \mathrm{Inv}(K^m_{\{\Delta_m\}})\\ &= \mathrm{Inv}(\widehat{K}^{(m)}),\end{aligned}$$

where $\Delta_m = \mathrm{Diag}(\Omega^m)$. Similarly, equality (3.5.4) can be verified with the help of formula (3.2.18) and statement (2) of Exercise 3.7.12. □

The equality in (3.5.3) holds if the group K is semiregular or a symmetric group (Exercises 3.7.41 and 3.7.42). In general, the equality is not necessarily attained. The smallest known example of this phenomenon is given below.

Example 3.5.4. *In* [**45**], *the* 2*-dimensional extension of the scheme $\mathcal{X}$ of a projective plane $\mathcal{P}$ of order q was explicitly calculated* (*Exercise* 2.7.43). *In turns out that if $q \ge 3$, then its rank does not depend on q and equals* 208. *Let $q = 5$ and $K = \mathrm{Aut}(\mathcal{X})$* (*in fact, $\mathcal{P}$ is a Galois plane and $K = \mathrm{Aut}(\mathcal{P})$, see Theorem* 2.5.3). *A computer computation shows that*

$$\mathrm{rk}(\mathrm{Inv}(\widehat{K}^{(2)})) = 224 > 208 = \mathrm{rk}(\widehat{\mathrm{Inv}(K)}^{(2)}).$$

Along with the m-dimensional extension of a coherent configuration, one can define the m-dimensional extension of an algebraic isomorphisms between two coherent configurations $\mathcal{X}$ on Ω and $\mathcal{Y}$ on Δ. Namely, let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{Y})$.

Definition 3.5.5. *An algebraic isomorphism $\psi \in \mathrm{Iso}_{\mathrm{alg}}(\widehat{\mathcal{X}}^{(m)}, \widehat{\mathcal{Y}}^{(m)})$ is called the m-dimensional extension of φ if the following two conditions are satisfied:*

(1) $\mathrm{Diag}(\Omega^m)^\psi = \mathrm{Diag}(\Delta^m)$;
(2) $\psi(s) = \varphi^m(s)$ *for all* $s \in S(\mathcal{X}^m)$,

where $\varphi^m \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}^m, \mathcal{Y}^m)$ is the algebraic isomorphism induced by φ (see statement (1) *of Exercise* 3.7.33*).*

Clearly, the m-dimensional extension of φ is uniquely determined (if exists); it is denoted by $\widehat{\varphi}^{(m)}$. It is also clear that every algebraic isomorphism has 1-dimensional extension and $\widehat{\varphi}^{(1)} = \varphi$.

The m-dimensional extension does not necessarily exist for $m \ge 2$. For example, a direct computation shows that if $\mathcal{X}$ is a unique antisymmetric coherent configuration of degree 15 and rank 3, then the algebraic automorphism φ of $\mathcal{X}$ induced by transposition has no 2-dimensional extension.

Example 3.5.6. *It was proved in* [**45**] *that any algebraic isomorphism between the schemes of projective planes has* 2*-dimensional extension.*

The m-dimensional extensions of coherent configurations and algebraic isomorphisms are usually difficult to find explicitly. However, some indirect information can be obtained from an analysis of known relations of the m-dimensional extension. Among them, there are relations of the form

$$\mathrm{Cyl}_s(i,j) = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : \ (\alpha_i, \beta_j) \in s\}, \tag{3.5.5}$$

where $s \subseteq \Omega^2$ and $i, j \in \{1, \ldots, m\}$. The following theorem shows that they are relations of the m-dimensional extension of each coherent configuration, having s as a relation, and that they are respected by the m-dimensional extension of any algebraic isomorphism.

Theorem 3.5.7. *Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration and $m \ge 1$ an integer. Then given a relation $s \in S^\cup$ and indices $i, j \in \{1, \ldots, m\}$,*

(1) $\mathrm{Cyl}_s(i,j)$ *is a relation of* $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$*;*
(2) *if $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{Y})$ and $\widehat{\varphi} = \widehat{\varphi}^{(m)}$, then $\widehat{\varphi}(\mathrm{Cyl}_s(i,j)) = \mathrm{Cyl}_{\varphi(s)}(i,j)$.*

Proof. For any $k \in \{1, \ldots, m\}$, set

$$e_k = e_k(\Omega) = (\Omega^2 \otimes \cdots \otimes \Omega^2 \otimes 1_\Omega \otimes \Omega^2 \otimes \cdots \otimes \Omega^2) \cdot 1_{\mathrm{Diag}(\Omega^m)},$$

where 1_Ω is located at the kth place. Clearly,

$$(\alpha, \beta) \in e_k \quad \Leftrightarrow \quad \beta = (\alpha_k, \ldots, \alpha_k).$$

Therefore,

$$\mathrm{Cyl}_s(i,j) = e_i \cdot s^m \cdot e_j^*,$$

where

$$s^m = s \otimes \cdots \otimes s.$$

Since e_i, s^m, and e_j^* are relations of $\widehat{\mathcal{X}}$, statement (1) follows from Proposition 2.1.4.

Next let Δ be the point set of the coherent configuration $\mathcal{Y}$. Then by the definition of the algebraic isomorphism $\widehat{\varphi}$, we have

$$\widehat{\varphi}(e_k(\Omega)) = e_k(\Delta) \quad \text{and} \quad \widehat{\varphi}(s^m) = \varphi(s)^m.$$

Thus,

$$\begin{aligned}\widehat{\varphi}(\mathrm{Cyl}_s(i,j)) &= \widehat{\varphi}(e_i(\Omega) \cdot s^m \cdot e_j^*(\Omega))\\ &= e_i(\Delta) \cdot \varphi(s)^m \cdot e_j^*(\Delta)\\ &= \mathrm{Cyl}_{\varphi(s)}(i,j),\end{aligned}$$

which proves statement (2). □

Let $g \in \mathrm{Sym}(m)$. Denote by $\widehat{g}$ the permutation of Ω^m defined by

$$\alpha^{\widehat{g}} = (\alpha_{1^g}, \ldots, \alpha_{m^g}), \qquad \alpha \in \Omega^m. \tag{3.5.6}$$

It is easily seen that $\widehat{g}$ is an isomorphism of the coherent configuration $\mathcal{X}^m$ and leaves the relation $1_{\mathrm{Diag}(\Omega^m)}$ fixed. By Theorem 2.6.4, this implies that $\widehat{g}$ is an isomorphism of the m-dimensional extension of $\mathcal{X}$. Moreover, since

$$\{(\alpha, \alpha^{\widehat{g}}) : \ \alpha \in \Omega^m\} = \bigcap_{i=1}^{m} \mathrm{Cyl}_{1_\Omega}(i, i^{g^{-1}}),$$

the statement below follows from Theorem 3.5.7.

Corollary 3.5.8. *The graph of any permutation $\widehat{g}$, $g \in \mathrm{Sym}(m)$, is a (thin) relation of the m-dimensional extension of any coherent configuration.*

The m-dimensional extension $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$ controls all the $(m-1)$-point extensions of the coherent configuration $\mathcal{X}$. To see this, denote by e the equivalence relation defined by the equality of the first $m-1$ coordinates, i.e.,

$$e = \underbrace{1_\Omega \otimes \cdots \otimes 1_\Omega}_{m-1} \otimes \Omega^2. \tag{3.5.7}$$

Clearly, e is a parabolic of $\widehat{\mathcal{X}}$, and each class $\Delta \in \Omega^m/e$ is uniquely determined by a suitable $(m-1)$ tuple $\delta \in \Omega^{m-1}$,

$$\Delta = \Delta_\delta = \{\alpha \in \Omega^m : \ \alpha_1 = \delta_1, \ldots, \alpha_{m-1} = \delta_{m-1}\}. \tag{3.5.8}$$

The m-tuples belonging to Δ are in one-to-one correspondence with the points of $\mathcal{X}$ via the natural bijection

$$\zeta : \Omega \to \Delta_\delta,\ \alpha \mapsto (\delta_1, \ldots, \delta_{m-1}, \alpha). \tag{3.5.9}$$

Corollary 3.5.9. *$(\mathcal{X}_{\delta_1,\ldots,\delta_{m-1}})^\zeta \le \widehat{\mathcal{X}}_{\Delta_\delta}$ for any $\delta \in \Omega^{m-1}$.*

Proof. By statement (1) of Theorem 3.5.7, for each $i \in \{1, \ldots, m-1\}$ the coherent configuration $\widehat{\mathcal{X}}$ contains the reflexive relation

$$s_i = 1_{\Omega^m} \cap \mathrm{Cyl}_{1_\Omega}(i, m),$$

consisting of all pairs (α, α) such that $\alpha_i = \alpha_m$. On the other hand, it is easily seen that

$$(s_i)_\Delta = (1_{\delta_i})^\zeta,$$

where $\Delta = \Delta_\delta$. Thus the coherent configuration $\widehat{\mathcal{X}}_\Delta$ contains the relation $(1_{\delta_i})^\zeta$ for each $i = 1, \ldots, m-1$. Furthermore for each $s \in S(\mathcal{X})$,

$$(1_\Omega \otimes \cdots \otimes 1_\Omega \otimes s)_\Delta = s^\zeta$$

is also a relation of $\widehat{\mathcal{X}}_\Delta$. Thus,

$$(\mathcal{X}_{\delta_1,\ldots,\delta_{m-1}})^\zeta = \mathrm{WL}(\mathcal{X}, 1_{\delta_1}, \ldots, 1_{\delta_{m-1}})^\zeta = \mathrm{WL}(S^\zeta, 1_{\delta_1}^\zeta, \ldots, 1_{\delta_{m-1}}^\zeta) \le \widehat{\mathcal{X}}_\Delta,$$

as required. □

One of the most important facts in the theory of multidimensional extensions is that for any m greater than the degree of a coherent configuration $\mathcal{X}$, the m-dimensional extension of $\mathcal{X}$ is schurian and separable. This is an immediate consequence of Theorem 3.3.19 and the following statement establishing a sufficient condition for the m-dimensional extension of $\mathcal{X}$ to be partly regular.

Theorem 3.5.10. *Let $\mathcal{X}$ be a coherent configuration and $m \ge 1$ an integer. Suppose that the extension of $\mathcal{X}$ with respect to some $m-1$ points is partly regular. Then the coherent configuration $\widehat{\mathcal{X}}^{(m)}$ is also partly regular.*

Proof. Let $\delta_1, \ldots, \delta_{m-1}$ be points with respect to which the extension of $\mathcal{X}$ is partly regular. Let $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$, and let $\Delta = \Delta_\delta$ be the set defined by formula (3.5.8). Then by Corollary 3.5.9, the coherent configuration $\widehat{\mathcal{X}}_\Delta$ has a partly regular fusion $(\mathcal{X}_{\delta_1,\ldots,\delta_{m-1}})^\zeta$, and so is partly regular (Exercise 3.7.28).

Let

$$\alpha = (\delta_1, \ldots, \delta_{m-1}, \alpha_m)$$

be a regular point of $\widehat{\mathcal{X}}_\Delta$. For each $i \in \{1, \ldots, m\}$, set e_i to be the equivalence relation on Ω^m defined by equality of the ith coordinates,

$$e_i = \mathrm{Cyl}_{1_\Omega}(i, i).$$

It is a parabolic of $\widehat{\mathcal{X}}$ by statement (1) of Theorem 3.5.7.

Lemma 3.5.11. *Each class of e_i equals αs for a suitable relation s of $\widehat{\mathcal{X}}$.*

Proof. Let Γ be a class of e_i. First, we assume that $i = m$. Take a point $\gamma \in \Omega$ such that

$$\Gamma = \{\beta \in \Omega^m : \ \beta_m = \gamma\}.$$

Denote by r the basis relation $r(\alpha, \gamma^\zeta)$ of the coherent configuration $\widehat{\mathcal{X}}$, where the mapping ζ is as in (3.5.9). Since α and γ^ζ lie in the class Δ, and α is a regular point of the coherent configuration $\widehat{\mathcal{X}}_\Delta$, this implies that $\alpha r = \{\gamma^\zeta\}$. Therefore,

$$\Gamma = \alpha s,$$

where $s = r \cdot e_m$.

Now let i be arbitrary. Take any permutation $g \in \mathrm{Sym}(m)$ taking i to m. Then the permutation $\widehat{g}$ defined by (3.5.6) is an isomorphism of the coherent configuration $\widehat{\mathcal{X}}$. It follows that $\alpha^{\widehat{g}}$ is a regular point of $\widehat{\mathcal{X}}_{\Delta^{\widehat{g}}}$,

$$(e_i)^{\widehat{g}} = e_m,$$

and $\Gamma^{\widehat{g}}$ is a class of e_m. The argument of the previous paragraph with α, Δ, and Γ replaced respectively by $\alpha^{\widehat{g}}$, $\Delta^{\widehat{g}}$, and $\Gamma^{\widehat{g}}$, shows that

$$\Gamma^{\widehat{g}} = \alpha^{\widehat{g}} s^{\widehat{g}}$$

for some relation s of the coherent configuration $\widehat{\mathcal{X}}$. Thus, $\Gamma = \alpha s$, as required. □

To complete the proof, let $\beta \in \Omega^m$. For each $i = 1, \ldots, m$, denote by Δ_i the class of e_i containing β. Then

$$\Delta_i = \alpha s_i$$

for a suitable relation s_i of the coherent configuration $\widehat{\mathcal{X}}$ (Lemma 3.5.11). It follows that

$$\{\beta\} = \Delta_1 \cap \ldots \cap \Delta_m = \alpha s_1 \cap \ldots \cap \alpha s_m = \alpha s,$$

where s is the intersection of all the s_i. Since s is a relation of $\widehat{\mathcal{X}}$, it can be replaced in the last formula by the basis relation $r(\alpha, \beta)$, which is of valency 1. Since β is an arbitrary point of $\widehat{\mathcal{X}}$, this shows that α is a regular point of $\widehat{\mathcal{X}}$. Thus this coherent configuration is partly regular. □

Corollary 3.5.12. *Let $\mathcal{X}$ be a coherent configuration and $m \geq b(\mathcal{X})+1$. Then the m-dimensional extension of $\mathcal{X}$ is schurian and separable.*

3.5.2 The m-dimensional closure

The aim of this subsection is to introduce a combinatorial analog of the series of approximations to a permutation group given in Corollary 2.2.24. To this end, we have to define the m-dimensional closure of a coherent configuration that plays the same role as the m-closure of a permutation group.

Let $\mathcal{X}$ be a coherent configuration on Ω and $m \geq 1$ an integer. The diagonal $\mathrm{Diag}(\Omega^m)$ is a homogeneity set of the m-dimensional extension of $\mathcal{X}$, and also the image of the diagonal mapping

$$\eta_m : \Omega \to \mathrm{Diag}(\Omega^m),\ \alpha \mapsto (\alpha, \dots, \alpha). \tag{3.5.10}$$

It follows that the restriction of $\widehat{\mathcal{X}}^{(m)}$ to the diagonal $\mathrm{Diag}(\Omega^m)$ is equal to the η_m-image of a certain coherent configuration $\overline{\mathcal{X}}^{(m)}$ on Ω, defined by the formula

$$\overline{\mathcal{X}}^{(m)} = ((\widehat{\mathcal{X}}^{(m)})_{\mathrm{Diag}(\Omega^m)})^{\eta_m^{-1}} \tag{3.5.11}$$

Definition 3.5.13. *The coherent configuration $\overline{\mathcal{X}}^{(m)}$ is called the m-dimensional closure or, briefly, the m-closure of $\mathcal{X}$.*

Taking the m-closure defines a closure operator on the set of all coherent configurations (Exercise 3.7.47). The following statement shows that the coherent configuration $\overline{\mathcal{X}}^{(m)}$ gives a better approximation to $\mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$ than $\mathcal{X}$.

Proposition 3.5.14. *For any integer $m \geq 1$,*

$$\mathcal{X} \leq \overline{\mathcal{X}}^{(m)} \leq \mathrm{Inv}(\mathrm{Aut}(\mathcal{X})). \tag{3.5.12}$$

In particular, $\overline{\mathcal{X}}^{(m)} = \mathcal{X}$ whenever $\mathcal{X}$ is schurian.

Proof. Clearly, the restriction of the coherent configuration $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$ to the diagonal $\Delta = \mathrm{Diag}(\Omega^m)$ is greater than or equal to the restriction of the coherent configuration $\mathcal{X}^m$ to Δ. However, the latter is equal to $\mathcal{X}^\eta$ with $\eta = \eta_m$. Thus,

$$\mathcal{X}^\eta = (\mathcal{X}^m)_\Delta \leq \widehat{\mathcal{X}}_\Delta = \overline{\mathcal{X}}^\eta,$$

where $\overline{\mathcal{X}} = \overline{\mathcal{X}}^{(m)}$. This proves the first inclusion in (3.5.12).

To prove the second inclusion, we make use of formulas (3.1.9) and (3.5.4) to get

$$\mathrm{Aut}(\overline{\mathcal{X}})^\eta = \mathrm{Aut}(\widehat{\mathcal{X}}_\Delta) \geq \mathrm{Aut}(\widehat{\mathcal{X}})^\Delta = \mathrm{Aut}(\mathcal{X})^\eta.$$

This implies that $\mathrm{Aut}(\overline{\mathcal{X}}) \geq \mathrm{Aut}(\mathcal{X})$. Thus in view of the Galois correspondence,

$$\overline{\mathcal{X}} \leq \mathrm{Inv}(\mathrm{Aut}(\overline{\mathcal{X}})) \leq \mathrm{Inv}(\mathrm{Aut}(\mathcal{X})),$$

as required. □

Corollary 3.5.15. $\mathrm{Aut}(\overline{\mathcal{X}}^{(m)}) = \mathrm{Aut}(\mathcal{X})$ *for all $m \geq 1$.*

The basis relations of the m-closure of a coherent configuration can explicitly be found from the fibers of its m-dimensional extension. Namely, the following statement holds.

Theorem 3.5.16. *Let $\mathcal{X}$ be a coherent configuration on Ω and $m \ge 1$ an integer. Then for any $\Lambda \in F(\widehat{\mathcal{X}}^{(m)})$ and any $i, j \in \{1, \ldots, m\}$, the relation*

$$\mathrm{pr}_{i,j}(\Lambda) = \{(\alpha_i, \alpha_j) \in \Omega^2 : \ \alpha \in \Lambda\}$$

belongs to $S(\overline{\mathcal{X}}^{(m)})$.

Proof. By statement (1) of Theorem 3.5.7,

$$s_i = \mathrm{Cyl}_{1_\Omega}(i, i) \quad \text{and} \quad s_j = \mathrm{Cyl}_{1_\Omega}(j, j)$$

are relations of the coherent configuration $\widehat{\mathcal{X}}^{(m)}$. A straightforward check shows that

$$\mathrm{pr}_{i,j}(\Lambda)^\eta = (s_i)_{\Delta,\Lambda} \cdot (s_j)_{\Lambda,\Delta},$$

where $\eta = \eta_m$ is the bijection (3.5.10) and $\Delta = \mathrm{Diag}(\Omega^m)$. Therefore, $\mathrm{pr}_{i,j}(\Lambda)$ is a relation of $\overline{\mathcal{X}} = \overline{\mathcal{X}}^{(m)}$.

Assume on the contrary that $\mathrm{pr}_{i,j}(\Lambda)$ is not a basis relation. Then it strictly contains some $s \in S(\overline{\mathcal{X}})$. Therefore,

$$\big((s_i)_{\Lambda,\Delta} \cdot s^\eta \cdot (s_j)_{\Delta,\Lambda}\big) \cap 1_\Lambda \ne 1_\Lambda.$$

Thus, Λ is not a fiber of $\widehat{\mathcal{X}}^{(m)}$, a contradiction. □

Definition 3.5.17. *The coherent configuration $\mathcal{X}$ is said to be m-closed if $\mathcal{X} = \overline{\mathcal{X}}^{(m)}$.*

Obviously, every coherent configuration is 1-closed. Every schurian coherent configuration is m-closed for all m (Proposition 3.5.14). However, for each m there exist non-schurian coherent configurations which are m-closed; the corresponding examples will be constructed later in Subsection 4.2.1.

Example 3.5.18. *Let $\mathcal{X}$ by a unique antisymmetric scheme of degree 15 and rank 3. A straightforward computation shows that*

$$\overline{\mathcal{X}}^{(2)} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$$

and this coherent configuration is not homogeneous. Thus, $\mathcal{X}$ is not 2-closed.

In a parallel way with the m-closure of a coherent configuration, one can define the m-closure of an algebraic isomorphism but only if it has the m-dimensional extension. Namely, let φ be an algebraic isomorphism from $\mathcal{X}$ to a coherent configuration $\mathcal{Y}$ on a set Δ.

Definition 3.5.19. *We say that φ is an algebraic m-dimensional isomorphism or briefly m-isomorphism if φ has the m-extension.*

The set of all m-isomorphisms from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathrm{Iso}_m(\mathcal{X},\mathcal{Y})$, and by $\mathrm{Iso}_m(\mathcal{X})$ if $\mathcal{X}=\mathcal{Y}$. Obviously, $\mathrm{Iso}_1(\mathcal{X},\mathcal{Y})=\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{Y})$, whereas not every algebraic isomorphism is even a 2-isomorphism.

Example 3.5.20. *Any isomorphism $f\in\mathrm{Iso}(\mathcal{X},\mathcal{Y})$ induces an isomorphism $\widehat{f}^{(m)}\in\mathrm{Iso}(\mathcal{X}^m,\mathcal{Y}^m)$ such that*

$$\mathrm{Diag}(\Omega^m)^{\widehat{f}^{(m)}}=\mathrm{Diag}(\Delta^m).$$

By Theorem 2.6.4, this implies that

$$\widehat{f}^{(m)}\in\mathrm{Iso}(\widehat{\mathcal{X}}^{(m)},\widehat{\mathcal{Y}}^{(m)}).$$

Thus the induced algebraic isomorphism $\varphi_f\in\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{Y})$ has the m-dimensional extension and hence is an m-isomorphism.

Let $\varphi\in\mathrm{Iso}_m(\mathcal{X},\mathcal{Y})$. The restriction of $\widehat{\varphi}=\widehat{\varphi}^{(m)}$ to the homogeneity set $\mathrm{Diag}(\Omega^m)$ induces an algebraic isomorphism

$$\widehat{\varphi}_{\mathrm{Diag}(\Omega^m)}\in\mathrm{Iso}_{\mathrm{alg}}(\widehat{\mathcal{X}}_{\mathrm{Diag}(\Omega^m)},\widehat{\mathcal{Y}}_{\mathrm{Diag}(\Delta^m)}),$$

where $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are the m-dimensional extensions of the coherent configurations $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Let $\eta=\eta_m$, and let

$$\epsilon:\Delta\to\mathrm{Diag}(\Delta^m)$$

be the bijection defined as η_m with Ω replaced by Δ. Then the composition of algebraic isomorphisms

$$\overline{\varphi}^{(m)}=\varphi_\eta\circ\widehat{\varphi}_{\mathrm{Diag}(\Omega^m)}\circ\varphi_{\epsilon^{-1}} \tag{3.5.13}$$

is an algebraic isomorphism from $\overline{\mathcal{X}}=\overline{\mathcal{X}}^{(m)}$ to $\overline{\mathcal{Y}}=\overline{\mathcal{Y}}^{(m)}$, where

$$\varphi_\eta\in\mathrm{Iso}_{\mathrm{alg}}(\overline{\mathcal{X}},\widehat{\mathcal{X}}_{\mathrm{Diag}(\Omega^m)})\quad\text{and}\quad\varphi_{\epsilon^{-1}}\in\mathrm{Iso}_{\mathrm{alg}}(\widehat{\mathcal{Y}}_{\mathrm{Diag}(\Delta^m)},\overline{\mathcal{Y}})$$

are the algebraic isomorphisms induced by the bijections η and ϵ^{-1}, respectively.

Definition 3.5.21. *The algebraic isomorphism (3.5.13) is called the m-dimensional closure or, briefly, the m-closure of φ.*

Clearly, the 1-closure of any algebraic isomorphism φ equals φ. However, the m-closure for $m\ge 2$ is defined for m-isomorphisms only.

Theorem 3.5.22. *For every coherent configurations $\mathcal{X}$ and $\mathcal{Y}$,*

$$\mathcal{X}=\overline{\mathcal{X}}^{(1)}\le\ldots\le\overline{\mathcal{X}}^{(m)}=\mathrm{Inv}(\mathrm{Aut}(\mathcal{X})), \tag{3.5.14}$$

$$\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{Y})=\mathrm{Iso}_1(\mathcal{X},\mathcal{Y})\supseteq\ldots\supseteq\mathrm{Iso}_m(\mathcal{X},\mathcal{Y})=\mathrm{Iso}_\infty(\mathcal{X},\mathcal{Y}), \tag{3.5.15}$$

where $m=b(\mathcal{X})+1$ and $\mathrm{Iso}_\infty(\mathcal{X},\mathcal{Y})$ is the set of all algebraic isomorphisms from $\mathcal{X}$ to $\mathcal{Y}$ induced by (combinatorial) isomorphisms.

Proof. Let $m \geq 2$ be an integer. Denote by π the quotient map π_e defined by formula (1.1.4) for the equivalence relation e, which is equal to the intersection of $\mathrm{Cyl}_{1_\Omega}(i,i)$, $i = 1, \dots, m-1$. Then

$$\pi(\mathrm{Diag}(\Omega^m)) = \mathrm{Diag}(\Omega^{m-1}) \quad \text{and} \quad \pi(\mathcal{X}^m) = \mathcal{X}^{m-1}, \tag{3.5.16}$$

where the first equality is obvious, whereas the second one follows from statement (3) of Exercise 3.7.33 for $\Omega_1 = \Omega^{m-1}$ and $\Omega_2 = \Omega$.

From formula (3.5.16), it follows that the quotient of $\widehat{\mathcal{X}}^{(m)}$ modulo the parabolic e is a fission of the coherent configuration $\widehat{\mathcal{X}}^{(m-1)}$:

$$\begin{aligned} \pi(\widehat{\mathcal{X}}^{(m)}) &= \pi(\mathrm{WL}(\mathcal{X}^m, 1_{\mathrm{Diag}(\Omega^m)}) \\ &\geq \mathrm{WL}(\pi(\mathcal{X}^m), \pi(1_{\mathrm{Diag}(\Omega^m)})) \\ &= \mathrm{WL}(\mathcal{X}^{m-1}, 1_{\mathrm{Diag}(\Omega^{m-1})}) \\ &= \widehat{\mathcal{X}}^{(m-1)}, \end{aligned} \tag{3.5.17}$$

see Exercise 3.7.13. It follows that

$$\begin{aligned} (\overline{\mathcal{X}}^{(m-1)})^{\eta_{m-1}} &= (\widehat{\mathcal{X}}^{(m-1)})_{\mathrm{Diag}(\Omega^{m-1})} \\ &\leq \pi(\widehat{\mathcal{X}}^{(m)})_{\pi(\mathrm{Diag}(\Omega^m))} \\ &= \pi((\widehat{\mathcal{X}}^{(m)})_{\mathrm{Diag}(\Omega^m)}), \end{aligned} \tag{3.5.18}$$

where the last equality follows from statement (2) of Exercise 3.7.9.

The intersection of any class of e with $\mathrm{Diag}(\Omega^m)$ consists of at most one point. Therefore, the restriction of π to $\mathrm{Diag}(\Omega^m)$ coincides with the bijection

$$\xi : \mathrm{Diag}(\Omega^m) \to \mathrm{Diag}(\Omega^{m-1}), \alpha \mapsto (\alpha_1, \dots, \alpha_{m-1}).$$

Taking into account that the mapping η_m equals the composition $\eta_{m-1} \circ \xi^{-1}$, we can combine (3.5.17) and (3.5.18) to get

$$\begin{aligned} \overline{\mathcal{X}}^{(m-1)} &\leq \pi(\widehat{\mathcal{X}}_{\mathrm{Diag}(\Omega^m)})^{\eta_{m-1}^{-1}} \\ &= (\widehat{\mathcal{X}}_{\mathrm{Diag}(\Omega^m)})^{\xi\eta_{m-1}^{-1}} \\ &= (\widehat{\mathcal{X}}_{\mathrm{Diag}(\Omega^m)})^{\eta_m^{-1}} \\ &= \overline{\mathcal{X}}^{(m)}, \end{aligned}$$

where $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$. This proves the inclusions in formula (3.5.14).

Next, let $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{Y})$. From formulas (3.5.16), it follows that the contraction of the induced algebraic isomorphism $(\widehat{\varphi}^{(m)})_{\Omega^m/e}$ to the $(m-1)$-dimensional extension of $\mathcal{X}$ is equal to the $(m-1)$-dimensional extension of φ. Therefore, $\varphi \in \mathrm{Iso}_{m-1}(\mathcal{X}, \mathcal{Y})$, which proves the inclusions in formula (3.5.15).

To prove the equalities in formulas (3.5.14) and (3.5.15), we note that by the definition of the base number, the coherent configuration $\mathcal{X}$ has a partly regular extension with respect to $m-1$ points. By Theorem 3.5.10, this implies that the coherent configuration $\widehat{\mathcal{X}}^{(m)}$ is partly regular, and hence is schurian and separable (Theorem 3.3.19). It follows that the m-closure $\overline{\mathcal{X}}^{(m)}$, which isomorphic to the restriction of $\widehat{\mathcal{X}}^{(m)}$ to $\mathrm{Diag}(\Omega^m)$, is also schurian. By Proposition 3.5.14, this implies that

$$\overline{\mathcal{X}}^{(m')} = \overline{\mathcal{X}}^{(m)} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$$

for all $m' \geq m$.

To complete the proof, let $\varphi \in \mathrm{Iso}_{m'}(\mathcal{X}, \mathcal{Y})$, where $m' \geq m$. By the first part of the theorem, φ is an m-isomorphism and hence has the m-dimensional extension $\widehat{\varphi}^{(m)}$. The separability of the coherent configuration $\widehat{\mathcal{X}}^{(m)}$ (see above) implies that there exists an isomorphism

$$\widehat{f} \in \mathrm{Iso}(\widehat{\mathcal{X}}^{(m)}, \widehat{\mathcal{Y}}^{(m)}, \widehat{\varphi}^{(m)}).$$

Now let $\overline{f} : \Omega \to \Delta$ be the composition $\eta \circ \widehat{f}_{\mathrm{Diag}(\Omega^m)} \circ \epsilon^{-1}$. Then by formula (3.5.13),

$$\overline{f} \in \mathrm{Iso}(\overline{\mathcal{X}}^{(m)}, \overline{\mathcal{Y}}^{(m)}, \overline{\varphi}^{(m)}).$$

Since $\overline{\varphi}^{(m)}$ extends φ, it follows that $\overline{f}$ induces φ. Thus, $\varphi \in \mathrm{Iso}_\infty(\mathcal{X}, \mathcal{Y})$. □

Corollary 3.5.23. *For any $l \leq m$,*

1. *any m-closed coherent configuration is l-closed;*
2. *any m-isomorphism of coherent configurations is an l-isomorphism.*

In general, it is difficult to verify that a (non-schurian) coherent configuration is m-closed for a given integer $m \geq 2$. A useful sufficient condition is given in the following statement.

Theorem 3.5.24. *Let $\mathcal{X}$ be an m-closed coherent configuration, $m \geq 1$. Then given a group $\Phi \leq \mathrm{Iso}_m(\mathcal{X})$, the algebraic fusion $\mathcal{X}^\Phi$ is also m-closed.*

Proof. Denote by $\widehat{\Phi}$ the set of the m-dimensional extensions $\widehat{\varphi}$ of the algebraic isomorphisms $\varphi \in \Phi$. It is easily seen that

$$\widehat{\Phi} \leq \mathrm{Aut}_{\mathrm{alg}}(\widehat{\mathcal{X}}),$$

where $\widehat{\mathcal{X}}$ is the m-dimensional extension of $\mathcal{X}$. Furthermore,

$$(\mathcal{X}^\Phi)^m \leq (\mathcal{X}^m)^{\widehat{\Phi}} \quad \text{and} \quad \Delta^{\widehat{\Phi}} = \Delta,$$

where $\Delta = \mathrm{Diag}(\Omega^m)$. This implies that $\widehat{\mathcal{X}^\Phi} \leq \widehat{\mathcal{X}}^{\widehat{\Phi}}$ and hence

$$\widehat{\mathcal{X}^\Phi}_\Delta \leq (\widehat{\mathcal{X}}^{\widehat{\Phi}})_\Delta = (\widehat{\mathcal{X}}_\Delta)^{\widehat{\Phi}_\Delta}.$$

Applying the inverse to the bijection η_m defined by formula (3.5.10) to both sides and taking into account that the coherent configuration $\mathcal{X}$ is

m-closed, we obtain

$$\overline{\mathcal{X}^\Phi} \leq \mathcal{X}^\Phi.$$

The reverse inclusion follows from Exercise 3.7.47. Thus the coherent configuration $\mathcal{X}^\Phi = \overline{\mathcal{X}}^\Phi$ is m-closed. □

3.5.3 2-closed coherent configurations

In this subsection, we show that the m-closedness condition for $m \geq 2$ enables us to get combinatorial analogs of some theorems on permutation groups that do not hold for general coherent configurations.

We begin with observation that for any $K \leq \mathrm{Sym}(\Omega)$, any $\alpha \in \Omega$, and any $\beta \in \alpha^K$,

$$F(\mathrm{Inv}(K_\alpha)) = \mathrm{Orb}(K_\alpha) \quad \text{and} \quad \mathrm{Inv}(K_\alpha) \cong \mathrm{Inv}(K_\beta).$$

The following statement generalizes these relations to 2-closed coherent configurations and gives a necessary condition for a coherent configuration to be 2-closed. It is not clear whether this condition is also sufficient.

Lemma 3.5.25. *Let $\mathcal{X} = (\Omega, S)$ be a 2-closed coherent configuration on Ω, $\alpha \in \Omega$, and β lies in the fiber containing α. Then*

(1) $F(\mathcal{X}_\alpha) = \{\alpha s : \ s \in S\}^\natural$;
(2) *there is $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ such that $\varphi(1_\alpha) = 1_\beta$ and $\varphi|_S = \mathrm{id}$.*

Proof. To prove statement (1), let α and $s \in S$ be such that $\alpha s \neq \varnothing$. From the 2-closedness of $\mathcal{X}$ and Theorem 3.5.16, it follows that s is a fiber of $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(2)}$. Therefore if $\Delta = \Delta_\delta$ with $\delta = (\alpha)$ is a class of the parabolic e of $\widehat{\mathcal{X}}$ defined by the equalities of the first coordinates (see formula (3.5.8)), then

$$s \cap \Delta \in F(\widehat{\mathcal{X}}_\Delta).$$

By Corollary 3.5.9, there exists a fiber $\Gamma \in F(\mathcal{X}_\alpha)$ such that

$$\Gamma^\zeta \supseteq s \cap \Delta,$$

where ζ is the bijection defined in (3.5.9). Since $(\alpha s)^\zeta = s \cap \Delta$, we have

$$\Gamma \supseteq (s \cap \Delta)^{\zeta^{-1}} = \alpha s.$$

However, αs is a homogeneity set of the coherent configuration $\mathcal{X}_\alpha$ (statement (1) of Lemma 3.3.5). Therefore, $\alpha s = \Gamma$ is a fiber of $\mathcal{X}$.

To prove statement (2), let $\Gamma = \Delta_\delta$ with $\delta = (\beta)$ be the class of the parabolic e. Then Δ and Γ are classes of the same indecomposable component of e: otherwise the pairs (α, α) and (β, β) lie in different fibers of $\widehat{\mathcal{X}}$ (Lemma 2.1.21) and hence α and β lie in different fibers of $\overline{\mathcal{X}}^{(2)} = \mathcal{X}$, a contradiction.

In accordance with Example 2.3.16, the mapping

$$\widehat{\varphi} : \widehat{\mathcal{X}}_\Delta \to \widehat{\mathcal{X}}_\Gamma, \ s_\Delta \mapsto s_\Gamma$$

is an algebraic isomorphism. We have

$$\{1_\alpha\}^{\widehat{\varphi}} = (\Delta \cap 1_\Omega)^{\widehat{\varphi}} = \Gamma \cap 1_\Omega = \{1_\beta\} \tag{3.5.19}$$

and

$$\widehat{\varphi}((1_\Omega \otimes s)_\Delta) = (1_\Omega \otimes s)_\Gamma \tag{3.5.20}$$

for all $s \in S$.

Now we define an injection $\xi : \Omega \to \Gamma$ in the same way as the mapping ζ (3.5.9) with Δ replaced by Γ. Then

$$\alpha^\zeta = (\alpha, \alpha),\ s^\zeta = (1_\Omega \otimes s)_\Delta \quad \text{and} \quad \beta^\xi = (\beta, \beta),\ s^\xi = (1_\Omega \otimes s)_\Gamma.$$

Together with formulas (3.5.19) and (3.5.20), this shows that the algebraic isomorphism

$$\varphi = \varphi_\zeta \circ \widehat{\varphi} \circ \varphi_{\xi^{-1}},$$

from $\mathcal{X}_\alpha$ to $\mathcal{X}_\beta$ takes 1_α to 1_β and leaves each $s \in S$ fixed, where φ_ζ and $\varphi_{\xi^{-1}}$ are the algebraic isomorphisms induced by ζ and ξ^{-1}, respectively. $\square$

In what follows, we establish several results on 2-closed primitive schemes that generalize the corresponding results on primitive permutation groups. In each case a "permutation group theorem" can be deduced from the corresponding "coherent configuration theorem" by using Proposition 3.1.4 and the simplest properties of 2-equivalent permutation groups. We begin with the following characterization of 2-closed primitive schemes, cf. [**125**, Theorem 8.2].

Theorem 3.5.26. *Let $\mathcal{X}$ be a 2-closed scheme on Ω. Then $\mathcal{X}$ is primitive if and only if given a point $\alpha \in \Omega$, the α-extension $\mathcal{X}_\alpha$ is a minimal proper fission of $\mathcal{X}$.*

Proof. To prove the necessity, we assume that $\mathcal{X}$ is primitive and $\mathcal{X}'$ is a coherent configuration on Ω such that

$$\mathcal{X} \le \mathcal{X}' < \mathcal{X}_\alpha.$$

Denote by Δ the fiber of $\mathcal{X}'$ that contains α. Then $\Delta \ne \{\alpha\}$, for otherwise $\mathcal{X}' \ge \mathcal{X}_\alpha$, a contradiction. Thus there exists $s \in S(\mathcal{X})^\#$ such that

$$\alpha s \cap \Delta \ne \varnothing. \tag{3.5.21}$$

By the 2-closedness of the scheme $\mathcal{X}$, statement (1) of Lemma 3.5.25 implies that $\alpha s \in F(\mathcal{X}_\alpha)$. It follows that αs is contained in some fiber of $\mathcal{X}'$. In view of (3.5.21), this fiber coincides with Δ, i.e.,

$$\alpha s \subseteq \Delta.$$

Since s_Δ is a relation of the scheme $(\mathcal{X}')_\Delta$, this implies that

$$\beta s \subseteq \Delta \quad \text{for all} \ \beta \in \Delta.$$

Consequently if $\beta \in \Delta$ and $\beta \xrightarrow{s} \gamma$, then $\gamma \in \Delta$. Since the relation s is strongly connected (Theorem 3.1.5), it follows that $\Delta = \Omega$, in particular, the coherent configuration $\mathcal{X}'$ is homogeneous.

It remains to verify that $\mathcal{X}' = \mathcal{X}$ or equivalently, that each $s' \in S(\mathcal{X}')$ is a basis relation of $\mathcal{X}$. To this end, we make use of the inclusion $\mathcal{X} \le \mathcal{X}'$ and denote by s the basis relation of $\mathcal{X}$ such that

$$s' \subseteq s.$$

Then $\alpha s' \subseteq \alpha s$ is a homogeneity set of $(\mathcal{X}')_\alpha \le \mathcal{X}_\alpha$ (statement (1) of Lemma 3.3.5). Since αs is a fiber of $\mathcal{X}_\alpha$ (statement (1) of Lemma 3.5.25), this is possible only if

$$\alpha s = \alpha s'.$$

Therefore, $n_s = n_{s'}$ and hence $|s| = |s'|$ by formula (2.1.11). Since $s' \subseteq s$, this shows that $s' = s$, as required.

To prove the sufficiency, we assume that any one-point extension of $\mathcal{X}$ is a minimal proper fission of $\mathcal{X}$. We have to verify that any parabolic $e \ne \Omega^2$ of $\mathcal{X}$ coincides with 1_Ω.

Note that for each $\Delta \in \Omega/e$ and each $\alpha \in \Delta$,

$$\mathcal{X} < \mathrm{WL}(\mathcal{X}, 1_\Delta) = \mathrm{WL}(\mathcal{X}, 1_{\alpha e}) \le \mathrm{WL}(\mathcal{X}, 1_\alpha) = \mathcal{X}_\alpha.$$

The minimality of $\mathcal{X}_\alpha$ implies that $\mathrm{WL}(\mathcal{X}, 1_\Delta) = \mathcal{X}_\alpha$. It follows that for each $\beta \in \Delta$,

$$\mathcal{X}_\beta = \mathrm{WL}(\mathcal{X}, 1_\Delta) = \mathcal{X}_\alpha,$$

i.e., $\{\beta\}$ is a fiber of $\mathcal{X}_\alpha$. By statement (1) of Lemma 3.5.25, the relation $r(\alpha, \beta)$ is thin. Consequently, $e \in S_1(\mathcal{X})^\cup$. In accordance with Exercise 3.7.50, this shows that

$$\Delta \in F(\mathrm{WL}(\mathcal{X}, 1_\Delta)) = F(\mathcal{X}_\alpha),$$

which implies that $|\Delta| = 1$. Thus, $e = 1_\Omega$, as required. □

As an almost immediate consequence of Theorem 3.5.26, we are able to generalize another well-known property of non-regular primitive groups, namely, any such group is generated by two distinct point stabilizers [**125**, Proposition 8.7].

Theorem 3.5.27. *Let $\mathcal{X}$ be a 2-closed primitive scheme on Ω, $\alpha, \beta \in \Omega$, and $\alpha \ne \beta$. Then $\mathcal{X} = \mathcal{X}_\alpha \cap \mathcal{X}_\beta$ unless $\mathcal{X}$ is a regular scheme of prime degree.*

Proof. Assume that $\mathcal{X} \ne \mathcal{X}_\alpha \cap \mathcal{X}_\beta$. Then by Theorem 3.5.26,

$$\mathcal{X}_\alpha \cap \mathcal{X}_\beta = \mathcal{X}_\alpha = \mathcal{X}_\beta.$$

Since the scheme $\mathcal{X}$ is 2-closed, statement (1) of Lemma 3.5.25 implies that

$$r(\alpha, \beta) \in S_1(\mathcal{X}),$$

which completes the proof by statement (1) of Theorem 3.1.6. □

In accordance with [**125**, Theorem 10.4], any 3/2-transitive group is either primitive or a Frobenius group. Below we generalize this result to equivalenced schemes, which are combinatorial analogs of 3/2-transitive groups (statement (1) of Corollary 2.2.6).

Theorem 3.5.28. *Let $\mathcal{X}$ be a 2-closed equivalenced scheme. Assume that $\mathcal{X}$ is imprimitive. Then*

(1) $\mathcal{X} = \mathrm{Inv}(K)$, *where K is either regular or a Frobenius group;*
(2) $b(\mathcal{X}) = b(K)$.

Proof. Let us verify that the group $K = \mathrm{Aut}(\mathcal{X})$ is transitive. To this end, we prove that any two points α and β of $\mathcal{X}$ lie in the same orbit of $\mathrm{Aut}(\mathcal{X})$.

Indeed, by statement (2) of Lemma 3.5.25, the 2-closedness of $\mathcal{X}$ implies that the trivial algebraic automorphism of $\mathcal{X}$ is extended to an algebraic isomorphism
$$\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}_\beta)$$
such that $\varphi(1_\alpha) = 1_\beta$. By Theorem 3.3.8, the coherent configurations $\mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ are partly regular and hence are separable (Theorem 3.3.19). Consequently, φ is induced by an automorphism $k \in K$. Thus
$$1_{\alpha^k} = (1_\alpha)^k = \varphi(1_\alpha) = 1_\beta,$$
whence $\alpha^k = \beta$, as required.

To prove that $\mathcal{X} = \mathrm{Inv}(K)$, it suffices to verify that αs is an orbit of K_α for any $s \in S(\mathcal{X})$ (the implication (3) $\Rightarrow$ (1) of Theorem 2.2.10). However, $\alpha s \in F(\mathcal{X}_\alpha)$ by statement (1) of Lemma 3.5.25. Furthermore in accordance with the previous paragraph, the coherent configuration $\mathcal{X}_\alpha$ is partly regular and hence schurian (Theorem 3.3.19). It follows that
$$\mathcal{X}_\alpha = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_\alpha)) = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X})_\alpha) = \mathrm{Inv}(K_\alpha),$$
see statement (1) of Proposition 3.3.3. Thus, $\alpha s \in \mathrm{Orb}(K_\alpha)$ by statement (1) of Proposition 2.2.5.

To complete the proof, let α and β be different points of $\mathcal{X}$. Note that the coherent configuration $\mathcal{X}_\alpha$ is partly regular and its restriction to $\Omega \setminus \{\alpha\}$ is half-homogeneous (see above). Consequently, each point other than α is regular. It follows that the singleton $\{\beta\}$ is a base of $\mathcal{X}_\alpha$.

By Exercise 3.7.27, this implies that $\{\alpha, \beta\}$ is a base of $\mathcal{X}$; in particular, $b(\mathcal{X}) \leq 2$. Thus,
$$K_{\alpha,\beta} = \mathrm{Aut}(\mathcal{X})_{\alpha,\beta} = \mathrm{Aut}(\mathcal{X}_{\alpha,\beta}) = \mathrm{Aut}(\mathcal{D}_\Omega),$$
i.e., the point stabilizer $K_{\alpha,\beta}$ is trivial. Consequently, K is regular or a Frobenius group. In the former case, $b(K) = b(\mathcal{X}) = 1$, whereas in the latter one,
$$2 = b(K) \leq b(\mathrm{Inv}(K)) = b(\mathcal{X}) \leq 2,$$
see formula (3.3.5). Thus, $b(K) = b(\mathcal{X})$. □

In Theorems 3.5.26, 3.5.27, and 3.5.28 the hypothesis for a primitive scheme to be 2-closed is essential: the corresponding examples are given by the antisymmetric scheme of degree 15 and rank 3 in the first two cases, and by the scheme of non-Desarguesian affine plane in the third one (Theorem 2.5.7).

3.6 Representation theory

The main focus of the representation theory of coherent configurations is to study the standard representation of a coherent algebra on the underlying linear space. Analysis of the basic invariants of this representation, like the degrees and multiplicities of irreducible characters, allows us to obtain some information not only on the intersection numbers but sometimes also on the structure of the coherent configuration in question.

In this section, we restrict ourselves to the main issues including the structure of standard representation of a general coherent configurations and the orthogonality relations in the homogeneous case. As an application, we prove an upper bound for the base number of a primitive scheme in terms of the multiplicities and degrees [**39**]; one more application concerning the Hanaki–Uno theorem on schemes of prime degree will be presented in Section 4.5. Most of the material concerning basic representation theory of general coherent configurations and schemes can be found in papers of D. Higman [**65, 66**] and A. Hanaki [**55**].

3.6.1 Standard representation

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration and $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ the adjacency algebra of $\mathcal{X}$; the linear space $\mathcal{L}_\Omega = \mathbb{C}\Omega$ spanned by Ω is considered as a left $\mathrm{Mat}_\Omega(\mathbb{C})$-module and hence $\mathcal{A}$-module with respect to natural multiplication of matrix by vector.

Definition 3.6.1. *The module $\mathcal{L}_\Omega$, the representation of $\mathcal{A}$ afforded by it, and the character π of this representation are said to be standard.*

Certainly,

$$\pi(I_\Omega) = \mathrm{tr}(I_\Omega) = |\Omega|. \tag{3.6.1}$$

The set of all irreducible characters of the adjacency algebra $\mathcal{A}$ of the coherent configuration $\mathcal{X}$ is denoted by $\mathrm{Irr}(\mathcal{X})$.

From the condition (A1) in Theorem 2.3.6, it follows that the algebra $\mathcal{A}$ is closed under the Hermitian conjugation. Therefore, any right ideal of $\mathcal{A}$ contains together with each matrix A a Hermitian matrix AA^*. If this ideal is nonzero, then one of the matrices AA^* is also nonzero (this follows from statement (1) of Exercise 2.7.6). Therefore the ideal in question cannot be nilpotent. This proves the following statement.

Proposition 3.6.2. *The adjacency algebra of a coherent configuration is semisimple.*

By the Wedderburn theorem on simple unitary rings, Proposition 3.6.2 implies that the standard character of $\mathcal{A}$ can be decomposed into the linear combination of irreducible characters,

$$\pi = \sum_{\xi \in \mathrm{Irr}(\mathcal{X})} m_\xi \xi, \tag{3.6.2}$$

where m_ξ is a nonnegative integer called the *multiplicity* of ξ.

In view of (3.6.1), taking the value of both sides of formula (3.6.2) at the identity matrix I_Ω, we get a useful relation

$$|\Omega| = \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} m_\xi n_\xi, \tag{3.6.3}$$

where $n_\xi = \xi(I_\Omega)$ is the *degree* of any irreducible representation corresponding to the character ξ.

In accordance with formula (3.6.2), there are the direct decompositions

$$\mathcal{L}_\Omega = \sum_{\xi\in\mathrm{Irr}(\mathcal{A})} \mathcal{L}_\xi \quad\text{and}\quad \mathcal{A} = \prod_{\xi\in\mathrm{Irr}(\mathcal{A})} \mathcal{A}_\xi \tag{3.6.4}$$

where $\mathcal{L}_\xi$ is the submodule of $\mathcal{L}_\Omega$ of dimension $m_\xi n_\xi$ that corresponds to the irreducible character ξ, and $\mathcal{A}_\xi$ is the image of the representation afforded by $\mathcal{L}_\xi$.

The Wedderburn theorem guarantees that $\mathcal{A}_\xi$ is isomorphic to the algebra of all $n_\xi \times n_\xi$ matrices. Comparing the dimensions of the algebras in the second equality of (3.6.4), we get

$$|S| = \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} n_\xi^2. \tag{3.6.5}$$

Let $\mathcal{X}$ be a regular scheme associated with regular representation of a group G. Then $\mathcal{A}$ is isomorphic to the group algebra of G (Example 2.3.4). Therefore, one can identify $\mathrm{Irr}(\mathcal{X})$ and $\mathrm{Irr}(G)$. Since the multiplicity and degree of any irreducible character of G are equal, we have $m_\xi = n_\xi$ for all $\xi \in \mathrm{Irr}(\mathcal{X})$. For a general scheme $\mathcal{X}$, a weaker statements holds.

Theorem 3.6.3. *For any scheme $\mathcal{X}$ and any character $\xi \in \mathrm{Irr}(\mathcal{X})$,*

$$n_\xi \le m_\xi. \tag{3.6.6}$$

The equality is simultaneously attained for all ξ if and only if $\mathcal{X}$ is regular.

Proof. For a point $\alpha \in \Omega$, denote by $\mathcal{L}_\alpha$ the $\mathcal{A}$-module spanned by the vectors $A_s\alpha = \underline{\alpha s^*}$, $s \in S$. Then the mapping

$$\mathcal{A} \to \mathcal{L}_\alpha,\ A_s \mapsto A_s\alpha$$

is an $\mathcal{A}$-module isomorphism. Denote by $\mathcal{L}_{\alpha,\xi}$ the image of the algebra $\mathcal{A}_\xi \subseteq \mathcal{A}$ with respect to this isomorphism. Then

$$\dim(\mathcal{A}_\xi) = \dim(\mathcal{L}_{\alpha,\xi}).$$

Since $\mathcal{L}_{\alpha,\xi} \subseteq \mathcal{L}_\xi$, this implies that

$$n_\xi^2 = \dim(\mathcal{A}_\xi) = \dim(\mathcal{L}_{\alpha,\xi}) \le \dim(\mathcal{L}_\xi) = m_\xi n_\xi.$$

This proves inequality (3.6.6).

To complete the proof of the theorem, we assume that $m_\xi = n_\xi$ for all $\xi \in \mathrm{Irr}(\mathcal{X})$. Then in view of formulas (3.6.5) and (3.6.3),

$$|S| = \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} n_\xi^2 = \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} m_\xi n_\xi = |\Omega|.$$

Thus the scheme $\mathcal{X}$ is regular by Theorem 2.1.29. □

Let $\Delta \subseteq \Omega$. The matrix J_Δ belongs to the algebra $\mathcal{A}$ whenever Δ is a homogeneity set of $\mathcal{X}$. It follows that $\mathcal{A}$ contains also the matrix

$$P_0 = \sum_{\Delta\in F} \frac{1}{|\Delta|} J_\Delta, \tag{3.6.7}$$

where $F = F(\mathcal{X})$. Note that $P_0^2 = P_0$, because $(J_\Delta)^2 = |\Delta|\, J_\Delta$.

Proposition 3.6.4. *The matrix P_0 is a central primitive idempotent of the algebra $\mathcal{A}$,*

$$\dim(P_0\mathcal{L}) = d \quad \text{and} \quad P_0\mathcal{A} \cong \mathrm{Mat}_d(\mathbb{C}),$$

where $d = |F|$.

Proof. Let $\Delta, \Gamma \in F$ and $s \in S_{\Delta,\Gamma}$. From formula (2.1.5), it follows that

$$n_s\, |\Delta| = n_{s^*}\, |\Gamma|.$$

Furthermore for any $\Lambda \in F$, we have

$$A_s J_\Lambda = n_s \delta_{\Lambda,\Gamma} J_{\Delta,\Gamma} \quad \text{and} \quad J_\Lambda A_s = n_{s^*} \delta_{\Lambda,\Delta} J_{\Delta,\Gamma}, \tag{3.6.8}$$

where $\delta(\cdot,\cdot)$ is the Kronecker delta. Consequently,

$$P_0 A_s = \sum_{\Delta\in F} \frac{1}{|\Delta|} J_\Delta A_s = \frac{n_{s^*}}{|\Delta|} J_{\Delta,\Gamma} = \frac{n_s}{|\Gamma|} J_{\Delta,\Gamma} = \sum_{\Delta\in F} \frac{1}{|\Delta|} A_s J_\Delta = A_s P_0.$$

This shows that P_0 is a central idempotent of the algebra $\mathcal{A}$.

Next, it is easily seen that given a point α, $P_0\alpha = \frac{1}{|\Delta|}\underline{\Delta}$, where Δ is the fiber of $\mathcal{X}$ that contains α. Therefore,

$$P_0\mathcal{L} = \mathrm{Span}\{\underline{\Delta} :\ \Delta \in F\}.$$

Since the vectors $\underline{\Delta}$, $\Delta \in F$, form a linear independent subset of $\mathcal{L}$, this implies that $\dim(P_0\mathcal{L}) = d$.

Let e be the parabolic of $\mathcal{X}$ such that $\Omega/e = F$. Then in the notation of Theorem 3.1.19, we have $P_0 = P_e$ and $\mathcal{X}_{\Omega/e} = \mathcal{D}_d$. By this theorem,

$$P_0\mathcal{A} \cong \mathrm{Adj}(\mathcal{X}_{\Omega/e}) = \mathrm{Adj}(\mathcal{D}_d) = \mathrm{Mat}_d(\mathbb{C}).$$

In particular, the idempotent P_0 is primitive. □

By Proposition 3.6.4, the linear space $P_0\mathcal{L}$ is an $|F|$-dimensional irreducible $\mathcal{A}$-submodule of the standard module. Denote by ξ_0 the irreducible character of $\mathcal{X}$ corresponding to this submodule. Then

$$m_{\xi_0} = 1 \quad \text{and} \quad n_{\xi_0} = |F|. \tag{3.6.9}$$

Definition 3.6.5. *The character ξ_0, module $\mathcal{L}_{\xi_0} = P_0\mathcal{L}$, and the representation afforded by ξ_0 are said to be principal; the set of all non-principal irreducible characters of $\mathcal{X}$ is denoted by $\mathrm{Irr}(\mathcal{X})^{\#}$.*

Example 3.6.6. *Let $\mathcal{X}$ be a discrete coherent configuration. Then, obviously, $|F| = |\Omega|$ and $\xi_0 = \pi$. Thus, $\mathrm{Irr}(\mathcal{X}) = \{\xi_0\}$.*

From formulas (3.6.7) and (3.6.8), one can easily find the values of the principal character on the basis matrices of $\mathcal{A}$.

Lemma 3.6.7. *For any $s \in S$,*

$$\xi_0(A_s) = \begin{cases} n_s, & \text{if } \Omega_-(s) = \Omega_+(s), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $\mathcal{X}$ is a scheme, then $\xi_0(s) = n_s$ for all $s \in S$.

As in the case of groups, one can form the character table of the coherent configuration $\mathcal{X}$: the rows and columns are indexed by the elements of the sets $\mathrm{Irr}(\mathcal{X})$ and S, respectively, and the entry at row ξ and column s is equal to $\xi(A_s)$. In contrast to the case of groups, the table is not necessarily a square one; in fact, the table is square if and only if $\mathcal{X}$ is commutative. Lemma 3.6.7 determines the first row of the character table. In the homogeneous case, the first column is also clear:

	1_Ω	...	s	...
ξ_0	1	...	n_s	...
...	...	...	...	...
ξ	n_ξ	...	$\xi(s)$	...
...	...	...	...	...

Example 3.6.8. *Let $\mathcal{X}$ be a trivial scheme of degree $n \geq 2$. Then $|S| = 2$ and equality (3.6.5) implies that*

$$\mathrm{Irr}(\mathcal{X}) = \{\xi_0, \xi\},$$

where ξ is a one-dimensional character. In view of (3.6.3), we have

$$\pi = \xi_0 + (n-1)\xi.$$

Thus the character table of $\mathcal{X}$ is as follows:

	1_Ω	$\Omega^2 \setminus 1_\Omega$
ξ_0	1	$n-1$
ξ	1	-1

Let $\xi \in \mathrm{Irr}(\mathcal{X})$. The coefficients in the decomposition of a matrix $A \in \mathcal{A}_\xi$ into the linear combination of the basis matrices

$$A = \sum_{s \in S} a_s A_s \tag{3.6.10}$$

can easily be calculated. Namely, the algebra $\mathcal{A}_\xi$ being an ideal of $\mathcal{A}$, contains the matrix $A_{s^*}A$ for all s. By formula (3.6.2), this implies that

$$\pi(A_{s^*}A) = m_\xi\, \xi(A_{s^*}A).$$

On the other hand, by (3.6.10), we have

$$\pi(A_{s^*}A) = \mathrm{tr}(A_{s^*}A) = a_s\, |\Gamma|\, c_{s^*s}^{1_\Gamma} = a_s\, |\Gamma|\, n_{s^*} = a_s\, |s^*|,$$

where $\Gamma = \Omega_-(s^*)$ (statement (1) of Exercise 2.7.6). Thus,

$$a_s\, |s^*| = \pi(A_{s^*}A) = m_\xi\, \xi(A_{s^*}A),$$

which gives an explicit formula for a_s,

$$a_s = \frac{m_\xi\, \xi(A_{s^*}A)}{|s^*|}. \tag{3.6.11}$$

Proposition 3.6.9. *Let P_ξ be the central primitive idempotent corresponding to the character $\xi \in \mathrm{Irr}(\mathcal{X})$. Then*

$$P_\xi = m_\xi \sum_{s \in S} \frac{\xi(A_{s^*})}{|s^*|} A_s. \tag{3.6.12}$$

If the coherent configuration $\mathcal{X}$ is homogeneous, then

$$P_\xi = \frac{m_\xi}{n} \sum_{s \in S} \frac{\xi(A_{s^*})}{n_{s^*}} A_s. \tag{3.6.13}$$

Proof. In formula (3.6.10), take $A = P_\xi$. Then the first statement follows from (3.6.11) after taking into account that in this case,

$$\xi(A_{s^*}A) = \xi(A_{s^*}).$$

The second statement follows from the first one and formula (2.1.11). □

Corollary 3.6.10. *Each central primitive idempotent of $\mathrm{Adj}(\mathcal{X})$ is a block diagonal matrix with blocks belonging to $\mathrm{Mat}_{\Delta,\Gamma}$, $\Delta, \Gamma \in F(\mathcal{X})$.*

We say that coherent configurations $\mathcal{X}$ and $\mathcal{X}'$ have the same character table if there exist bijections $S(\mathcal{X}) \to S(\mathcal{X}')$, $s \mapsto s'$ and $\mathrm{Irr}(\mathcal{X}) \to \mathrm{Irr}(\mathcal{X}')$, $\xi \mapsto \xi'$ such that for all s and ξ,

$$\xi(A_s) = \xi'(A_{s'}). \tag{3.6.14}$$

The following statement implies that in this sense the character table of a commutative scheme determines this scheme uniquely up to algebraic isomorphisms.

Proposition 3.6.11. *Any two algebraically isomorphic coherent configurations have the same character table. The reverse statement is also true if the coherent configurations are commutative.*

Proof. Every algebraic isomorphism $\varphi : S \to S$, $s \mapsto s'$, induces a matrix algebra isomorphism

$$\widetilde{\varphi} : \mathrm{Adj}(\mathcal{X}) \to \mathrm{Adj}(\mathcal{X}'),\ A_s \mapsto A_{s'},$$

see (2.3.14). This isomorphism induces a bijection between the central primitive idempotents of the adjacency algebras and hence a bijection

$$\mathrm{Irr}(\mathcal{X}) \to \mathrm{Irr}(\mathcal{X}'),\ \xi \mapsto \xi',$$

such that for all $\xi \in \mathrm{Irr}(\mathcal{X})$,

$$\widetilde{\varphi}(P_\xi) = P_{\xi'}. \tag{3.6.15}$$

The isomorphism $\widetilde{\varphi}$ preserves the second decomposition in (3.6.4), and hence $n_\xi = n_{\xi'}$. It follows that

$$m_\xi = \mathrm{tr}(P_\xi)/n_\xi = \mathrm{tr}(P_{\xi'})/n_{\xi'} = m_{\xi'}.$$

Thus in view of (3.6.12) and (3.6.15), we obtain

$$\begin{aligned} m_{\xi'} \sum_{s'\in S'} \frac{\xi'(A_{(s')^*})}{|(s')^*|} A_{s'} = P_{\xi'} &= \widetilde{\varphi}(P_\xi) \\ &= m_\xi \sum_{s\in S} \frac{\xi(A_{s^*})}{|s^*|} \widetilde{\varphi}(A_s) \\ &= m_{\xi'} \sum_{s'\in S'} \frac{\xi(A_{s^*})}{|(s')^*|} A_{s'}. \end{aligned}$$

Comparing the coefficients at $A_{s'}$, we come to equality (3.6.14) showing that the coherent configurations $\mathcal{X}$ and $\mathcal{X}'$ have the same character table.

Let $\mathcal{X}$ and $\mathcal{X}'$ be commutative schemes. Since the central primitive idempotents of a commutative algebra form a linear basis of this algebra,

$$A_s = \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} a_{\xi,s} P_\xi \quad \text{and} \quad A_{s'} = \sum_{\xi'\in\mathrm{Irr}(\mathcal{X}')} a_{\xi',s'} P_{\xi'}$$

for all $s \in S$, $s' \in S'$, and some complex numbers $a_{\xi,s}$ and $a'_{\xi',s'}$.

Now assume that $\mathcal{X}$ and $\mathcal{X}'$ have the same character table, i.e., relation (3.6.14) holds for some bijections $\varphi : s \mapsto s'$ and $\psi : \xi \mapsto \xi'$. Note that φ induces a linear isomorphism (2.3.14) preserving the Hadamard multiplication, and also $\varphi(P_\xi) = P_{\xi'}$ for all ξ (this follows from formula (3.6.13) with taking into account formula (3.6.26) proved later). Furthermore, the orthogonality of central primitive idempotents implies that for any $r, s \in S$,

$$\begin{aligned}\varphi(A_r A_s) &= \varphi\Big(\sum_{\xi\in\mathrm{Irr}(\mathcal{X})} a_{\xi,r}P_\xi \sum_{\eta\in\mathrm{Irr}(\mathcal{X})} a_{\eta,s}P_\eta\Big)\\ &= \varphi\Big(\sum_{\xi\in\mathrm{Irr}(\mathcal{X})} a_{\xi,r}a_{\xi,s}P_\xi\Big)\\ &= \sum_{\xi'\in\mathrm{Irr}(\mathcal{X}')} a_{\xi,r}a_{\xi,s}P_{\xi'}\\ &= \sum_{\xi'\in\mathrm{Irr}(\mathcal{X}')} a_{\xi,r}P_{\xi'} \sum_{\eta'\in\mathrm{Irr}(\mathcal{X}')} a_{\eta,s}P_{\eta'}\\ &= \varphi(A_r)\varphi(A_s).\end{aligned}$$

Thus, φ is an algebraic isomorphism by Proposition 2.3.17. □

Two noncommutative coherent configurations having the same character table are not necessarily algebraically isomorphic. For example, the schemes corresponding of regular groups isomorphic to D_8 and Q_8 are not algebraically isomorphic by Corollary 2.3.34. However, these schemes have the same character table, because this is true for the groups D_8 and Q_8.

The fact that the structure constants c_{rs}^t of the algebra $\mathcal{A}$ are integral, implies that $\xi(A_s)$ is an algebraic integer for all $\xi \in \mathrm{Irr}(\mathcal{X})$ and $s \in S$. We make use of this observation in the proof of the following statement obtained by D. Higman in [**65**] (see also [**123**, p. 68]).

Theorem 3.6.12. *Every scheme of rank at most* 5 *is commutative.*

Proof. Let $\mathcal{X} = (\Omega, S)$ be a noncommutative scheme of rank at most 5. By formulas (3.6.5) and (3.6.9), we may assume that

$$|S| = 5 \quad \text{and} \quad \mathrm{Irr}(\mathcal{X}) = \{\xi_0, \xi\},$$

where $n_\xi = 2$. In view of (3.6.2), this implies that $\pi = \xi_0 + m_\xi \xi$. Therefore, by Lemma 3.6.7 and formula (2.1.13),

$$\begin{aligned}m_\xi \xi(J) &= \pi(J) - \xi_0(J)\\ &= |\Omega| - \sum_{s\in S}\xi_0(A_s)\\ &= |\Omega| - \sum_{s\in S} n_s = 0,\end{aligned} \tag{3.6.16}$$

where $J = J_\Omega$. On the other hand, if $s \neq 1_\Omega$, then by the same lemma we have

$$0 = \pi(A_s) = \xi_0(A_s) + m_\xi\, \xi(A_s) = n_s + m_\xi\, \xi(A_s).$$

This implies that the algebraic integer $\xi(A_s) = -n_s/m_\xi$ is less than or equal to -1. Thus by formula (3.6.16),

$$\begin{aligned} 0 = \xi(J) &= \xi(\sum_{s\in S} A_s) \\ &= \xi(I_\Omega) + \sum_{s\neq 1_\Omega} \xi(A_s) \\ &\leq n_\xi - (|S|-1) = -2, \end{aligned}$$

a contradiction. □

The smallest noncommutative scheme of rank 6 is the scheme of a regular group isomorphic to Sym(3). More interesting examples are given in Exercise 2.7.43. The non-commutative schemes of rank 6 are intensively studied by many researchers; we refer the reader to paper [**59**] and references therein.

The following result is well known in the commutative case and was proved in general case in [**102**]. It gives a combinatorial definition of pseudocyclic schemes to be studied in Subsection 4.3.4.

Theorem 3.6.13. *For any scheme $\mathcal{X}$ and an integer $k \geq 1$, the following two statements are equivalent:*

(1) $k = \frac{m_\xi}{n_\xi}$ *for all* $\xi \in \mathrm{Irr}(\mathcal{X})^\#$*;*
(2) $k = n_s = c(s) + 1$ *for all* $s \in S(\mathcal{X})^\#$*,*

where $c(s)$ is the indistinguishing number of s, see (2.1.15)*.*

Proof. Let ρ and π be the regular and standard characters of the adjacency algebra of the scheme $\mathcal{X}$. For any s in $S = S(\mathcal{X})$, we have

$$\rho(A_s) = \sum_{t\in S} c_{ts}^t = n_s + \sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#} n_\xi \xi(A_s), \tag{3.6.17}$$

$$\pi(A_s) = \delta_{s,1_\Omega} n = n_s + \sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#} m_\xi\, \xi(A_s). \tag{3.6.18}$$

Assume that $\frac{m_\xi}{n_\xi} = k$ for all $\xi \in \mathrm{Irr}(\mathcal{X})^\#$. Then for each $s \in S^\#$ formulas (3.6.17) and (3.6.18) yield

$$\begin{aligned} -n_s &= \sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#} m_\xi\, \xi(A_s) \\ &= \sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#} \frac{m_\xi}{n_\xi} n_\xi\, \xi(A_s) \\ &= k \sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#} n_\xi\, \xi(A_s) \\ &= -k(n_s - \rho(A_s)), \end{aligned}$$

or equivalently,

$$n_s - \rho(A_s) = \frac{n_s}{k}. \tag{3.6.19}$$

Since $n_s > 0$, this together with equality (3.6.17) implies that $n_s - \rho(A_s)$ is greater than or equal to 1 and hence $n_s \geq k$. Using formulas (3.6.3) and (3.6.5), we obtain

$$\begin{aligned} n-1 = \sum_{s\in S^{\#}} n_s &\geq |S^{\#}|\, k \\ &= \sum_{\xi\in \mathrm{Irr}(\mathcal{X})^{\#}} n_\xi^2\, k \\ &= \sum_{\xi\in \mathrm{Irr}(\mathcal{X})^{\#}} n_\xi^2 \frac{m_\xi}{n_\xi} \\ &= \sum_{\xi\in \mathrm{Irr}(\mathcal{X})^{\#}} m_\xi n_\xi = n-1. \end{aligned}$$

Thus, $n_s = k$ for all $s \in S^{\#}$, and $\mathcal{X}$ is an equivalenced scheme of valency k. After replacing n_s with k in formula (3.6.19), we conclude that $\rho(A_s) = k-1$. Finally using formula (2.1.14), we have

$$\rho(A_s) = \sum_{t\in S} c_{ts}^t = \sum_{t\in S} \frac{n_s}{n_{t^*}} c_{t^*t}^{s^*} = \sum_{t\in S} c_{tt^*}^s = c(s).$$

It follows that $c(s) = k-1$ for all $s \in S^{\#}$. This completes the proof of the implication (1) $\Rightarrow$ (2).

Now assume that $n_s = k$ and $c(s) = k-1$ for all $s \in S^{\#}$. Then $c_{ts}^t = c_{t^*t}^{s^*}$, see formula (2.1.14). Therefore,

$$\rho(A_s) = c(s) = k-1.$$

According to Exercise 3.7.59, this implies that

$$\begin{aligned} n \sum_{\xi\in \mathrm{Irr}(\mathcal{X})} \frac{n_\xi}{m_\xi} P_\xi = \sum_{s\in S} \frac{\rho(A_s)}{n_s} A_s &\\ = |S| I + \frac{k-1}{k} \sum_{s\in S^{\#}} A_s &\\ = (|S| - \frac{k-1}{k}) I + \frac{k-1}{k} J,& \end{aligned}$$

where $I = I_\Omega$ and $J = J_\Omega$. Now, let $\xi \in \mathrm{Irr}(\mathcal{X})^{\#}$. After multiplying each side of the above equality by P_ξ with taking into account that $P_\xi P_\eta = \delta_{\xi,\eta} P_\xi$ for all $\eta \in \mathrm{Irr}(\mathcal{X})$, we obtain

$$n \frac{n_\xi}{m_\xi} P_\xi = (|S| - \frac{k-1}{k}) P_\xi.$$

It follows that

$$\frac{n_\xi}{m_\xi} = \frac{(|S|-1)k+1}{kn} = \frac{(n-1)+1}{kn} = \frac{1}{k}.$$

Thus, $\frac{m_\xi}{n_\xi} = k$, as required. □

3.6.2 Irreducible characters of a homogeneous component

In this subsection, we are interested in the relationship between the irreducible characters of a coherent configuration $\mathcal{X} = (\Omega, S)$ and those of a homogeneous component of it. In what follows $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$.

For any character $\xi \in \mathrm{Irr}(\mathcal{X})$ and any fiber $\Delta \in F(\mathcal{X})$, denote by ξ_Δ the restriction of ξ to the algebra $\mathcal{A}_\Delta = \mathrm{Adj}(\mathcal{X}_\Delta)$: more precisely, for $s \in S(\mathcal{X}_\Delta)$, we set

$$\xi_\Delta(A_s) = \xi(A_{s'}),$$

where s' is the same relation as s but considered as a relation on Ω. Since $\xi(I_\Omega) \neq 0$ and I_Ω is the sum of the matrices I_Δ, $\Delta \in F(\mathcal{X})$, the sets

$$\mathrm{Supp}_{\mathcal{X}}(\xi) = \{\Delta \in F(\mathcal{X}) : \ \xi_\Delta \neq 0\} \tag{3.6.20}$$

and

$$\mathrm{Irr}_\Delta(\mathcal{X}) = \{\xi \in \mathrm{Irr}(\mathcal{X}) : \ \xi_\Delta \neq 0\} \tag{3.6.21}$$

are not empty. If $\mathcal{X}$ is a scheme, then obviously $\mathrm{Supp}_{\mathcal{X}}(\xi) = \{\Omega\}$ and $\mathrm{Irr}_\Omega(\mathcal{X}) = \mathrm{Irr}(\mathcal{X})$.

Example 3.6.14. *Let $\mathcal{X}$ be a semiregular coherent configuration. For any fibers Δ, Γ of $\mathcal{X}$ and relations $s \in S_\Delta$ and $t \in S_{\Delta,\Gamma}$,*

$$r = t^* \cdot s \cdot t \in S_\Gamma.$$

It follows that the matrices A_s and A_r are conjugate via the adjacency matrix of the relation $t \cup t^* \cup 1_{\Omega \setminus (\Delta \cup \Gamma)}$. This implies that

$$\xi(A_r) = \xi(A_s), \quad \xi \in \mathrm{Irr}(\mathcal{X}).$$

Since the fibers Δ and Γ were arbitrary, we have

$$\mathrm{Supp}_{\mathcal{X}}(\xi) = F(\mathcal{X}) \quad \text{and} \quad \mathrm{Irr}_\Delta(\mathcal{X}) = \mathrm{Irr}(\mathcal{X})$$

for all ξ and Δ.

Example 3.6.15. *Let $\mathcal{X}$ be the direct sum of schemes. Then $\Delta \times \Gamma \in S$ for any distinct fibers Δ, Γ of $\mathcal{X}$. Furthermore,*

$$P_0 A_{\Delta \times \Gamma} = P_0 A_{\Delta \times \Gamma} P_0 = A_{\Delta \times \Gamma}.$$

Therefore for any character $\xi \in \mathrm{Irr}(\mathcal{X})^\#$ and any $\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)$, the set $I_\Delta \mathcal{A}_\xi I_\Delta$ is a nonzero ideal of the algebra $\mathcal{A}_\xi$. Since this algebra is simple, this implies that

$$|\,\mathrm{Supp}_{\mathcal{X}}(\xi)| = 1 \quad \text{and} \quad |\,\mathrm{Irr}_\Delta(\mathcal{X})| = |\,\mathrm{Irr}(\mathcal{X}_\Delta)|$$

for all ξ and Δ.

Let $\xi \in \mathrm{Irr}(\mathcal{X})$ and $\mathcal{M}_\xi$ an irreducible submodule of the module $\mathcal{L}_\xi$. Then for any nonzero $v \in \mathcal{M}_\xi$,

$$\mathcal{M}_\xi = \mathcal{A}v.$$

Now if $\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)$, then $I_\Delta v = v$, because I_Δ is the unit of the algebra $\mathcal{A}_\Delta$. It follows that

$$I_\Delta M_\xi = I_\Delta \mathcal{A} v = I_\Delta \mathcal{A} I_\Delta v = \mathcal{A}_\Delta v.$$

Consequently, $I_\Delta M_\xi$ is an irreducible $\mathcal{A}_\Delta$-module. This shows that

$$\xi_\Delta \in \mathrm{Irr}(\mathcal{X}_\Delta).$$

The decomposition of the identity matrix I_Ω into the sum of central primitive idempotents of $\mathcal{A}$ gives the corresponding decomposition of I_Δ in the algebra $\mathcal{A}_\Delta$,

$$I_\Delta = I_\Omega I_\Delta = \Big(\sum_{\xi \in \mathrm{Irr}(\mathcal{X})} P_\xi\Big) I_\Delta = \sum_{\xi \in \mathrm{Irr}(\mathcal{X}_\Delta)} P_\xi I_\Delta = \sum_{\xi \in \mathrm{Irr}(\mathcal{X}_\Delta)} P_{\xi_\Delta}.$$

Summarizing the above, we arrive at the following statement.

Theorem 3.6.16. *For any coherent configuration $\mathcal{X}$ and any fiber Δ of $\mathcal{X}$,*

(1) $\xi_\Delta \in \mathrm{Irr}(\mathcal{X}_\Delta)$ *for all* $\xi \in \mathrm{Irr}(\mathcal{X})$ *such that* $\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)$*;*
(2) *the mapping* $\mathrm{Irr}_\Delta(\mathcal{X}) \to \mathrm{Irr}(\mathcal{X}_\Delta),\ \xi \mapsto \xi_\Delta$ *is a bijection.*

The degrees and multiplicities of irreducible representations of a coherent configuration can be expressed via those of its homogeneous components as follows.

Corollary 3.6.17. *For any* $\xi \in \mathrm{Irr}(\mathcal{X})$*,*

$$n_\xi = \sum_{\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)} n_{\xi_\Delta} \tag{3.6.22}$$

and given $\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)$*,*

$$m_\xi = m_{\xi_\Delta}. \tag{3.6.23}$$

Proof. Since $\xi(I_\Delta) = \xi_\Delta(I_\Delta)$ for all $\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)$, we have

$$\begin{aligned} n_\xi = \xi(I_\Omega) = \xi\Big(\sum_{\Delta \in F} I_\Delta\Big) &= \sum_{\Delta \in F} \xi(I_\Delta) \\ &= \sum_{\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)} \xi_\Delta(I_\Delta) \\ &= \sum_{\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)} n_{\xi_\Delta}, \end{aligned}$$

where $F = F(\mathcal{X})$. This proves equality (3.6.22). Now let us consider a decomposition of $\mathcal{L}_\xi$ into the direct sum of irreducible $\mathcal{A}$-submodules,

$$\mathcal{L}_\xi = \sum_{i=1}^{m_\xi} \mathcal{M}_i.$$

Then for any $\Delta \in F$,

$$P_{\xi_\Delta}\mathcal{L}_\Delta = P_\xi I_\Delta \mathcal{L}_\xi = I_\Delta P_\xi \mathcal{L}_\xi = \sum_{i=1}^{m_\xi} I_\Delta \mathcal{M}_i.$$

If $\Delta \in \mathrm{Supp}_{\mathcal{X}}(\xi)$, then, as we saw before, $I_\Delta \mathcal{M}_i$ is an irreducible $\mathcal{A}_\Delta$-module. This proves equality (3.6.23). □

We complete the subsection by a useful application of the representation theory technique that was proved in [**123**, p. 86].

Corollary 3.6.18. *Let $\mathcal{X}$ be a coherent configuration, $S = S(\mathcal{X})$, and Δ and Γ are fibers of $\mathcal{X}$. Then*

$$|S_{\Delta,\Gamma}| \le \frac{|S_\Delta| + |S_\Gamma|}{2}.$$

In particular, if $\mathcal{X}_\Delta$ and $\mathcal{X}_\Gamma$ are trivial, then $|S_{\Delta,\Gamma}| \le 2$.

Proof. Without loss of generality, we may assume that Δ and Γ are the only fibers of $\mathcal{X}$. Denote by $\mathcal{A}_{\Delta,\Gamma}$ and $\mathcal{A}_{\Gamma,\Delta}$ the linear spaces $I_\Delta \mathcal{A} I_\Gamma$ and $I_\Gamma \mathcal{A} I_\Delta$, respectively, where $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$. Then

$$|S_{\Delta,\Gamma}| = \dim(\mathcal{A}_{\Delta,\Gamma}) \quad \text{and} \quad |S_{\Gamma,\Delta}| = \dim(\mathcal{A}_{\Gamma,\Delta}). \tag{3.6.24}$$

As a linear space the algebra $\mathcal{A}$ is the direct sum of the spaces $\mathcal{A}_\Delta$, $\mathcal{A}_\Gamma$, $\mathcal{A}_{\Delta,\Gamma}$, and $\mathcal{A}_{\Gamma,\Delta}$. Therefore for any character $\xi \in \mathrm{Irr}(\mathcal{X})$, the sum

$$\mathcal{A}_\xi = P_\xi \mathcal{A}_\Delta + P_\xi \mathcal{A}_\Gamma + P_\xi \mathcal{A}' \tag{3.6.25}$$

is also direct, where

$$\mathcal{A}' = \mathcal{A}_{\Delta,\Gamma} + \mathcal{A}_{\Gamma,\Delta}.$$

It is easily seen that $P_\xi \mathcal{A}' = 0$ unless both Δ and Γ belong to $\mathrm{Supp}_{\mathcal{X}}(\xi)$. In the latter case, equality (3.6.22) implies that

$$\dim(\mathcal{A}_\xi) = n_\xi^2 = (n_{\xi_\Delta} + n_{\xi_\Gamma})^2.$$

On the other hand, taking the dimensions of the summands on the right-hand side of decomposition (3.6.25), we obtain

$$\dim(\mathcal{A}_\xi) = n_{\xi_\Delta}^2 + n_{\xi_\Gamma}^2 + \dim(P_\xi \mathcal{A}').$$

Thus,

$$\dim(P_\xi \mathcal{A}') = 2 n_{\xi_\Delta} n_{\xi_\Gamma} \le n_{\xi_\Delta}^2 + n_{\xi_\Gamma}^2,$$

where we set n_{ξ_Δ} (respectively, n_{ξ_Γ}) to be zero if $\Delta \notin \mathrm{Supp}_{\mathcal{X}}(\xi)$ (respectively, $\Gamma \notin \mathrm{Supp}_{\mathcal{X}}(\xi)$). Taking into account that $|S_{\Delta,\Gamma}| = |S_{\Gamma,\Delta}|$ and using formulas (3.6.5) and (3.6.24), we have

$$\begin{aligned}
2|S_{\Delta,\Gamma}| &= |S_{\Delta,\Gamma}| + |S_{\Gamma,\Delta}| \\
&= \dim(\mathcal{A}_{\Delta,\Gamma}) + \dim(\mathcal{A}_{\Gamma,\Delta}) \\
&= \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} \dim(P_\xi \mathcal{A}') \\
&\le \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} (n^2_{\xi_\Delta} + n^2_{\xi_\Gamma}) \\
&\le \sum_{\xi'\in\mathrm{Irr}(\mathcal{X}_\Delta)} n^2_{\xi'} + \sum_{\xi'\in\mathrm{Irr}(\mathcal{X}_\Gamma)} n^2_{\xi'} \\
&= |S_\Delta| + |S_\Gamma|,
\end{aligned}$$

which completes the proof. □

One can prove that the arithmetic mean in Corollary 3.6.18 can be replaced by the geometric mean, see [**100**].

3.6.3 The orthogonality relations and the Frame number

One of the basic statements in the representation theory of coherent configurations is given by the following theorem which is an analog of the orthogonality relations for the irreducible characters of groups. In view of the results of the previous subsection and to simplify the formulas, we restrict ourselves to the homogeneous case only.

Theorem 3.6.19. *For any scheme $\mathcal{X}$ of degree n,*

$$\sum_{r,s\in S} \frac{c_{r^*t}^{s}}{n_{r^*}} \eta(A_{r^*})\xi(A_{s^*}) = \delta_{\eta,\xi} \frac{n\,\eta(A_{t^*})}{m_\xi}, \qquad t \in S(\mathcal{X}), \quad \eta, \xi \in \operatorname{Irr}(\mathcal{X}).$$

In particular,

$$\sum_{s\in S} \frac{1}{n_s} \eta(A_s)\xi(A_{s^*}) = \delta_{\eta,\xi} \frac{n\,n_\xi}{m_\xi}. \tag{3.6.26}$$

Proof. The second statement follows from the first one for $t = 1_\Omega$. To prove the first one, let P_η and P_ξ be the central primitive idempotents corresponding to the irreducible characters η and ξ, respectively. Using formula (3.6.13), we obtain

$$\begin{aligned} n^2 P_\eta\, P_\xi &= \left(\sum_{r\in S} \frac{m_\eta\, \eta(A_{r^*})}{n_{r^*}} A_r\right) \left(\sum_{s\in S} \frac{m_\xi\, \xi(A_{s^*})}{n_{s^*}} A_s\right) \\ &= \sum_{r,s\in S} \frac{m_\eta\, m_\xi\, \eta(A_{r^*})\, \xi(A_{s^*})}{n_{r^*}\, n_{s^*}} A_r\, A_s \\ &= \sum_{r,s\in S} \frac{m_\eta\, m_\xi\, \eta(A_{r^*})\, \xi(A_{s^*})}{n_{r^*}\, n_{s^*}} \left(\sum_{t\in S} c_{rs}^{t} A_t\right) \\ &= \sum_{t\in S} \left(\sum_{r,s\in S} \frac{m_\eta\, m_\xi\, \eta(A_{r^*})\, \xi(A_{s^*})}{n_{r^*}\, n_{s^*}} c_{rs}^{t}\right) A_t. \end{aligned}$$

On the other hand, by the orthogonality of P_η and P_ξ we have

$$P_\eta P_\xi = \delta_{\eta,\xi} P_\eta = \delta_{\eta,\xi} \frac{m_\eta}{n} \sum_{t\in S} \frac{\eta(A_{t^*})}{n_{t^*}} A_t.$$

Thus equating the coefficients at A_t in the two above expressions and taking into account that $c_{rs}^{t} = \frac{n_{s^*}}{n_t} c_{t^*r}^{s^*} = \frac{n_{s^*}}{n_t} c_{r^*t}^{s}$ (see formulas (3.6.13) and (2.1.3)), we have

$$n^2 \delta_{\eta,\xi} \frac{m_\eta}{n} \frac{\eta(A_{t^*})}{n_{t^*}} = \sum_{r,s\in S} \frac{m_\eta\, m_\xi\, \eta(A_{r^*})\, \xi(A_{s^*})}{n_{r^*}\, n_{s^*}} \frac{n_{s^*}}{n_t} c_{r^*t}^{s}$$

which proves the required equality. □

An analog of formula (3.6.26) for the non-homogeneous case was proved in [**66**, p. 220]. Namely, let $\mathcal{X}$ be an arbitrary coherent configuration. For each character $\xi \in \mathrm{Irr}(\mathcal{X})$, fix an irreducible representation ρ_ξ of the algebra $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ corresponding to ξ.

Given a triple $\lambda = (\xi, i, j)$ with $1 \le i, j \le n_\xi$, we define a linear functional

$$\rho_\lambda : \mathcal{A} \to \mathbb{C},\ A \mapsto \rho_\xi(A)_{ij}, \tag{3.6.27}$$

and set $\lambda' = (\xi, j, i)$. The basis of the underlying linear space $\mathcal{L}_\Omega$ can be chosen so that for any two triples λ and μ,

$$\sum_{s\in S} \frac{1}{n_s} \rho_\lambda(A_{s^*})\rho_\mu(A_s) = \frac{n}{m_\xi}\delta_{\lambda,\mu'}. \tag{3.6.28}$$

An important property of a scheme $\mathcal{X}$ that connects, on the one hand, the valencies of its basis relations, and, on the other hand, the degree and multiplicity of its irreducible characters, can be expressed via the *Frame number* of $\mathcal{X}$ defined as follows:

$$\mathrm{Fr}(\mathcal{X}) = n^{|S(\mathcal{X})|} \frac{\prod\limits_{s\in S(\mathcal{X})} n_s}{\prod\limits_{\xi\in\mathrm{Irr}(\mathcal{X})} m_\xi^{n_\xi^2}}.$$

The proof of the theorem below is taken from [**2**], where the corresponding statement was proved under a weaker assumption.

Theorem 3.6.20. *The Frame number of a scheme is a rational integer.*

Proof. Let $\mathcal{X}$ be a scheme and $S = S(\mathcal{X})$. Denote by Λ the set of all triples

$$\lambda = (\xi, i, j), \qquad \xi \in \mathrm{Irr}(\mathcal{X}),\ 1 \le i, j \le n_\xi.$$

This set is closed under the transposition $\lambda \mapsto \lambda'$, where λ' is as above.

Denote by B and C_Λ the $\Lambda \times S$- and $\Lambda \times \Lambda$-matrices defined by

$$B_{\lambda,s} = \rho_\lambda(A_s) \quad \text{and} \quad (C_\Lambda)_{\lambda,\mu} = \delta_{\lambda,\mu'},$$

where ρ_λ is the linear functional (3.6.27).

A straightforward computation shows that for all $r, s \in S$,

$$(B^T C_\Lambda B)_{r,s} = \sum_{\lambda\in\Lambda} \rho_\lambda(A_r)\rho_{\lambda'}(A_s) = \sum_{\xi\in\mathrm{Irr}(\mathcal{X})} \xi(A_r A_s).$$

The number on the right-hand side of this equality is an algebraic integer, because so are the values of an irreducible character and the intersection numbers. It follows that the entries of the matrix $B^T C_\Lambda B$ are algebraic integers. This immediately implies that

$$\det(B)^2 = \pm\det(B^T C_\Lambda B)$$

is an algebraic integer; here, we use the fact that C_Λ is a permutation matrix.

Denote by C_S, D_Λ, and D_S, respectively, the $S\times S$-, $\Lambda\times\Lambda$-, and $S\times S$-matrices defined by

$$(C_S)_{r,s}=\delta_{r,s^*},\qquad (D_\Lambda)_{\lambda,\mu}=\delta_{\lambda,\mu}\frac{m_\xi}{n},\qquad (D_S)_{r,s}=\delta_{r,s}n_r.$$

In this notation, formula (3.6.28) exactly means that

$$B\,C_S\,(D_S)^{-1}\,B^T=(D_\Lambda)^{-1}\,C_\Lambda.$$

Passing to the determinants on the right- and left-hand sides, we obtain

$$\det(B)^2\det(C_S)\left(\prod_{s\in S}n_s\right)^{-1}=\det(C_\Lambda)\prod_{\xi\in\mathrm{Irr}(\mathcal{X})}\left(\frac{n}{m_\xi}\right)^{n_\xi^2}.$$

Note that C_Λ is a permutation matrix and hence $\det(C_\Lambda)=\pm1$. Thus using formula (3.6.5), we conclude that

$$\mathrm{Fr}(\mathcal{X})=\pm\det(B)^2.$$

By the first part of the proof, this implies that the rational number $\mathrm{Fr}(\mathcal{X})$ is an algebraic integer. Thus it is a rational integer, as required. □

It is worth mentioning a result in [**54**] showing that if one defines the algebra $\mathrm{Adj}(\mathcal{A})$ of a scheme $\mathcal{X}$ over a field of positive characteristic p rather than over $\mathbb{C}$, then $\mathcal{A}$ is semisimple if and only if p does not divide the Frame number of $\mathcal{X}$.

One more application of the Frame number found in [**102**] gives the following sufficient condition for the commutativity of the schemes from Theorem 3.6.13.

Theorem 3.6.21. *Let $\mathcal{X}$ be an scheme. Suppose that the numbers*

$$k=\frac{m_\xi}{n_\xi}\quad\text{and}\quad a=n_\xi$$

do not depend on the choice of the character $\xi\in\mathrm{Irr}(\mathcal{X})^\#$. Then $a=1$, i.e., $\mathcal{X}$ is commutative.

Proof. By Theorem 3.6.13, the assumption implies that $\mathcal{X}$ is an equivalenced scheme of valency k. In accordance with formulas (3.6.3) and (3.6.9), we have

$$n-1=\sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#}m_\xi n_\xi=\sum_{\xi\in\mathrm{Irr}(\mathcal{X})^\#}\frac{m_\xi}{n_\xi}n_\xi^2=|\,\mathrm{Irr}(\mathcal{X})^\#|\,k\,a^2.$$

It follows that a is coprime to n and

$$|\,\mathrm{Irr}(\mathcal{X})^\#|\,a^2=\frac{n-1}{k}=d-1,$$

where $d=|S(\mathcal{X})|$.

On the other hand, since $(m_\xi)^{n_\xi^2} = (ka)^{a^2}$ for all $\xi \in \mathrm{Irr}(\mathcal{X})^\#$, the Frame number of the scheme $\mathcal{X}$ is equal to

$$\mathrm{Fr}(\mathcal{X}) = \frac{n^d\, k^{d-1}}{(k\,a)^{|\,\mathrm{Irr}(\mathcal{X})^\#|\,a^2}} = \frac{n^d\, k^{d-1}}{k^{d-1}a^{d-1}} = \frac{n^d}{a^{d-1}}.$$

By Theorem 3.6.20, this number is an integer. Taking into account that a is coprime to n, we conclude that $a = 1$, as required. □

3.6.4 The base number of a primitive scheme

In this subsection we establish an upper bound for the base number of a primitive scheme in terms of the multiplicities and degrees of irreducible characters; for permutation groups a similar bound was found in [**112**].

The theorem below was proved in [**39**] and can be considered as a far-reaching generalization of the trivial observation that the base number of a regular scheme $\mathcal{X}$ is equal to $1 = \frac{m_\xi}{n_\xi}$, where $\xi \in \mathrm{Irr}(\mathcal{X})$ (Theorem 3.6.3).

Theorem 3.6.22. *Let $\mathcal{X}$ be a primitive scheme. Then*

$$b(\mathcal{X}) \le \min_{\xi \in \mathrm{Irr}(\mathcal{X})^\#} \frac{m_\xi}{n_\xi}.$$

Moreover, the same upper bound holds for the size of each irredundant base of $\mathcal{X}$.

The estimate in Theorem 3.6.22 is sharp: the equality is attained if, for example, $\mathcal{X}$ is a trivial scheme of degree at least 2. Indeed, in this case

$$\frac{m_\xi}{n_\xi} = n - 1 = b(\mathcal{X}),$$

where n is the degree of $\mathcal{X}$ and ξ is the unique nonprincipal irreducible character of $\mathcal{X}$, see Example 3.6.8.

The primitivity assumption in Theorem 3.6.22 is essential. Indeed, let $\mathcal{X}$ be the imprimitive scheme of rank 6 from Exercise 2.7.43 for $q = 2$. One can verify that $\mathcal{X}$ has an irredundant base of size 4, whereas $\frac{m_\xi}{n_\xi} = 3$ for its irreducible character ξ of degree 2 and multiplicity 6.

Proof of Theorem 3.6.22. We need the following lemma.

Lemma 3.6.23. *Let $\mathcal{X}$ be a coherent configuration on Ω, $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$, and $A \in \mathcal{A}$. Suppose that the nonzero columns of the matrix A are pairwise distinct. Then given $m \ge 1$ and $\alpha, \alpha_1, \dots, \alpha_m \in \Omega$,*

$$A\underline{\alpha} \in \sum_{i=1}^{m} \mathcal{A}\underline{\alpha_i} \quad \Rightarrow \quad \{\alpha\} \in F(\mathcal{X}_{\alpha_1,\dots,\alpha_m}).$$

Proof. Denote by B the matrix in Mat_Ω with the βth column, $\beta \in \Omega$, defined by

$$B\underline{\beta} = \begin{cases} A\underline{\alpha}, & \text{if } \beta \in \{\alpha_1, \dots, \alpha_m\}, \\ A\underline{\beta}, & \text{otherwise.} \end{cases}$$

Claim 1. *B belongs to the algebra $\mathcal{A}' = \mathrm{Adj}(\mathcal{X}')$ with $\mathcal{X}' = \mathcal{X}_{\alpha_1,\dots,\alpha_m}$.*

Proof. By the assumption there exist matrices $A_1, \dots, A_m$ belonging to $\mathcal{A} \subset \mathcal{A}'$ and such that

$$A\underline{\alpha} = A_1\underline{\alpha_1} + \cdots + A_m\underline{\alpha_m}. \qquad (3.6.29)$$

Since $\mathcal{A}'$ also contains the matrix of any permutation of Ω leaving every point of the set $\Omega' = \Omega \setminus \{\alpha_1, \ldots, \alpha_m\}$ fixed, the matrices

$$B_i = \sum_{j=1}^{m} P_{ij}^{-1} A_j I_{\{\alpha_j\}} P_{ij}, \quad i = 1, \ldots, m,$$

belong to $\mathcal{A}'$, where P_{ij} is the matrix of the transposition (α_i, α_j). Note that in view of (3.6.29), the α_jth column of B_i is equal to $A\underline{\alpha}$ if $i = j$, and 0 otherwise. Consequently,

$$B = B_1 + \cdots + B_m + AI_{\Omega'},$$

which proves the Claim 1. □

Denote by $e = e(B)$ the partial equivalence relation defined by formula (2.7.1). Then $e \in E(\mathcal{X}')$ by Claim 1 and Exercise 2.7.24. Since the nonzero columns of the matrix A are pairwise distinct,

$$\Delta := \{\alpha_1, \ldots, \alpha_m, \alpha\} \in \Omega/e.$$

However, $\{\alpha_i\}$ is a fiber of $\mathcal{X}'$ for all i. Thus, $\Delta = \alpha_i e$ is a homogeneity set of $\mathcal{X}'$. Consequently,

$$\{\alpha\} = \Delta \setminus \{\alpha_1, \ldots, \alpha_m\}$$

is a fiber of $\mathcal{X}'$. This completes the proof of Lemma 3.6.23. □

Let $\{\alpha_1, \ldots, \alpha_b\}$ be an irredundant base of $\mathcal{X}$ and $\xi \in \mathrm{Irr}(\mathcal{X})^{\#}$.

Lemma 3.6.24. *For $m = 1, \ldots, b$, the sum*

$$\mathcal{L}_m = \sum_{i=1}^{m} \mathcal{A}_\xi \underline{\alpha_i} \tag{3.6.30}$$

is direct.

Proof. It suffices to verify that $\mathcal{A}_\xi \underline{\alpha_m} \cap \mathcal{L}_{m-1} = \{0\}$ for each m. Assume on the contrary that there exists a nonzero matrix $A \in \mathcal{A}_\xi$ such that

$$A\underline{\alpha_m} \in \mathcal{L}_{m-1}.$$

The singleton $\{\alpha_m\}$ is not a fiber of the coherent configuration $\mathcal{X}_{\alpha_1,\ldots,\alpha_{m-1}}$: this is true for $m = 1$, because $\mathcal{X}$ is a primitive scheme of degree at least 2, and also for $m > 1$, because the base $\{\alpha_1, \ldots, \alpha_b\}$ is irredundant.

By Lemma 3.6.23, this implies that the nonzero columns of the matrix A are not pairwise distinct. It follows that

$$e(A) \neq 1_\Omega,$$

where $e(A)$ is the partial equivalence relation defined by formula (2.7.1). On the other hand, $e(A) \in E(\mathcal{X})$ by Exercise 2.7.24. Thus,

$$e(A) = \Omega^2$$

by the primitivity of $\mathcal{X}$. This shows that the matrix A is a multiple of J_Ω. Consequently, A belongs the intersection $\mathcal{A}_\xi \cap \mathcal{A}_{\xi_0} = 0$, a contradiction. $\square$

To complete the proof, we note that the mapping $\mathcal{A} \to \mathcal{A}\underline{\alpha_i}$, $A \mapsto A\underline{\alpha_i}$ is an $\mathcal{A}$-module isomorphism, $i = 1, \ldots, b$. Therefore,

$$\dim(\mathcal{A}_\xi \underline{\alpha_i}) = \dim(\mathcal{A}_\xi) = n_\xi^2.$$

On the other hand, $\mathcal{L}_b \subseteq \mathcal{L}_\xi$ and $\dim(\mathcal{L}_\xi) = m_\xi n_\xi$. Thus the direct decomposition (3.6.30) from Lemma 3.6.24 with $m = b$ yields the inequality

$$bn_\xi^2 = \sum_{i=1}^{b} \dim(\mathcal{A}_\xi \underline{\alpha_i}) = \dim(\mathcal{L}_b) \le \dim(\mathcal{L}_\xi) = m_\xi n_\xi,$$

i.e., $b \le \frac{m_\xi}{n_\xi}$ as required. $\square$

Let $\mathcal{X}$ be a primitive scheme of rank d and base number b. From formulas (3.6.5) and (3.6.3), and Theorems 3.6.3 and 3.6.22, it immediately follows that

$$\begin{aligned} b\,(d-1) &= \sum_{\xi \in \mathrm{Irr}(\mathcal{X})^\#} b\, n_\xi^2 \\ &\le \sum_{\xi \in \mathrm{Irr}(\mathcal{X})^\#} \frac{m_\xi}{n_\xi}\, n_\xi^2 \\ &= \sum_{\xi \in \mathrm{Irr}(\mathcal{X})^\#} m_\xi\, n_\xi \\ &= n-1. \end{aligned}$$

The number $\frac{n-1}{d-1}$ equals the average valency of the scheme $\mathcal{X}$. Thus we arrive at the following consequence of Theorem 3.6.22.

Corollary 3.6.25. *The base number of a primitive coherent configuration is less than or equal to its average valency.*

It should be noted that if the Babai conjecture 3.1.7 was true, then the base number of a primitive scheme would be bounded from above by a function on n_{min}. At present such a result is known in few cases, e.g., the base number of a schurian primitive antisymmetric scheme is at most 3 [**109**]. In the non-schurian case, no nontrivial upper bound is known.

3.7 Exercises

In what follows, unless otherwise stated, $\mathcal{X}$ is a coherent configuration on Ω and $S = S(\mathcal{X})$, $F = F(\mathcal{X})$, and $E = E(\mathcal{X})$. The notations $\mathcal{X}'$ and Ω', S', F', and E' have the same meaning. The number m denotes a positive integer and $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$, $\overline{\mathcal{X}} = \overline{\mathcal{X}}^{(m)}$, etc.

3.7.1 Let $\mathcal{X}$ be a fusion of an affine scheme of degree q^2. Then

(1) for each $s \in S^\#$, $n_s = a_s(q-1)$ for some integer $a_s \geq 1$;
(2) $\mathcal{X}$ is primitive if and only if $a_s \geq 2$ for all $s \in S^\#$.

3.7.2 A coherent configuration of a disconnected graph is either nonhomogeneous or imprimitive.

3.7.3 Let $\mathcal{X}$ be a primitive nonregular scheme. Then given $s \in S^\#$, there exists a positive integer m such that $s^m = S$, where

$$s^m = \underbrace{s\,s\,\cdots\,s}_{m} \qquad \text{(complex product).}$$

3.7.4 Let $\mathcal{X}$ and $\mathcal{X}'$ be algebraically isomorphic coherent configurations. Then $\mathcal{X}$ is primitive (respectively, imprimitive) if and only if $\mathcal{X}'$ is primitive (respectively, imprimitive).

3.7.5 [**17**, Theorem 4.2.1] Let $\mathcal{X}$ be the scheme of a distance-regular graph of diameter d and valency at least 3. Then $\mathcal{X}$ is imprimitive only if s_1 is a bipartite graph or s_d is the disjoint union of cliques (here, s_1 and s_d are defined by formula (2.6.6)).

3.7.6 Let e be a parabolic of $\mathcal{X}$ with indecomposable components e_i, $i \in I$, and π_e the mapping (1.1.4). Then

$$F(\mathcal{X}_{\Omega/e}) = \{\Omega(\pi_e(e_i)) :\ i \in I\}.$$

In particular, $\mathcal{X}_{\Omega/e}$ is homogeneous if and only if e is indecomposable.

3.7.7 Let $e \in E$ be such that $\Omega/e = F$. Then $\mathcal{X}_{\Omega/e} = \mathcal{D}_F$.

3.7.8 Let $\mathcal{X} \leq \mathcal{X}'$ and $e \in E$. Then $\mathcal{X}_{\Omega/e} \leq \mathcal{X}'_{\Omega/e}$.

3.7.9 Let $e_0, e_1 \in E$ be such that $e_0 \subseteq e_1$. Then

(1) the quotient of $\mathcal{X}_{\Omega/e_0}$ modulo $\pi_{e_0}(e_1)$ is canonically isomorphic to the quotient $\mathcal{X}_{\Omega/e_1}$;
(2) for any $\Delta \in \Omega/e_1$, the quotient of $\mathcal{X}_\Delta$ modulo $(e_0)_\Delta$ is canonically isomorphic to the restriction of $\mathcal{X}_{\Omega/e_0}$ to $\pi_{e_0}(\Delta)$.

3.7.10 Let $\mathcal{X}$ be a semiregular coherent configuration, and let e be the union of all relations in a system of distinct representative of $\{S_{\Delta,\Gamma}\}_{\Delta,\Gamma\in F}$ given in statement (3) of Exercise 2.7.13. Then

(1) e is an indecomposable parabolic of $\mathcal{X}$;
(2) given $\Delta \in F$ and $\Gamma \in \Omega/e$, we have $\Delta \cap \Gamma = \{\alpha_{\Delta,\Gamma}\}$ for some $\alpha_{\Delta,\Gamma}$;
(3) for any $\Delta \in F$, the mapping $f : \Omega/e \to \Delta$, $\Gamma \mapsto \alpha_{\Delta,\Gamma}$ is a bijection;
(4) $f \in \mathrm{Iso}(\mathcal{X}_{\Omega/e}, \mathcal{X}_\Delta)$.

3.7.11 A scheme is schurian if and only if it is isomorphic to the quotient of a regular scheme.

3.7.12 Let $\Delta \subseteq \Omega$. Then

(1) $\mathrm{WL}(\mathrm{Inv}(K), 1_\Delta) \le \mathrm{Inv}(K_{\{\Delta\}})$ for any $K \le \mathrm{Sym}(\Omega)$;
(2) $\mathrm{Aut}(\mathrm{WL}(\mathcal{X}, 1_\Delta)) = \mathrm{Aut}(\mathcal{X})_{\{\Delta\}}$.

3.7.13 Let S be a set of relations on Ω, and let e be an equivalence relation on Ω. Then $\mathrm{WL}(S_{\Omega/e}) \le \mathrm{WL}(S)_{\Omega/e}$.

3.7.14 Let e be a residually thin parabolic of $\mathcal{X}$. Then

(1) $s \cdot s^* \subseteq e$ for any $s \in S$;
(2) $\mathcal{X}_e = \mathrm{WL}(\mathcal{X}, 1_\Delta)$ for any $\Delta \in \Omega/e$.

3.7.15 Let $e \in E$ and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. Then e is residually thin if and only if $e' = \varphi(e)$ is residually thin.

3.7.16 The thin residue parabolic of a scheme $\mathcal{X}$ is equal to the minimal parabolic of $\mathcal{X}$ containing $s \cdot s^*$ for any $s \in S$.

3.7.17 [110] Let p be a prime. A scheme $\mathcal{X}$ is called a *p-scheme* if $|s|$ is a p-power for each $s \in S$. For such a scheme,

(1) $|\Omega|$ is a p-power;
(2) the thin radical parabolic of $\mathcal{X}$ is not equal to 1_Ω unless $|\Omega| = 1$;
(3) if $|\Omega| = p$, then $\mathcal{X}$ is regular;
(4) any quotient of $\mathcal{X}$ is a p-scheme;
(5) the thin residue parabolic of $\mathcal{X}$ is not equal to Ω^2 unless $|\Omega| = 1$.

3.7.18 [72] Any quasiregular coherent configuration $\mathcal{X}$ with all non-singleton fibers of the same prime cardinality is the direct sum of semiregular coherent configurations. In particular, $\mathcal{X}$ is schurian and separable.

3.7.19 [105] A coherent configuration $\mathcal{X}$ is said to be *quasitrivial* if

$$\mathrm{Aut}(\mathcal{X})^\Delta = \mathrm{Sym}(\Delta) \quad \text{for all } \Delta \in F,$$

and *semitrivial* if, in addition, the group $\mathrm{Aut}(\mathcal{X})^{\Delta \cup \Gamma}$ is isomorphic to both $\mathrm{Sym}(\Delta)$ and $\mathrm{Sym}(\Gamma)$ for all $\Delta, \Gamma \in F$. Prove that every quasitrivial coherent configuration is the direct sum of semitrivial coherent configurations.

3.7.20 A coherent configuration with all fibers of cardinality at most 3 is the direct sum of the coherent configurations isomorphic to $\mathcal{Y} \otimes \mathcal{D}_{m_\mathcal{Y}}$, where $\mathcal{Y}$ is a scheme of degree at most 3 and $m_\mathcal{Y} \ge 1$. In particular, $\mathcal{X}$ is schurian and separable.

3.7.21 [43, Theorem 2.2] Let $\mathcal{X}$ be a commutative subtensor product on $\Omega = \Omega_1 \times \Omega_2$, and let e_1 and e_2 be the parabolics of $\mathcal{X}$ defined by formula (3.2.5). Then

(1) for each $\Delta \in \Omega/e_1$, the mapping $\tau_\Delta : \Delta \to \Omega/e_2$, $\alpha \mapsto \alpha e_2$ is a bijection;
(2) $\tau_\Delta \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_{\Omega/e_2})$ and also $(s_\Delta)^{\tau_\Delta} = s_{\Omega/e_2}$ for all $s \in S$;
(3) if $\Gamma \in \Omega/e_1$, then $\tau_\Delta \tau_\Gamma^{-1} \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_\Gamma, \varphi_{\Delta,\Gamma})$ (for $\varphi_{\Delta,\Gamma}$, see Example 2.3.16).

3.7.22 Let X be a Cayley scheme over a group G. Then the following two statements are equivalent:

(1) $\mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k$ for some $k \geq 1$;
(2) $G = G_1 \times \cdots \times G_k$, where G_i is an $\mathcal{X}$-group such that $\mathcal{X}_{G_i} = \mathcal{X}_i$, and $\mathrm{rk}(\mathcal{X}) = \mathrm{rk}(\mathcal{X}_1) \cdots \mathrm{rk}(\mathcal{X}_k)$.

Moreover, if one of these statements holds, then $\mathcal{X}$ is normal if and only if $\mathcal{X}_i$ is a normal Cayley scheme over G_i for all i.

3.7.23 The extension of trivial coherent configuration $\mathcal{T}_\Omega$ with respect to the points of a set $\Delta \subseteq \Omega$, is equal to $\mathcal{D}_\Delta \boxplus \mathcal{T}_{\Omega \setminus \Delta}$.

3.7.24 Let $\mathcal{X}$ be a scheme such that the coherent configuration $\mathcal{X}_\alpha$ is schurian and separable for some $\alpha \in \Omega$. Assume that for every $\alpha' \in \Omega'$ and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, there exists algebraic isomorphism

$$\varphi_{\alpha,\alpha'} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'})$$

extending φ. Then $\mathcal{X}$ is separable, and schurian if $F(\mathcal{X}_\alpha) = \{\alpha s : \ s \in S\}$.

3.7.25 Let $\alpha \in \Omega$ and

$$T_\alpha = \{r_{u,v} : \ r \in S, \ u, v \in S \setminus S_1\}^\natural,$$

where $r_{u,v} = r \cap (\alpha u \times \alpha v)$. Then the pair

$$\mathcal{X}_\alpha^\perp = (\alpha S_{1'}, T_\alpha)$$

with $S_{1'} = \{s \in S : \ n_s > 1\}$, is a rainbow and

$$\mathcal{X}_\alpha = \mathcal{D}_{\alpha S_1} \boxplus \mathrm{WL}(\mathcal{X}_\alpha^\perp).$$

3.7.26 [71, Theorem 3.1] Any primitive scheme admitting a one-point extension with exactly one non-singleton fiber, is trivial.

3.7.27 Let $\alpha \in \Omega$ and Δ a base of $\mathcal{X}_\alpha$. Then $\{\alpha\} \cup \Delta$ is a base of $\mathcal{X}$. In particular,

$$b(\mathcal{X}) \leq 1 + b(\mathcal{X}_\alpha)$$

with equality if α belongs to a base of cardinality $b(\mathcal{X})$.

3.7.28 The class of all partly regular coherent configurations is closed under taking fissions and tensor products.

3.7.29 Let Ω_1 and Ω_2 be sets. Then the only proper fusion of the wreath product $\mathcal{T}_{\Omega_1} \wr \mathcal{T}_{\Omega_2}$ is the trivial scheme $\mathcal{T}_{\Omega_1 \times \Omega_2}$.

3.7.30 Let $\mathcal{X}$ be a scheme and $\mathcal{Y} = \mathrm{Inv}(K, \Delta)$, where Δ is a set and $K \leq \mathrm{Sym}(\Delta)$ is a transitive group. Then K acts as a group of isomorphisms of the direct sum $\mathcal{X}'$ of $|\Delta|$ copies of $\mathcal{X}$, and $\mathcal{X} \wr \mathcal{Y} \cong (\mathcal{X}')^K$.

3.7.31 Let $\mathcal{X}_1 = (\Omega_1, S_1)$ and $\mathcal{X}_2 = (\Omega_2, S_2)$ be schemes and Φ a family of the algebraic isomorphisms

$$\varphi_\alpha \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}_{1\alpha}), \quad \alpha \in \Omega_2,$$

where $\mathcal{X}_{1\alpha}$ is a scheme on the set $\Omega_\alpha = \Omega_1 \times \{\alpha\}$. Define a rainbow $\mathcal{X}$ on the set $\Omega = \Omega_1 \times \Omega_2$ with $S(\mathcal{X}) = S^{(1)} \cup S^{(2)}$, where

$$S^{(1)} = \{ \bigcup_{\alpha \in \Omega_2} \varphi_\alpha(s_1) : \; s_1 \in S_1\} \quad \text{and} \quad S^{(2)} = \{ \bigcup_{(\alpha,\beta) \in s_2} \Omega_\alpha \times \Omega_\beta : \; s_2 \in S_2^{\#}\}.$$

Then $\mathcal{X}$ is a scheme, called the *wreath product of* $\mathcal{X}_1$ *by* $\mathcal{X}_2$ *with respect to the family* Φ; it is denoted by $\mathcal{X}_1 \wr_\Phi \mathcal{X}_2$. Moreover,

(1) the equivalence relation e with classes Ω_α, $\alpha \in \Omega_2$, is an indecomposable parabolic of $\mathcal{X}$;
(2) if for each α, the algebraic isomorphism φ_α is induced by the bijection $\beta \mapsto (\beta, \alpha)$, $\beta \in \Omega_1$, then $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$;
(3) $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$ is isomorphic to $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1) \times \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_2)$.

3.7.32 Let $\mathcal{X}$ be a scheme on $\Omega_1 \times \Omega_2$, and let e be the equivalence relation with classes $\Omega_\alpha = \Omega_1 \times \{\alpha\}$, $\alpha \in \Omega_2$. Assume that e is an indecomposable parabolic of $\mathcal{X}$. Take an arbitrary $\alpha \in \Omega_2$ and set

$$\Phi = \{\varphi_{\Omega_\alpha, \Omega_\beta} : \; \beta \in \Omega_2\},$$

where $\varphi_{\Omega_\alpha,\Omega_\beta}$ is the algebraic isomorphism defined in Example 2.3.16. Then $\mathcal{X}$ is a fission of the scheme $\mathcal{X}_1 \wr_\Phi \mathcal{X}_2$, where $\mathcal{X}_1 = \mathcal{X}_{\Omega_\alpha}$ and $\mathcal{X}_2 = \mathcal{X}_{\Omega/e}$.

3.7.33 Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be coherent configurations on Ω_1 and Ω_2, respectively, and let $\square$ denote $\boxplus$ or $\otimes$ or $\wr$; in the latter case, $\mathcal{X}_1$ and $\mathcal{X}_2$ are schemes. Then

(1) for any $\varphi_1 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}_1')$ and $\varphi_2 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}_2')$, there exists a unique
$$\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1 \square \mathcal{X}_2, \mathcal{X}_1' \square \mathcal{X}_2')$$
such that $\varphi_{\Omega_1} = \varphi_1$ and $\varphi_{\Omega_2} = \varphi_2$;
(2) the inclusion
$$\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1 \square \mathcal{X}_2) \geq \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1) \times \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_2)$$
holds with equality if $\mathcal{X}_1$ and $\mathcal{X}_2$ are not algebraically isomorphic;[5]
(3) for any $e_1 \in E(\mathcal{X}_1)$,
$$(\mathcal{X}_1 \square \mathcal{X}_2)_{\Omega/e} = (\mathcal{X}_1)_{\Omega_1/e_1} \square \mathcal{X}_2,$$
where $e = e_1$ if $\square = \boxplus$, and $e = e_1 \otimes 1_{\Omega_2}$ otherwise.

3.7.34 Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be coherent configurations. Then

(1) $b(\mathcal{X}_1 \boxplus \mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2)$;
(2) $b(\mathcal{X}_1 \otimes \mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2) - 1$ unless $\min\{b(\mathcal{X}_1), b(\mathcal{X}_2)\} = 0$; in the latter case, $b(\mathcal{X}_1 \otimes \mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2)$;
(3) if $\mathcal{X}_1$ and $\mathcal{X}_2$ are schemes, then $b(\mathcal{X}_1 \wr \mathcal{X}_2) = |\Omega_2|\, b(\mathcal{X}_1)$.

3.7.35 [**48**, Corollary 5.2] Let $\mathfrak{X}$ be a graph with connected components $\mathfrak{X}_{ij}$, where $i = 1, \ldots, a$ and $j = 1, \ldots, a_i$ for each i. Assume that the

[5] For $\square = \wr$, this condition is superfluous.

indices are chosen so that the graphs $\mathfrak{X}_{ij}$ and $\mathfrak{X}_{i'j'}$ are isomorphic if and only if $i = i'$. Then

$$\mathrm{WL}(\mathfrak{X}) \cong \boxplus_{i=1}^{a} \mathrm{WL}(\mathcal{X}_{i1}) \wr \mathcal{T}_{a_i}.$$

3.7.36 The exponentiation respects the partial orders of coherent configurations and permutation groups:

(1) if $\mathcal{Y} \le \mathcal{X}$, then $\mathcal{Y} \uparrow K \le \mathcal{X} \uparrow K$ for any K;
(2) if $L \le K$, then $\mathcal{X} \uparrow L \ge \mathcal{X} \uparrow K$ for any $\mathcal{X}$.

3.7.37 Let $\mathcal{X}$ be the scheme of the Hamming graph $H(d, q)$, where $d \ge 1$ and $q \ge 2$. Then

$$\mathcal{X} = \mathcal{T}_q \uparrow \mathrm{Sym}(d) \quad \text{and} \quad \mathrm{Aut}(\mathcal{X}) = \mathrm{Sym}(q) \uparrow \mathrm{Sym}(d).$$

3.7.38 Let $\mathcal{X}$ be a Cayley scheme over G. Assume that $\mathcal{X}$ is the U/L-wreath product. Then

(1) if $\mathcal{X}' \le \mathcal{X}$, and L and U are $\mathcal{X}'$-groups, then $\mathcal{X}'$ is the U/L-wreath product;
(2) if $L' \le L$ and $U' \ge U$ are $\mathcal{X}$-subgroups and $L' \trianglelefteq G$, then $\mathcal{X}$ is the U'/L'-wreath product;
(3) if $H \ge L$ is a normal $\mathcal{X}$-subgroup of G, then $\mathcal{X}_{G/H}$ is the HU/HL-wreath product.

3.7.39 Let $\mathcal{X}$ be a Cayley scheme over a group $G = L \times H \times V$, where L, H, and V are $\mathcal{X}$-groups. Assume that $\mathcal{X}$ is the U/L-wreath product, where $U = HL$. Then

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathcal{T}_L \wr \mathcal{X}_{G/L}) \cap \mathrm{Aut}(\mathcal{X}_U \wr \mathcal{T}_V).$$

3.7.40 [42] Suppose that we are given

(1) primes p_1, p_2, p_3, p_4 such that $\{p_1, p_2\} \cap \{p_3, p_4\} = \varnothing$;
(2) a positive integer d dividing $\mathrm{GCD}(p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1)$;
(3) an isomorphism $f_{ij} \in \mathrm{Iso}(M_i, M_j)$, $(i, j) \in \{(1, 3), (2, 3), (2, 4), (1, 4)\}$,

where M_i is the subgroup of $\mathrm{Aut}(C_{p_i})$ of order d.

Denote by $\mathcal{X}_{ij}$ the cyclotomic Cayley scheme $\mathrm{Cyc}(M_{ij}, C_{p_i p_j})$, where

$$M_{ij} = \{(x, y) \in M_i \times M_j : \; f_{ij}(x) = y\}.$$

Let us consider the generalized wreath product

$$\mathcal{X}(d) = (\mathcal{X}_{13} \wr_{p_3} \mathcal{X}_{23}) \wr_{p_1 p_2} (\mathcal{X}_{14} \wr_{p_4} \mathcal{X}_{24}),$$

where the subscript at the sign $\wr$ denotes the number $|U/L|$ in the corresponding U/L-wreath product: for example, $\mathcal{X}_{13} \wr_{p_3} \mathcal{X}_{23}$ is a Cayley scheme over $C_{p_1 p_2 p_3}$ that is the U/L-wreath product with $|U| = p_1 p_3$ and $|L| = p_1$.

Then

(1) if the automorphism $f = f_{13} \circ f_{23}^{-1} \circ f_{24} \circ f_{14}^{-1}$ of the group K_1 is not trivial, then the Cayley scheme $\mathcal{X}(d)$ is not schurian;

(2) if, in addition, for some d' dividing d the automorphism f is identical on the subgroup of order d' and the factorgroup modulo it, then the scheme $\mathcal{X}(d')$ is not separable.

3.7.41 Let $\mathcal{X}$ be semiregular and $K = \mathrm{Aut}(\mathcal{X})$. Then

$$\widehat{\mathcal{X}} = \mathrm{Inv}(K, \Omega^m). \tag{3.7.1}$$

In particular, the m-dimensional extension of any semiregular coherent configuration is also semiregular.

3.7.42 Let $\mathcal{X} = \mathcal{T}_\Omega$ and $K = \mathrm{Sym}(\Omega)$. Then

(1) $S(\widehat{\mathcal{X}}) = \mathrm{Orb}(K, \Omega^m \times \Omega^m)$; in particular, equality (3.7.1) holds;
(2) $\Delta_i = \{\alpha \in \Omega^m : \ |\{\alpha_1, \ldots, \alpha_m\}| = i\}$ is a homogeneity set of $\widehat{\mathcal{X}}$ for all $i = 1, \ldots, m$;
(3) the equivalence relation $\sim$ on Δ_m defined by

$$\alpha \sim \beta \quad \Leftrightarrow \quad \{\alpha_1, \ldots, \alpha_m\} = \{\beta_1, \ldots, \beta_m\}$$

is a partial parabolic of $\widehat{\mathcal{X}}$;
(4) $\widehat{\mathcal{X}}_{\Omega_m/\sim}$ is isomorphic to the scheme of the Johnson graph $J(n, m)$.

3.7.43 Let $\mathfrak{X} = \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_k$ be the direct product of the graphs $\mathfrak{X}_1, \ldots, \mathfrak{X}_k$, i.e., the graph with vertex set

$$\Omega(\mathfrak{X}) = \Omega(\mathfrak{X}_1) \times \cdots \times \Omega(\mathfrak{X}_k)$$

whose arcs are the pairs (α, β) for which there is an index i such that $(\alpha_i, \beta_i) \in D_i$ and $\alpha_j = \beta_j$ for all $j \neq i$. Then $D(\mathfrak{X}) = s_1 \cup \cdots \cup s_k$, where

$$s_i = \bigcap_{j=1}^{k} \mathrm{Cyl}_{t_{i,j}}(j, j)$$

with $t_{i,j} = D(\mathfrak{X}_i)$ or $1_{\Omega(\mathfrak{X}_i)}$ depending on whether $i = j$ or not.

3.7.44 Let $\mathcal{X}' \geq \mathcal{X}$ and $\mathcal{Y}' \geq \mathcal{Y}$. Then

(1) $\widehat{\mathcal{X}'} \geq \widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}'} \geq \widehat{\mathcal{Y}}$;
(2) if $\psi \in \mathrm{Iso}_m(\mathcal{X}', \mathcal{Y}')$ extends $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{Y})$, then $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{Y})$ and $\widehat{\psi}$ extends $\widehat{\varphi}$.

3.7.45 For any $\Delta \in F^{\cup}$, we have $\widehat{\mathcal{X}_\Delta} \leq \widehat{\mathcal{X}}_{\Delta^m}$.

3.7.46 [40, Lemma 6.2] Let $s \in \widehat{S}$. Then for any indices $i, j \in \{1, \ldots, 2m\}$ the following two statements hold:

(1) $\mathrm{pr}_{i,j}(s) = \{((\alpha \cdot \beta)_i, (\alpha \cdot \beta)_j)) : \ (\alpha, \beta) \in s\}$ is a basis relation of $\overline{\mathcal{X}}$;
(2) if φ is an m-isomorphism from $\mathcal{X}$ to another coherent configuration, then

$$\mathrm{pr}_{i,j}(\widehat{\varphi}(s)) = \overline{\varphi}(\mathrm{pr}_{i,j}(s)).$$

3.7.47 The mapping $\mathcal{X} \mapsto \overline{\mathcal{X}}$ is a closure operator, i.e.,

(1) $\mathcal{X} \leq \overline{\mathcal{X}}$;
(2) if $\mathcal{X} \leq \mathcal{Y}$, then $\overline{\mathcal{X}} \leq \overline{\mathcal{Y}}$;

(3) $\overline{\mathcal{X}}$ is m-closed.

3.7.48 For fixed sets Ω and Ω', we define a partial order on the set of all algebraic isomorphisms $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, where $\mathcal{X}$ and $\mathcal{X}'$ are coherent configurations on Ω and Ω', respectively. Namely, if $\psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}')$, then

$$\varphi \le \psi \quad \Leftrightarrow \quad \mathcal{X} \le \mathcal{Y},\ \mathcal{X}' \le \mathcal{Y}',\ \text{and}\ \psi \text{ extends } \varphi.$$

Then the mapping taking an m-isomorphism φ to $\mathrm{cl}(\varphi) = \overline{\varphi}$, is a closure operator on the m-isomorphisms, i.e.,

(1) $\varphi \le \mathrm{cl}(\varphi)$;
(2) if $\varphi \le \psi$, then $\mathrm{cl}(\varphi) \le \mathrm{cl}(\psi)$;
(3) $\mathrm{cl}(\mathrm{cl}(\varphi)) = \mathrm{cl}(\varphi)$.

3.7.49 [**38**, Theorems 7.5 and 7.6] Let $\mathcal{X} = \mathcal{X}_1 \boxplus \cdots \boxplus \mathcal{X}_k$, $k \ge 1$. Then

(1) $\overline{\mathcal{X}} = \overline{\mathcal{X}}_1 \boxplus \cdots \boxplus \overline{\mathcal{X}}_k$;
(2) $\mathcal{X}$ is m-closed if and only if so are $\mathcal{X}_1, \ldots, \mathcal{X}_k$;
(3) if $\mathcal{X}' = \mathcal{X}'_1 \boxplus \cdots \boxplus \mathcal{X}'_k$ and $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ is induced by certain $\varphi_i \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_i, \mathcal{X}'_i)$, $i = 1, \ldots, k$, then $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}')$ if and only if $\varphi_i \in \mathrm{Iso}_m(\mathcal{X}_i, \mathcal{X}'_i)$ for all i.

3.7.50 [**39**, Corollary 5.4] Let $\mathcal{X}$ be a 2-closed scheme and e is a parabolic of $\mathcal{X}$. Suppose that $e \subseteq S_1$. Then each $\Delta \in \Omega/e$ is a fiber of the coherent closure $\mathrm{WL}(\mathcal{X}, 1_\Delta)$.

3.7.51 [**39**, Theorem 5.9] Let $\mathcal{X}$ be a 2-closed primitive scheme. For a fixed $\alpha \in \Omega$, denote by Δ the set of all fibers $\Gamma \in F(\mathcal{X}_\alpha)$ such that the scheme $(\mathcal{X}_\alpha)_\Delta$ is imprimitive. Then

(1) if $\Delta \ne \varnothing$, then the union of all $\Gamma \in \Delta$ is a base of $\mathcal{X}$;
(2) if $\Delta = \varnothing$, then any fiber of $\mathcal{X}_\alpha$ other than $\{\alpha\}$ is a base of $\mathcal{X}$.

3.7.52 [**119**] Let G be an abelian group and $\widehat{G}$ the group of all irreducible complex characters of G. For an S-ring $\mathfrak{A}$ over G, define an equivalence relation $\sim$ on $\widehat{G}$ so that

$$\xi \sim \eta \quad \Leftrightarrow \quad \xi(\underline{X}) = \eta(\underline{X}) \quad \text{for all}\ X \in \mathcal{S}(\mathfrak{A}).$$

Then the partition $\widehat{\mathcal{S}}$ of the group $\widehat{G}$ into the classes of $\sim$ satisfies the conditions (SR1), (SR2), and (SR3) at p. 69; in particular,

$$\widehat{\mathfrak{A}} = \mathrm{Span}\,\underline{\widehat{\mathcal{S}}}$$

is an S-ring over $\widehat{G}$. Moreover, $\mathrm{rk}(\mathfrak{A}) = \mathrm{rk}(\widehat{\mathfrak{A}})$.

3.7.53 In the conditions and notation of Exercise 3.7.52, given a group $H \le G$ denote by $H^\perp$ the group of all characters $\xi \in \widehat{G}$ such that $\ker(\xi) \ge H$. Then

(1) the mapping $\mathcal{E}(\mathfrak{A}) \to \mathcal{E}(\widehat{\mathfrak{A}})$, $H \mapsto H^\perp$ is a lattice antiisomorphism;
(2) $\widehat{\mathfrak{A}_H} = \widehat{\mathfrak{A}}_{\widehat{G}/H^\perp}$ for each $H \in \mathcal{E}(\mathfrak{A})$;
(3) $\widehat{\mathfrak{A}_{G/H}} = \widehat{\mathfrak{A}}_{H^\perp}$ for each $H \in \mathcal{E}(\mathfrak{A})$.

3.7.54 [**47**, Sec. 2.3] In the conditions and notation of Exercise 3.7.52,

(1) $\mathfrak{A} = \mathrm{Cyc}(K, G)$ for $K \le \mathrm{Aut}(G)$ if and only if $\widehat{\mathfrak{A}} = \mathrm{Cyc}(K, \widehat{G})$;
(2) $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ if and only if $\widehat{\mathfrak{A}} = \widehat{\mathfrak{A}_1} \otimes \widehat{\mathfrak{A}_2}$;
(3) $\mathfrak{A}$ is the U/L-wreath product if and only if $\widehat{\mathfrak{A}}$ is the $L^\perp/U^\perp$-wreath product.

3.7.55 Let $\xi \in \mathrm{Irr}(\mathcal{X})$. Then

$$n_\xi \le |\,\mathrm{Supp}_{\mathcal{X}}(\xi)|\, m_\xi,$$

and the equality is simultaneously attained for all irreducible characters ξ if and only if $\mathcal{X}$ is quasiregular.

3.7.56 [**37**, Theorem 3] There exists a constant $c > 0$ such that given a primitive scheme $\mathcal{X}$,

$$n_{min} \le 2^{c m_{\min}},$$

where n_{min} is the minimal valency of irreflexive basis relation of $\mathcal{X}$ and m_{min} is the minimal multiplicity of nonprincipal irreducible character of $\mathcal{X}$.

3.7.57 Let $G = G_1 \times G_2 \times G_3$ be an abelian group, where $|G_1| = |G_2| = |G_3|$. Denote by K the permutation group induced by the action of G by multiplications on the set

$$\Omega = G/G_1 \,\cup\, G/G_2 \,\cup\, G/G_3,$$

and set $\mathcal{X} = \mathrm{Inv}(K, \Omega)$. Then

(1) $F(\mathcal{X}) = \{G/G_1, G/G_2, G/G_3\}$;
(2) $m_\xi = 1$ and $n_\xi = |\,\mathrm{Supp}_{\mathcal{X}}(\xi)|$ for all $\xi \in \mathrm{Irr}(\mathcal{X})$;
(3) the mapping $\xi \mapsto \mathrm{Supp}_{\mathcal{X}}(\xi)$ induces a surjection from $\mathrm{Irr}(\mathcal{X})$ to the nonempty homogeneity sets of $\mathcal{X}$.

3.7.58 [**10**, Theorem 3.6(ii)] Let $\mathcal{X}$ be a commutative scheme of degree n, and $r, s, t \in S$. Then

$$c_{rs}^t = \frac{n_r n_s}{n} \sum_{\xi \in \mathrm{Irr}(\mathcal{X})} \frac{1}{m_\xi^2}\, \xi(r)\xi(s)\overline{\xi(t)}.$$

3.7.59 [**2**, Proposition 3.4(i), Lemma 3.8(i)] Let $\mathcal{X}$ be a scheme and ρ the regular character of $\mathcal{X}$, see formula (3.6.17). Then

$$n \sum_{\xi \in \mathrm{Irr}(\mathcal{X})} \frac{n_\xi}{m_\xi} P_\xi = \sum_{s \in S} \frac{\rho(A_s)}{n_s} A_s.$$

CHAPTER 4

Developments

The purpose of this chapter is to present several directions in the theory of coherent configurations, which on the one hand are quite well studied, and on the other hand, leave room for further research. The topics that we have selected, range from very special to fairly general. For example, the Hanaki–Uno theorem on schemes of prime degree is the main subject of Section 4.5, whereas the schurity and separability numbers are defined in Section 4.2 for arbitrary coherent configurations.

Three other classes, studied in Sections 4.1, 4.3, and 4.4 are quasiregular coherent configurations, the schemes with at most two different valencies, and cyclotomic schemes, respectively. In Section 4.6, we introduce the Weisfeiler–Leman method which reveals a deep connection between some natural combinatorial algorithms for testing isomorphism of graphs and so called multidimensional extensions of coherent configurations and algebraic isomorphisms.

4.1 Quasiregular coherent configurations

At first glance, it seems that quasiregular coherent configurations are close to schurian, but this is not true at all. Even if the homogeneous components of a quasiregular coherent configuration $\mathcal{X}$ are commutative, the schurity of $\mathcal{X}$ is in a sense equivalent to the existence of a certain amalgam of the groups corresponding to the homogeneous components of $\mathcal{X}$ [**72**].

Difficulties in identifying schurian quasiregular coherent configurations arise already when $\mathcal{X}$ is a *Klein configuration*, i.e., one each homogeneous component of which is a regular scheme of the Klein group. This situation is studied in Subsections 4.1.2 and 4.1.3. In general case, we describe a method that allows at least in principle to construct all quasiregular coherent configurations with a given set of homogeneous components (Section 4.1.1).

4.1.1 Systems of linked quotients

The material of this subsection is taken from [**72**].

Let I be a set, and let $\mathfrak{G}$ be a family of groups G_i with identity e_i, $i \in I$. It is assumed that each G_i acts regularly on $\Omega_i := G_i$. The permutations of G_i are treated as thin relations on Ω_i (the graphs of the corresponding permutations). In particular, (Ω_i, G_i) is a regular scheme isomorphic to $\mathrm{Inv}(G_i, \Omega_i)$, see Theorem 2.2.11.

Denote by Ω the disjoint union of the sets Ω_i, $i \in I$. We are interested in the class of all coherent configurations $\mathcal{X}$ on Ω such that

$$F(\mathcal{X}) = \{\Omega_i\}_{i\in I} \quad \text{and} \quad \mathcal{X} \geq \boxplus_{i\in I} \mathrm{Inv}(G_i, \Omega_i).$$

Any such $\mathcal{X}$ is said to be a $\mathfrak{G}$-configuration on Ω. By Theorem 2.2.11, a coherent configuration is quasiregular if and only if it is isomorphic to a $\mathfrak{G}$-configuration for a suitable family $\mathfrak{G}$.

Let $\mathcal{X}$ be a $\mathfrak{G}$-configuration and $S = S(\mathcal{X})$. For any indices $i, j \in I$, set

$$S_{ij} = S_{\Omega_i,\Omega_j}.$$

In particular, $S_{ii} = G_i$. Since all relations belonging to $G_i \cup G_j$ are thin, Lemma 2.1.24 implies that

$$s^* S_{ij} = S_{ij} = S_{ij} t, \qquad s \in G_i,\ t \in G_j.$$

Therefore the mappings

$$\pi_{ij} : G_i \to \mathrm{Sym}(S_{ij}) \quad \text{and} \quad \rho_{ij} : G_j \to \mathrm{Sym}(S_{ij})$$

defined by the formulas

$$x^{\pi_{ij}(s)} = s^* \cdot x,\ x \in S_{ij}, \quad \text{and} \quad x^{\rho_{ij}(t)} = x \cdot t,\ x \in S_{ij},$$

respectively, are group homomorphisms.

Lemma 4.1.1. *For all $i, j \in I$,*

(1) $\mathrm{Im}(\pi_{ij})$ *and* $\mathrm{Im}(\rho_{ij})$ *are regular groups centralizing each other;*
(2) $\ker(\pi_{ij}) = xx^*$ *for all* $x \in S_{ij}$, *and* $\ker(\rho_{ij}) = x^*x$ *for all* $x \in S_{ij}$.

Proof. Let $x, y \in S_{ij}$. Then any basis relations $s \in xy^*$ and $t \in x^*y$ belong to G_i and G_j, respectively. Therefore they are thin and hence

$$s^* \cdot x = y = x \cdot t.$$

This proves that the groups $\mathrm{Im}(\pi_{ij})$ and $\mathrm{Im}(\rho_{ij})$ are transitive. Obviously each of them centralizes the other.

Next assume that $s^* \cdot x = x$ for some $s \in G_i$ and $x \in S_{ij}$. Since the group $\mathrm{Im}(\rho_j)$ is transitive, any $y \in S_{ij}$ is of the form $x \cdot t$ for a suitable $t \in G_j$. Therefore,

$$s^* \cdot y = s^* \cdot x \cdot t = x \cdot t = y.$$

Thus the group $\mathrm{Im}(\pi_{ij})$ is regular. Similarly, one can prove that the group $\mathrm{Im}(\rho_{ij})$ is also regular. This completes the proof of statement (1).

By the regularity of $\mathrm{Im}(\pi_{ij})$, we have $s \in \ker(\pi_{ij})$ if and only if $s^* \cdot x = x$ if and only if $s \in xx^*$ for some (and hence for all) $x \in S_{ij}$. Similarly, $t \in \ker(\rho_{ij})$ if and only if $t \in x^*x$ for some (and hence for all) $x \in S_{ij}$. $\square$

An important role in the subsequent analysis is played by the family $\mathfrak{S}$ of the groups

$$G_{ij} = \ker(\pi_{ij}) = \ker(\rho_{ji}), \qquad i, j \in I. \tag{4.1.1}$$

Obviously,

$$G_{ii} = e_i \quad \text{and} \quad G_{ij} \trianglelefteq G_i, \qquad i, j \in I. \tag{4.1.2}$$

By statement (1) of Lemma 4.1.1, the quotients G_i/G_{ij} and G_j/G_{ji} treated as permutation groups on S_{ij} are the left and right representations of the same group. In particular, these groups are isomorphic. To write an explicit isomorphism, denote by α_i the point of Ω_i corresponding to e_i (recall that $\Omega_i = G_i$ and e_i is the identity of G_i). Set

$$s_{ij} := r(\alpha_i, \alpha_j), \quad i, j \in I.$$

Clearly, $s_{ii} = 1_{\Omega_i}$ and $s_{ij}^* = s_{ji}$ for all i and j.

Lemma 4.1.2. *For all $i, j \in I$, the mapping*

$$f_{ij} : G_i/G_{ij} \to G_j/G_{ji},\ G_{ij}s \mapsto s_{ij}^*(G_{ij}s)s_{ij} \tag{4.1.3}$$

is a group isomorphism, and $(f_{ij})^{-1} = f_{ji}$.

Proof. Set $x = s_{ij}$. By statement (1) of Lemma 4.1.1 for any $s \in G_i$, there exists $t \in G_j$ such that $sx = xt$. Moreover, the element t depends on the coset $G_{ij}s$ only. By statement (2) of Lemma 4.1.1 and Exercise 2.7.14, we have

$$x^*(G_{ij}s)x = x^*(xx^*)sx = x^*sx = x^*xt = G_{ji}t \in G_j/G_{ji}. \tag{4.1.4}$$

Thus the mapping f_{ij} is well-defined. It is a homomorphism, because for any $s, s' \in G_i$,

$$\begin{aligned} f_{ij}(G_{ij}s\, G_{ij}s') &= f_{ij}(G_{ij}ss') \\ &= x^*(ss')x \\ &= (x^*sx)(x^*s'x) \\ &= f_{ij}(G_{ij}s)f_{ij}(G_{ij}s'). \end{aligned}$$

Now assume that $f_{ij}(G_{ij}s) = G_{ji}$ for some $s \in G_i$. Then the element $t \in G_j$ in formula (4.1.4) belongs to G_{ji}. Therefore,

$$sx = xt \subseteq xx^*x = \{x\}.$$

It follows that $s^* \in G_{ij}$ and hence $s \in G_{ij}$. Thus the homomorphism f_{ij} is a monomorphism.

By statement (1) of Lemma 4.1.1, for each $t \in G_j$ there exists $s \in G_i$ such that $sx = xt$. Then formula (4.1.4) implies that $f_{ij}(G_{ij}s) = G_{ji}t$. Thus, f_{ij} is an epimorphism and hence an isomorphism.

To complete the proof, it suffices to see that

$$f_{ij}f_{ji}(G_{ji}t) = f_{ij}(xtx^*) = x^*xtx^*x = G_{ji}t.$$

Thus the mapping $f_{ij}f_{ji}$ is identical. □

An important relation between the isomorphisms f_{ij} and the groups G_{ij} is established in the lemma below.

Lemma 4.1.3. *For any $i, j, k \in I$,*

$$f_{ik}(G_{ij}G_{ik}/G_{ik}) = G_{kj}G_{ki}/G_{ki}. \tag{4.1.5}$$

Proof. From the definition of s_{ij}, it follows that $s_{ab} \in s_{ac}s_{cb}$ for all $a, b, c \in I$. Using statement (2) of Lemma 4.1.1 we obtain

$$\begin{aligned} G_{kj} &= s_{kj}s_{kj}^* \\ &\subseteq (s_{ki}s_{ij})(s_{ki}s_{ij})^* \\ &= s_{ki}s_{ij}\, s_{ij}^*s_{ki}^* \\ &= s_{ik}^*s_{ij}s_{ij}^*s_{ik} \\ &= s_{ik}^*G_{ij}s_{ik}. \end{aligned}$$

Since $G_{ki} = s_{ik}^*G_{ik}s_{ik}$, this implies that

$$s_{ik}^*G_{ij}G_{ik}s_{ik} = s_{ik}^*G_{ij}s_{ik}\, s_{ik}^*G_{ik}s_{ik} \supseteq G_{kj}G_{ki}. \tag{4.1.6}$$

Using this inclusion and the inclusion with i and k interchanged, we obtain

$$\begin{aligned} G_{ij}G_{ik} &= (s_{ik}s_{ik}^*)G_{ij}G_{ik}(s_{ik}s_{ik}^*) \\ &= s_{ik}(s_{ik}^*G_{ij}G_{ik}s_{ik})s_{ik}^* \\ &\supseteq s_{ki}^*G_{kj}G_{ki}s_{ki} \\ &\supseteq G_{ij}G_{ik}. \end{aligned}$$

This shows that inclusion (4.1.6) is, in fact, an equality. Since it is equivalent to equality (4.1.5), we are done. □

From Lemma 4.1.3, it follows that for all $i, j, k \in I$, the isomorphism f_{ik} induces an isomorphism

$$f_{ijk} : G_i/G_{ij}G_{ik} \to G_k/G_{ki}G_{kj}. \tag{4.1.7}$$

Lemma 4.1.4. *For any $i, j, k \in I$,*

$$f_{jki}f_{kij}f_{ijk} = \mathrm{id}\,. \tag{4.1.8}$$

Proof. For any $s \in G_i$, we have

$$\begin{aligned} f_{jki}f_{kij}f_{ijk}(G_{ij}G_{ik}s) &= f_{jki}f_{kij}(G_{ki}G_{kj}s_{ik}^*s\, s_{ik}) \\ &= f_{jki}(G_{ji}G_{jk}s_{kj}^*s_{ik}^*s\, s_{ik}s_{kj}) \\ &= G_{ij}G_{ik}s_{ji}^*s_{kj}^*s_{ik}^*s\, s_{ik}s_{kj}s_{ji}. \end{aligned}$$

By the definition of the relations s_{ij}, both sets $s_{ji}^*s_{kj}^*s_{ik}^*$ and $s_{ik}s_{kj}s_{ji}$ contains the diagonal 1_{Ω_i}. Therefore the right-hand side of the above equality equals the coset of $G_{ij}G_{ik}s$. Thus the composition on the left-hand side of (4.1.8) is the identity map, as required. $\square$

To summarize all the above, let us introduce a concept of system of linked quotients. Namely, suppose we are given

(F1) a family $\mathfrak{G}$ of groups G_i, $i \in I$;
(F2) a family $\mathfrak{S}$ is subgroups $G_{ij} \trianglelefteq G_i$ with $G_{ii} = e_i$, $i, j \in I$;
(F3) a family $\mathfrak{F}$ of group isomorphisms $f_{ij} : G_i/G_{ij} \to G_j/G_{ji}$ with $f_{ii} = f_{ij}f_{ji} = \mathrm{id}$, $i, j \in I$.

The triple $(\mathfrak{G}, \mathfrak{S}, \mathfrak{F})$ is called a *system of linked quotients* based on $\mathfrak{G}$ if for all $i, j, k \in I$, relations (4.1.5) and (4.1.8) hold. By Lemmas 4.1.2, 4.1.3, and 4.1.4, we have the following statement.

Theorem 4.1.5. *Let $\mathfrak{G} = \{G_i\}_{i \in I}$ be a family of groups and $\mathcal{X}$ a $\mathfrak{G}$-configuration. Denote by $\mathfrak{S}$ and $\mathfrak{F}$ the families of the groups G_{ij} and isomorphisms f_{ij} defined by formulas* (4.1.1) *and* (4.1.3)*, respectively. Then*

$$\mathcal{T}(\mathcal{X}) := (\mathfrak{G}, \mathfrak{S}, \mathfrak{F})$$

is a system of linked quotients.

Let us give two examples of $\mathfrak{G}$-configurations $\mathcal{X}$, where the families $\mathfrak{S}$ and $\mathfrak{F}$ have especially simple forms.

Example 4.1.6. *Let $\mathcal{X}$ be a direct sum of schemes. Then for all $i, j \in I$, the group G_{ij} equals G_i and the isomorphism f_{ij} is trivial.*

Example 4.1.7. *Let $\mathcal{X}$ be semiregular. Then the groups G_i are pairwise isomorphic, the groups G_{ij} are the identity ones, and the isomorphisms f_{ij} are of the form f_s, where s runs over a system of distinct representatives defined in statement* (3) *Exercise 2.7.13.*

Let $\mathcal{T}$ be a system of linked quotients defined by the conditions (F1), (F2), and (F3). In what follows, we find a $\mathfrak{G}$-configuration $\mathcal{X}$ such that

$$\mathcal{T} = \mathcal{T}(\mathcal{X}).$$

Recall that Ω is the disjoint union of $\Omega_i = G_i$ and α_i is a point of Ω_i corresponding to the identity e_i of G_i, $i \in I$. Set

$$s_{ij} = \bigcup_{s \in G_i} \alpha_i s \times \alpha_j f_{ij}(G_{ij}s), \qquad i, j \in I. \tag{4.1.9}$$

Clearly, $s_{ij} \subseteq \Omega_i \times \Omega_j$ and $s_{ii} = 1_{\Omega_i}$. Moreover, given $x \in G_i$,

$$x \cdot s_{ij} = \bigcup_{s \in G_i} \alpha_i(xs) \times \alpha_j f_{ij}(G_{ij}xs). \tag{4.1.10}$$

Lemma 4.1.8. *For any $i, j \in I$, the relations $r \cdot s_{ij}$, $r \in G_i$, form a partition of $\Omega_i \times \Omega_j$.*

Proof. We have to verify that if $x, y \in G_i$, then the relations $x \cdot s_{ij}$ and $y \cdot s_{ij}$ are disjoint or coincide. To this end, we assume that $x \cdot s_{ij}$ and $y \cdot s_{ij}$

have a common pair (α,β). Then formula (4.1.10) implies that there exists $s\in G_i$ such that

$$\{\alpha\}=\alpha_i s \quad\text{and}\quad \beta\in\beta_j f_{ij}(G_{ij}xs)\cap\beta_j f_{ij}(G_{ij}ys).$$

It follows that the two right cosets $f_{ij}(G_{ij}xs)$ and $f_{ij}(G_{ij}ys)$ of the same group $G_{ji}\le G_j$ are intersected. However, this is possible only if $G_{ij}x=G_{ij}y$. But then $x=y\cdot z$ for some $z\in G_{ij}$. Taking into account that $z\cdot s_{ij}=s_{ij}$, we obtain

$$x\cdot s_{ij}=(y\cdot z)\cdot s_{ij}=y\cdot(z\cdot s_{ij})=y\cdot s_{ij}$$

as required. □

Denote by S the union of the sets

$$S_{ij}=\{x\cdot s_{ij}\mid x\in G_i\},\qquad i,j\in I.$$

From Lemma 4.1.8, it follows that S forms a partition of Ω^2. In fact, the pair $\mathcal{X}=(\Omega,S)$ is a rainbow. Indeed, the condition (CC1) is satisfied, because 1_Ω is the disjoint union of e_i, $i\in I$.

To verify the condition (CC2), it suffices to verify that $s_{ij}^*=s_{ji}$ for all $i,j\in I$. However this is true, because if $\alpha\in\Omega_i$ and $\beta\in\Omega_j$, then

$$(\alpha,\beta)\in s_{ij}\ \Leftrightarrow\ f_{ij}(G_{ij}s)=G_{ji}t \quad\text{and}\quad (\beta,\alpha)\in s_{ji}\ \Leftrightarrow\ f_{ji}(G_{ji}t)=G_{ij}s$$

with $s=r(\alpha_i,\alpha)$ and $t=r(\alpha_j,\beta)$, and the right-hand sides of the implications are equivalent by the condition (F3).

Lemma 4.1.9. *The rainbow $\mathcal{X}$ is a quasiregular $\mathfrak{G}$-configuration such that $\mathcal{T}(\mathcal{X})=(\mathfrak{G},\mathfrak{S},\mathfrak{F})$.*

Proof. We need to verify the condition (CC3) only, or equivalently, that given $s=x\cdot s_{ij}$ and $r=y\cdot s_{jk}$, the matrix A_sA_r is a linear combination of the matrices A_t, $t\in S$. Take $z\in S_k$ so that $ys_{jk}=s_{jk}z$. Then

$$A_sA_r=A_{xs_{ij}}A_{s_{jk}z}=A_x(A_{s_{ij}}A_{s_{jk}})A_z.$$

Since obviously $A_xA_tA_z=A_{x\cdot t\cdot z}$ for all $t\in S_{ik}$, it suffices to prove that $A_{s_{ij}}A_{s_{jk}}$ is a linear combination of the matrices A_t, $t\in S_{ik}$.

Let $t\in S_{ik}$. We have to verify that the number $|\alpha s_{ij}\cap\beta s_{kj}|$ does not depend on $(\alpha,\beta)\in t$. By the definition of the relations s_{ij} and s_{kj}, we have

$$\alpha s_{ij}=\alpha_j G_{ji}u \quad\text{and}\quad \beta s_{kj}=\alpha_j G_{jk}v,$$

where the relations $u,v\in G_j$ are such that $f_{ij}(G_{ij}r(\alpha_i,\alpha))=G_{ji}u$ and $f_{jk}(G_{jk}r(\alpha_k,\beta))=G_{jk}v$. Thus,

$$|\alpha s_{ij}\cap\beta s_{kj}|=|G_{ji}u\cap G_{jk}v|.$$

The number on the right-hand side is equal to $|G_{ji}\cap G_{jk}|$ whenever $\alpha s_{ij}\cap\beta s_{kj}$ is not empty. Thus it remains to verify that the latter property does not depend on $(\alpha, \beta) \in t$, or more generally, that

$$s_{ij} \cdot s_{jk} = \bigcup_{x \in G_{ij}} x \cdot s_{ik}, \tag{4.1.11}$$

Using the fact that $f_{jk} f_{ij}$ and f_{ik} coincide on the cosets of $G_{ik} G_{ij}$ in G_i (see identity (4.1.8)), we have

$$\begin{aligned} \alpha\, s_{ij} s_{jk} &= (\alpha s_{ij}) s_{jk} = (\alpha_j f_{ij}(G_{ij} w)) s_{jk} \\ &\subseteq \alpha_k f_{jk}(f_{ij}(G_{ij} w)) = \alpha_k f_{ik}(G_{ij} w) \\ &= \alpha_k f_{ik}(G_{ik} G_{ij} w) = \alpha \bigcup_{x \in G_{ij}} x \cdot s_{ik}, \end{aligned}$$

where $w = r(\alpha_i, \alpha)$. Thus the left-hand side of formula (4.1.11) is contained in the right-hand side. Conversely, if $x \in G_{ij}$, then

$$x \cdot s_{ik} \in s_{ij} s_{ji}\, s_{ik} \subseteq s_{ij} \bigcup_{y \in G_{ji}} y s_{jk} = \Big(\bigcup_{y \in G_{ji}} s_{ij} y \Big) s_{jk} = s_{ij} s_{jk},$$

which completes the proof of the claim. □

Corollary 4.1.10. *The mapping $\mathcal{X} \mapsto \mathcal{T}(\mathcal{X})$ from the $\mathfrak{G}$-configurations to the systems of linked quotients based on $\mathfrak{G}$ is one-to-one.*

4.1.2 General Klein configurations

Let G be a Klein group (an elementary abelian group of order 4). A $\mathfrak{G}$-configuration $\mathcal{X}$ is said to be *Klein* if $G_i = G$ for all i. It is convenient to consider a semiregular action of G on Ω such that

$$\operatorname{Orb}(G, \Omega) = F(\mathcal{X}) = \{\Omega_i : \ i \in I\},$$

where the Ω_i are as in Subsection 4.1.1. Thus, $\mathcal{X}$ is a quasiregular coherent configuration and the homogeneous component $\mathcal{X}_{\Omega_i}$ is a regular scheme associated with the group G^{Ω_i}, $i \in I$.

Example 4.1.11. *Up to isomorphism, there are exactly three distinct Klein configurations of degree* 8*: semiregular of rank* 16*, the direct sum of rank* 10*, and the coherent configuration of rank* 12*.*

Let us analyze the system $\mathcal{T}(\mathcal{X})$ of linked quotients that is defined by the Klein configuration $\mathcal{X}$. The first three statements of the lemma below are immediate consequences of the fact that $|G| = 4$ and the condition (F2); the fourth one immediately follows from Exercise 4.7.3.

Lemma 4.1.12. *Given $i, j \in I$, we have*

1. $|G_{ij}| = 2$ *or* $G_{ij} \in \{e_i, G\}$*;*
2. $|G_{ij}| = |G_{ji}|$*;*
3. f_{ij} *is uniquely determined unless* $G_{ij} = e_i$*;*
4. $|S_{ij}| = [G : G_{ij}]$ *and* $n_s = |G_{ij}|$ *for* $s \in S_{ij}$*.*

For each $i, j \in I$, the explicit form of a relation belonging to the set S_{ij} is given in Lemma 4.1.8 in terms of the relation s_{ij} defined by formula (4.1.9). The graph of s_{ij} has one of the forms depicted in Fig. 4.1.

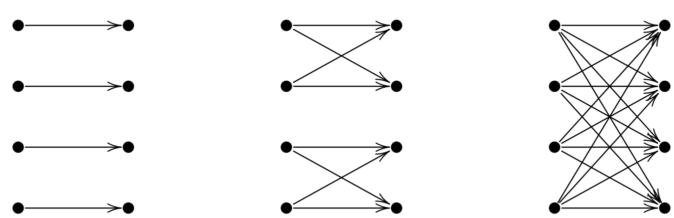


FIGURE 4.1 The relations s_{ij} for $|S_{ij}| = 1$, 2, and 4, respectively.

Example 4.1.13. *Let $|S_{ij}| = 2$. Then*

$$G_{ij} = \{e_i, s\} \quad \text{and} \quad G_{ji} = \{e_j, t\}$$

for some irreflexive relations $s \in G_i$ and $t \in G_j$. Moreover, $e_i \cup s$ and $e_j \cup t$ are parabolics of the schemes $\mathcal{X}_{\Omega_i}$ and $\mathcal{X}_{\Omega_j}$, respectively; denote the classes of these two parabolics by Δ_1, Δ_2 and Γ_1, Γ_2. Then

$$S_{ij} = \{\Delta_1 \times \Gamma_1 \,\cup\, \Delta_2 \times \Gamma_2,\ \Delta_1 \times \Gamma_2 \,\cup\, \Delta_2 \times \Gamma_1\}.$$

Let us define a relation $\sim$ on the set I so that

$$i \sim j \quad \Leftrightarrow \quad G_{ij} = e_i \qquad (\Leftrightarrow \quad |S_{ij}| = 4). \tag{4.1.12}$$

This relation is obviously reflexive and symmetric. If $i \sim j$ and $j \sim k$, then S_{ij} and S_{jk} consist of thin relations (statement (4) of Lemma 4.1.12). Therefore the set $S_{ij} \cdot S_{jk}$ consists of thin relations (Lemma 2.1.24). Thus, $i \sim k$ again by statement (4) of Lemma 4.1.12. Consequently, the relation $\sim$ is transitive and hence is an equivalence relation.

Let $J \subset I$ be a system of distinct representatives of the classes of the equivalence relation $\sim$. Then the union

$$\Omega_J = \bigcup_{i \in J} \Omega_i \tag{4.1.13}$$

is a homogeneity set of $\mathcal{X}$. In accordance with Exercise 4.7.5, the Klein configuration $\mathcal{X}_{\Omega_J}$ does not depend (up to isomorphism) on the choice of J. In what follows, $\mathcal{X}_{\Omega_J}$ is called a *reduction* of $\mathcal{X}$.

Lemma 4.1.14. *Every Klein configuration $\mathcal{X}$ is schurian (respectively, separable) if so is a reduction of $\mathcal{X}$.*

Proof. Let J be as above. Then for any $i \in I \setminus J$, there exists $j \in J$ such that $i \sim j$ and hence S_{ij} consists of thin relations. It follows that $\mathcal{X}$ satisfies condition (3.3.14) for $\Delta = \Omega_J$. Thus the required statement immediately follows from statement (2) of Lemma 3.3.20. □

Lemma 4.1.14 shows that if we are interested in the schurity or separability problems for the Klein configurations, then without loss of generality we can restrict ourselves to reduced configurations defined as follows.

Definition 4.1.15. *A Klein configuration is said to be reduced if it coincides with any of its reductions.*

Certainly, $\mathcal{X}$ is reduced if and only if the equivalence relation $\sim$ is trivial, or equivalently, $|S_{ij}| \le 2$ for all distinct $i, j \in I$. Note that the class of reduced Klein configurations is closed under algebraic isomorphisms and taking restrictions and direct sums.

Proposition 4.1.16. *Every reduced Klein configuration is uniquely determined by the family of the associated groups $G_{ij} \le G$, $i, j \in I$. These groups satisfy the relations*

$$G_{ij} = G_{ik},\ |G_{ij}| = 2,\ j \ne k \qquad \Rightarrow \qquad G_{ji} = G_{jk} \tag{4.1.14}$$

for all $i, j, k \in I$.

Proof. Let $\mathcal{X}$ and $\mathcal{X}'$ be reduced Klein configurations. Assume that $G_{ij} = G'_{ij}$ for all $i, j \in I$. Then by statements (1) and (4) of Lemma 4.1.12,

$$|S_{ij}| = |S'_{ij}| \in \{1, 2, 4\}, \qquad i, j \in I.$$

Now if $|S_{ij}| = 4$, then $i = j$ (because $\mathcal{X}$ is reduced) and hence $S_{ij} = G_i = S'_{ij}$. Furthermore, if $|S_{ij}| = 1$, then $S_{ij} = \{\Omega_i \times \Omega_j\} = S'_{ij}$. Finally, if $|S_{ij}| = 2$, then the required statement follows from Example 4.1.13.

To prove relation (4.1.14), we assume that $G_{ij} = G_{ik}$, $|G_{ik}| = 2$, and $j \neq k$. In view of condition (4.1.5),

$$[G_i : G_{ij}G_{ik}] = [G_j : G_{ji}G_{jk}].$$

Since $|G_i| = 4 = |G_j|$, it follows that

$$2 = |G_{ij}| = |G_{ij}G_{ik}| = |G_{ji}G_{jk}|.$$

Taking into account that $|G_{ji}| = |G_{ij}|$, we conclude that $G_{jk} \leq G_{ji}$ which completes the proof unless $G_{jk} = e_j$. In the latter case, the fact that the Klein configuration in question is a reduced one implies $j = k$, in contrary to the assumption. □

With any reduced Klein configuration, one can associate a graph $\mathfrak{X}$ with vertex set I and arc set

$$D = \{(i, j) \in I \times I : \ |S_{ij}| = 2\}.$$

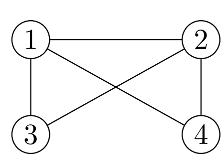
Since $|S_{ij}| = |S_{ji}|$, this graph can be considered as an undirected one. Each algebraic isomorphism between two reduced Klein configuration induces an isomorphism of the corresponding graphs. As the following example shows, the converse is not true.

Example 4.1.17. *Let $I = \{1, 2, 3, 4\}$, and let $\mathcal{X}$ be a Klein configuration of degree 16 represented by a family $\mathfrak{S} = \{G_{ij}\}$ defined by*

$$\mathfrak{S} = \begin{pmatrix} 1 & A & A & B \\ A & 1 & A & C \\ A & A & 1 & G \\ B & C & G & 1 \end{pmatrix},$$

where A, B, and C are the proper non-identity subgroups of G. The graph $\mathfrak{X}$ associated with $\mathcal{X}$ is depicted below and has an automorphism interchanging vertices 3 and 4.

The scheme $\mathcal{X}_{\Omega_3}$ has a unique parabolic with two classes of the form ss^* for some $s \in S(\mathcal{X})$, namely A. However, the scheme $\mathcal{X}_{\Omega_4}$ has two such parabolics, namely B and C. Thus no algebraic automorphism of $\mathcal{X}$ interchanges the fibers Ω_3 and Ω_4.



The structure of the graph $\mathfrak{X}$ imposes some restrictions to the Klein configuration $\mathcal{X}$. For example, the latter is the direct sum of the Klein configurations corresponding to the connected components of $\mathfrak{X}$. More information about $\mathcal{X}$ can be obtained from the cliques of $\mathfrak{X}$. Let us discuss this in detail.

Let $J \subseteq I$ be a clique of $\mathfrak{X}$, or equivalently, $|S_{ij}| = 2$ for all distinct $i, j \in J$. The clique J is said to be *regular* if for all $i \in J$,

$$G_{ij} = G_{ik} \text{ for all } j, k \in J \setminus \{i\}. \tag{4.1.15}$$

Clearly, any edge of $\mathfrak{X}$ forms a regular clique. In Example 4.1.17, the set $\{1, 2, 3\}$ is a regular clique, whereas $\{1, 2, 4\}$ is not.

Some properties of regular cliques are given in the following statement, where $\mathcal{L} = \mathcal{L}(\mathcal{X})$ denotes the set of all maximal regular cliques of the graph $\mathfrak{X}$ associated with a reduced Klein configuration $\mathcal{X}$.

Lemma 4.1.18. *Suppose that the graph $\mathfrak{X}$ has no isolated vertices. Then*

(1) $|J| \geq 2$ *for all* $J \in \mathcal{L}$*;*
(2) $|J \cap J'| \leq 1$ *for all distinct* $J, J' \in \mathcal{L}$*;*
(3) *any vertex of* $\mathfrak{X}$ *belongs to at most three cliques from* $\mathcal{L}$*.*

Proof. Statement (1) is obvious. To prove statement (2), we assume that cliques $J, J' \in \mathcal{L}$ have at least two distinct common vertices, say a and b. We have to check that $J = J'$, or equivalently, that given $i \in J \setminus J'$ and $i' \in J' \setminus J$, we have

$$G_{ix} = G_{ii'} \quad \text{and} \quad G_{i'i} = G_{i'x}$$

for each $x \in \{a, b\}$.

By formula (4.1.14), it suffices to verify the first equality. Without loss of generality, we may assume that $x = a$. Then by that formula,

$$G_{ia} = G_{ib} \;\Rightarrow\; G_{ai} = G_{ab} \quad \text{and} \quad G_{i'a} = G_{i'b} \;\Rightarrow\; G_{ai'} = G_{ab}.$$

It follows that $G_{ai} = G_{ai'}$. By formula (4.1.14) again, $G_{ia} = G_{ii'}$, as required.

To prove statement (3), let $i \in I$. Given a clique $J \in \mathcal{L}$ containing i, the group $G(J) = G_{ij}$ does not depend on $j \in J \setminus \{i\}$. By formula (4.1.14), the mapping $J \mapsto G(J)$ is injective. Since the image of this mapping consists of proper subgroups of the Klein group G, the vertex i belongs to at most three cliques from $\mathcal{L}$. □

Let $\mathcal{X}$ be a reduced Klein configuration. In the notation and assumption of Lemma 4.1.18, denote by $\mathcal{G} = \mathcal{G}(\mathcal{X})$ an incidence structure with point set I, line set $\mathcal{L}$, and incidence relation containing all pairs $i \in I$ and $J \in \mathcal{L}$ such that $i \in J$. By statements (1) and (2) of that lemma, any line contains at least two points, and two distinct points belong to at most one common line. Thus, $\mathcal{G}$ is a *partial linear space* in the sense of [**117**]. The following statement is an immediate corollary of statement (3) of Lemma 4.1.18.

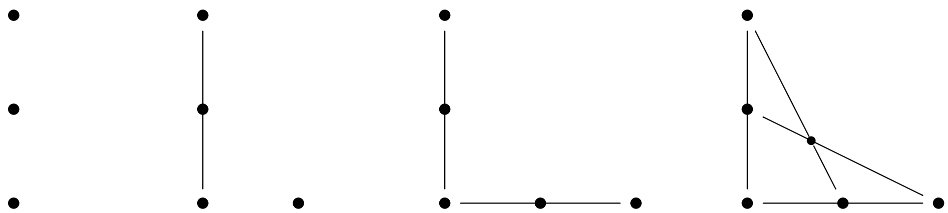


FIGURE 4.2 Exceptional partial linear spaces; all 2-point lines are excluded.

Corollary 4.1.19. *Let $\mathcal{X}$ be a reduced Klein configuration. Assume that the associated graph has no isolated vertices. Then, $\mathcal{G}(\mathcal{X})$ is a partial linear space in which any point belongs to at most three lines.*

Especially simple situation arises when the graph associated with a reduced Klein configuration $\mathcal{X}$ is complete; in this case the Klein configuration $\mathcal{X}$ is said to be *geometric*. The name is justified by the following statement.

Theorem 4.1.20. *Let $\mathcal{X}$ be a geometric Klein configuration. Then the partial linear space $\mathcal{G}(\mathcal{X})$ is a projective or affine plane of order* 2*, or one of the four linear spaces in Fig.* 4.2*, or has exactly one line.*

Proof. By the hypothesis, for any two distinct points of $\mathcal{G}(\mathcal{X})$ there is a unique line containing both of them. Without loss of generality we may assume that $\mathcal{G}(\mathcal{X})$ has at least two lines. Then it is a linear space in the sense of monograph [**11**]. We claim that this linear space satisfies the following conditions:

(LS1) the number of points is at most 7;
(LS2) each point belongs to at most 3 lines;
(LS3) each line contains at most 3 points.

Indeed, (LS2) follows from Corollary 4.1.19, and (LS1) is a consequence of (LS2) and (LS3). To prove (LS3), we assume the contrary. Then any point not in a line containing at least 4 points belongs to at least 4 lines. However, this is impossible by (LS2).

Checking the list of all small linear spaces given in [**11**, p. 190–191], one can find those satisfying the conditions (LS1)–(LS3). Among them, there are an affine plane of order 2 and the Fano plane; the other four are presented in Fig. 4.2. □

4.1.3 Klein configurations from cubic graphs

A reduced Klein configuration $\mathcal{X}$ is said to be *cubic* if the following condition is satisfied for all $i, j, k \in I$:

$$|G_{ij}| = 2 \text{ and } G_{ij} = G_{ik} \quad \Rightarrow \quad j = k. \tag{4.1.16}$$

In other words, for a fixed $i \in I$ all the groups G_{ij} of order 2 must be different. In particular, the Klein configuration from Example 4.1.17 is not cubic. The following lemma shows how to construct all cubic configurations.

Lemma 4.1.21. *An undirected graph $\mathfrak{X}$ is the graph associated with a cubic configuration if and only if the valency of any vertex of $\mathfrak{X}$ is at most* 3.

Proof. The necessity immediately follows from condition (4.1.16) and the fact that the Klein group G has exactly three subgroups of order 2. Conversely, let $\mathfrak{X}$ be an undirected graph in which the valency of each vertex is at most 3. For each $i \in I$,

- set $G_i = G$ and $G_{ii} = e_i$;
- for any neighbor j of i in $\mathfrak{X}$, choose a proper non-identity subgroup $G_{ij} \leq G$ so that $G_{ij} \neq G_{ik}$ for $j \neq k$;
- put $G_{ij} = G$ for all j other than i and not adjacent with i.

The obtained families $\mathfrak{G} = \{G_i\}$ and $\mathfrak{S} = \{G_{ij}\}$ obviously satisfy the conditions (F1) and (F2), and the relation (4.1.14) holds trivially. Hence by Exercise 4.7.7, there exists a reduced Klein configuration $\mathcal{X}$ such that $\mathcal{T}(\mathcal{X}) = (\mathfrak{G}, \mathfrak{S}, \mathfrak{F})$ for a certain family $\mathfrak{F}$. By the construction, this configuration is cubic and $\mathfrak{X}$ is the graph associated with $\mathcal{X}$. □

Let us study the isomorphisms of a cubic configuration $\mathcal{X}$ that leave each fiber fixed. For an arc (i, j) of the associated graph $\mathfrak{X}$, let $e_{ij} \in \mathrm{Sym}(\Omega_i)$ be a unique involution of the group G_{ij}. Then

$$g_{(i,j)} = \prod_{k \in I} h_k, \quad \text{where} \quad h_k = \begin{cases} e_{ij}, & \text{if } k = i, \\ e_{ji}, & \text{if } k = j, \\ e_k, & \text{otherwise,} \end{cases} \tag{4.1.17}$$

is a permutation of the set Ω.

Lemma 4.1.22. *Given $i, j, a, b \in I$, the permutation $g_{(i,j)}$ leaves each relation in S_{ab} fixed unless*

$$(a, b) \in D \quad \text{and} \quad |\{a, b\} \cap \{i, j\}| = 1. \tag{4.1.18}$$

In the latter case, $g_{(i,j)}$ interchanges the two relations in S_{ab}. In particular,

$$g_{(i,j)} \in \mathrm{Iso}(\mathcal{X}).$$

Proof. Given $s \in S_{ab}$, the definition of $g_{(i,j)}$ implies that

$$(a, b) \notin D \quad \text{or} \quad \{a, b\} \cap \{i, j\} = \varnothing \quad \Rightarrow \quad s^{g_{(i,j)}} = s,$$

where $D = D(\mathfrak{X})$.

Let $(a, b) \in D$. Then the set S_{ab} contains exactly two relations, say s_{ab} and t_{ab}, see Example 4.1.13. If $\{i, j\} = \{a, b\}$, then obviously,

$$(s_{ab})^{g_{(i,j)}} = s_{ab} \quad \text{and} \quad (t_{ab})^{g_{(i,j)}} = t_{ab}.$$

Let $\{a, b\} \cap \{i, j\}$ be a singleton, say $a = i$ and $b \neq j$. Then in the notation of Example 4.1.13, the permutation $g_{(i,j)}$ interchanges the sets Δ_1 and Δ_2, and leaves the sets Γ_1 and Γ_2 fixed (as sets). Thus,

$$(s_{ab})^{g_{(i,j)}} = t_{ab} \quad \text{and} \quad (t_{ab})^{g_{(i,j)}} = s_{ab},$$

as required. □

Let $P = (i_0, \ldots, i_d)$ be a path of length d connecting the vertices i_0 and i_d in the graph $\mathfrak{X}$. Then by Lemma 4.1.22, the permutation

$$g_P = \prod_{k=1}^{d} g_{(i_{k-1}, i_k)} \tag{4.1.19}$$

is an isomorphism of the Klein configuration $\mathcal{X}$. It is easily seen that

$$g_{P^{-1}} = (g_P)^{-1} \quad \text{and} \quad g_{P \cdot Q} = g_P\, g_Q, \tag{4.1.20}$$

where $P^{-1} = (i_d, \ldots, i_0)$ and $P \cdot Q$ is the path obtained by the composition of P and Q (providing that the last vertex of P coincides with the first vertex of Q).

Denote by φ_P the algebraic automorphism of $\mathcal{X}$ induced by g_P,

$$\varphi_P = \varphi_{g_P}. \tag{4.1.21}$$

Obviously, $\varphi_P \in \mathrm{Sym}(S)$ is the identity or an involution identical on each homogeneous component of $\mathcal{X}$.

Lemma 4.1.23. *Let $\mathcal{X}$ be a cubic Klein configuration, $\mathfrak{X}$ the associated graph, and P a path of $\mathfrak{X}$. Then*

1. *if $P = (i, j)$, then $\varphi_P|_{S_{ab}} \neq \mathrm{id}$ if and only if relation (4.1.18) holds;*
2. *if P is a closed path, then $g_P \in \mathrm{Aut}(\mathcal{X})$.*

Proof. Statement (1) follows from Lemma 4.1.22. If the path P is closed, then by statement (1) and the second formula in (4.1.20) the algebraic automorphism φ_P acts trivially on S. This means that $g_P \in \mathrm{Aut}(\mathcal{X})$. □

By statement (2) of Lemma 4.1.23, the automorphism group of a cubic configuration $\mathcal{X}$ contains a subgroup generated by the permutations g_P, where P is a closed path of the graph $\mathfrak{X}$ associated with $\mathcal{X}$. However, this group is trivial if the graph contains no cycles. At the same time, the group $\mathrm{Aut}(\mathcal{X})$ can be as large as possible, for example, if $\mathfrak{X}$ is the empty graph.

There are many non-schurian and non-separable cubic Klein configurations. Any of them has at least four fibers (Exercise 2.7.34). The smallest example of a non-schurian Klein configuration is given below; concerning non-separable examples, see Lemma 4.2.6.

Example 4.1.24. *Let $I = \{1, 2, 3, 4\}$, and let $\mathcal{X}$ be a cubic Klein configuration of degree* 16 *represented by the family $\mathfrak{S} = \{G_{ij}\}$ defined by*

$$\mathfrak{S} = \begin{pmatrix} 1 & C & A & B \\ C & 1 & B & A \\ A & B & 1 & G \\ B & A & G & 1 \end{pmatrix},$$

where A, B, and C are the proper subgroups of G. The graph $\mathfrak{X}$ associated with $\mathcal{X}$ is the same as in Example 4.1.17.

A straightforward check shows that any pair $(\alpha, \beta) \in \Omega_1 \times \Omega_2$ forms a base of the coherent configuration $\mathcal{X}$. This implies that

$$\operatorname{Aut}(\mathcal{X})_{\alpha,\beta} = \operatorname{Aut}(\mathcal{X}_{\alpha,\beta}) = 1.$$

Therefore,

$$|\operatorname{Aut}(\mathcal{X})| \le |\Omega_1|\, n_s = 8 < 16 = |\Omega_3 \times \Omega_4|,$$

where $s = r(\alpha, \beta)$. Consequently, the basis relation $\Omega_3 \times \Omega_4$ of $\mathcal{X}$ cannot be a 2*-orbit of the group* $\operatorname{Aut}(\mathcal{X})$*. Thus the Klein configuration $\mathcal{X}$ is not schurian.*

We complete the subsection by establishing a sufficient condition for a cubic Klein configuration to be schurian; one more condition is given in Exercise 4.7.10).

Theorem 4.1.25. *A cubic Klein configuration is schurian whenever the associated graph is* 3*-connected.*

Proof. We make use of the following property of a 3-connected graph: any its edge $\{i, j\}$ lies in a cycle not containing a given vertex other than i and j. This is true, because the subgraph obtained from the graph in question by removing the vertex is 2-connected and so has a cycle containing the edge $\{i, j\}$.

Let $\mathcal{X}$ be a cubic Klein configuration and $\mathfrak{X} = (I, D)$ the associated graph. Assume that $\mathfrak{X}$ is 3-connected. Given distinct vertices $i, j \in I$, we choose a set $C_{i,j}$ of cycles of $\mathfrak{X}$ as follows:

- if $\{i, j\}$ is an edge of $\mathfrak{X}$, then $C_{i,j}$ consists of a cycle passing through i but not through j, a cycle passing through j but not through i, and a cycle passing through the edge $\{i, j\}$;

- if $\{i, j\}$ is not an edge of $\mathfrak{X}$, then $C_{i,j}$ consists of four distinct cycles, each of which contains an edge $\{x, y\}$ with $x \in \{i, j\}$ and $y \notin \{i, j\}$, and does not contain the vertex in $\{i, j\}$ other than x.

Since the graph $\mathfrak{X}$ is cubic, we may assume that the cycles in $C_{i,j}$ are chosen so that each of them contains an edge not belonging to the other cycles.

Denote by K the group generated by the permutations g_P with $P \in C_{i,j}$ and $i \neq j$. Then $K \leq \mathrm{Aut}(\mathcal{X})$ by statement (2) of Lemma 4.1.23. Therefore,

$$K_{ij} \leq \mathrm{Aut}(\mathcal{X}_{ij}), \tag{4.1.22}$$

where $K_{ij} = K^{\Omega_{ij}}$ and $\mathcal{X}_{ij} = \mathcal{X}_{\Omega_{ij}}$ with $\Omega_{ij} = \Omega_i \cup \Omega_j$.

By the choice of the sets $C_{i,j}$ and formulas (4.1.17) and (4.1.19), we have

$$|K_{ij}| \geq 2^{|C_{i,j}|} = \begin{cases} 8, & \text{if } (i,j) \in D, \\ 16, & \text{otherwise.} \end{cases} \tag{4.1.23}$$

On the other hand, the homogeneous components of $\mathcal{X}_{ij}$ are regular. Therefore any points $\alpha \in \Omega_i$ and $\beta \in \Omega_j$ form a base of the group $\mathrm{Aut}(\mathcal{X}_{ij})$. Consequently if $s = r(\alpha, \beta)$, then

$$|\mathrm{Aut}(\mathcal{X}_{ij})| \leq |\Omega_i|\, n_s = |s| = \begin{cases} 8, & \text{if } (i,j) \in D, \\ 16, & \text{otherwise.} \end{cases}$$

In view of formulas (4.1.23) and (4.1.22), this implies that

$$\mathrm{Aut}(\mathcal{X}_{ij}) = K_{ij}.$$

Since the coherent configuration $\mathcal{X}_{ij}$ is schurian (Exercise 2.7.34), this shows that

$$S_{ij} = \mathrm{Orb}(\mathrm{Aut}(\mathcal{X}_{ij}), \Omega_{ij}) = \mathrm{Orb}(K_{ij}, \Omega_{ij}) = \mathrm{Orb}(K, \Omega_{ij}),$$

for all $i, j \in I$. Taking into account that $K \leq \mathrm{Aut}(\mathcal{X})$, we conclude that the coherent configuration $\mathcal{X}$ is schurian. □

4.2 Highly closed coherent configurations

Theory of multidimensional extensions developed in Section 3.5 leaves a question: to what extent a non-schurian coherent configuration or an algebraic isomorphism not induced by a combinatorial one, can be closed? The answer we obtain in this section is that there exists a constant $\varepsilon > 0$ such that for a sufficiently large n, there is a non-schurian $\lfloor cn \rfloor$-closed coherent configuration of degree n and an $\lfloor cn \rfloor$-automorphism of it that is not induced by a combinatorial one. In the proof of this result we follow [**38**].

In fact, the constant c is less than $1/3$ [**40**]. This inequality is proved with the help of a useful technique generalizing the concept of the t-condition for colored graphs. In the last part of this section, this technique is used to show that the classical schemes of distance-regular graphs are "almost" schurian and separable.

4.2.1 High non-schurity and non-separability

Let us introduce a terminology which is convenient to formulate results in the theory of multidimensional extensions. To this end, let $m \ge 1$ be an integer, $\mathcal{X}$ a coherent configuration, and φ an algebraic isomorphism from $\mathcal{X}$ to another coherent configuration. Theorem 3.5.22 says that the greater m is, the more $\overline{\mathcal{X}}^{(m)}$ and $\overline{\varphi}^{(m)}$ look like a schurian coherent configuration and an algebraic isomorphism induced by isomorphism, respectively.

Definition 4.2.1. *A coherent configuration $\mathcal{X}$ is said to be*

(1) *m-schurian if $\overline{\mathcal{X}}^{(m)} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$;*
(2) *m-separable if $\mathrm{Iso}_m(\mathcal{X}, \mathcal{X}') = \mathrm{Iso}_\infty(\mathcal{X}, \mathcal{X}')$ for all $\mathcal{X}'$.*

From Theorem 3.5.22, it follows that an m-schurian (respectively, m-separable) coherent configuration is m'-schurian (respectively, m'-separable) for all $m' \ge m$. In particular, any schurian coherent configuration is m-schurian for all m, and any separable coherent configuration is m-separable for all m. Furthermore by the same theorem, any coherent configuration of degree n is m-schurian and m-separable for all $m \ge n$.

Definition 4.2.2. *The minimal number m for which $\mathcal{X}$ is m-schurian (respectively, m-separable) is called the schurity number (respectively, the separability number) and denoted by $t(\mathcal{X})$ (respectively, $s(\mathcal{X})$).*

By Theorem 3.5.22,

$$1 \le t(\mathcal{X}) \le b(\mathcal{X}) + 1 \quad \text{and} \quad 1 \le s(\mathcal{X}) \le b(\mathcal{X}) + 1. \tag{4.2.1}$$

Obviously, $t(\mathcal{X}) = 1$ (respectively, $s(\mathcal{X}) = 1$) if and only if the coherent configuration $\mathcal{X}$ is schurian (respectively, separable).

The example of a trivial scheme shows that $s(\mathcal{X})$ and $t(\mathcal{X})$ can be rather far from $b(\mathcal{X})$. On the other hand, there are nontrivial examples for which the equalities are attained. Indeed, let $\mathcal{X}$ be the scheme of a strongly regular graph $\mathfrak{X}$ on 26 points of valency 10 marked as #4 in [**123**, p.176]. Then a straightforward check shows that $b(\mathcal{X}) = 1$. Since the group $\mathrm{Aut}(\mathcal{X})$ is not transitive, the scheme $\mathcal{X}$ is not schurian and hence $t(\mathcal{X}) \geq 2$. In addition, $s(\mathcal{X}) \geq 2$, because there exist several strongly regular graphs with the same parameters as $\mathfrak{X}$.

In some cases, the upper bounds for $t(\mathcal{X})$ and $s(\mathcal{X})$ can be reduced by one. This is an almost immediate consequence of Theorems 3.5.10 and 3.3.19.

Theorem 4.2.3. *Let $\mathcal{X}$ be a coherent configuration admitting a partly regular extension with respect to $m-1$ points ($m \geq 1$). Then $t(\mathcal{X}) \leq m$ and $s(\mathcal{X}) \leq m$.*

Proof. Let $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$ and $\overline{\mathcal{X}} = \overline{\mathcal{X}}^{(m)}$. By Theorem 3.5.10, the coherent configuration $\widehat{\mathcal{X}}$ is partly regular. This implies that $\widehat{\mathcal{X}}$ is schurian and separable (Theorem 3.3.19). Since $\overline{\mathcal{X}}$ is isomorphic to the restriction of $\widehat{\mathcal{X}}$ to a homogeneity set $\Delta = \mathrm{Diag}(\Omega^m)$, the schurity of $\widehat{\mathcal{X}}$ implies the schurity of $\overline{\mathcal{X}}$ (statement (3) of Exercise 2.7.21). Thus, $\mathcal{X}$ is m-schurian and $t(\mathcal{X}) \leq m$.

Let $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}')$ for some $\mathcal{X}'$. Then φ is induced by the algebraic isomorphism

$$\widetilde{\varphi} = \varphi_\eta \, \widehat{\varphi}_\Delta \, \varphi_{\eta'}^{-1}$$

from $\overline{\mathcal{X}}$ to $\overline{\mathcal{X}}'$, where η and η' are the injections defined by formula (3.5.10) for Ω and Ω', and $\widehat{\varphi}$ is the m-dimensional extension of φ. Thus, it suffices to verify that $\widetilde{\varphi}$ is induced by an isomorphism. However, this follows from the separability of $\widehat{\mathcal{X}}$. Thus,

$$\varphi \in \mathrm{Iso}_\infty(\mathcal{X}, \mathcal{X}').$$

Consequently, $\mathcal{X}$ is m-separable and $s(\mathcal{X}) \leq m$. □

In the rest of the subsection, we prove Theorem 4.2.4 below, showing that the inequalities in Theorem 4.2.3 are asymptotically optimal in the sense that the schurity and separability numbers of a coherent configuration $\mathcal{X}$ can be close to its degree up to a linear factor.

Theorem 4.2.4. *There exist a positive real $c < 1$, an infinite sequence of integers $n \geq 1$, and coherent configurations $\mathcal{X}_n$ and $\mathcal{Y}_n$ of degrees n and $2n$, respectively, such that*

$$s(\mathcal{X}_n) \geq \lfloor cn \rfloor \quad \textit{and} \quad t(\mathcal{Y}_n) \geq \lfloor cn \rfloor. \tag{4.2.2}$$

More careful analysis shows that the coherent configurations satisfying these inequalities, exist for all sufficiently large n [**38**] and can be chosen homogeneous (Exercise 4.7.15). The proof of Theorem 4.2.4 is given in the end of the subsection.

First, we establish a sufficient condition for an algebraic isomorphism to be an m-isomorphism. To this end, denote by $F_k(\mathcal{X})$ the set of all unions of at most k fibers of the coherent configuration $\mathcal{X}$.

Lemma 4.2.5. *Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ and k a positive integer. Suppose that*

(i) *for any* $\Delta \in F_k(\mathcal{X})$, *there exists* $h(\Delta) \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ *such that*

$$(\varphi_{h(\Delta)})_\Delta = \varphi_\Delta,$$

where $\varphi_{h(\Delta)}$ *is the algebraic isomorphism induced by* $h(\Delta)$*;*

(ii) *for any* $\Delta, \Gamma \in F_k(\mathcal{X})$, *there exists* $h(\Delta, \Gamma) \in \mathrm{Aut}(\mathcal{X})$ *such that*

$$h(\Delta)^{\Delta \cap \Gamma} = h(\Delta, \Gamma)^{\Delta \cap \Gamma}\, h(\Gamma)^{\Delta \cap \Gamma}.$$

Then for any positive integer m *such that* $3m \le k$, *we have* $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}')$.

Proof. Let s be a basis relation of the coherent configuration $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$. Then there exist $s_1, \ldots, s_m \in S$ such that $s \subseteq s_1 \otimes \cdots \otimes s_m$. It follows that $s \in S(\widehat{\mathcal{X}})_{\Delta^m}$, where $\Delta = \Omega(s_1) \cup \cdots \Omega(s_m)$. In particular,

$$\Delta \in F_{2m}(\mathcal{X}) \subseteq F_k(\mathcal{X}).$$

Now if $\Omega(s) \subseteq \Gamma^m$ for some $\Gamma \in F_k(\mathcal{X})$, then the condition (ii) implies that

$$s^{h(\Delta)} = s^{h(\Delta,\Gamma) h(\Gamma)} = s^{h(\Gamma)}.$$

Thus the basis relation $s' = s^{h(\Delta)}$ of the coherent configuration $\widehat{\mathcal{X}'} = \widehat{\mathcal{X}'}^{(m)}$ does not depend on $\Delta \in F_k(\mathcal{X})$. This enables us to define a mapping

$$\psi : S(\widehat{\mathcal{X}}) \to S(\widehat{\mathcal{X}'}),\ s \mapsto s'.$$

Obviously, ψ is a bijection.

Now let $r, s \in S(\widehat{\mathcal{X}})$. Assuming $r \cdot s \ne \varnothing$, one can find a homogeneity set $\Delta \in F_{3m}(\mathcal{X})$ such that $r, s \in S(\widehat{\mathcal{X}})_{\Delta^m}$. Since $3m \le k$, the condition (i) yields

$$\psi'(A_r\, A_s) = (A_r\, A_s)^{h(\Delta)} = (A_r)^{h(\Delta)}\, (A_s)^{h(\Delta)} = \psi'(A_r)\, \psi'(A_s),$$

where $\psi' : \mathrm{Adj}(\widehat{\mathcal{X}}) \to \mathrm{Adj}(\widehat{\mathcal{X}'})$ is a linear isomorphism induced by ψ as in (2.3.14). By Proposition 2.3.17, this implies that

$$\psi \in \mathrm{Iso}_{\mathrm{alg}}(\widehat{\mathcal{X}}, \widehat{\mathcal{X}'}).$$

In view of the condition (i), the restriction of the algebraic isomorphism ψ to Δ^m extends the m-extension of the algebraic isomorphism φ_Δ for all $\Delta \in F_k(\mathcal{X})$. Therefore, ψ is the m-extension of φ. □

We are going to apply Lemma 4.2.5 to a cubic Klein configuration $\mathcal{X}$ with fibers Ω_a, $a \in I$, and an algebraic automorphism of $\mathcal{X}$ defined as follows.

Denote by $\mathfrak{X}$ the graph associated with $\mathcal{X}$, see p. 234. For each arc (a, b) of this graph, the set

$$S_{ab} = S(\mathcal{X})_{\Omega_a, \Omega_b}$$

consists of exactly two relations; if s is one of them, then the other is denoted by s'. It is easily seen that the mapping $\psi_{a,b} : S(\mathcal{X}) \to S(\mathcal{X})$ such that

$$\psi_{a,b}(s) = \begin{cases} s', & \text{if } s \in S_{ij} \text{ and } \{i,j\} = \{a,b\}, \\ s, & \text{otherwise} \end{cases}$$

is an algebraic automorphism of the coherent configuration $\mathcal{X}$. It follows that so is the mapping

$$\psi_a = \prod_{b \in \mathfrak{X}(a)} \psi_{a,b}, \tag{4.2.3}$$

where $\mathfrak{X}(a)$ is the neighborhood of a in $\mathfrak{X}$. Clearly, this algebraic isomorphism is an involution (which is the product of at most three transpositions).

In the following lemma presenting important properties of ψ_a, under a separator of the graph $\mathfrak{X}$, we mean a set J of its vertices such that each connected component of the graph $\mathfrak{X} - J$ (obtained from $\mathfrak{X}$ by removing all the vertices of J) contains at most half vertices of $\mathfrak{X}$.

Lemma 4.2.6. *In the above notation, suppose that the graph $\mathfrak{X}$ is cubic. Then for any vertex a of this graph,*

(1) *the algebraic isomorphism ψ_a is not induced by a permutation;*
(2) *$\psi_a \in \mathrm{Iso}_m(\mathcal{X})$ if $\mathfrak{X}$ is connected and has no separators of cardinality less than or equal to $3m$.*

Proof. Denote by Φ the set of algebraic automorphisms of $\mathcal{X}$ leaving each fiber of $\mathcal{X}$ fixed. For any $\varphi \in \Phi$, set $t(\varphi)$ to be the cardinality of the set

$$T(\varphi) = \{\{i,j\} \subset I : \ \varphi^{S_{ij} \cup S_{ji}} \ne \mathrm{id}\}.$$

One can easily see that

$$t(\psi_a) = 3.$$

Therefore to prove statement (1), it suffices to verify that for any permutation $g \in \mathrm{Sym}(G^I)$ (recall that G is the Klein group associated with $\mathcal{X}$) such that $\varphi_g \in \Phi$, we have

$$t(\varphi_g) = 0 \pmod 2. \tag{4.2.4}$$

However, if $g, h \in \mathrm{Sym}(G^I)$ are such that $\varphi_g, \varphi_h \in \Phi$, then

$$t(\varphi_{gh}) = t(\varphi_g) + t(\varphi_h) - 2|T(\varphi_g) \cap T(\varphi_h)|,$$

which implies that

$$t(\varphi_{gh}) = t(\varphi_g) + t(\varphi_h) \pmod 2.$$

On the other hand, since $\mathfrak{X}$ is a cubic graph, the definition of cubic Klein configuration implies that $t(\varphi_g) = 2$ for each involution $g \in G^I$ such that $g_i \ne \mathrm{id}$ for exactly one $i \in I$. Thus formula (4.2.4) holds for every $\varphi_g \in \Phi$.

To prove statement (2), it suffices to verify that the conditions (i) and (ii) of Lemma 4.2.5 are satisfied for $\varphi = \psi_a$ and $k = 3m$. Let $\Delta \in F_k(\mathcal{X})$. Denote by C_Δ the vertex set of a largest connected component of the graph $\mathfrak{X} - J_\Delta$, where

$$J_\Delta = \{i \in I :\ \Omega_i \subset \Delta\}.$$

Choose arbitrarily a vertex $b_\Delta \in C_\Delta$ and a path $P = P_\Delta$ connecting a and b in the graph $\mathfrak{X}$. Then obviously, $\Omega_b \not\subset \Delta$. So by statement (1) of Lemma 4.1.23,

$$\varphi_P = \psi_a \psi_b,$$

where φ_P is the algebraic isomorphism defined by formula (4.1.21). Therefore,

$$(\varphi_P)_\Delta = (\psi_a)_\Delta.$$

Thus the condition (i) is satisfied for the isomorphisms $h(\Delta) := g_P$.

Let $\Delta, \Gamma \in F_k(\mathcal{X})$. By the hypothesis of statement (2), the sets J_Δ and J_Γ are not separators of the graph $\mathfrak{X}$. Therefore, both C_Δ and C_Γ contain more than half vertices of $\mathfrak{X}$. Consequently,

$$C_\Delta \cap C_\Gamma \neq \varnothing.$$

It follows that there exists a path P' connecting the vertices b_Δ and b_Γ in the subgraph of $\mathfrak{X}$ induced by the set $C_\Delta \cup C_\Gamma$. In particular, no vertex of P' belongs to $J_\Delta \cap J_\Gamma$ and hence

$$(g_{P'})^{\Delta\cap\Gamma} = \mathrm{id}\,.$$

Now if $P = P_\Delta \cdot P' \cdot P_\Gamma^{-1}$, then in view of (4.1.20), we have

$$(g_P)^{\Delta\cap\Gamma} = (g_{P_\Delta} g_{P'} g_\Gamma^{-1})^{\Delta\cap\Gamma} = h(\Delta)^{\Delta\cap\Gamma}\,(h(\Gamma)^{-1})^{\Delta\cap\Gamma},$$

Thus the condition (ii) is satisfied for $h(\Delta, \Gamma) := g_P$, because P is a closed path and hence $g_P \in \mathrm{Aut}(\mathcal{X})$ (statement (2) of Lemma 4.1.23). □

To apply Lemma 4.2.6, we need to find cubic graphs without small separators. To this end, let $\mathfrak{X}$ be an undirected graph with vertex set Ω, and let $\varepsilon > 0$ be a real number.

We say that $\mathfrak{X}$ is an *ε-expander* if

$$|\partial\Gamma| \geq \varepsilon|\Gamma| \tag{4.2.5}$$

for every set $\Gamma \subset \Omega$ containing at most half of the vertices of $\mathfrak{X}$, where $\partial\Gamma$ is the set of all vertices outside Γ and adjacent to at least one vertex of Γ. Some explicit constructions of infinite families of ε-expanders for a fixed ε can be found in [**29**].

Lemma 4.2.7. *Let $\mathfrak{X}$ be a ε-expander with n vertices. Then $\mathfrak{X}$ has no separator of cardinality less than $\lfloor c_\varepsilon n \rfloor$, where $c_\varepsilon = \frac{\varepsilon}{4+\varepsilon}$.*

Proof. Let Δ be a separator of $\mathfrak{X}$. Denote by k the minimal number of classes in the partition $\Omega \setminus \Delta$ into the sets $\Gamma_1, \ldots, \Gamma_k$ such that for each index $i = 1, \ldots, k$,

$$\partial\Gamma_i \subseteq \Delta \quad \text{and} \quad |\Gamma_i| \le n/2. \tag{4.2.6}$$

Note that k is less than or equal to the number of connected components of the graph $\mathfrak{X} - \Delta$. Furthermore among the Γ_i, there are no two distinct sets each of cardinality less than $n/4$, for otherwise one can replace them by their union. Consequently, $k \le 4$.

On the other hand, conditions (4.2.6) imply that

$$|\Delta| \ge |\partial\Gamma_i| \ge \varepsilon|\Gamma_i|$$

for all $i = 1, \ldots, k$. Thus,

$$4\,|\Delta| \ge \sum_{i=1}^{k} |\Delta| \ge \varepsilon \sum_{i=1}^{k} |\Gamma_i| = \varepsilon(n - |\Delta|).$$

It follows that $(4+\varepsilon)\,|\Delta| \ge \varepsilon n$, as required. □

Proof of Theorem 4.2.4. In accordance with [**97**, Theorem 5.13], for every integer $d \ge 1$ there exists a cubic vertex-transitive graph $\mathfrak{X}_d$ with $n(d) = 2^{3d} - 2^d$ vertices that is an ε-expander for a positive real constant $\varepsilon < 1$.[1] Our goal is to verify that inequalities (4.2.2) hold for the constant

$$c = c_\varepsilon/12 \quad \text{and} \quad n = 4n(d),$$

the cubic coherent configuration $\mathcal{X} := \mathcal{X}_n$ associated with the graph $\mathfrak{X} = \mathfrak{X}_d$, and the coherent configuration $\mathcal{Y} := \mathcal{Y}_n$, which will be defined later.

Denote by φ the algebraic isomorphism ψ_a given by formula (4.2.3), where a is a vertex of $\mathfrak{X}$. By Lemma 4.2.7, this graph has no separators of size less than or equal to $k = \lfloor c_\varepsilon\, n/4 \rfloor - 1$. Therefore by Lemma 4.2.6,

$$\varphi \in \mathrm{Iso}_m(\mathcal{X}) \setminus \mathrm{Iso}_\infty(\mathcal{X}), \tag{4.2.7}$$

where $m = \lfloor k/3 \rfloor = \lfloor cn \rfloor$, that proves the first inequality in (4.2.2).

The graph $\mathfrak{X}$ being cubic and vertex-transitive, must be 3-connected [**122**, Lemma 4.1]. By Theorem 4.1.25, this implies that the coherent configuration $\mathcal{X}$ is schurian. Set

$$\mathcal{Y} = (\mathcal{X}_1 \boxplus \mathcal{X}_2)^{\langle\psi\rangle},$$

where the coherent configurations $\mathcal{X}_1 = (\Omega_1, S_1)$ and $\mathcal{X}_2 = (\Omega_2, S_2)$ are disjoint copies of $\mathcal{X}$ and the algebraic automorphism ψ of the direct sum is induced by the algebraic isomorphisms

$$\psi_1 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}_2) \quad \text{and} \quad \psi_2 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}_1)$$

[1] In fact, these graphs are cubic Ramanujan graphs and, in particular, are ε-expanders for $\varepsilon \ge (3 - 2\sqrt{2})/2$, see, e.g., [**98**, Sec. 4].

that coincide with φ. Note that the coherent configuration $\mathcal{X}_1 \boxplus \mathcal{X}_2$ is schurian by Corollary 3.2.6, and

$$\psi \in \mathrm{Iso}_m(\mathcal{X}_1 \boxplus \mathcal{X}_2)$$

by statement (3) of Exercise 3.7.49. Thus the coherent configuration $\mathcal{Y}$ is m-closed by Theorem 3.5.24.

It remains to verify that $\mathcal{Y}$ is not schurian. To this end, denote by e the equivalence relation with classes Ω_1 and Ω_2. It is easily seen that e is a parabolic of $\mathcal{Y}$. Assume on the contrary that the coherent configuration $\mathcal{Y}$ is schurian. Then

$$\Omega_1^f = \Omega_2 \quad \text{for some } \ f \in \mathrm{Aut}(\mathcal{Y}),$$

because $\mathrm{Aut}(\mathcal{Y})$ acts transitively on each fiber of $\mathcal{Y}$ and such a fiber intersects both Ω_1 and Ω_2.

To get a contradiction, let $s \in S(\mathcal{X}_1) \subseteq S(\mathcal{X})$. Then

$$s^{\langle\psi\rangle} = s \cup \psi(s).$$

is a basis relation of $\mathcal{Y}$. The automorphism f preserves $s^{\langle\psi\rangle}$. It follows that

$$s^f \subseteq (s \cup \psi(s)) \cap \Omega_2^2 = \psi(s).$$

Since $|s^f| = |s| = |\psi(s)|$, this implies that $s^f = \psi(s) = \varphi(s)$. This implies that f induces φ contrary to (4.2.7). $\square$

4.2.2 Highly regular graphs and the t-condition

The aim of this subsection is twofold. First, we show that the basis relations of the multidimensional extension of a coherent configuration are highly regular in a combinatorial sense, which is close to the concept of the t-condition. Using the developed technique, we can improve a little bit the trivial upper bounds (4.2.1) for the schurity and separability numbers of a coherent configuration. More precisely, the following statement is true.

Theorem 4.2.8. *For any coherent configuration $\mathcal{X}$ of degree n,*

$$t(\mathcal{X}) \le \lceil n/3 \rceil \quad \textit{and} \quad s(\mathcal{X}) \le \lceil n/3 \rceil.$$

Theorem 4.2.8 will be proved in end of the subsection and the proof is based on the following concept generalizing the concept of the t-condition, see p. 99. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be colored graphs.

Definition 4.2.9. *An injective mapping $h : \Omega(\mathfrak{Y}) \to \Omega(\mathfrak{X})$ is called an embedding of $\mathfrak{Y}$ into $\mathfrak{X}$ if*

$$D(\mathfrak{Y})^h \subseteq D(\mathfrak{X}) \quad \textit{and} \quad c_{\mathfrak{X}}(\alpha^h, \beta^h) = c_{\mathfrak{Y}}(\alpha, \beta) \text{ for all } (\alpha, \beta) \in D(\mathfrak{Y}),$$

where $c_{\mathfrak{X}}$ and $c_{\mathfrak{Y}}$ are the coloring of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. The set of all embeddings of $\mathfrak{Y}$ into $\mathfrak{X}$ is denoted by $\mathrm{Emb}(\mathfrak{Y}, \mathfrak{X})$.

Given a mapping g from a subset of $\Omega(\mathfrak{Y})$ to $\Omega(\mathfrak{X})$, we define a number

$$q_{\mathfrak{X}}(\mathfrak{Y}, g) = |\{h \in \mathrm{Emb}(\mathfrak{Y}, \mathfrak{X}) : \ h|_{\mathrm{Dom}(g)} = g\}|, \tag{4.2.8}$$

where $\mathrm{Dom}(g)$ is the domain of g. Thus the number $q_{\mathfrak{X}}(\mathfrak{Y}, g)$ is equal to the number of ways to extend g to an embedding of $\mathfrak{Y}$ to $\mathfrak{X}$.

Let $\mathfrak{X}$ be a colored graph on Ω and m a positive integer. Given a colored graph $\mathfrak{Y}$, its subgraph $\mathfrak{Z}$ with

$$\Omega(\mathfrak{Y}) \subseteq \{1, \ldots, 3m\} \quad \text{and} \quad \Omega(\mathfrak{Z}) \subseteq \{1, \ldots, 2m\}, \tag{4.2.9}$$

and an integer $d \ge 0$, denote by $s_{\mathfrak{X}}(\mathfrak{Y}, \mathfrak{Z}, d)$ a relation consisting of all pairs $(\alpha, \beta) \in \Omega^m \times \Omega^m$ such that there exists $g \in \mathrm{Emb}(\mathfrak{Z}, \mathfrak{X})$ for which

$$q_{\mathfrak{X}}(\mathfrak{Y}, g) = d \quad \text{and} \quad (\alpha \cdot \beta)_i = i^g \text{ for all } i \in \Omega(\mathfrak{Z}). \tag{4.2.10}$$

Definitely, the relation $s_{\mathfrak{X}}(\mathfrak{Y}, \mathfrak{Z}, d)$ is empty if $|\Omega|$ less than $|\Omega(\mathfrak{Y})|$.

If $m = 1$, $\mathfrak{X} = \mathfrak{X}(\mathcal{X})$ is a colored graph of a coherent configuration $\mathcal{X}$, and the graphs $\mathfrak{Z}$ and $\mathfrak{Y}$ have two and three vertices, respectively, then $s_{\mathfrak{X}}(\mathfrak{Y}, \mathfrak{Z}, d)$ is the union (possibly empty) of color classes of $\mathfrak{X}$: indeed, the numbers $q_{\mathfrak{X}}(\cdot, \cdot)$ are equal to sums of the intersection numbers of $\mathcal{X}$. The following statement generalizes this observation to an arbitrary m, and gives one more way to construct relations of the m-dimensional extension.

Lemma 4.2.10.

(1) $s_{\mathfrak{X}}(\mathfrak{Y}, \mathfrak{Z}, d)$ *is a relation of the coherent configuration* $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$*;*

(2) *if φ is an m-isomorphism from $\mathcal{X}$ to another coherent configuration, then*
$$s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z},d)^{\widehat{\varphi}} = s_{\mathfrak{X}^{\varphi}}(\mathfrak{Y},\mathfrak{Z},d)$$
where $\widehat{\varphi} = \widehat{\varphi}^{(m)}$ and $\mathfrak{X}^{\varphi}$ is as in Exercise 2.7.52.

Proof. First, we assume that $\mathfrak{Y} = \mathfrak{Z}$ and $d = 1$. Then $s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z},d)$ is equal to the intersection of the relations

$$\widehat{s}_{ij} = \{(\alpha,\beta) \in \Omega^m \times \Omega^m : \ ((\alpha\cdot\beta)_i, (\alpha\cdot\beta)_j) \in s_{ij}\}$$

with $(i,j) \in D(\mathfrak{Y})$, where $s_{ij} = c_{\mathfrak{X}}^{-1}(c_{\mathfrak{Y}}(i,j))$. Thus the required statements are immediate consequences of the corresponding statements of Exercise 3.7.46.

Now let the graphs $\mathfrak{Y}$, $\mathfrak{Z}$, and integer d be arbitrary. Denote by Q the set of all triples $(\alpha,\beta,\gamma) \in \Omega^m \times \Omega^m \times \Omega^m$ such that

$$c_{\mathfrak{X}}((\alpha\cdot\beta\cdot\gamma)_i, (\alpha\cdot\beta\cdot\gamma)_j) = c_{\mathfrak{Y}}(i,j)$$

for all $i,j \in \Omega(\mathfrak{Y})$, and by $\mathrm{pr}_{a,b}(Q) \subseteq \Omega^m \times \Omega^m$ the projection of Q to the coordinates a and b, where $a,b \in \{1,2,3\}$. Thus,

$$u := \mathrm{pr}_{1,2}(Q), \quad v := \mathrm{pr}_{2,3}(Q), \quad w := \mathrm{pr}_{1,3}(Q)$$

are relations on Ω^m. Each of them is of the form $s_{\mathfrak{X}}(\mathfrak{Y}',\mathfrak{Y}',1)$, where $\mathfrak{Y}'$ is a colored graph on $\{1,\ldots,2m\}$. Therefore, both statements of the lemma hold true for u, v, and w due to the first part of the proof.

Next, denote by u' and v' the intersection of the relations $\mathrm{Cyl}_{1_\Omega}(i,i)$ for all $i \in \Omega(\mathfrak{Z})$ and for all i with $i+m \in \Omega(\mathfrak{Z})$, respectively. Then the relation $s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z},d)$ consists of exactly those pairs $(\mu,\nu) \in \Omega^m \times \Omega^m$ for which the number of triples $(\alpha,\beta,\gamma) \in \Omega^m \times \Omega^m \times \Omega^m$ such that

$$(\mu,\alpha) \in u', \quad (\alpha,\beta) \in u, \quad (\beta,\gamma) \in v, \quad (\gamma,\nu) \in v', \quad (\alpha,\gamma) \in w$$

is equal to the number $k = dn^{3m-n_{\mathfrak{Y}}}$, where $n_{\mathfrak{Y}} = |\Omega(\mathfrak{Y})|$: indeed each embedding $h \in \mathrm{Emb}(\mathfrak{Y},\mathfrak{X})$ corresponds to exactly $n^{3m-n_{\mathfrak{Y}}}$ triples such that

$$i^h = (\alpha\cdot\beta\cdot\gamma)_i, \quad i \in \Omega(\mathfrak{Y}).$$

On the other hand, it is easily seen that

$$k = ((A_{u'}A_uA_vA_{v'}) \circ A_w)_{\mu,\nu}.$$

Note that the adjacency matrices A_u, A_v, A_w and $A_{u'}$, $A_{v'}$ belong to the algebra $\mathrm{Adj}(\widehat{\mathcal{X}})$: the first three by the above paragraph, and the second two by statement (1) of Theorem 3.5.7. This proves statement (1) by the Wielandt principle (Proposition 2.3.10), and also statement (2) again by the first part of the proof and statement (2) of Theorem 3.5.7. □

Let us introduce a terminology convenient to deal with relations from Lemma 4.2.10. Suppose that inclusions (4.2.9) hold for a colored graph $\mathfrak{Y}$ and its subgraph $\mathfrak{Z}$.

Denote by $s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z})$ the union of all relations $s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z},d)$, where d runs over nonnegative integers. The (colored) graph $\mathfrak{X}$ on Ω is said to be $(\mathfrak{Y},\mathfrak{Z})$-regular of degree $d \geq 0$ with respect to a relation r on Ω^m if

$$r \cap s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z}) \subseteq s_{\mathfrak{X}}(\mathfrak{Y},\mathfrak{Z},d).$$

Note that if the left-hand side is nonempty, then the number d is uniquely determined by the graphs $\mathfrak{Y}$ and $\mathfrak{Z}$, and the relation r; if the left-hand side is empty, then any nonnegative integer can be taken as d. This enables us to omit mentioning the degree, and say briefly that $\mathfrak{X}$ is $(\mathfrak{Y},\mathfrak{Z})$-regular with respect to r. In these terms Lemma 4.2.10 can be formulated as follows.

Corollary 4.2.11. *Given $r \in S(\widehat{\mathcal{X}})$ and for all admissible graphs $\mathfrak{Y},\mathfrak{Z}$,*

(1) *the graph $\mathfrak{X}$ is $(\mathfrak{Y},\mathfrak{Z})$-regular with respect to r;*
(2) *if $\mathfrak{X}$ is $(\mathfrak{Y},\mathfrak{Z})$-regular with respect to r, then $\mathfrak{X}^\varphi$ is $(\mathfrak{Y},\mathfrak{Z})$-regular of the same degree with respect to $\widehat{\varphi}(r)$.*

Let $\mathfrak{X}$ be an $(\mathfrak{Y},\mathfrak{Z})$-regular colored graph of degree d with respect to the relation $r = \Omega^m \times \Omega^m$. Then obviously,

$$q_{\mathfrak{X}}(\mathfrak{Y},g) = d \quad \text{for all } \ g \in \mathrm{Emb}(\mathfrak{Z},\mathfrak{X}).$$

Any colored graph $\mathfrak{X}$ satisfying this condition for a colored graph $\mathfrak{Y}$, its subgraph $\mathfrak{Z}$,[2] and a nonnegative integer d, is said to be $(\mathfrak{Y},\mathfrak{Z})$-*regular* of degree d.

Example 4.2.12. *Let $\mathfrak{X}$ be an undirected graph, and let $c_{\mathfrak{X}}$ be a coloring of $\mathfrak{X}$ such that $c_{\mathfrak{X}}(\alpha,\beta) = 1$ or 2 depending on whether or not (α,β) is an arc of $\mathfrak{X}$ or not. Then $\mathfrak{X}$ is d-regular if and only if the corresponding colored graph is $(\mathfrak{Y},\mathfrak{Z})$-regular of degree d, where*

$$\Omega(\mathfrak{Z}) = \{1\}, D(\mathfrak{Z}) = \varnothing, \quad \Omega(\mathfrak{Y}) = \{1,2\}, D(\mathfrak{Y}) = \{(1,2)\}, \quad c_{\mathfrak{Y}}(1,2) = 1.$$

The concept of $(\mathfrak{Y},\mathfrak{Z})$-regularity enables us to give an equivalent definition for the t-condition introduced at p. 100 for colored graphs. Namely, the following statement is straightforward.

Lemma 4.2.13. *Let $t \geq 2$ be an integer and $\mathfrak{X}$ a complete colored graph. Then $\mathfrak{X}$ satisfies the t-condition if and only if it is $(\mathfrak{Y},\mathfrak{Z})$-regular for any colored graph $\mathfrak{Y}$ with vertex set $\{1,\dots,t\}$ and any of its subgraphs $\mathfrak{Z}$ of the form*

$$\Omega(\mathfrak{Z}) = \{i,j\} \quad \textit{and} \quad D(\mathfrak{Z}) = \{(i,j)\}, \tag{4.2.11}$$

where $1 \leq i,j \leq t$.

We arrive at the main result of this section establishing a relationship between the m-closed coherent configurations and coherent configurations whose colored graphs satisfy the m-condition.

[2]We do not assume here that conditions (4.2.9) are satisfied.

Theorem 4.2.14. *A colored graph of an m-closed coherent configuration satisfies the $3m$-condition.*

Proof. Let $\mathcal{X}$ be an m-closed coherent configuration. By Lemma 4.2.13, it suffices to verify that the complete colored graph $\mathfrak{X} = \mathfrak{X}(\mathcal{X})$ is $(\mathfrak{Y}, \mathfrak{Z})$-regular for any colored graph $\mathfrak{Y}$ with

$$\Omega(\mathfrak{Y}) = \{1, \ldots, 3m\}$$

and any of its subgraphs $\mathfrak{Z}$ having the form (4.2.11). By statement (1) of Corollary 4.2.11, the graph $\mathfrak{X}$ is $(\mathfrak{Y}, \mathfrak{Z})$-regular of a certain degree $d(r)$ with respect to any $r \in \widehat{S}$, where $\widehat{S}$ is the set of basis relations of the m-extension of $\mathcal{X}$.

Let $g \in \mathrm{Emb}(\mathfrak{Z}, \mathfrak{X})$. In the notation of Exercise 3.7.46, set

$$R_g = \{r \in \widehat{S} : \ (i^g, j^g) \in \mathrm{pr}_{i,j}(r)\},$$

recall that i and j are the vertices of $\mathfrak{Z}$. Since the coherent configuration $\mathcal{X}$ is m-closed, statement (1) of that exercise implies that R_g does not depend on g. It follows that the number

$$q_{\mathfrak{X}}(\mathfrak{Y}, g) = \sum_{r \in R_g} d(r) =: d$$

also does not depend on g. Thus the graph $\mathfrak{X}$ is $(\mathfrak{Y}, \mathfrak{Z})$-regular of degree d, as required. □

Proof of Theorem 4.2.8. Let $\mathfrak{X}$ be a colored graph of a coherent configuration $\mathcal{X}$ of degree n. Denote by $\mathfrak{Y}$ a colored graph on $\{1, \ldots, n\}$ isomorphic to $\mathfrak{X}$, and by $\mathfrak{Z}$ the empty subgraph of $\mathfrak{X}$ with exactly one vertex. By statement (1) of Corollary 4.2.11, the graph $\mathfrak{X}$ is $(\mathfrak{Y}, \mathfrak{Z})$-regular of positive degree with respect to a basis relation r of the m-extension of $\mathcal{X}$, where

$$m = \lceil n/3 \rceil.$$

By Theorem 4.2.14 the colored graph of the m-closure $\overline{\mathcal{X}}$ of $\mathcal{X}$ satisfies the $3m$-condition and hence the n-condition. Consequently, the coherent configuration $\overline{\mathcal{X}}$ is schurian (Exercise 2.7.62). Thus,

$$t(\mathcal{X}) \leq m = \lceil n/3 \rceil.$$

Now let φ be an algebraic isomorphism from $\mathcal{X}$ to another coherent configuration. Then by statement (2) of Corollary 4.2.11, the graph $\mathfrak{X}^{\varphi}$ is $(\mathfrak{Y}, \mathfrak{Z})$-regular of the same positive degree as $\mathcal{X}$ with respect to $\varphi(r)$. The definition of the $(\mathfrak{Y}, \mathfrak{Z})$-regularity implies that that $\mathfrak{X}^{\varphi}$ is isomorphic to $\mathfrak{Y}$. It follows that the composition isomorphism from $\mathfrak{X}$ to $\mathfrak{X}^{\varphi}$ via $\mathfrak{Y}$ belongs to the set $\mathrm{Iso}(\mathcal{X}, \mathcal{X}^{\varphi}, \varphi)$ (Exercise 2.7.52). Thus,

$$s(\mathcal{X}) \leq m = \lceil n/3 \rceil,$$

and we are done. □

4.2.3 Schurity and separability numbers of classical schemes

In this subsection, we calculate the schurity and separability numbers for several classes of coherent configurations. Some of the results here are not so deep: they are just interpretations of what is known on the objects in question (projective planes and distance-regular graphs). However, we decide to include this material to show how the theory of multidimensional coherent configurations can be used to establish the results of such a kind in a uniform way. The theorems below are taken from [**45, 40**].

Theorem 4.2.15. *Let $\mathcal{X}$ be the coherent configuration associated with a projective plane $\mathcal{P}$ of order q. Then*

(1) $t(\mathcal{X}) \le O(\log\log q)$ *and* $s(\mathcal{X}) \le O(\log\log q)$*;*
(2) $t(\mathcal{X}) = 1$ *if and only if* $\mathcal{P}$ *is a Galois plane;*
(3) $s(\mathcal{X}) \le 3$ *whenever* $\mathcal{P}$ *is a Galois plane.*

Proof. Let P and L be the sets of points and lines of $\mathcal{P}$, respectively. A subset of $\Omega := P \cup L$ is said to be closed if it contains each line (respectively, each point) incident to two different points (respectively, two different lines) belonging to the subset. The following statement is straightforward.

Lemma 4.2.16. *Any closed subset of $\mathcal{P}$ that contains a quadrangle forms a subplane of $\mathcal{P}$.*

If a minimal closed set containing $\Delta \subseteq \Omega$ coincides with Ω, then Δ is called a generating set of $\mathcal{P}$. In accordance with Exercise 4.7.19, any generating set of $\mathcal{P}$ is a base of the coherent configuration $\mathcal{X}$.

From [**31**, Theorem 3.2.17], it follows that a Galois plane is generated by a quadrangle and a suitable point on one of its sides. So if $\mathcal{P}$ is a Galois plane, then

$$b(\mathcal{X}) \le 5.$$

Let $\mathcal{P}$ be a projective plane and Ω' the minimal closed subset containing a quadrangle Q. If $\Omega' = \Omega$, then Q is a generating set of $\mathcal{P}$ and hence $b(\mathcal{X}) \le 4$. Otherwise $\mathcal{P}$ contains a proper subplane $\mathcal{P}'$ (Lemma 4.2.16).

Without loss of generality we may assume that $\mathcal{P}'$ is a maximal proper subplane of $\mathcal{P}$. Then any generating set of $\mathcal{P}'$ together with any element in $\Omega \setminus \Omega'$ form a generating set of $\mathcal{P}$. This implies that

$$b(\mathcal{X}) \le 1 + b(\mathcal{X}'), \tag{4.2.12}$$

where $\mathcal{X}'$ is the coherent configuration associated with $\mathcal{P}'$.

On the other hand, from [**31**, Theorem 3.2.18], it follows that given a projective plane $\mathcal{P}$ of order q, any proper closed set of $\mathcal{P}$ containing a quadrangle is a subplane $\mathcal{P}'$ of $\mathcal{P}$ of order at most $\sqrt{q}$. By induction this fact and inequality (4.2.12) imply that

$$b(\mathcal{X}) \le O(\log\log q)$$

and statement (1) follows from inequalities (4.2.1).

Statement (2) follows from Theorem 2.5.3.

To prove statement (3), we find certain relations of the 3-dimensional extension $\widehat{\mathcal{X}}$ of the coherent configuration $\mathcal{X}$. All of them are obtained from the basis relations (2.5.1) and (2.5.2) of $\mathcal{X}$ with the help of statement (1) of Theorem 3.5.7. In what follows, $Q \in \{P, L\}$.

First, we denote by Δ_Q the set of all triangles formed by triples of non-collinear elements of Q,

$$\Delta_Q = \{\alpha \in Q^3 |\text{ for no }\ \beta \in Q' :\ (\alpha_i, \beta) \in s_Q,\ 1 \le i \le 3\},$$

where $Q' = L$ or P and $s_Q = s_5$ or s_6 depending on whether $Q = P$ or L. Then

$$\Delta_P, \Delta_L \in F(\widehat{\mathcal{X}})^{\cup},$$

because Δ_Q is equal to the complement of $\Omega_-(t_Q)$ in Q^3, where

$$t_Q = (Q^3 \times \mathrm{Diag}(\Omega^3)) \cap \mathrm{Cyl}_{s_Q}(1,1) \cap \mathrm{Cyl}_{s_Q}(2,1) \cap \mathrm{Cyl}_{s_Q}(3,1).$$

Denote by r_Q a relation on Ω^3 consisting of all $(\alpha, \beta) \in \Delta_Q \times \Delta_Q$ such that there exists $\gamma \in Q$ for which

$$\gamma \in \alpha_i\beta_i, \quad i = 1, 2, 3,$$

where $\alpha_i\beta_i$ is the element of Ω incident to both α_i and β_i. Thus any pair belonging to r_P (respectively, r_L) consists of two triangles formed by three non-collinear points (respectively, three lines without common point) that are in perspective from some point (respectively, line), see Fig. 2.4.

One can see that r_Q is a relation of $\widehat{\mathcal{X}}$, because

$$r_Q = x_Q \cap y_Q, \tag{4.2.13}$$

where

$$x_Q = \Delta_Q \times \Omega_-(t_{Q'}) \quad \text{and} \quad y_Q = \mathrm{Cyl}_{s_Q}(1,1) \cap \mathrm{Cyl}_{s_Q}(2,2) \cap \mathrm{Cyl}_{s_Q}(3,3).$$

Finally, let u be the graph of the natural bijection $\Delta_P \to \Delta_L$ taking a triple of points to the triple of lines through any two of these points,

$$u = \{(\alpha, \beta) \in \Delta_P \times \Delta_L :\ \beta_1 = \alpha_2\alpha_3,\ \beta_2 = \alpha_3\alpha_1,\ \beta_3 = \alpha_1\alpha_2\}.$$

Then u is a relation of $\widehat{\mathcal{X}}$, because

$$u = \Big(\bigcap_{1 \le i \ne j \le 3} \mathrm{Cyl}_{s_5}(i,j)\Big)_{\Delta_P, \Delta_L}. \tag{4.2.14}$$

Lemma 4.2.17. *$\mathcal{P}$ is a Galois plane if and only if $r_P = u \cdot r_L \cdot u^*$.*

Proof. In the above notation, the Desargues theorem for the projective plane $\mathcal{P}$ can be reformulated as follows: $\alpha \in \Delta_P$ if and only if $\alpha^f \in \Delta_Q$, where f is the bijection defining the relation u. Since this is true if and only if $r_P = u \cdot r_L \cdot u^*$, we are done. □

Now let $\mathcal{P}$ be a Galois plane, $\mathcal{X}'$ a coherent configuration, and

$$\varphi \in \mathrm{Iso}_3(\mathcal{X}, \mathcal{X}').$$

Without loss of generality, we assume that $\mathcal{X}'$ is the coherent configuration of a projective plane $\mathcal{P}'$ (Exercise 2.7.44). Since $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$ is a group of order 2 and the nontrivial algebraic automorphism of $\mathcal{X}$ is induced by the isomorphism from $\mathcal{P}$ to its dual plane, we may also assume that

$$P^{\varphi} = P' \quad \text{and} \quad L^{\varphi} = L',$$

where P' and L' are the point and line sets of $\mathcal{P}'$, respectively (recall that P, L are the fibers of $\mathcal{X}$, and P', L' are the fibers of $\mathcal{X}'$).

Comparing the valencies of basis relations, one can easily see that

$$\varphi(s_i) = s_i', \quad i = 1, \ldots, 8, \tag{4.2.15}$$

where the s_i' are the basis relations of $\mathcal{X}'$ defined by (2.5.1) and (2.5.2).

Denote by $\widehat{\varphi}$ the 3-dimensional extension of φ. Then using statement (2) of Theorem 3.5.7, formula (4.2.15), and statement (2) of Theorem 2.1.4, we obtain

$$\widehat{\varphi}(r_P) = r_P', \quad \widehat{\varphi}(r_L) = r_L', \quad \widehat{\varphi}(u) = u',$$

where r_P', r_L', and u' are the relations of the 3-dimensional extension of $\mathcal{X}'$ defined by formulas (4.2.13) and (4.2.14). Thus, $\mathcal{P}'$ is a Galois plane by Lemma 4.2.17 for $\mathcal{P} = \mathcal{P}'$.

Now, $\mathcal{P}$ and $\mathcal{P}'$ are Galois planes of the same order. Consequently, they are isomorphic. The corresponding isomorphism induces an isomorphism $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ that takes P to P'. Therefore,

$$(s_i)^f = s_i' = \varphi(s_i), \quad i = 1, \ldots, 8,$$

see (4.2.15). This means that f induces φ. Thus,

$$\mathrm{Iso}_3(\mathcal{X}, \mathcal{X}') = \mathrm{Iso}_\infty(\mathcal{X}, \mathcal{X}')$$

and then $s(\mathcal{X}) \leq 3$. □

It should be noted that in accordance with Examples 3.5.4 and 3.5.6, the separability number of the coherent configuration $\mathcal{X}$ associated with a projective plane $\mathcal{P}$ cannot equal 2. By statement (3) of Theorem 4.2.15, this implies that if $\mathcal{P}$ is a Galois plane, then $s(\mathcal{X}) = 1$ or 3. Of course, the former case occurs if and only if the Galois plane is a unique plane having the same order as $\mathcal{P}$.

Let $\mathcal{X}$ be the scheme of the Johnson graph $\mathfrak{X} = J(n, k)$. It is schurian and hence $t(\mathcal{X}) = 1$. Moreover, in accordance with [**17**, Section 9.1.B] the graph $\mathfrak{X}$ is uniquely determined by parameters unless

$$(n, k) = (8, 2). \tag{4.2.16}$$

In the last case, any distance-regular graph with parameters of $J(n,k)$ is isomorphic either to $\mathfrak{X}$ or to one of the three Chang graphs which are not distance-transitive [**17**, p.105].

Theorem 4.2.18. *Let $\mathcal{X}$ be the scheme of a distance-regular graph $\mathfrak{X}$ with parameters of some Johnson graph. Then $t(\mathcal{X}) \le 2$ and $s(\mathcal{X}) \le 2$. More exactly,*

(1) *if $\mathfrak{X} = J(n,k)$, then $t(\mathcal{X}) = 1$ for all n,k, and $s(\mathcal{X}) = 1$ unless condition (4.2.16) is satisfied;*
(2) *if $\mathfrak{X}$ is a Chang graph, then $t(\mathcal{X}) = s(\mathcal{X}) = 2$.*

Proof. Without loss of generality, we may assume that $\mathfrak{X}$ is a Chang graph, see Exercise 4.7.12. In this case, a computer computation shows that the 2-closure of $\mathcal{X}$ is a schurian coherent configuration of rank 11, 12, or 14. It follows that $t(\mathcal{X}) = 2$ and the scheme of a Chang graph is not 2-isomorphic to the scheme of another Chang graph or the graph $J(8,2)$. Thus, $s(\mathcal{X}) = 2$. □

Let $\mathcal{X}$ be the scheme of the Hamming graph $\mathfrak{X} = H(d,q)$. It is schurian and hence $t(\mathcal{X}) = 1$. Moreover, in accordance with [**17**, Section 9.2.B] the graph $\mathfrak{X}$ is uniquely determined by parameters unless

$$q = 4 \quad \text{and} \quad d \ge 2. \tag{4.2.17}$$

Let $q = 4$. Then any distance-regular graph with the same parameters as $H(d,q)$ is isomorphic to the *Doob graph*

$$D_{a,b} = \underbrace{\mathfrak{S} \times \cdots \times \mathfrak{S}}_{a} \times \underbrace{\mathfrak{T} \times \cdots \times \mathfrak{T}}_{b},$$

where $\mathfrak{S}$ is the Shrikhande graph from Example 3.4.15, $\mathfrak{T}$ is a complete graph on 4 vertices (for the direct product of graphs, see Example 3.7.43), and a, b are nonnegative integers. In particular,

$$2a + b = d, \quad D(1,0) = \mathfrak{S}, \quad D(0,1) = \mathfrak{T}.$$

Obviously, $\mathfrak{X} = D_{0,d}$. If $a \ge 1$, then the Doob graph $D_{a,b}$ is not distance-transitive.

Theorem 4.2.19. *Let $\mathcal{X}$ be the scheme of a distance-regular graph $\mathfrak{X}$ with parameters of some Hamming graph. Then $t(\mathcal{X}) \le 2$ and $s(\mathcal{X}) \le 2$. More exactly,*

(1) *if $\mathfrak{X} = H(d,q)$, then $t(\mathcal{X}) = 1$ for all d,q, and $s(\mathcal{X}) = 1$ unless condition (4.2.17) is satisfied;*
(2) *if $\mathfrak{X} = D(a,b)$, then $t(\mathcal{X}) = 2$ unless $a = 0$, and $s(\mathcal{X}) = 2$ for all a,b.*

Proof. Without loss of generality, we may assume that $\mathfrak{X} = D(a,b)$ for some nonnegative integers a and b, see Exercise 4.7.12. Set

$$r_{a,b} = D(\mathfrak{X}) \quad \text{and} \quad 1_{a,b} = 1_{\Omega(\mathfrak{X})}.$$

Again without loss of generality, we may assume that the graph $\mathfrak{X}$ is colored.

Denote by $\mathfrak{Y}$ the complete colored graph on $\{1, \ldots, 4\}$ such that

$$c_{\mathfrak{Y}}(i, j) = c_{\mathfrak{X}}(r_{a,b})$$

for all $1 \le i \ne j \le 4$, where $c_{\mathfrak{X}}(r_{a,b})$ is the color of any pair belonging to $r_{a,b}$.

It is easily seen that given $g \in \mathrm{Emb}(\mathfrak{Y}, \mathfrak{X})$, the vertices in $\mathrm{Dom}(g)$ differ in exactly one fixed coordinate. Therefore,

$$q_{\mathfrak{X}}(\mathfrak{Y}, g) = \begin{cases} q_{\mathfrak{X}}(\mathfrak{X}_{1,0}, g), & \text{if } (1^g, 2^g) \in r_{a,0} \otimes 1_{0,b}, \\ q_{\mathfrak{X}}(\mathfrak{X}_{0,1}, g), & \text{if } (1^g, 2^g) \in 1_{a,0} \otimes r_{0,b}, \end{cases}$$

see (4.2.8), where $\mathfrak{X}_{1,0} = D(1,0)$ and $\mathfrak{X}_{0,1} = D(0,1)$. A straightforward check shows that

$$q_{\mathfrak{X}}(\mathfrak{X}_{1,0}, g) = 0 \quad \text{and} \quad q_{\mathfrak{X}}(\mathfrak{X}_{0,1}, g) = 2.$$

Consequently if $\mathfrak{Z}$ is the subgraph of $\mathfrak{Y}$ induced by the set $\{1, 2\}$ and

$$s_0 = s(\mathfrak{Y}, \mathfrak{Z}, 0) \quad \text{and} \quad s_2 = s(\mathfrak{Y}, \mathfrak{Z}, 2)$$

are the relations defined by formula (4.2.10), then

$$r_{a,0} \otimes 1_{0,b} = \mathrm{pr}_{1,2}(s_0) \cap r_{a,b}, \quad 1_{a,0} \otimes r_{0,b} = \mathrm{pr}_{1,2}(s_2) \cap r_{a,b}, \tag{4.2.18}$$

where $\mathrm{pr}_{i,j}(r)$ is the relation defined as in statement (1) of Exercise 3.7.46. Thus the lemma below follows from the first statements of Theorem 4.2.10 and Exercise 3.7.46.

Lemma 4.2.20. *$r_{a,0} \otimes 1_{0,b}$ and $1_{a,0} \otimes r_{0,b}$ are relations of $\overline{\mathcal{X}} = \overline{\mathcal{X}}^{(2)}$.*

The Shrikhande graph $D_{1,0}$ is edge-transitive and the edge set of its complement graph splits into the two 2-orbits of the group $\mathrm{Aut}(\mathfrak{X}_{1,0})$: $s_{1,0}$ of valency 6, and $t_{1,0}$ of valency 3. Let $s_{a,0}$ (respectively, $t_{a,0}$) be the edge set of the direct product of a copies of the graph with the edge set $s_{1,0}$ (respectively, $t_{1,0}$).

Lemma 4.2.21. *$s_{a,b} = s_{a,0} \otimes 1_{0,b}$ and $t_{a,b} = t_{a,0} \otimes 1_{0,b}$ are relations of $\overline{\mathcal{X}}$.*

Proof. Denote by $\mathfrak{Y}'$ the graph obtained from $\mathfrak{Y}$ by recoloring the pairs $(1,2)$ and $(2,1)$ in the color of the relation $r'_{a,b}$ "to be at distance 2 in the graph $D_{a,b}$". As above, one can see that given $g \in \mathrm{Emb}(\mathfrak{Y}', \mathfrak{X})$,

$$q_{\mathfrak{X}}(\mathfrak{Y}', g) = \begin{cases} 2, & \text{if } (1^g, 2^g) \in s_{a,b}, \\ 0, & \text{if } (1^g, 2^g) \in r'_{a,b} \setminus s_{a,b}. \end{cases}$$

Let $\mathfrak{Z}'$ be the induced subgraph of $\mathfrak{Y}'$ with vertices 1 and 2, and let $s'_2 = s(\mathfrak{Y}', \mathfrak{Z}', 2)$. Then

$$s_{a,b} = \mathrm{pr}_{1,2}(s'_2) \cap r'_{a,b}.$$

Thus, $s_{a,b}$ is a relation of $\overline{\mathcal{X}}$. A straightforward computation shows that

$$A_{r'_{a,b}} \circ (A_{r_{a,0}\otimes 1_{0,b}} \cdot A_{s_{a,b}}) = 2aA_{s_{a,b}} + 4aA_{t_{a,b}} + A_{t'}$$

where $t' = r'_{a,b}\setminus(s_{a,b}\cup t_{a,b})$. The left-hand side belongs to the algebra $\mathrm{Adj}(\overline{\mathcal{X}})$ by Lemma 4.2.20. Therefore, the matrix $A_{t_{a,b}}$ also belongs to it and hence $t_{a,b}$ is a relation of $\overline{\mathcal{X}}$. □

Denote by $\mathcal{X}_{a,b}$ the scheme of the graph $D_{a,b}$ and set $K_{a,b} = \mathrm{Aut}(\mathcal{X}_{a,b})$. In the following lemma, we show that the 2-closure of this scheme for $b = 0$ is schurian and coincides with the coherent closure of the set

$$U = \{r_{a,0}, s_{a,0}, t_{a,0}\}.$$

Lemma 4.2.22. $\overline{\mathcal{X}}_{a,0} = \mathrm{WL}(U) = \mathrm{Inv}(K_{a,0})$.

Proof. It is easily seen that $u = u^{\mathrm{Sym}(a)}$ for each $u \in U$. It follows that

$$\overline{\mathcal{X}}_{1,0} \uparrow \mathrm{Sym}(a) \geq \mathrm{WL}(U).$$

On the other hand,

$$\mathrm{WL}(U) \geq \mathrm{WL}(r_{a,0}) = \mathcal{X}_{1,0} \uparrow \mathrm{Sym}(a).$$

Thus taking into account that no scheme lies strictly between $\overline{\mathcal{X}}_{1,0}$ and $\mathcal{X}_{1,0}$ and the fact that $\mathrm{WL}(U)^{\mathrm{Sym}(a)} = \mathrm{WL}(U)$, we conclude that

$$\mathrm{WL}(U) = \overline{\mathcal{X}}_{1,0} \uparrow \mathrm{Sym}(a).$$

In particular, $\mathrm{WL}(U)$ is a schurian scheme (Theorem 3.4.14). However by Lemmas 4.2.20 and 4.2.21,

$$\overline{\mathcal{X}}_{a,0} \geq \mathrm{WL}(U) \geq \mathcal{X}_{a,0}.$$

After taking the 2-closure of each coherent configuration in this formula, we obtain $\overline{\mathcal{X}}_{a,0} = \mathrm{WL}(U)$ (statement (3) of Exercise 3.7.47), which implies the required statement. □

By Lemmas 4.2.20, 4.2.21, and 4.2.22, the scheme $\overline{\mathcal{X}}$ contains the product $s \otimes 1_{0,b}$, where s is a basis relation of $\mathrm{Inv}(K_{a,0})$. On the other hand, by the second of the equalities (4.2.18) and the distance-transitivity of $D_{0,b}$, it also contains the products $1_{a,0} \otimes s$, where s is a basis relation of $\mathrm{Inv}(K_{0,b})$. Thus,

$$\overline{\mathcal{X}} \geq \mathrm{Inv}(K_{a,0}) \otimes \mathrm{Inv}(K_{0,b}).$$

Since the reverse inclusion is obvious, the scheme $\overline{\mathcal{X}}$ is schurian. It follows that $t(\mathcal{X}) = 2$ for $a > 0$.

To prove that $s(\mathcal{X}) = 2$, let φ be a 2-isomorphism from $\mathcal{X}$ to another coherent configuration $\mathcal{X}'$. Then by statement (2) of Exercise 2.7.55, $\mathcal{X}'$ is a scheme of a distance-regular graph $\mathfrak{X}'$ and also $\mathrm{IA}(\mathfrak{X}) = \mathrm{IA}(\mathfrak{X}')$. It follows that $\mathfrak{X}' = D_{a',b'}$ for some nonnegative integers a' and b', and

$$\varphi(r_{a,b}) = r_{a',b'}.$$

In view of equalities (4.2.18), the second statements of Theorem 4.2.10 and Exercise 3.7.46 imply that

$$\overline{\varphi}(r_{a,0} \otimes 1_{0,b}) = r_{a',0} \otimes 1_{0,b'}, \quad \overline{\varphi}(1_{a,0} \otimes r_{0,b}) = 1_{a',0} \otimes r_{0,b'},$$

where $\overline{\varphi} = \overline{\varphi}^{(2)}$. Taking into account that the valencies of the relations $r_{a,0} \otimes 1_{0,b}$ and $1_{a,0} \otimes r_{0,b}$ are equal to $6a$ and $3b$, respectively, we conclude that

$$(a, b) = (a', b'),$$

see Corollary 2.3.20. Therefore, $\mathcal{X} = \mathcal{X}'$ and $\overline{\varphi}$ is induced by the identity isomorphism. □

We complete the subsection by making some remarks on the scheme $\mathcal{X}$ of the Grassmann graph $J_q(n, k)$. A characterization of this graph by local structure, given in [**17**, Section 9.3], can easily be translated in the language of $(\mathfrak{Y}, \mathfrak{Z})$-regular graphs.

Using a technique similar to that in the proof of Theorem 4.2.19, one can verify that $s(\mathcal{X}) \leq 2$ for all q, n, k, see [**40**, Theorem 7.7]. Moreover, for $k > 2$ the graph $J_q(n, k)$ is uniquely determined by parameters unless

$$n \in \{2k + 1, 2k + 2\} \quad \text{or} \quad (n, q) \in \{(2k + 2, 2), (2k + 2, 3), (2k + 3, 2)\},$$

see [**96**]. Thus except for these cases, $s(\mathcal{X}) = 1$ for all q, n and $k > 2$.

On the other hand, there is a number of non-Grassmann distance-regular graphs with parameters of a Grassmann graph. For example, given a finite group K there exists a strongly regular graph with the same parameters as $J_2(n, 2)$ for some n and the automorphism group isomorphic to K [**95**]. In all these cases, $s(\mathcal{X}) = 2$.

4.3 Two-valenced schemes

Every scheme, in which the valency of a basis relation takes exactly one value, is regular and so is schurian and separable. However, this is not true if the scheme is *two-valenced*, i.e., if the valency of a basis relation takes exactly two values; when they are 1 and k, the term $\{1,k\}$-*valenced* scheme is also used.

In this section, a sufficient condition for a two-valenced scheme to be schurian and separable is established. This condition consists of two parts. They are introduced and studied in Subsections 4.3.1 and 4.3.2, respectively. The first part says that the scheme in question has sufficiently many intersection numbers equal to 1, whereas the second one ensures that the basis relations form a good geometric structure.

In the last two subsections, it is proved that the condition is satisfied for all $\{1,2\}$-valenced schemes, and asymptotically for all equivalenced schemes. The most part of material is taken from [**102, 103, 27, 73**].

4.3.1 Saturation condition

Throughout this subsection, $k>1$ is an integer and $\mathcal{X}=(\Omega,S)$ is a scheme. By technical reasons, the basis relations of $\mathcal{X}$ are mainly denoted below by x,y,z rather than r,s,t.

Our primary goal is to define a graph with vertex set

$$S_k=\{x\in S:\ n_x=k\}$$

that accumulates information about intersection numbers equal to 1. The following simple lemma indicates a way how we do this.

Lemma 4.3.1. *Given $x,y\in S_k$,*

$$|x^*y|=n_x=n_y\quad\Leftrightarrow\quad c_{xs}^y=1\ \text{ for all }\ s\in x^*y. \tag{4.3.1}$$

Proof. We have $n_x=n_{x^*}=n_y=n_{y^*}=k$. By formulas (2.1.8), (2.1.14), and (2.1.3), this implies that

$$k^2=n_{x^*}n_y=\sum_{s\in x^*y}n_sc_{x^*y}^s=\sum_{s\in x^*y}n_{y^*}c_{s^*x^*}^{y^*}=k\sum_{s\in x^*y}c_{xs}^y.$$

Since $c_{xs}^y\ge 1$ for all $s\in x^*y$, we are done. □

Let us define a relation $\sim$ on S_k by setting $x\sim y$ if the right-hand or left-hand side in formula (4.3.1) holds. This relation is symmetric, because

$$c_{xs}^y=\frac{n_{x^*}}{n_y}c_{sy^*}^{x^*}=c_{ys^*}^x$$

for all $x,y\in S_k$. The undirected graph with vertex set S_k and adjacency relation $\sim$ is denoted by $\mathfrak{X}=\mathfrak{X}_k$. Note that this graph can have loops.

Definition 4.3.2. *The scheme $\mathcal{X}$ is said to be k-saturated if for any set $T\subseteq S$ with at most four elements, the set*

$$N(T) = \{y \in S_k : \ y \sim x \quad \text{for all} \ \ x \in T\} \tag{4.3.2}$$

is not empty.

Under the saturation condition, any two vertices of the graph $\mathfrak{X}$ are connected by a path of length at most two. A k-saturated two-valenced scheme is said to be *saturated* and the mention of k is omitted. In what follows, we also write $N(x, y, \ldots)$ instead of $N(\{x, y, \ldots\})$.

Example 4.3.3. *Let $\mathcal{A}$ be a finite affine space with point set Ω and line set L, see* [**19**]. *Denote by $\mathcal{P}$ the set of parallel classes of lines. The lines belonging to a class $P \in \mathcal{P}$ form a partition of Ω; the corresponding equivalence relation with removed diagonal is denoted by e_P.*

Using the axioms of affine space, one can verify that the set $S = S_{\mathcal{A}}$ of all the e_P together with 1_Ω form a commutative scheme such that

$$c_{e_P e_P}^s = \begin{cases} q-1, & \text{if } s = 1, \\ q-2, & \text{if } s = e_P, \\ 0, & \text{otherwise}, \end{cases} \tag{4.3.3}$$

where q is the cardinality of a line of $\mathcal{A}$, and if $P \neq Q$, then

$$c_{e_P e_Q}^s = \begin{cases} 1, & \text{if } s \in e_P e_Q \setminus \{e_P, e_Q\}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.3.4}$$

We say that $\mathcal{X} = (\Omega, S)$ is the scheme associated with the affine space $\mathcal{A}$. Formula (4.3.3) *implies that $\mathcal{X}$ is a $\{1, q-1\}$-valenced scheme and the graph $\mathfrak{X}$ is complete and loopless. In particular, the scheme $\mathcal{X}$ is $(q-1)$-saturated whenever the rank of $\mathcal{X}$ is at least* 6.

The following statement provides a sufficient condition for a scheme $\mathcal{X}$ to be k-saturated in terms of its *indistinguishing number* c which is defined as the maximum of the indistinguishing numbers $c(s)$, $s \in S^\#$ (see (2.1.15)).

Theorem 4.3.4. *A scheme $\mathcal{X}$ with indistinguishing number c is k-saturated whenever*

$$|S_k| > 4c(k-1).$$

Theorem 4.3.4 is an immediate consequence of Lemma 4.3.5 below, that establishes a lower bound for the number of common neighbors of a subset of vertices of the graph $\mathfrak{X}$.

Lemma 4.3.5. *For each $T \subseteq S_k$,*

$$|N(T)| \geq |S_k| - c(k-1)|T|.$$

Proof. It suffices to verify that for any $x \in S_k$,

$$|C_x| \leq c(k-1), \tag{4.3.5}$$

where C_x the complement of $N(x)$ in S_k. Indeed, from the above inequality, it follows that

$$|N(T)| \geq |S_k \setminus \bigcup_{x\in T} C_x| \geq |S_k| - |T| \max_{x\in T} |C_x| \geq |S_k| - c(k-1)|T|.$$

To verify (4.3.5), fix $\alpha \in \Omega$. By the definition of the relation $\sim$, an element y belongs to C_x only if $c_{xs}^y > 1$ for some $s \in x^*y$. In this case, for each $\beta \in \alpha y$, there exists two distinct elements $\gamma, \delta \in \alpha x$ such that

$$r(\gamma,\beta) = s = r(\delta,\beta). \tag{4.3.6}$$

It follows that the set $T_{x,y}$ of all triples (β,γ,δ) satisfying this condition, contains at least $|\alpha y| = n_y = k$ elements. Therefore, the union of all $T_{x,y}$ with $y \in C_x$ contains at least $k|C_x|$ triples.

On the other hand, if s_x is the set of all pairs of distinct points of αx, then

$$\bigcup_{y\in C_x} T_{x,y} = \bigcup_{e\in s_x} T_e,$$

where T_e is the set of all (β,γ,δ) belonging to the union on the left-hand side and such that $(\gamma,\delta) = e$. Moreover, the number of nonempty summands on the right-hand side is at most $|s_x| = k(k-1)$. Thus there exists a pair $e \in s_x$ such that

$$|T_e| \geq \frac{1}{k(k-1)} \sum_{y\in C_x} |T_{x,y}| \geq \frac{k|C_x|}{k(k-1)}.$$

In view of (4.3.6) given a pair $e = (\gamma,\delta)$ belonging to the relation s_x, the set of the first two positions of elements in T_e is contained in the set

$$\Omega_{\gamma,\delta} = \{\alpha \in \Omega : \ r(\gamma,\alpha) = r(\delta,\alpha)\}. \tag{4.3.7}$$

The number $|\Omega_{\gamma,\delta}|$ is equal to the indistinguishing number of the relation $r(\gamma,\delta)$ and hence less than or equal to c. Thus,

$$c \geq |\Omega_{\gamma,\delta}| \geq |T_e| \geq \frac{|C_x|}{k-1},$$

which proves formula (4.3.5). □

The saturation condition enables us to get information on a one-point extension of the scheme in question. More precisely, let $\alpha \in \Omega$ and set

$$S_{\alpha S_k} = \{s_{x,y} : \ x, y \in S_k, \ s \in x^*y\},$$

where $s_{x,y} = s \cap (\alpha x \times \alpha y)$. From statement (1) of Lemma 3.3.5 it follows that αS_k is a homogeneity set of the coherent configuration $\mathcal{X}_\alpha$, and the restriction of it to this set is a fission of the rainbow

$$\mathcal{X}_{\alpha S_k} = (\alpha S_k, S_{\alpha S_k}). \tag{4.3.8}$$

The fibers of this rainbow are exactly the sets αx, and the set $S_{\alpha S_k}$ of its basis relations is equal to the disjoint union of the sets

$$S_{xy} = \{s_{x,y} : \ s \in x^*y\},$$

where $x, y \in S_k$. When $x \sim y$, the set S_{xy} consists of k disjoint matchings. The union of all such S_{xy} is denoted by $M_{\alpha S_k}$,

$$M_{\alpha S_k} = \bigcup_{\substack{x,y \in S_k,\\ x \sim y}} S_{xy}. \tag{4.3.9}$$

If the scheme $\mathcal{X}$ is k-saturated, then $M_{\alpha S_k}$ is enough large, and under an additional condition the coherent closure of the rainbow $\mathcal{X}_{\alpha S_k}$ can explicitly be found.

Theorem 4.3.6. *Let $\mathcal{X}$ be a k-saturated scheme, $\alpha \in \Omega$, and $\mathcal{Y} = \mathrm{WL}(\mathcal{X}_{\alpha S_k})$. Assume that for all $x, y, z \in S_k$,*

$$x \sim y \sim z \sim x \quad \Rightarrow \quad S_{xy} = S_{xz} \cdot S_{zy}. \tag{4.3.10}$$

Then

(1) $S(\mathcal{Y}) = (M \cdot M)^\natural$, *where* $M = M_{\alpha S_k}$*;*
(2) $F(\mathcal{Y}) = \{\alpha x : \ x \in S_k\}$;
(3) $\mathcal{Y}$ *is semiregular.*

Proof. Recall that two vertices x and y of the graph $\mathfrak{X} = \mathfrak{X}_k$ are adjacent if and only if the set S_{xy} consists of k matchings. Since the composition of two matchings s and t with $\Omega_+(s) = \Omega_-(t)$ is also matching, for any path $P = (x_1, \ldots, x_{d+1})$ of this graph, the set

$$S_P = S_{x_1x_2} \cdot \cdots \cdot S_{x_dx_{d+1}}$$

consists of matchings contained in $\alpha x_1 \times \alpha x_{d+1}$.

Lemma 4.3.7. *The set S_P is a partition of $\alpha x_1 \times \alpha x_{d+1}$ into k classes. Moreover, this partition does not depend on the choice of the path connecting x_1 and x_{d+1} in $\mathfrak{X}$.*

Proof. To prove the first statement, it suffices to verify that if $d = 2$, then

$$a \cdot S_{x_2x_3} = S_{x_1x_2} \cdot S_{x_2x_3} = S_{x_1x_2} \cdot b \tag{4.3.11}$$

for any $a \in S_{x_1x_2}$ and $b \in S_{x_2x_3}$. Note that if the first equality is true for all paths P of length 2 and elements a, then it is true for the path $P^* = (x_3, x_2, x_1)$ and hence

$$S_{x_1x_2} \cdot b = (b^* \cdot S_{x_2x_1})^* = (S_{x_3x_2} \cdot S_{x_2x_1})^* = S_{x_1x_2} \cdot S_{x_2x_3}.$$

Thus it suffices to verify the first equality in (4.3.11), or equivalently, that for every $a' \in S_{x_1x_2}$ and $b' \in S_{x_2x_3}$ there exists $b \in S_{x_2x_3}$ such that

$$a \cdot b = a' \cdot b'. \tag{4.3.12}$$

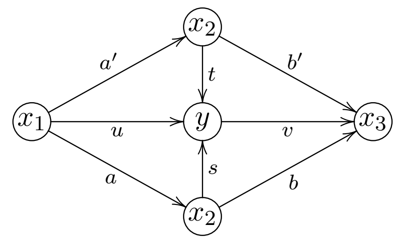


FIGURE 4.3 Configuration in Lemma 4.3.7.

To prove this statement, we make use of the saturation property to find a vertex $y \in S_k$ adjacent to each of the vertices x_1, x_2, x_3 in the graph $\mathfrak{X}$. Then

$$x_1 \sim y \sim x_2 \sim x_1 \quad \text{and} \quad x_2 \sim y \sim x_3 \sim x_2.$$

By condition (4.3.10), this implies that

$$S_{x_1 y} \cdot S_{y x_2} = S_{x_1 x_2}, \qquad S_{x_2 y} \cdot S_{y x_3} = S_{x_2 x_3}. \tag{4.3.13}$$

Using these equalities, we successively find $u \in S_{x_1 y}$ and $t \in S_{x_2 y}$ such that $a' \cdot t = u$, and then $v \in S_{y x_3}$ such that $t \cdot v = b'$. Then

$$a' \cdot b' = (u \cdot t^*) \cdot (t \cdot v) = u \cdot v. \tag{4.3.14}$$

Using equalities (4.3.13) again, we first find $s \in S_{x_2 y}$ such that $a \cdot s = u$, and then $b \in S_{x_2 x_3}$ such that $s^* \cdot b = v$ (the obtained configuration is depicted at Fig. 4.3). Thus from (4.3.14), it follows that

$$a' \cdot b' = u \cdot v = (a \cdot s) \cdot (s^* \cdot b) = a \cdot b,$$

which proves (4.3.12). This completes the proof of (4.3.11) and hence, the first statement.

To prove the second statement of Lemma 4.3.7, let P and P' be paths of length d and d' that connect the vertices

$$u = x_1 = x'_1 \quad \text{and} \quad v = x_{d+1} = x'_{d'+1}.$$

Without loss of generality, we may assume that $d + d' \geq 3$. By formula (4.3.10), the required statement holds for $d + d' = 3$. Suppose that

$$d + d' \geq 4.$$

Now by the saturation property, there exists a vertex $w \in S_k$ adjacent to each of the vertices $\{u, x_2, x_3, x'_2\}$ in the graph $\mathfrak{X}$. By formula (4.3.11) applied for (u, w, x_2) and (x_2, w, x_3), we have

$$\begin{aligned} S_P =& S_{ux_2} \cdot S_{x_2x_3} \cdot S_{Q_1} \\ =& (S_{uw} \cdot a) \cdot (a^* \cdot S_{wx_3}) \cdot S_{Q_1} \\ =& S_{uw} \cdot S_{wx_3} \cdot S_{Q_1} \\ =& S_{uw} \cdot S_Q, \end{aligned}$$

where $Q_1 = (x_3, \ldots, v)$, $Q = (w, x_3, \ldots, v)$, and $a \in S_{wx_2}$. Similarly, one can prove that

$$S_{P'} = S_{uw} \cdot S_{Q'},$$

where $Q' = (w, x'_2, \ldots, v)$. Note that the path Q is of length $d - 1$, the path Q' is of length d', and both paths connect w and v. By the induction hypothesis, this implies that $S_Q = S_{Q'}$ and hence

$$S_P = S_{uw} \cdot S_Q = S_{uw} \cdot S_{Q'} = S_{P'}$$

as required. □

At this point, we need one more auxiliary lemma. Namely, let us write the set $T = (M \cdot M)^\natural$ as the union of the sets

$$T(x, y) = \bigcup_{z \in N(x,y)} S_{xz} \cdot S_{zy}, \quad x, y \in S_k.$$

Note that by the saturation property, $T(x, y)$ is not empty for all x and y. Moreover, if $z \in N(x, y)$ and $P = (x, z, y)$, then $S_{xz} \cdot S_{zy} = S_P$. By Lemma 4.3.7, this implies that $T(x, y)$ is the partition of $\alpha x \times \alpha y$ into matchings. In particular for $u \in S_{xz}$,

$$1_{\alpha x} = u \cdot u^* \in S_{xz} \cdot S_{zx} = T(x, x). \tag{4.3.15}$$

Thus, T is a partition of $(\alpha S_k) \times (\alpha S_k)$ into matchings and $1_{\alpha S_k} \in T^\cup$. Since also $T^* = T$, the pair

$$\mathcal{Y}' = (\alpha S_k, T)$$

is a rainbow.

Lemma 4.3.8. *The rainbow $\mathcal{Y}'$ is a coherent configuration.*

Proof. It suffices to verify that if $u, v \in T$ and $u \cdot v \neq \varnothing$, then $u \cdot v \in T$. To this end, let $u \in S_P$ and $v \in S_Q$, where P and Q are paths of length 2 in the graph $\mathfrak{X}$. Since $u \cdot v \neq \varnothing$, the last vertex of P coincides with the first vertex of Q. By Lemma 4.3.7, this implies that

$$S_P \cdot S_Q = S_{P \cdot Q},$$

where $P \cdot Q$ is the path of length 4 in $\mathfrak{X}$ consisting of the vertices of P followed by the vertices of Q (the last vertex of P is identified with the first vertex of Q). Thus, $u \cdot v$ belongs to $S_{P \cdot Q}$ and hence to T, as required. □

The coherent configuration $\mathcal{Y}'$ defined in Lemma 4.3.8 is obviously a fission of the coherent closure $\mathcal{Y}$. Indeed, let $x \sim y$. Then

$$S_{xy} = 1_{\alpha x} \cdot S_{xy} \subseteq T.$$

This implies that $M \subseteq T \subseteq T^{\cup}$. Thus the claim follows, because $\mathcal{Y}$ is the smallest coherent configuration for which every relation of $S_{\alpha S_k}$ is the union of basis relations.

Conversely, every relation in the set $S_{\alpha S_k} \cdot S_{\alpha S_k}$ is contained in $S(\mathcal{Y})^{\cup}$. By the definition of T, this implies that

$$T \subseteq S(\mathcal{Y})^{\cup},$$

i.e., $\mathcal{Y}' \leq \mathcal{Y}$. Thus, $\mathcal{Y}' = \mathcal{Y}$. This proves statement (1), and in view of (4.3.15) also statement (2). By statement (1), the set $S(\mathcal{Y})$ consists of matchings. This proves statement (3). □

Statement (3) of Theorem 4.3.6 remains true even one removes condition (4.3.10) (see Exercise 4.7.23). However, statements (1) and (2) cease to be fair, for example, if the affine space in Example 4.3.3 is non-Desarguesian.

Corollary 4.3.9. *Under the conditions of Theorem 4.3.6, suppose that $\mathcal{X}$ is a two-valenced scheme. Then*

$$\mathcal{X}_\alpha = \mathcal{D}_{\alpha S_1} \boxplus \mathcal{Y} \quad \text{and} \quad F(\mathcal{X}_\alpha) = \{\alpha s : \ s \in S\}.$$

Moreover, the coherent configuration $\mathcal{X}_\alpha$ is schurian and separable.

Proof. The first equality is a consequence of Exercise 3.7.25, whereas the second one follows from statement (2) of Theorem 4.3.6. Next, by statement (3) of this theorem, the coherent configuration $\mathcal{X}_\alpha$ is partly regular. This completes the proof by Theorem 3.3.19. □

4.3.2 Desarguesian two-valenced schemes

In this subsection, we introduce the property of a two-valenced scheme to be Desarguesian and prove that any two-valenced saturated scheme having this property is schurian and separable. Throughout this subsection, $k > 1$ is an integer and $\mathcal{X} = (\Omega, S)$ is a $\{1, k\}$-scheme. We also keep notation of Subsection 4.3.1.

Let us define a noncommutative geometry associated with the scheme $\mathcal{X}$ as follows: the points are elements of S, the lines are the sets x^*y, $x, y \in S$, and the incidence relation is given by inclusion. Thus the point $z \in S$ belongs to the line x^*y if and only if $z \in x^*y$. The geometry is extremely unusual: the line x^*y does not necessarily contain the points x, y, and can be different from y^*x. However, in terms of this geometry, one can define Desarguesian configurations, see below.

Assume that we are given two "triangles" with vertices $x, y, z \in S$ and $u, v, w \in S$, respectively, that are perspective with respect to a point q, i.e.,

$$u \in x^*q, \quad v \in y^*q, \quad w \in z^*q, \tag{4.3.16}$$

see the configuration depicted in Fig. 4.4; note that the intersections of lines is not necessarily consists of a unique point, and even may be empty. However, if

$$x^*z \cap uw^* = \{r\}, \quad z^*y \cap wv^* = \{s\}, \quad x^*y \cap uv^* = \{t\} \tag{4.3.17}$$

for some $r, s, t \in S$, then, as in the case of Desargues' theorem, we would like that the point t would lie on the line rs. When this is true, the obtained 10-element configuration is said to be *Desarguesian*. In what follows we are going to study $\{1, k\}$-schemes with enough many Desarguesian configurations. The exact definition is as follows.

Definition 4.3.10. *The ten relations in Fig. 4.4 forms a Desarguesian configuration if conditions (4.3.16) and (4.3.17) are satisfied and $t \in rs$.*

Let $x, y, z \in S_k$ and $r, s \in S$ be basis relations of the scheme $\mathcal{X}$. We say that they form an *initial* configuration if

$$x \sim z \sim y \quad \text{and} \quad r \in x^*z, \;\; s \in z^*y. \tag{4.3.18}$$

In geometric language, this means that the points r and s belong to the lines x^*z and z^*y, respectively, and each of these lines consists of exactly k points.

Definition 4.3.11. *The relations r and s are said to be linked with respect to (x, y, z) if the initial configuration lies in a Desarguesian configuration, namely, there exist*

$$q \in N(x, y, z), \qquad u, v, w \in S, \qquad t \in rs,$$

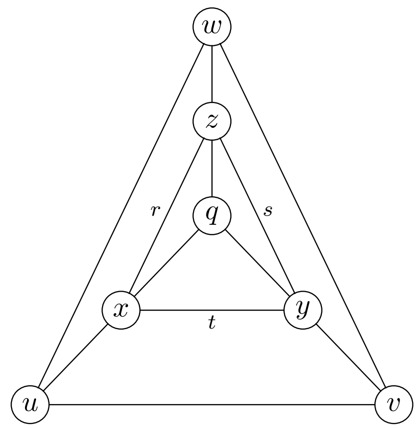


FIGURE 4.4 The Desarguesian configuration.

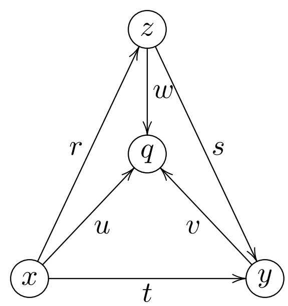


FIGURE 4.5 The relations r and s are linked with respect to (x, y, z).

for which conditions (4.3.16) *and* (4.3.17) *are satisfied (a more compact picture of the linked relations is given in Fig.*4.5*).*

Assume that the relations r and s are linked with respect to (x, y, z). Then the relation t is uniquely determined by the third of equalities (4.3.17). The following statement shows that in this case, t does not depend on the choice of q and u, v, w.

Lemma 4.3.12. *Assume that r and s are linked with respect to (x, y, z). Then*

$$r_{x,z} \cdot s_{z,y} \subseteq t_{x,y}, \tag{4.3.19}$$

with equality if $x \sim y$.

Proof. By formulas (4.3.17), we have

$$u_{x,q} \cdot w^*_{q,z} \subseteq r_{x,z}, \quad w_{z,q} \cdot v^*_{q,y} \subseteq s_{z,y}, \quad u_{x,q} \cdot v^*_{q,y} \subseteq t_{x,y}. \tag{4.3.20}$$

The relations $u_{x,q} \cdot w^*_{q,z}$ and $r_{x,z}$ are matchings, because $x \sim q \sim z$, and $x \sim z$. By the first inclusion in (4.3.20), this implies the first of the following two

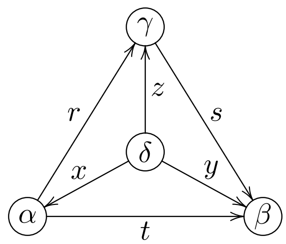


FIGURE 4.6 The triangle (α, β, γ) corresponding the triple (x, y, z).

equalities, and the second one is proved similarly:

$$u_{x,q} \cdot w^*_{q,z} = r_{x,z} \quad \text{and} \quad w_{z,q} \cdot v^*_{q,y} = s_{z,y}.$$

Now the third inclusion in (4.3.20) yields

$$r_{x,z} \cdot s_{z,y} = (u_{x,q} \cdot w^*_{q,z}) \cdot (w_{z,q} \cdot v^*_{q,y}) = u_{x,q} \cdot v^*_{q,y} \subseteq t_{x,y},$$

which proves (4.3.19). If $x \sim y$, then the relation $t_{x,y}$ is a matching and we are done by the above argument. □

We need an auxiliary statement proved in [**102**, Theorem 5.1]. In the geometric language, the conclusion of this statement means that the lines rs and x^*y have a unique common point.

Lemma 4.3.13. *Let $x, y, z \in S_k$ be such that*

$$(xx^*\, yy^*) \cap zz^* = \{1\}. \tag{4.3.21}$$

Then $|rs \cap x^*y| = 1$ for all $r \in x^*z$ and $s \in z^*y$.

Proof. Let $r \in x^*z$ and $s \in z^*y$. Then obviously, $z \in xr \cap ys^*$. This implies that $|rs \cap x^*y| \geq 1$. It suffices to verify that $|rs \cap x^*y| \leq 1$.

Claim. *In the above notation, given $t \in rs \cap x^*y$ and $\alpha, \beta, \gamma \in \Omega$ such that*

$$r(\alpha, \beta) = t, \quad r(\alpha, \gamma) = r, \quad r(\gamma, \beta) = s,$$

there exists a unique $\delta \in \Omega$ for which

$$r(\delta, \alpha) = x, \quad r(\delta, \beta) = y, \quad r(\delta, \gamma) = z,$$

see Fig. 4.6.

Proof. Since $r \in x^*z$, $s \in z^*y$, and $t \in x^*y$, there exist λ, μ, $\nu \in \Omega$ such that

$$r(\lambda, \alpha) = x,\ r(\lambda, \gamma) = z, \quad r(\mu, \beta) = y,\ r(\mu, \gamma) = z, \quad r(\nu, \alpha) = x,\ r(\nu, \beta) = y$$

(see Fig. 4.7). By the lemma hypothesis, this implies that

$$r(\lambda, \mu) \in (xx^*\, yy^*) \cap zz^* = \{1\}.$$

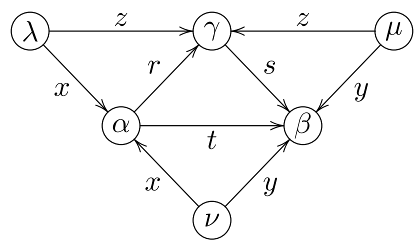


FIGURE 4.7 The point configuration from the claim.

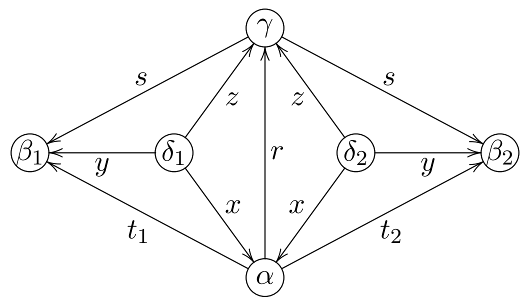


FIGURE 4.8 The point configuration for Lemma 4.3.13.

Thus, $\lambda = \mu$. Denote this point by δ. Then for δ the statement of the claim holds.

To prove the uniqueness, we assume that there are two configurations in Fig. 4.6 with $\delta = \delta_1$ and $\delta = \delta_2$. Then the relation $r(\delta_1, \delta_2)$ belongs to the set

$$zz^* \cap xx^* \cap yy^* = \{1\}.$$

It follows that $\delta_1 = \delta_2$, as required. □

To complete the proof of Lemma 4.3.13, we assume that $rs \cap x^* y$ contains two distinct elements t_1 and t_2. Then there exist $(\alpha, \gamma) \in r$ and distinct $\beta_1, \beta_2 \in \Omega$ such that

$$r(\gamma, \beta_i) = s \quad \text{and} \quad r(\alpha, \beta_i) = t_i, \qquad i = 1, 2.$$

By the claim for $t = t_i$, there exists $\delta_i \in \Omega$ such that

$$r(\delta_i, \alpha) = x, \quad r(\delta_i, \beta_i) = y, \quad r(\delta_i, \gamma) = z$$

(see Fig. 4.8). Now the relation $r(\delta_1, \delta_2)$ belongs to $zz^* \cap xx^* = \{1\}$ (see (4.3.21)). Therefore, $\delta_1 = \delta_2$. Denote this point by δ. Then

$$\beta_1, \beta_2 \in \delta y \cap \gamma s.$$

Since $\beta_1 \neq \beta_2$ and $r(\delta, \gamma) = z$, this implies that $c_{ys^*}^z \geq 2$, contrary to the lemma hypothesis implying $y \sim z$. □

The statement below establishes two sufficient conditions for relations r and s to be linked with respect to (x, y, z).

Corollary 4.3.14. *Let $x, y, z \in S_k$ and $r, s \in S$ form an initial configuration. Assume that at least one of the following conditions is satisfied:*

(L1) *z is a loop of the graph $\mathfrak{X}$ and equality (4.3.21) holds;*
(L2) *there exists $q \in S_k$ such that*

$$qq^* \cap (xx^*yy^* \cup xx^*zz^* \cup zz^*yy^*) = \{1\}. \tag{4.3.22}$$

Then r and s are linked with respect to (x, y, z).

Proof. Under the condition (L1), Lemma 4.3.13 implies that there exists $t \in S$ such that $rs \cap x^*y = \{t\}$. Then condition (4.3.17) is obviously satisfied for $q = z$ and $u = r$, $v = s^*$, $w = 1$.

Now assume that the condition (L2) is satisfied. We claim that there exist $t \in x^*q$, $u \in q^*z$, and $v \in q^*y$ such that

$$r \in tu \cap x^*z, \quad s \in u^*v \cap z^*y, \quad rs \cap tv \cap x^*y \neq \varnothing. \tag{4.3.23}$$

Indeed, fix a point α. Since $r \in x^*z$ and $s \in z^*y$, one can find points $\beta \in \alpha x$, $\gamma \in \alpha z$, and $\delta \in \alpha y$ such that

$$(\beta, \gamma) \in r \quad \text{and} \quad (\gamma, \delta) \in s.$$

Now take $\epsilon \in \alpha q$ and set

$$t := r(\beta, \epsilon), \quad u := r(\epsilon, \gamma), \quad v := r(\epsilon, \delta).$$

Then formula (4.3.23) holds, because $r(\beta, \delta) \in (tv \cap rs \cap x^*y)$.

To complete the proof, we note that by Lemma 4.3.13, each of the sets

$$tu \cap x^*z, \quad u^*v \cap z^*y, \quad tv \cap x^*y$$

has at most one element. Therefore the first and the second sets are singletons $\{r\}$ and $\{s\}$, and also the third one is a singleton since $rs \cap tv \cap x^*y \neq \varnothing$. Thus r and s are linked with respect to (x, y, z). □

Now we arrive at the main definition in this subsection.

Definition 4.3.15. *The scheme $\mathcal{X}$ is said to be Desarguesian if for all $x, y, z \in S_k$ and all $r, s \in S$ satisfying (4.3.18), the elements r and s are linked with respect to (x, y, z).*

Example 4.3.16. *Let $\mathcal{X}$ be the scheme on the affine space $\mathcal{A}$ of order q and dimension at least 3. Then from formulas (4.3.3) and (4.3.4) it follows that for any three parallel classes P, Q, and R there exists a parallel class T such that*

$$e_T \notin (e_P e_Q \cup e_P e_R \cup e_R e_Q).$$

Consequently, the statement (L2) of Corollary 4.3.14 holds for $q = e_T$, $x = e_P$, $y = e_Q$, and $z = e_R$. Therefore any relations $r \in e_P e_R$ and $s \in e_R e_Q$ are linked with respect to (e_P, e_Q, e_R).

Thus, the scheme $\mathcal{X}$ is Desarguesian. If the space $\mathcal{A}$ is an affine plane, i.e., affine space of dimension 2*, then from* [**102**, Theorem 4.1], *it follows that $\mathcal{X}$ is Desarguesian if and only if $\mathcal{A}$ is Desarguesian* (*Exercise* 4.7.26).

We arrive to the main result of this subsection.

Theorem 4.3.17. *Let $\mathcal{X}$ be a two-valenced scheme. Suppose that $\mathcal{X}$ is saturated and Desarguesian. Then $\mathcal{X}$ is schurian and separable.*

Proof. A key point in the proof is to extend an algebraic isomorphism between any two schemes satisfying the hypothesis of the theorem, to an algebraic isomorphism between their one-point extensions. In the sequel, we assume that $\mathcal{X}$ is a saturated and Desarguesian $\{1, k\}$-valenced scheme, $\mathcal{X}'$ an arbitrary scheme on Ω', and

$$\varphi : S(\mathcal{X}) \to S(\mathcal{X}'),\ s \mapsto s'$$

an algebraic isomorphism. The lemma below immediately follows from Corollary 2.3.20 and Exercises 4.7.21 and 4.7.24.

Lemma 4.3.18. *$\mathcal{X}'$ is $\{1, k\}$-valenced, saturated, and Desarguesian.*

Let us fix points $\alpha \in \Omega$ and $\alpha' \in \Omega'$. The algebraic isomorphism φ induces a bijection from S_k onto S'_k, where $S' = S(\mathcal{X}')$. This bijection is obviously extended to the bijection

$$S_{\alpha S_k} \to S'_{\alpha' S'_k},\ r_{x,y} \mapsto r'_{x',y'}, \tag{4.3.24}$$

denoted also by φ. One can easily see that it takes S_{xy} to $S'_{x'y'}$ for all $x, y \in S_k$. Moreover, since $x \sim y$ if and only if $x' \sim y'$,

$$\varphi(M_{\alpha S_k}) = M_{\alpha' S'_k},$$

where the sets $M = M_{\alpha S_k}$ and $M' = M_{\alpha' S'_k}$ are defined by formula (4.3.9).

Finally, since the schemes $\mathcal{X}$ and $\mathcal{X}'$ are Desarguesian, Lemma 4.3.12 implies that

$$S_{xy} = S_{xz} \cdot S_{zy} \quad \text{and} \quad S'_{x'y'} = S'_{x'z'} \cdot S'_{z'y'} \tag{4.3.25}$$

for all $x, y, z \in S_k$ such that $x \sim y \sim z \sim x$. In particular, these schemes satisfy the conditions of Theorem 4.3.6. In what follows, this theorem and Corollary 4.3.9 are used with no further explanations.

Lemma 4.3.19. *The mapping*

$$\psi : M \cdot M \to M' \cdot M', \quad b \cdot c \mapsto b' \cdot c' \tag{4.3.26}$$

is a well-defined bijection.

Proof. Assume that $b_1 \cdot c_1 = b_2 \cdot c_2$ for some $b_1, c_1, b_2, c_2 \in M$. Then

$$b_i \in S_{xz_i} \quad \text{and} \quad c_i \in S_{z_i y}, \qquad i = 1, 2,$$

for some $x, y, z_1, z_2 \in S_k$ such that

$$x \sim z_1 \sim y \quad \text{and} \quad x \sim z_2 \sim y.$$

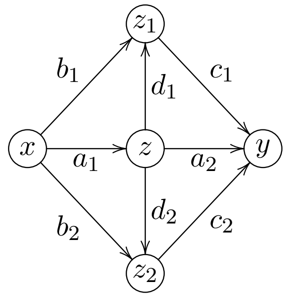


FIGURE 4.9 The configuration for Lemma 4.3.19.

By the saturation property, there exists a vertex z of the graph $\mathfrak{X}_k$ that is adjacent to each of the vertices x, z_1, z_2, y. The first equality in (4.3.25) implies that

$$S_{xz_i} = S_{xz} \cdot S_{zz_i} \quad \text{and} \quad S_{zy} = S_{zz_i} \cdot S_{z_iy}$$

for each i. Take any $a_1 \in S_{xz}$. Then the above equalities show respectively that

$$d_1 := a_1^* \cdot b_1 \in S_{zz_1} \quad \text{and} \quad d_2 := a_1^* \cdot b_2 \in S_{zz_2}, \tag{4.3.27}$$

and

$$a_2 := d_1 \cdot c_1 \in S_{zy}. \tag{4.3.28}$$

Thus,

$$a_1 \cdot a_2 = (a_1 \cdot d_1) \cdot (d_1^* \cdot a_2) = b_1 \cdot c_1 = b_2 \cdot c_2 = a_1 \cdot d_2 \cdot c_2, \tag{4.3.29}$$

whence $c_2 = d_2^* \cdot a_2$ (see Fig. 4.9).

Now since $x \sim y \sim z \sim x$, the first equality in (4.3.25) implies that $S_{xz} = S_{xy} \cdot S_{yz}$. Taking into account that $a \in S_{xy}$ and $b \in S_{yz}$, we conclude that $(a \cdot b)' = a' \cdot b'$. By formulas (4.3.27), (4.3.28) and (4.3.29), this yields

$$\begin{aligned} b_1' \cdot c_1' &= (a_1 \cdot d_1)' \cdot (d_1^* \cdot a_2)' \\ &= a_1' \cdot (d_1' \cdot (d_1')^*) \cdot a_2' \\ &= a_1' \cdot a_2' \\ &= (b_2 \cdot d_2^*)' \cdot (d_2 \cdot c_2)' \\ &= b_2' \cdot (d_2' \cdot (d_2^*)') \cdot c_2' \\ &= b_2' \cdot c_2', \end{aligned}$$

which proves that ψ is a well-defined bijection. □

Set

$$\mathcal{Y} = \mathrm{WL}(\mathcal{X}_{\alpha S_k}) \quad \text{and} \quad \mathcal{Y}' = \mathrm{WL}(\mathcal{X}'_{\alpha' S_k'}).$$

Then by statement (1) of Theorem 4.3.6,

$$S(\mathcal{Y}) = (M \cdot M)^\natural \quad \text{and} \quad S(\mathcal{Y}') = (M' \cdot M')^\natural. \tag{4.3.30}$$

By Lemma 4.3.19, the mapping ψ induces a bijection from $S(\mathcal{Y})$ onto $S(\mathcal{Y}')$.

Lemma 4.3.20. $\psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}')$.

Proof. Note that given $a, b \in S(\mathcal{Y})$, either $a \cdot b \in S(\mathcal{Y})$ or $a \cdot b = \varnothing$. Thus it suffices to verify that if $a \cdot b \neq \varnothing$, then

$$(a \cdot b)^\psi = a^\psi \cdot b^\psi. \tag{4.3.31}$$

Take such a and b. In view of (4.3.30), there exist $x, z_1, z', z_2, y \in S_k$ for which

$$a \in S_{xz_1} \cdot S_{z_1 z'} \quad \text{and} \quad b \in S_{z' z_2} \cdot S_{z_2 y}.$$

By the saturation property, there exists $z'' \in N(x, z', y)$. Since $a \in x^* z'$ and $b \in (z')^* y$, we may assume that $z_1 = z_2 = z''$; denote this element by z.

In view of formula (4.3.11), one can find $c_1 \in S_{xz}$, $c_2 \in S_{zz'}$, and $c_3 \in S_{zy}$ such that

$$c_1 \cdot c_2 = a \quad \text{and} \quad c_2 \cdot b = c_3,$$

see Fig. 4.10. Furthermore by the definition of ψ, we have $(c_1 \cdot c_3)^\psi = c_1^\varphi \cdot c_3^\varphi$.

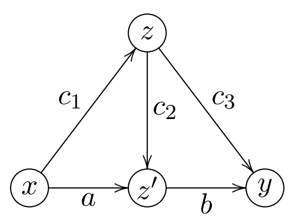


FIGURE 4.10 The configuration for Lemma 4.3.20.

It follows that

$$\begin{aligned}
(a \cdot b)^\psi &= (c_1 \cdot c_2 \cdot c_2^* \cdot c_3)^\psi \\
&= (c_1 \cdot c_3)^\psi \\
&= c_1^\varphi \cdot c_3^\varphi \\
&= c_1^\varphi \cdot c_2^\varphi \cdot (c_2^*)^\varphi \cdot c_3^\varphi \\
&= (c_1 \cdot c_2)^\psi \cdot (c_2^* \cdot c_3)^\psi \\
&= a^\psi \cdot b^\psi,
\end{aligned}$$

which completes the proof of (4.3.31). □

To continue, we make use of Corollary 4.3.9 to see that

$$\mathcal{X}_\alpha = \mathcal{D}_{\alpha S_1} \boxplus \mathcal{Y} \quad \text{and} \quad \mathcal{X}'_{\alpha'} = \mathcal{D}_{\alpha' S'_1} \boxplus \mathcal{Y}'. \tag{4.3.32}$$

It follows that

$$(\mathcal{X}_\alpha)_{\alpha S_1} = \mathcal{D}_{\alpha S_1} \quad \text{and} \quad (\mathcal{X}'_{\alpha'})_{\alpha' S'_1} = \mathcal{D}_{\alpha' S'_1}.$$

Therefore there is a unique algebraic isomorphism

$$\psi_1 \in \mathrm{Iso}_{\mathrm{alg}}((\mathcal{X}_\alpha)_{\alpha S_1}, (\mathcal{X}'_{\alpha'})_{\alpha' S'_1})$$

extending φ. Now by statement (1) of Exercise 3.7.33 for $\square = \boxplus$, $\varphi_1 = \psi_1$ and $\varphi_2 = \psi$, there exists a unique algebraic isomorphism

$$\varphi_{\alpha,\alpha'} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'})$$

such that

$$(\varphi_{\alpha,\alpha'})_{\alpha S_1} = \psi_1 \quad \text{and} \quad (\varphi_{\alpha,\alpha'})_{\alpha S_k} = \psi.$$

Lemma 4.3.21. *The algebraic isomorphism $\varphi_{\alpha,\alpha'}$ extends φ.*

Proof. Let $c \in S(\mathcal{X}_\alpha)$. Denote by t the basis relation of $\mathcal{X}$ that contains c. In view of (4.3.32), we have

$$S(\mathcal{X}_\alpha) = S(\mathcal{X}_{\alpha S_1}) \boxplus S(\mathcal{Y}),$$

see (3.2.1). It follows that if $c \notin (M \cdot M)^\natural$, then $c = t_{x,y}$ for some $x, y \in S$. By the definition of $\varphi_{\alpha,\alpha'}$, this implies that

$$\varphi_{\alpha,\alpha'}(c) = t'_{x',y'},$$

as required.

Thus we may assume that $c = r_{x,z} \cdot s_{z,y}$ for some $x, y, z \in S_k$ that make up, together with r and s, an initial configuration. Since $\mathcal{X}$ is Desarguesian, this implies that the relations r and s are linked with respect to (x, y, z), i.e., there exist relations q, u, v, and w for which conditions (4.3.16) and (4.3.17) are satisfied. Therefore,

$$(x')^* z' \cap u'(w')^* = \{r'\}, \quad (z')^* y' \cap w'(v')^* = \{s'\}, \quad (x')^* y' \cap u'(v')^* = \{t'\}$$

for suitable relations $u' \in (x')^* q'$, $v' \in (y')^* q'$, and $w' \in (z')^* q'$. Thus, r' and s' are linked with respect to (x', y', z'). Now by Lemma 4.3.12,

$$\psi(c) = \psi(r_{x,z} \cdot s_{z,y}) = r'_{x',z'} \cdot s'_{z',y'} \subseteq t'_{x',y'},$$

as required. □

To complete the proof, we make use of Corollary 4.3.9 to conclude that the coherent configuration $\mathcal{X}_\alpha$ is schurian and separable, and moreover $F(\mathcal{X}_\alpha) = \{\alpha s : \ s \in S\}$. By Lemma 4.3.21, this implies that the scheme $\mathcal{X}$ satisfies all the conditions of Exercise 3.7.24. Thus, $\mathcal{X}$ is schurian and separable. □

A careful analysis of the proof of Theorem 4.3.17 shows that the conclusion remains valid if the property of the coherent configuration $\mathcal{X}$ "to be

Desarguesian” is replaced by a weaker condition, in which not each initial configuration lies in a Desarguesian configuration.

Definition 4.3.22. *A $\{1,k\}$-valenced scheme $\mathcal{X}$ is said to be weakly Desarguesian if*
(*) for any $x, y, z \in S_k$ with $x \sim z \sim y$, there exist $a \in x^*z$ and $b \in z^*y$ such that any $r \in x^*z \setminus \{a\}$ and $s \in z^*y \setminus \{b\}$ are linked with respect to (x, y, z).

Clearly, any Desarguesian scheme is weakly Desarguesian and the latter property is preserved with respect to algebraic isomorphisms. The proof of the theorem below repeats literally the proof of Theorem 4.3.17 except for two places indicated in the next paragraph.

The first one concerns formula (4.3.25), which is proved in this case with the help of Exercise 4.7.22. The second one is the proof of Lemma 4.3.21: it goes smoothly if, in the notation of this lemma,

$$r_{x,z} \neq a \quad \text{and} \quad s_{z,y} \neq b,$$

where a and b are as in (*), and the proof of the remaining cases is obtained with the help of Exercise 4.7.25.

Theorem 4.3.23. *The conclusion of Theorem 4.3.17 remains true if the scheme $\mathcal{X}$ is saturated and weakly Desarguesian.*

4.3.3 Quasi-thin schemes

In this subsection, we study the schurity and separability problems for $\{1,2\}$-schemes also called *quasi-thin*.[3] Many (but not all) schemes of this type are of the form $\mathrm{Inv}(K)$, where K is a transitive permutation group of even order with pointwise stabilizer of order 2 (see statement (4) of Theorem 2.2.7).

Let $\mathcal{X} = (\Omega, S)$ be a quasi-thin scheme. From formula (2.1.6), it immediately follows that

$$|\Omega| = |S_1| + 2\,|S_2|.$$

In particular, the numbers of thin and non-thin basis relations are uniquely determined by the degree and an *index* of $\mathcal{X}$, which is defined to be the ratio $|\Omega|/|S_1|$. The smallest examples of non-schurian quasi-thin schemes are of index 4 and 7, see below.

Example 4.3.24. *Let $\mathcal{X}_{16}$ and $\mathcal{X}'_{16}$ (respectively, $\mathcal{X}_{28}$ and $\mathcal{X}'_{28}$) be the schemes #173 and #172 (respectively, #176 and #175) from the Hanaki–Miyamoto list [**56**] of association schemes of degree 16 (respectively, of degree 28). A straightforward computation with GAP shows that*

- *each of the schemes $\mathcal{X}_{16}$, $\mathcal{X}'_{16}$, $\mathcal{X}_{28}$, and $\mathcal{X}'_{28}$ is quasi-thin with thin radical isomorphic to the Klein group;*
- *the schemes $\mathcal{X}'_{16}$ and $\mathcal{X}'_{28}$ are schurian, whereas the schemes $\mathcal{X}_{16}$ and $\mathcal{X}_{28}$ are non-schurian;*
- *the schemes $\mathcal{X}_{16}$ and $\mathcal{X}_{28}$ are algebraically isomorphic to the schemes $\mathcal{X}'_{16}$ and $\mathcal{X}'_{28}$, respectively;*
- *the index of $\mathcal{X}_{16}$ (respectively, $\mathcal{X}_{28}$) is equal to 4 (respectively, 7).*

The schurity problem for quasi-thin schemes was studied in a series of papers [**70, 75, 74, 104**], and then (together with the separability problem) was completely solved in [**103**]; as it turned out essentially all non-schurian and all non-separable quasi-thin schemes are closely related with the above examples.

To come to the general statement, we introduce and study a class of Kleinian schemes; they are associated with Klein coherent configurations considered in Subsection 4.1.2 and include the schemes from Example 4.3.24. Then we prove that all non-Kleinian quasi-thin schemes are schurian and separable.

[3] In [**103**], the class of quasi-thin schemes includes regular schemes.

Kleinian schemes

A quasi-thin scheme is said to be *Kleinian* if its thin residue consists of four thin basis relations that form the Klein group with respect to the composition. In particular, all the schemes in Example 4.3.24 are Kleinian. For each Kleinian scheme $\mathcal{X}$, infinitely many Kleinian schemes can be constructed as the tensor product $\mathcal{X} \otimes \mathcal{Y}$, where $\mathcal{Y}$ is a regular scheme.

Example 4.3.25. *Let K be a permutation group of degree 12 induced by the action of $C_2 \times \mathrm{Alt}(4)$ on the right cosets of a subgroup generated by the product of two involutions, one in C_2 and another one in $\mathrm{Alt}(4)$. Then $\mathrm{Inv}(K)$ is a quasi-thin scheme (#49 from the Hanaki–Miyamoto list [**56**]) of index 3, in which the thin radical equals the thin residue, and is the Klein group. Thus, $\mathrm{Inv}(K)$ is a Kleinian scheme.*

The following statement reveals a geometry underlying a Kleinian scheme. This geometry coincides with the partial linear space of a geometric Klein configuration (Theorem 4.1.20). In what follows, the number of points of a partial linear space is called an *order*, and the first partial linear space in Fig. 4.2 is called a *near-pencil*.

Proposition 4.3.26. *The residually thin extension of a Kleinian scheme $\mathcal{X}$ is a Klein configuration. Moreover, if $\mathcal{Y}$ is a reduction of this configuration, then*

1. *$\mathcal{Y}$ is a geometric Klein configuration;*
2. *the index of $\mathcal{X}$ equals the order of the partial linear space $\mathcal{G} := \mathcal{G}(\mathcal{Y})$;*
3. *$\mathcal{G}$ is a near-pencil of order 3, or a projective or affine plane of order 2.*

Proof. Denote by e the thin residue parabolic of $\mathcal{X}$ and consider the extension $\mathcal{X}_e$ of $\mathcal{X}$ with respect to e (see Subsection 3.1.3). Then $\mathcal{X}_e$ is a Klein configuration by statement (1) of Theorem 3.1.26. It immediately follows that the coherent configuration $\mathcal{Y}$ being the restriction of $\mathcal{X}_e$ is also a Klein configuration.

Assume on the contrary that $\mathcal{Y}$ is not geometric. Then there exist fibers $\Delta, \Gamma \in F(\mathcal{Y})$ such that

$$s = \Delta \times \Gamma$$

is a basis relation of $\mathcal{Y}$. However, then the basis relation of the scheme $\mathcal{X}$ that contains s has valency ≥ 4, which is impossible in a quasi-thin scheme. This proves statement (1).

Let e_1 be the thin radical parabolic of $\mathcal{X}$. Then given $i, j \in I$,

$$i \sim j \quad \Leftrightarrow \quad \exists \Delta \in \Omega/e_1 : \ \Omega_i \cup \Omega_j \subseteq \Delta,$$

where the equivalence relation $\sim$ is defined by formula (4.1.12); at this point we make use of the notation from Subsection 4.1.2 for the coherent configuration $\mathcal{X}_e$. It follows that the classes of e_1 are in a one-to-one correspondence with the classes of $\sim$. Consequently, the number $|\Omega/\sim|$ is equal to the index of $\mathcal{X}$. Since this number equals by definition the order of the partial linear space $\mathcal{G}$, statement (2) follows.

Let us prove statement (3). By statement (3) of Theorem 3.1.26, the group $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_e)$ acts transitively on $F(\mathcal{X}_e)$ and preserves the relation $\sim$. Therefore, the group $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{Y})$ acts transitively on the classes of $\sim$ and hence on $F(\mathcal{Y})$. It follows that the partial linear space $\mathcal{G}$ admits a point-transitive automorphism group. Since the last three spaces in Fig. 4.2 are not point-transitive, the required statement follows from Theorem 4.1.20 unless $\mathcal{G}$ has exactly one line. Let us verify that the latter is impossible.

By statement (1) of Exercise 3.7.14, the parabolic e contains $s \cdot s^*$ for all $s \in S$. Since the scheme $\mathcal{X}$ is Kleinian, there exist distinct irreflexive basis relations $s, t \subseteq e$ such that

$$s \in xx^* \quad \text{and} \quad t \in yy^*$$

for some $x, y \in S$. By statement (1) of Theorem 3.1.26, there are $i, j, k \in I$ such that the relations $x_{ij} = x_{\Omega_i,\Omega_j}$ and $y_{ik} = y_{\Omega_i,\Omega_k}$ are nonempty. Then

$$G_{ij} = x_{ij}x_{ij}^* = \{e_i, s_i\} \quad \text{and} \quad G_{ik} = y_{ik}y_{ik}^* = \{e_i, t_i\},$$

where $s_i = s_{\Omega_i}$ and $t_i = t_{\Omega_i}$. It follows that $G_{ij} \neq G_{ik}$. By the definition of $\mathcal{Y}$, this implies that in the graph associated with $\mathcal{Y}$, the vertex i lies in at least two regular cliques (corresponding to G_{ij} and G_{ik}). But this exactly means that the point i of $\mathcal{G}$, lies in two distinct lines, a contradiction. □

Depending on the isomorphism type of the partial linear space $\mathcal{G}$ from Proposition 4.3.26, we say that $\mathcal{X}$ is a scheme over a near-pencil, an affine plane, or a projective plane. One can check that the schemes $\mathcal{X}_{16}$, $\mathcal{X}'_{16}$ and $\mathcal{X}_{28}$, $\mathcal{X}'_{28}$ from Example 4.3.24 are schemes over an affine plane and over a projective plane, respectively.

Theorem 4.3.27. *Any Kleinian scheme $\mathcal{X}$ has index $3, 4,$ or 7. In these cases, $\mathcal{X}$ is a scheme over a near-pencil, an affine plane, or a projective plane, respectively. Moreover, in the first case, $\mathcal{X}$ is schurian and separable.*

Proof. All the statements except for the last one easily follow from Proposition 4.3.26. Let $\mathcal{X}$ be a scheme over a near-pencil and e the thin residue parabolic of $\mathcal{X}$. Then this scheme is schurian and separable if and only if so is $\mathcal{X}_e$ (Theorem 3.1.29) which is true if the reduction $\mathcal{Y}$ of $\mathcal{X}_e$ is schurian and separable (Lemma 4.1.14). However, in our case the index of $\mathcal{X}$ is equal to 3. Thus, $\mathcal{Y}$ has exactly three fibers. By Exercise 2.7.34, this implies that $\mathcal{Y}$ is schurian and separable, and we are done. □

Orthogonals

Let $\mathcal{X}$ be a quasi-thin scheme. By Exercise 4.7.28, given $s \in S_2$, there exists a uniquely determined relation $s^\perp \in S^\#$ such that

$$ss^* = \{1, s^\perp\}.$$

This relation is called the *orthogonal* of s. Note that any orthogonal is an irreflexive symmetric relation. For any $T \subseteq S_2$, we set

$$T^{\perp} = \{s^{\perp} : \ s \in T\}.$$

Any element of the set $S^{\perp}$ is called the *orthogonal* of $\mathcal{X}$. Obviously, any quasi-thin scheme has at least one orthogonal.

Theorem 4.3.28. *Any quasi-thin scheme with exactly one orthogonal is schurian and separable.*

Proof. Let $\mathcal{X}$ be a quasi-thin scheme. Assume that $S^{\perp} = \{s\}$ for some $s \in S^{\#}$. Then, $s = s^*$ and the parabolic

$$e = s^* \cdot s = \langle s \rangle$$

coincides with the thin residue parabolic of $\mathcal{X}$ (Exercise 3.7.16). It follows that each fiber of the extension $\mathcal{X}_e$ of $\mathcal{X}$ with respect to e has cardinality

$$n_s + 1 = \begin{cases} 2, & \text{if } s \in S_1, \\ 3, & \text{if } s \in S_2. \end{cases}$$

By Exercise 3.7.20, this implies that $\mathcal{X}_e$ is schurian and separable. Consequently, $\mathcal{X}$ is schurian and separable by Theorem 3.1.29. □

A Kleinian scheme contains at most 3 orthogonals, and cannot have exactly one orthogonal (Exercise 3.7.16). On the other hand, we assume that a quasi-thin scheme $\mathcal{X}$ has exactly two thin orthogonals, say u and v. Then for any $s \in S$ such that $s^{\perp} = u$,

$$\{1, u, v\} \supseteq (vs)(vs)^* = v(ss^*)v = v\{1, u\}v = \{1, v \cdot u \cdot v\}.$$

Therefore, $v \cdot u \cdot v = u$, i.e., $u \cdot v = v \cdot u$. Furthermore, $u \cdot u = v \cdot v = 1$, because u and v are thin symmetric relations. Thus the thin residue of $\mathcal{X}$ consists of 1_{Ω}, u, v, and $u \cdot v$. This gives the following characterization of the Kleinian schemes in terms of the orthogonals.

Lemma 4.3.29. *A quasi-thin scheme $\mathcal{X}$ is Kleinian if and only if $S^{\perp}$ consists of two or three thin orthogonals, and in the latter case they generate a Klein group.*

Saturation

Lemma 4.3.30. *Any non-Kleinian quasi-thin scheme of degree at least* 9 *and with at least two orthogonals, is saturated.*

Proof. Let $\mathcal{X}$ be a scheme satisfying the lemma hypothesis. To prove the saturation condition let a set $T \subseteq S_2$ contain at most 4 elements.

Claim 1. *The set $N(T)$ is nonempty unless the following statements hold:*

(1) $T^{\perp} = S^{\perp}$;
(2) $|T \cap S_x| = 2$ *for each* $x \in S^{\perp}$, *where* $S_x = \{s \in S_2 : \ s^{\perp} = x\}$*;*

(3) $|S^{\perp}| = 2$.

Proof. By the definition of $N(T)$ (formula (4.3.2) for $k = 2$) and Exercise 2.7.27, a relation $a \in S_2$ lies in $N(T)$ whenever

$$aa^* \cap tt^* = \{1\} \quad \text{for all } \ t \in T,$$

or equivalently $a^{\perp} \notin T^{\perp}$. Thus without loss of generality, we may assume that statement (1) holds. In particular,

$$|S^{\perp}| = |T^{\perp}| \le |T| \le 4.$$

Assume that $|T \cap S_x| \le 1$ for some $x \in S_2$. Then any element $a \in S_x$ lies in the set $N(T')$, where T' contains all the elements of T, except at most one. Taking into account that $a \sim a$ (as $k = 2$), we may assume that

$$|T \cap S_x| \ge 2 \quad \text{for each } \ x \in S^{\perp}.$$

Since $S^{\perp}$ has at least two elements, say $x^{\perp}$ and $y^{\perp}$, this implies that T contains at least two elements of S_x and S_y. In view of the above inequality, this proves statements (2) and (3). □

To complete the proof of Lemma 4.3.30, it suffices to verify that statements (1), (2), and (3) of Claim 1 lead to a contradiction with the lemma hypothesis. Assume on the contrary that they hold true.

Claim 2. *$S^{\perp} = \{u, v\}$, where $u \in S_2$ is such that $S_u \subseteq T$, and $v \in S_1$.*

Proof. By statement (3) of Claim 1,

$$S^{\perp} = \{u, v\}$$

for distinct relations $u, v \in S$. If they are thin, then $\mathcal{X}$ is a Kleinian scheme with two thin orthogonals (Lemma 4.3.29), a contradiction.

Now let $u, v \in S_2$. Then from statements (2) and (3) of Exercise 4.7.29, it follows that given $x, y \in S_2$,

$$c_{xs}^{y} = 1 \ \text{ for all } \ s \in x^* y.$$

This means that any two vertices of the graph $\mathfrak{X}_2$ associated with the scheme $\mathcal{X}$ (see Subsection 4.3.1) are adjacent. However, no two distinct vertices of S_u as well of S_v are adjacent. So, $|S_u| = |S_v| = 1$, contrary to statement (2) of Claim 1.

Thus we may assume that

$$u \in S_2 \quad \text{and} \quad v \in S_1.$$

Then $S_u \subseteq T$, because by statements (2) and (3) of Exercise 4.7.29, any vertex of $S_u \setminus T$ is adjacent in $\mathfrak{X}_2$ to each vertex of S_2. □

Claim 3. *$S_1 = \{1, v\}$ and $S_u = \{t, v \cdot t\}$ for some $t \in T$.*

Proof. From statement (2) of Claim 1 and Claim 2, it follows that $|S_u| = 2$. Take any $t \in S_u$. Then given $r, s \in S_1$,

$$rt = st \quad \Leftrightarrow \quad r = s.$$

Indeed, the "if" part is obvious. Conversely, we assume that $rt = st$. Then

$$s^* r \subseteq tt^* = \{1, u\}.$$

Since also $n_u = 2$, this implies that $s^* \cdot r = 1$.

Next, it is easily seen that $S_1 S_u = S_u$. Since $|S_u| = 2$,

$$|S_1| = |S_1 t| \leq |S_1 S_u| = |S_u| = 2,$$

which proves the claim. □

Claim 4. *The set $R := S_1 \cup S_v$ is closed.*

Proof. Given $r \in S_v$, we have $r^\perp = (r^*)^\perp$, for otherwise $rr^* \subseteq S_2$ by statement (3) of Exercise 4.7.29. It follows that $(S_v)^* = S_v$ and hence $R^* = R$. It is also easily seen that

$$S_1 S_v = S_v S_1 = S_v,$$

It remains to prove that $S_v S_v \subseteq R$. Let $r, s \in S_v$. Since $r^* \in S_v$ and $v \in S_1$, we have

$$v \cdot r^* = (r^*)^\perp \cdot r^* = r^*.$$

This implies that

$$rs\,(rs)^* = r(ss^*)r^* = r\{1, v\}r^* = rr^* = \{1_\Omega, v\}.$$

Thus, $rs \subseteq S_v$. □

To get a final contradiction, set

$$Q = \begin{cases} S_1 \cup S_u, & \text{if } u^\perp = u, \\ S_1 \cup S_u \cup \{u\}, & \text{if } u^\perp = v. \end{cases}$$

Since $S = R \cup Q$, Claim 4 implies that $Q = Q^*$. Using Claim 3 and the fact that $Qv = Q$, one can easily verify that Q is closed. Thus, S is the union of two closed subsets Q and R. Consequently, one of them coincides with S. This leads to a contradiction if $u^\perp = u$. In the remaining case, $S = S_u$ and hence

$$|\Omega| = n_S = n_Q = 8,$$

contrary to the lemma hypothesis. □

The Desargus condition

Lemma 4.3.31. *Any non-Kleinian quasi-thin scheme is weakly Desarguesian.*

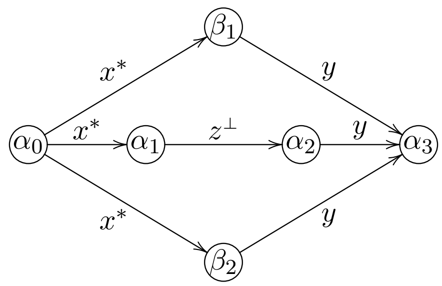


FIGURE 4.11 A six points configuration for the claim.

Proof. Let $\mathcal{X}$ be a non-Kleinian quasi-thin scheme, and let $x, y, z \in S_2$ be such that $x \sim z \sim y$. Then by Exercise 2.7.27,

$$x^{\perp} \neq z^{\perp} \quad \text{and} \quad z^{\perp} \neq y^{\perp}.$$

Assume on the contrary that $\mathcal{X}$ is not weakly Desarguesian. Then the intersection $rs \cap x^*y$ is not a singleton. Indeed, otherwise any $r \in x^*z$ is linked with any $s \in z^*y$ with respect to (x, y, z), because formulas (4.3.16) and (4.3.17) are trivially satisfied for $q = z$, $u = r$, $v = s^*$, and $w = 1$.

On the other hand, the number $|rs \cap x^*y|$ is at most $|x^*y| \leq 2$. Thus,

$$|rs \cap x^*y| = 2 \quad \text{and} \quad rs = x^*y, \qquad r \in x^*z,\ s \in z^*y. \tag{4.3.33}$$

It follows that $x \sim y$ and hence $x^{\perp} \neq y^{\perp}$. Thus,

$$x^{\perp} \neq z^{\perp} \neq y^{\perp} \neq x^{\perp}. \tag{4.3.34}$$

Claim. $x^{\perp} \cdot y^{\perp} \cdot z^{\perp} = 1$.

Proof. In view of relations (4.3.33), statement (3) of Exercise 4.7.29 implies that $x^*y \subset S_2$ and $A_{rs} = A_{x^*y}$ for all $r \in x^*z$ and $s \in z^*y$. It follows that

$$\begin{aligned} 2A_{x^*}A_y + \frac{2}{n_{z^{\perp}}}A_{x^*}A_{z^{\perp}}A_y &= A_{x^*}(2I_{\Omega} + \frac{2}{n_{z^{\perp}}}A_{z^{\perp}})A_y \\ &= A_{x^*}(A_zA_{z^*})A_y \\ &= (A_{x^*}A_z)(A_{z^*}A_y) \\ &= \sum_{\substack{r \in x^*z \\ s \in z^*y}} A_rA_s = 4A_{x^*}A_y, \end{aligned}$$

implying

$$A_{x^*}A_{z^{\perp}}A_y = n_{z^{\perp}}A_{x^*}A_y. \tag{4.3.35}$$

First, we assume that $n_{z^{\perp}} = 2$. Then there exist six points that form a configuration depicted in Fig.4.11, in which the points β_1 and β_2 are distinct. Since $n_{x^*} = 2$, it follows that either $\alpha_1 = \beta_1$ or $\alpha_1 = \beta_2$. In both cases,

$$z^{\perp} \in yy^{*} = \{1, y^{\perp}\},$$

contrary to (4.3.34).

Thus, $n_{z^{\perp}} = 1$, i.e., $z^{\perp} \in S_1$. Then (4.3.35) yields

$$A_{x^{*}} A_{z^{\perp}} A_{y} = A_{x^{*}} A_{y}.$$

It follows that

$$\langle A_{x^{*}} A_{z^{\perp}} A_{y}, A_{x^{*}} A_{y} \rangle = \langle A_{x^{*}} A_{y}, A_{x^{*}} A_{y} \rangle = 4,$$

where the scalar product $\langle \cdot, \cdot \rangle$ is as in Exercise 2.7.26. By the associativity of this scalar product and formula (4.3.34),

$$\begin{aligned} 4 &= \langle A_x A_{x^{*}} A_{z^{\perp}}, A_y A_{y^{*}} \rangle \\ &= \langle 2A_{z^{\perp}} + \frac{n_x}{n_{x^{\perp}}} A_{x^{\perp}} A_{z^{\perp}}, 2I + \frac{n_y}{n_{y^{\perp}}} A_{y^{\perp}} \rangle \\ &= \frac{n_x n_y}{n_{x^{\perp}} n_{y^{\perp}}} \langle A_{x^{\perp}} A_{z^{\perp}}, A_{y^{\perp}} \rangle. \end{aligned}$$

In particular, the right-hand side of this equality is nonzero. With taking into account that $z^{\perp} \in S_1$ and $n_x = n_y = 2$, this is possible only if

$$x^{\perp} \cdot z^{\perp} = y^{\perp} \quad \text{and} \quad n_{x^{\perp}} = n_{z^{\perp}} = 1.$$

But then $x^{\perp} \cdot y^{\perp} \cdot z^{\perp} = 1$. □

From the Claim, it follows that $x^{\perp}$, $y^{\perp}$, and $z^{\perp}$ are pairwise distinct thin relations and the product of any two of them is equal to the third. Together with 1 they form a Klein group contained in the thin residue of $\mathcal{X}$. Since the scheme $\mathcal{X}$ is not Kleinian, there exists an element $q \in S$ such that $q^{\perp}$ is different from $x^{\perp}$, $y^{\perp}$, and $z^{\perp}$. It follows that

$$(4.3.36) \quad (xx^{*}\,yy^{*}) \cap qq^{*} = \{1\},\ (xx^{*}\,zz^{*}) \cap qq^{*} = \{1\},\ (yy^{*}\,zz^{*}) \cap qq^{*} = \{1\}.$$

By the condition (L2) of Corollary 4.3.14, this implies that r and s are linked with respect to (x, y, z), a contradiction. □

Schurity and separability

We arrive at the main result of this subsection that is [**103**, Theorem 1.1.]; it should be noted that the proof in [**103**] is direct in the sense that it uses neither saturation nor Desargues condition.

Theorem 4.3.32. *Any non-schurian or non-separable quasi-thin scheme is a Kleinian scheme of index* 4 *or* 7*. Moreover, there exist infinitely many Kleinian schemes of index* 4 *and of index* 7 *that are both non-schurian and non-separable.*

Proof. Let $\mathcal{X}$ be a quasi-thin scheme. Assume that $\mathcal{X}$ is non-schurian or non-separable. Then

- the degree of $\mathcal{X}$ is at least 9 (a straightforward check);
- $|S^{\perp}| \geq 2$ (Theorem 4.3.28).

It follows that $\mathcal{X}$ is Kleinian. Indeed, otherwise it is saturated (Lemma 4.3.30) and weakly Desarguesian (Lemma 4.3.31), and thus schurian and separable (Theorem 4.3.17), a contradiction. Now the index of $\mathcal{X}$ is equal to 4 or 7 by Theorem 4.3.27.

To prove the second part, we note that the schemes $\mathcal{X}_{16}$ and $\mathcal{X}_{28}$ from Example 4.3.24 are non-schurian and non-separable Kleinian schemes of indices 4 and 7, respectively. Let $\mathcal{X}$ be one of these schemes, and let $\mathcal{Y}$ be an arbitrary regular scheme. Then $\mathcal{X} \otimes \mathcal{Y}$ is obviously a Kleinian scheme of the same index as $\mathcal{X}$ (see statement (3) of Exercise 4.7.27). Since this scheme is non-schurian (Corollary 3.2.22) and non-separable (Corollary 3.2.24), we are done. □

Corollary 4.3.33. *Any non-Kleinian quasi-thin scheme is schurian and separable.*

It would be interesting to extend the results of this subsection in two directions. First, in a natural way one can define quasi-thin coherent configurations and look for generalization of Theorem 4.3.32 to the class of all of them. Second, the characterization of Kleinian schemes in Theorem 4.3.27 is in a sense implicit. The question is how to characterize them up to isomorphism.

4.3.4 Pseudocyclic schemes

A wide subclass of two-valenced schemes is formed by nonregular equivalenced schemes. The schurian equivalenced schemes come from 3/2-transitive groups which are completely classified in [**92**]. Many of these schemes are Frobenius and this is always true in the imprimitive case (Corollary 3.3.9). One of the goals of this subsection is to show that, at least "asymptotically", the imprimitivity condition here is superfluous. The most part of the material is taken from [**102**].

A scheme $\mathcal{X}$ is said to be *pseudocyclic* if the ratio m_ξ/n_ξ does not depend on the choice of the irreducible character $\xi \in \mathrm{Irr}(\mathcal{X})^\#$. The following combinatorial characterization of pseudocyclic schemes immediately follows from Theorem 3.6.13.

Proposition 4.3.34. *A scheme $\mathcal{X}$ is pseudocyclic if and only if $\mathcal{X}$ is an equivalenced scheme of valency $k \geq 1$ and $c(s) = k - 1$ for all $s \in S(\mathcal{X})^\#$.*

Not every equivalenced scheme is pseudocyclic but only a few such examples are known. One of them is given below.

Example 4.3.35. [**17**, p.48] *The scheme $\mathcal{X}$ of the Johnson graph $J(7, 2)$ has degree 21, rank 3, and is an equivalenced scheme of valency 10. On the other hand, one can find that $c(s) = 11$ or 7 for an irreflexive basis relation s, i.e., the scheme $\mathcal{X}$ is not pseudocyclic.*

Any regular scheme (not necessarily commutative) is pseudocyclic because in this case $m_\xi = n_\xi$ for all irreducible characters ξ. A less trivial example below comes from a Frobenius group with non-abelian kernel.

Example 4.3.36. [**107**, p.187-189] *Let q be a prime power and $n > 1$ an odd integer. Denote by H a subgroup of $\mathrm{GL}(3, q^n)$ that consists of all matrices*

$$A(a,b) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^q \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b \in \mathbb{F}_{q^n}\,.$$

Take $c \in \mathbb{F}_{q^n}$ such that the multiplicative order of c equals $(q^n - 1)/(q - 1)$. Then the mapping

$$\sigma : A(a, b) \mapsto A(ca, c^{1+q}b)$$

is a fixed point free automorphism of H. It follows that $K = \langle H, \sigma\rangle$ is a Frobenius group with non-abelian kernel H and complement $\langle\sigma\rangle$. The scheme

$$\mathcal{X} = \mathrm{Inv}(K, H)$$

is of degree $|H| = q^{2n}$ and rank $q^{n+1} - q^n + q$. This scheme is equivalenced of valency $\frac{q^n-1}{q-1}$.

A straightforward calculation for $(q, n) = (2, 3)$ shows that $\mathrm{Irr}(\mathcal{X})$ consists of four characters, the multiplicities and degrees of which are as follows:

$$(m_0, n_0) = (1, 1),\ (m_1, n_1) = (7, 1),\ (m_2, n_2) = (m_3, n_3) = (14, 2).$$

In particular, $\mathcal{X}$ is a noncommutative pseudocyclic scheme: $m_\xi/n_\xi = 7$ for all characters $\xi \in \mathrm{Irr}(\mathcal{X})^\#$.

Theorem 4.3.37. *Any Frobenius scheme is pseudocyclic. Moreover, it is commutative if the corresponding Frobenius group has abelian kernel.*

Proof. Let $\mathcal{X}$ be a Frobenius scheme on Ω, $K \le \mathrm{Sym}(\Omega)$ the corresponding Frobenius group, and

$$\pi : \mathbb{C}K \to \mathrm{Mat}_\Omega(\mathbb{C}),\ k \mapsto P_k$$

is the permutation representation of K. By Proposition 2.3.5, each of the algebras $\mathcal{A} = \mathrm{Adj}(\mathcal{X})$ and $\pi(\mathbb{C}K)$ is equal to the centralizer of the other algebra in $\mathrm{Mat}_\Omega(\mathbb{C})$ (see also [**28**, p.178]). Therefore these two algebras have the same center,

$$Z(\mathcal{A}) = Z(\pi(\mathbb{C}K)) = \mathcal{A} \cap \pi(\mathbb{C}K).$$

Denote by $\mathrm{Irr}(K)$ the set of irreducible characters of K entering to the permutation representation. The above equality implies that there is a bijection $\mathrm{Irr}(\mathcal{X}) \to \mathrm{Irr}(K)$, $\xi \mapsto \zeta$ such that

$$n_\xi = m_\zeta \quad \text{and} \quad m_\xi = \zeta(1), \quad \xi \in \mathrm{Irr}(\mathcal{X}).$$

Thus the first part of the theorem is a consequence of the following statement proved in [**102**, Theorem 2.4]: if K is a nonregular transitive permutation group with point stabilizer L, and

$$\pi = \sum_{\zeta \in \mathrm{Irr}(K)} m_\zeta \zeta$$

is the decomposition of the permutation character π of K into irreducibles, then K is a Frobenius group if and only if

$$\frac{\zeta(1)}{m_\zeta} = |L| \quad \text{for each } \zeta \in \mathrm{Irr}(K)^\#.$$

The second part of the theorem follows from Proposition 2.4.5 and the fact that any Frobenius scheme is a Cayley scheme over the kernel of the corresponding Frobenius group. □

Remark 4.3.38. *In fact, the commutativity of a Frobenius scheme implies that the corresponding Frobenius group has abelian kernel, see* [**102**, Theorem 3.1].

There are a lot of equivalenced schemes for which the group of algebraic isomorphisms acts transitively on irreflexive basis relations. These schemes were studied in [**77**] and include the cyclotomic schemes over finite fields and the affine schemes. The following statement shows that all of them are pseudocyclic.

Corollary 4.3.39. *Let $\mathcal{X}$ be an equivalenced scheme. Suppose that the group $\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X})$ acts transitively on $S^\#$. Then $\mathcal{X}$ is a pseudocyclic scheme.*

Proof. By the hypothesis, the number $c(s)$ does not depend on $s \in S^{\#}$, and by Lemma 2.1.14, this number is one less that the valency of $\mathcal{X}$. Thus the scheme $\mathcal{X}$ is pseudocyclic by Proposition 4.3.34. $\square$

Not every pseudocyclic scheme is Frobenius. Indeed, any scheme from Example 2.6.15 is pseudocyclic (Corollary 4.3.39). On the other hand, many of these schemes are non-schurian and hence non-Frobenius. A schurian example of a pseudocyclic scheme, which is not Frobenius, is given below.

Example 4.3.40. [**17**, p. 390] *Let $q > 4$ be a 2-power. Denote by Ω the set of cyclic subgroups of order $q+1$ in the group $\mathrm{PSL}(2,q)$. This group acts transitively on Ω by conjugation and the scheme of the induced permutation group is of degree*

$$|\Omega| = \frac{q^2 - q}{2}.$$

This scheme is symmetric and pseudocyclic of valency $q + 1$. Some of its algebraic fusions are also pseudocyclic and were studied in [**76**].

In all the examples we have seen so far, the degree of a non-schurian pseudocyclic scheme is bounded from above by a quadratic function on the valency, e.g., the scheme of a (non-Desarguesian) affine plane of order q is of degree q^2 and valency $q-1$. The same is true for non-separable pseudocyclic schemes. The following theorem shows that, in a sense, this reflects a general situation.

Theorem 4.3.41. *There exists a function f such that any pseudocyclic scheme of valency $k > 1$ and degree at least $f(k)$ is schurian and separable.*

Proof. Set $f(x) = 3x^6 + 1$. By Theorem 4.3.17, it suffices to verify that for any pseudocyclic scheme $\mathcal{X}$ of degree n and valency k the lemma below holds.

Lemma 4.3.42. *If $n > 3k^6$, then the scheme $\mathcal{X}$ is saturated and Desarguesian.*

Proof. By Proposition 4.3.34, $n_s = k$ and $c(s) = k - 1$ for any $s \in S^{\#}$. It follows that for $k \geq 2$,

$$|S_k| = \frac{n-1}{k} > \frac{3k^6 - 1}{k} > 4k(k-1) = 4ck.$$

Thus, $\mathcal{X}$ is saturated by Corollary 4.3.4.

To prove that $\mathcal{X}$ is Desarguesian, let $x, y, z \in S_k$ (here $S_k = S^{\#}$ and we do not assume that $x \sim z \sim y$). By the condition (L2) of Corollary 4.3.14, it suffices to find $q \in S_k$ such that

$$qq^* \cap \underbrace{(xx^*\, yy^* \cup yy^*\, zz^* \cup zz^*\, xx^*)}_{T} = \{1\}.$$

Assume on the contrary that no $q \in S_k$ satisfies this condition. Then for a fixed $\alpha \in \Omega$ and each $q \in S_k$, there exists $\beta_q \in \alpha T$ other than α and such that $r(\alpha, \beta_q)$ belongs to qq^*, or equivalently,

$$\Omega_{\alpha,\beta_q} \cap \alpha q \neq \varnothing, \tag{4.3.37}$$

where the set Ω_{α,β_q} is as in (4.3.7). Since

$$|\alpha T| \leq n_x^2 n_y^2 + n_y^2 n_z^2 + n_z^2 n_x^2 = 3k^4 \quad \text{and} \quad |S_k| = \frac{n-1}{k} \geq \frac{3k^6}{k} = 3k^5,$$

there exists a point $\beta \neq \alpha$ such that $\beta = \beta_q$ for at least k relations $q \in S_k$. In view of (4.3.37), this implies that for $s = r(\alpha, \beta)$ we have

$$k - 1 = c(s) \geq |\Omega_{\alpha,\beta}| \geq k,$$

a contradiction. □

The example of affine scheme and the proof of Theorem 4.3.41 show that the function f from this theorem satisfies the inequalities

$$(k+1)^2 < f(k) \leq 3k^6 + 1.$$

A more subtle argument shows that $f(k) \leq 1 + 6k(k-1)^2$, see [**27**]. It seems that this upper bound could be improved to a quadratic function. An indirect confirmation of this statement is the following theorem showing together with Theorem 4.3.41 that, at least "asymptotically", all pseudocyclic schemes are Frobenius.

Theorem 4.3.43. *Let $\mathcal{X}$ be a schurian pseudocyclic scheme of valency $k > 1$ and degree greater than $(k-1)(2k-1)$. Then $\mathcal{X}$ is a Frobenius scheme and* $\mathrm{Aut}(\mathcal{X})$ *is a Frobenius group.*

Proof. Let g be an automorphism of $\mathcal{X}$, and let $\mathrm{fix}(g)$ be the set of its fixed points. Then obviously,

$$r(\alpha, \beta) = r(\alpha^g, \beta) \quad \text{for all } \alpha \in \Omega,\ \beta \in \mathrm{fix}(g).$$

By the assumption on $\mathcal{X}$ and Proposition 4.3.34, this implies that

$$|\,\mathrm{fix}(g)| \leq |\{\beta \in \Omega :\ r(\alpha, \beta) = r(\alpha^g, \beta)\}| = c(s) = k - 1,$$

where $s = r(\alpha, \alpha^g)$. Thus if the permutation g is non-identity, then

$$|\,\mathrm{fix}(g)| \in \{0, \ldots, k-1\}. \tag{4.3.38}$$

At this point, we make use of a result from [**24**, Proposition 1] stating that if for all non-identity elements g of a permutation group the number $|\,\mathrm{fix}(g)|$ belongs to a fixed set X of nonnegative integers, then this group has at most $2\max(X) - 1$ nonregular orbits.

In view of (4.3.38), this result, applied for a point stabilizer of the group $K = \mathrm{Aut}(\mathcal{X})$ and $X = \{0, \ldots, k-1\}$, implies that

$$|\{\Delta \in \mathrm{Orb}(K_\alpha) :\ \Delta \text{ is nonregular}\}| \leq 2(k-1)$$

for all points α.

On the other hand, since $\mathcal{X}$ is equivalenced of valency k, we have

$$|\operatorname{Orb}(K_\alpha)| = |S(\mathcal{X})| = \frac{n-1}{k} + 1 > \frac{(k-1)(2k-1)}{k} + 1 \geq 2k-1,$$

where n is the degree of $\mathcal{X}$. Thus at least one orbit and hence all orbits of the group K_α are regular, i.e., $\operatorname{Aut}(\mathcal{X}) = K$ is a Frobenius group. □

4.4 Cyclotomic and circulant schemes

In this section we study cyclotomic schemes over a finite field. They are, of course, schurian. Moreover, being pseudocyclic, they are also asymptotically (in the sense of Theorem 4.3.41) separable. But what happens if such a scheme has small rank, e.g., is the scheme of the Paley graph or tournament? This question was answered in [**41**], where it was proved that every cyclotomic scheme over a finite field is 3-separable or equivalently, is determined up to isomorphism by the 3-dimensional intersection numbers. Below we prove this result; in the presentation of the material we follow [**41**].

The key idea of the proof is to show that a one-point extension of a cyclotomic scheme $\mathcal{X}$ is schurian and hence inequality (3.3.5) becomes an equality for $K = \mathrm{Aut}(\mathcal{X})$. Then the upper bound for the separability number of $\mathcal{X}$ follows from Theorem 4.2.3.

In the process of the proof it turns out that the one-point extension of a cyclotomic scheme over a field $\mathbb{F}$ is closely related with a Cayley scheme over the group $\mathbb{F}^{\times}$. This group is cyclic which makes it possible to study this scheme with the help of the Leung–Man theory on S-rings over cyclic group [**89, 90**].

4.4.1 Reduction of cyclotomic schemes to circulant schemes

In what follows, a Cayley scheme over a cyclic group is said to be *circulant*. Let $\mathcal{X}$ be a circulant scheme, and let G be the underlying cyclic group. By the Schur theorem on multipliers (Theorem 2.4.10),

$$\mathrm{Iso}_{cay}(\mathcal{X}) = \mathrm{Aut}(G).$$

The group $\mathrm{Aut}(G)$ acts transitively on the generators of G and hence on the relations $s = r(\alpha, \beta)$, where α and β are the identity and a generator of G, respectively. Therefore, the parabolic

$$e = \mathrm{rad}(s)$$

does not depend on the choice of the generator β.

The subgroup of G associated with the parabolic e, i.e., one defined by formula (2.4.6), is called the *radical* of $\mathcal{X}$ and denoted by $\mathrm{rad}(\mathcal{X})$. Some properties of this group can be found in Exercise 4.7.35.

The following important result on circulant schemes is known as the Leung–Man theorem. It was proved in [**89, 90**] (see also [**41**]) in terms of S-rings over a cyclic group. Using the one-to-one correspondence between Cayley schemes and S-rings established in Theorems 2.4.16 and 2.4.17, one can reformulate the Leung–Man theorem as follows.

Theorem 4.4.1. *Let $\mathcal{X}$ be a circulant scheme. Then*

(1) $\mathrm{rad}(\mathcal{X}) \neq 1$ *if and only if $\mathcal{X}$ is a proper generalized wreath product;*
(2) $\mathrm{rad}(\mathcal{X}) = 1$ *if and only if $\mathcal{X}$ is Cayley isomorphic to the tensor product of a cyclotomic circulant scheme with trivial radical and trivial schemes.*

Theorem 4.4.1 shows that any circulant scheme can be constructed from trivial and cyclotomic circulant schemes with the help of tensor and generalized wreath products. Another consequence of the Leung–Man theorem is that it strongly limits the structure of the automorphism group of a circulant scheme with trivial radical.

Corollary 4.4.2. *Let $\mathcal{X}$ be a circulant scheme with trivial radical. Then the degree of $\mathcal{X}$ admits a decomposition into the product of pairwise coprime factors $n_0, n_1, \ldots, n_k$ such that*

$$\mathcal{X} = \mathcal{X}_0 \otimes \mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}, \tag{4.4.1}$$

where $\mathcal{X}_0$ is a cyclotomic scheme over C_{n_0} with trivial radical. In particular,

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathcal{X}_0) \times \mathrm{Sym}(n_1) \times \cdots \times \mathrm{Sym}(n_k). \tag{4.4.2}$$

Proof. See statement (2) of Theorem 4.4.1 and formula (3.2.18). □

Below, we are interested in normal circulant schemes with trivial radical. In principle, Theorem 4.4.1 enables us to characterize them explicitly up to Cayley isomorphism, see Exercise 4.7.37. However for our purposes, the following statement is sufficient.

Corollary 4.4.3. *Every normal circulant scheme with trivial radical is cyclotomic.*

Proof. Let $\mathcal{X}$ be a circulant scheme with trivial radical. Then the group $\mathrm{Aut}(\mathcal{X})$ is of the form (4.4.2). If $\mathcal{X}$ is normal, then $\mathrm{Aut}(\mathcal{X})$ is also isomorphic to a subgroup of the holomorph of the underlying group. However, the holomorph of a cyclic group C_m, $m \geq 4$, has no subgroups isomorphic to $\mathrm{Sym}(m)$. By formula (4.4.1) this implies that $\mathcal{X}$ is the tensor product of circulant cyclotomic schemes. Thus, $\mathcal{X}$ is cyclotomic. □

Another surprising corollary of the Leung–Man theorem is that any nonregular circulant scheme has nontrivial automorphisms.

Corollary 4.4.4. *Let $\mathcal{X}$ be a Cayley scheme over a cyclic group G. Then*

$$\mathrm{Aut}(\mathcal{X}) = G_{right} \quad \Leftrightarrow \quad \mathcal{X} = \mathrm{Inv}(G_{right}).$$

Proof. The sufficiency follows from Theorem 2.2.11. Conversely, assume that

$$\mathrm{Aut}(\mathcal{X}) = G_{right}.$$

Then $\mathcal{X}$ is a normal circulant scheme. By Corollary 3.4.22, $\mathcal{X}$ cannot be a proper generalized wreath product. Consequently, it is a normal circulant scheme with trivial radical (Theorem 4.4.1). Thus,

$$\mathcal{X} = \mathrm{Cyc}(M, C_n)$$

for some $M \le \mathrm{Aut}(G)$ (Corollary 4.4.3). In view of the assumption, $M = 1$, and we are done. □

Now let $\mathcal{X}$ be a cyclotomic scheme over a finite field $\mathbb{F}$. In what follows, the zero element of $\mathbb{F}$ is denoted by α. Consider the restriction of the α-extension of $\mathcal{X}$ to the set $\mathbb{F}^\times = \mathbb{F} \setminus \{\alpha\}$,

$$\mathcal{X}' = (\mathcal{X}_\alpha)_{\mathbb{F}^\times}. \tag{4.4.3}$$

The group G induced by the multiplicative group of $\mathbb{F}$ that acts on $\mathbb{F}$ by right (or left) multiplications, becomes a subgroup of $\mathrm{Iso}(\mathcal{X})$ (Exercise 2.7.16). This subgroup leaves the point α fixed and hence can be identified with a subgroup of the group $\mathrm{Iso}(\mathcal{X}')$ (Theorem 2.6.4).

Thus the algebraic fusion

$$\mathcal{X}^* = (\mathcal{X}')^G \tag{4.4.4}$$

is a Cayley scheme over the group G. This group is cyclic and hence the scheme $\mathcal{X}^*$ is circulant.

Lemma 4.4.5. *In the above notation,*

(1) $\mathrm{Aut}(\mathcal{X}^*) \le G\,\mathrm{Aut}(\mathbb{F})$ *whenever the scheme* $\mathcal{X}$ *is nontrivial;*
(2) $\mathcal{X}^*$ *is schurian if and only if so is* $\mathcal{X}_\alpha$.

Proof. First, the hypothesis of Lemma 3.3.20 is obviously satisfied for the coherent configuration $\mathcal{X}_\alpha$ and $\Delta = \mathbb{F}^\times$. Therefore by this lemma

$$\mathrm{Aut}(\mathcal{X}') = \mathrm{Aut}((\mathcal{X}_\alpha)_{\mathbb{F}^\times}) = \mathrm{Aut}(\mathcal{X}_\alpha)^{\mathbb{F}^\times}$$

and the coherent configurations $\mathcal{X}_\alpha$ and $\mathcal{X}'$ are schurian or not simultaneously.

From statement (1) of Proposition 3.3.3 and Theorem 2.2.4, it follows that

$$\mathrm{Aut}(\mathcal{X}_\alpha)^{\mathbb{F}^\times} = (\mathrm{Aut}(\mathcal{X})_\alpha)^{\mathbb{F}^\times} \le G \rtimes \mathrm{Aut}(\mathbb{F}).$$

Thus,

$$\mathrm{Aut}(\mathcal{X}') \le G \rtimes \mathrm{Aut}(\mathbb{F})$$

and to prove statements (1) and (2) it suffices to verify that the hypothesis of Theorem 3.1.30 is satisfied for the coherent configuration $\mathcal{X}'$ and the group G.

Denote by M the subgroup of the multiplicative group of the field $\mathbb{F}$ such that

$$\mathcal{X} = \mathrm{Cyc}(M, \mathbb{F}).$$

By the definition of cyclotomic scheme,

$$\alpha s \in \mathrm{Orb}(M, \mathbb{F}), \quad s \in S(\mathcal{X}).$$

By statement (1) of Theorem 3.3.7, this implies that $F(\mathcal{X}_\alpha) = \mathrm{Orb}(M, \mathbb{F})$. Thus,

$$F(\mathcal{X}') = \mathrm{Orb}(M, \mathbb{F}^\times).$$

Clearly, the group G regularly acts on the fibers of $\mathcal{X}'$ and the kernel of this action coincides with $M \le \mathrm{Aut}(\mathcal{X}')$. Thus the hypothesis of Lemma 3.1.30 is satisfied, and we are done. □

One can see that the scheme $\mathcal{X}^*$ is trivial if and only if so is $\mathcal{X}$. The following statement together with Corollary 4.4.3 show that in the nontrivial case, $\mathcal{X}^*$ is also schurian.

Lemma 4.4.6. *Assume that the scheme $\mathcal{X}$ is not trivial. Then $\mathcal{X}^*$ is a normal circulant scheme with trivial radical.*

Proof. Clearly, $\mathrm{Aut}(\mathbb{F}) \le \mathrm{Aut}(G)$. Therefore, the scheme $\mathcal{X}^*$ is normal by Theorem 2.4.12 and statement (1) of Lemma 4.4.5. Assume on the contrary that

$$\mathrm{rad}(\mathcal{X}^*) \ne 1.$$

By statement (1) of Theorem 4.4.1, this implies that $\mathcal{X}^*$ is a proper generalized wreath product, say the U/L-wreath product, where L and U are proper subgroups of G such that $L \le U$.

Let $\Lambda \in (G/U)^\#$ and $g \in L^\#$. Then by Corollary 3.4.21 for $\mathcal{X} = \mathcal{X}^*$, there exists an automorphism $k \in \mathrm{Aut}(\mathcal{X}^*)$ such that

$$\lambda^k = \begin{cases} \lambda g, & \text{if } \lambda \in \Lambda, \\ \lambda, & \text{if } \lambda \in G \setminus \Lambda. \end{cases}$$

In particular, k leaves each element of $\mathbb{F}$ that is not in Λ fixed. Taking into account that

$$|\Lambda| = |U| \le \frac{q-1}{2},$$

where $q = |\mathbb{F}|$, we obtain

$$|\,\mathrm{fix}(k)| = 1 + |F^\times \setminus \Lambda| \ge 1 + \frac{q-1}{2},$$

where $\mathrm{fix}(k)$ is the set of all fixed points of the permutation $k \in \mathrm{Sym}(\mathbb{F})$.

On the other hand, this set contains the identity of G. By statement (1) of Lemma 4.4.5, this implies that $k \in \mathrm{Aut}(\mathbb{F})$. It follows that the elements of $\mathrm{fix}(k)$ form a proper subfield of $\mathbb{F}$, say $\mathbb{F}_0$. Thus,

$$q = |\mathbb{F}| \ge |\mathbb{F}_0|^2 = |\,\mathrm{fix}(k)|^2 \ge \big(\frac{q-1}{2} + 1\big)^2 = \frac{(q-1)^2}{4} + q,$$

a contradiction. □

4.4.2 A point extension of a normal circulant scheme

Lemmas 4.4.5 and 4.4.6 reduce the study of the one-point extension of a cyclotomic scheme to that of a normal circulant scheme with trivial radical. The aim of this subsection is to prove that every point extension of such a scheme is schurian and separable. More precisely, the following statement holds.

Theorem 4.4.7. *Any m-point extension of a normal circulant scheme with trivial radical is partly regular, $m \geq 1$.*

The arguments to prove Theorem 4.4.7 are based on two statements (Theorems 4.4.9 and 4.4.10). The first of them shows that the result is true in the prime power case, whereas the second one provides a reduction of the general case. In what follows,

$$\mathcal{X} = \mathrm{Cyc}(M, G), \tag{4.4.5}$$

where G is a cyclic group and $M \leq \mathrm{Aut}(G)$. We start with an auxiliary statement giving a necessary and sufficient condition for a point extension of $\mathcal{X}$ to be partly regular. In what follows, α denotes the identity of G.

Lemma 4.4.8. *$\mathcal{X}_\alpha$ is partly regular if and only if $\mathcal{X}_\alpha = \mathrm{Inv}(M, G)$.*

Proof. Each generator of G forms a base of the group $M \leq \mathrm{Sym}(G)$. Therefore,

$$b(M) \leq 1.$$

Thus the sufficiency follows from Theorem 3.3.18.

Conversely, we assume that the coherent configuration $\mathcal{X}_\alpha$ is partly regular. Then it is schurian by Theorem 3.3.19. By formula (3.3.4), this implies that

$$\mathcal{X}_\alpha = \mathrm{Inv}(K, G)_\alpha \leq \mathrm{Inv}(K_\alpha, G), \tag{4.4.6}$$

where $K = \mathrm{Aut}(\mathcal{X})$. In particular, the coherent configuration $\mathrm{Inv}(K_\alpha, G)$ is partly regular (Exercise 3.7.28).

By Theorem 3.3.18, this implies that $b(K_\alpha) \leq 1$, i.e., there exists $\beta \in G$ such that the group $K_{\alpha,\beta}$ is trivial. Therefore,

$$|K_\alpha| = |\beta^{K_\alpha}|.$$

On the other hand, in view of (4.4.5), the groups M and K_α have the same orbits on G. Consequently, $\beta^{K_\alpha} \in \mathrm{Orb}(M, G)$. It follows that

$$|M| \geq |\beta^{K_\alpha}| = |K_\alpha|.$$

Since $M \leq K_\alpha$, this implies that $M = K_\alpha$, and we are done by (4.4.6). □

Throughout the rest of the subsection, p is a prime divisor of $|G|$, and G_p and $G_{p'}$ are the Sylow p-subgroup and the complement of G_p in G, respectively. Thus,

$$G = G_p \times G_{p'}.$$

The fact that the scheme $\mathcal{X}$ is cyclotomic, implies that the equivalence relations

$$e_p = (G_p)^\rho \quad \text{and} \quad e_{p'} = (G_{p'})^\rho \tag{4.4.7}$$

are parabolics of $\mathcal{X}$, where the mapping ρ is defined by formula (2.4.10) (see statement (6) of Exercise 1.4.16).

Theorem 4.4.9. *Assume that the scheme $\mathcal{X}$ is nontrivial,* $\operatorname{rad}(\mathcal{X})$ *is trivial, and G is a p-group. Then*

$$\mathcal{X}_\alpha = \operatorname{Inv}(M, G).$$

Proof. By the theorem hypothesis and Theorem 4.4.1, the scheme $\mathcal{X}$ is not a proper generalized wreath product. So by Proposition 3.4.24,

$$p \ne 2,\ m \,|\, p-1 \quad \text{or} \quad p = 2,\ m \le 2,\ M \ne \{1, \sigma_{1+n/2}\}, \tag{4.4.8}$$

where $m = |M|$ and $n = |G|$. Thus by Lemma 4.4.8, it suffices to verify that in each of these two cases, the coherent configuration $\mathcal{X}_\alpha$ is partly regular.

Let $p > 2$ and $n \ne p$. Then $m \,|\, p-1$ and hence the group

$$M = (GM)_\alpha$$

acts half transitively on $G^\#$. It follows that the group GM is 3/2-transitive. By Corollary 2.2.6, this shows that the scheme $\mathcal{X}$ is equivalenced. It is also imprimitive, because $\mathcal{X}$ has a parabolic with classes of cardinality p, and this parabolic is nontrivial (since $n \ne p$). Thus the coherent configuration $\mathcal{X}_\alpha$ is partly regular by Theorem 3.3.8.

Let $p > 2$ and $|n| = p$. In this case, we may identify $\mathcal{X}$ with a nontrivial cyclotomic scheme over a field $\mathbb{F}$ of order p. Then by Lemma 4.4.6, the scheme $\mathcal{X}^*$ defined in (4.4.4) is a normal Cayley scheme over the cyclic group $\mathbb{F}^\times$.

Moreover, since the group $\operatorname{Aut}(\mathbb{F})$ is trivial, from statement (1) of Lemma 4.4.5, it follows that

$$\operatorname{Aut}(\mathcal{X}^*) \le \mathbb{F}^\times \operatorname{Aut}(\mathbb{F}) = \mathbb{F}^\times.$$

By Theorem 4.4.4, this shows that the scheme $\mathcal{X}^*$ is regular. Consequently, the coherent configuration

$$(\mathcal{X}_\alpha)_{\mathbb{F}^\times} \ge \mathcal{X}^*$$

is semiregular (statement (2) of Exercise 2.7.12). It follows that $\mathcal{X}_\alpha$ is partly regular, as required.

Let $p = 2$. Then in accordance with (4.4.8), we have

$$M = 1_G \quad \text{or} \quad M = \{\sigma_1, \sigma_{-1}\} \quad \text{or} \quad M = \{\sigma_1, \sigma_{-1+n/2}\}. \tag{4.4.9}$$

It suffices to verify that any generator β of the group G is a regular point of the coherent configuration $\mathcal{X}_\alpha$.

By the definition of a cyclotomic scheme, the basis relations of $\mathcal{X}$ are of the form

$$s_c = \{(\beta^a, \beta^b) : \ b - a = \mu c \pmod n), \ \mu \in M, \ a, b \in \mathbb{Z}\}$$

for some $c \in \mathbb{Z}$; by statement (1) of Theorem 2.2.5, the fibers of $\mathcal{X}_\alpha$ are of the form

$$\Delta_c = \{\beta^a : \ a = \mu c \pmod n), \ \mu \in Mc\}.$$

Now a straightforward computation shows that

$$|\beta s_c \cap \Delta_d| \le 1$$

for each group M in (4.4.9) and all the pairs (s_c, Δ_d), $d \in \mathbb{Z}$. On the other hand, by Lemma 3.3.5 we have

$$s_c \cap (\Delta_1 \times \Delta_d) \in S(\mathcal{X}_\alpha)^\cup.$$

Thus, β is a regular point of $\mathcal{X}_\alpha$. □

Given a prime p dividing $|G|$ and a relation $s \subseteq G \times G$, set

$$\mathrm{pr}_p(s) = \{(\beta_p, \gamma_p) : \ (\beta, \gamma) \in s\} \tag{4.4.10}$$

where β_p and γ_p are the p-components of the elements β and γ of the group G. Since $e_{p'}$ is a parabolic of $\mathcal{X}$, one can consider the quotient scheme

$$\mathrm{pr}_p(\mathcal{X}) := \mathcal{X}_{G/e_{p'}}.$$

The parabolics e_p and $e_{p'}$ are obviously orthogonal. Therefore under the natural identification of G_p and $G/e_{p'}$, we may assume that $\mathrm{pr}_p(\mathcal{X})$ is a Cayley scheme over G_p, the basis relations of which are defined by formula (4.4.10) with $s \in S(\mathcal{X})$.

Finally, set

$$M_p = M \cap \mathrm{Aut}(G_p) \quad \text{and} \quad \mathcal{X}_p = \mathrm{Inv}(K_p, G_p),$$

where $\mathrm{Aut}(G_p)$ is treated as a subgroup of $\mathrm{Aut}(G) = \mathrm{Aut}(G_p) \times \mathrm{Aut}(G_{p'})$, and $K_p = G_p M_p$.

Theorem 4.4.10. *Under the above assumptions and notation suppose, in addition, that $\mathcal{X}$ is normal, $\mathrm{rad}(\mathcal{X})$ is trivial, and $(\mathcal{X}_p)_{\alpha_p} = \mathrm{Inv}(M_p, G_p)$. Then*

$$\mathrm{pr}_p(\mathcal{X}_\alpha) = \mathrm{Inv}(\pi_p(M), G_p),$$

where $\pi_p : \mathrm{Aut}(G) \to \mathrm{Aut}(G_p)$ is the projection epimorphism.

Proof. In what follows, set $S = S(\mathcal{X})$ and denote by β a generator of $G_{p'}$. First, we prove several auxiliary statements.

Lemma 4.4.11.

$$\mathrm{Orb}(K_p, G_p \times G_p\beta) = \{s_{G_p, G_p\beta} : \ s \in S\}^\natural.$$

Proof. It suffices to check that any nonempty relation $s_{G_p,G_p\beta}$ is an orbit of the group K_p acting on $G_p \times G_p\beta$. To this end, fix an element $(\gamma, \delta\beta)$ of this set. Since s is a 2-orbit of the group $K = GM$, we have

$$s = \{(\gamma^\sigma, \delta^\sigma\beta^\sigma) : \ \sigma \in K\}.$$

Let us write $\sigma \in G\,\mathrm{Aut}(G)$ as the product of $\sigma_p \in G_p\,\mathrm{Aut}(G_p)$ and $\sigma_{p'} \in G_{p'}\,\mathrm{Aut}(G_{p'})$. Note that

$$(\gamma^\sigma, \delta^\sigma\beta^\sigma) \in s_{G_p,G_p\beta} \quad \Leftrightarrow \quad (G_p)^\sigma = G_p \ \text{and} \ (G_p\beta)^\sigma = G_p\beta.$$

The latter two equalities hold true if and only if $\sigma_{p'}$ leaves the identity of $G_{p'}$ and the element $\beta_{p'}$ fixed. But this means that $\sigma_{p'}$ is identity, or in other words, $\sigma \in K_p$ (see the definition of the group M_p). Thus,

$$s_{G_p,G_p\beta} = (\gamma, \delta\beta)^{K_p},$$

as required. □

Lemma 4.4.12.

$$\mathrm{Inv}(K_p, G_p\beta) \le (\mathcal{X}_\alpha)_{G_p\beta}.$$

Proof. Let r be a basis relation of the scheme $\mathrm{Inv}(K_p, G_p\beta)$, i.e., a 2-orbit of the group K_p acting on $G_p\beta$. Then the set

$$t = \{(\gamma_p, \gamma') : \ (\gamma, \gamma') \in r\}$$

is an orbit of K_p acting on $G_p \times G_p\beta$.

By Lemma 4.4.11, this implies that $t = s_{G_p,G_p\beta}$ for some $s \in S$. It follows that

$$r = (e_{p'} \cdot 1_{G_p} \cdot s)_{G_p\beta},$$

where $e_{p'}$ is as in formula (4.4.7). Since, $e_{p'}$ and 1_{G_p} are relations of $\mathcal{X}_\alpha$, we conclude that so is the composition $e_{p'} \cdot 1_{G_p} \cdot s$. Thus, r is a relation of the coherent configuration $(\mathcal{X}_\alpha)_{G_p\beta}$. □

Lemma 4.4.13.

$$(\mathcal{X}_\alpha)_{G_p\beta} = \mathrm{Inv}(M_p, G_p\beta).$$

Proof. The parabolic e_p of the scheme $\mathcal{X}$ is M_p-invariant. Furthermore,

$$G_p\beta = G_p\beta_{p'} = \beta_{p'}e_p,$$

is a class of e_p, containing the point $\beta_{p'}$ fixed by M_p. Therefore, the set $G_p\beta$ is invariant with respect to the group $M_p \le \mathrm{Aut}(\mathcal{X}_\alpha)$. Thus,

$$(\mathcal{X}_\alpha)_{G_p\beta} \le \mathrm{Inv}(M_p, G_p\beta). \tag{4.4.11}$$

Conversely, the intersection of $G_p\beta$ and the class of $e_{p'}$ containing α is equal to the singleton $\{\beta_{p'}\}$. Since $G_p\beta_{p'} = G_p\beta_p$, this singleton is the fiber of the coherent configuration $(\mathcal{X}_\alpha)_{G_p\beta}$.

By Lemma 4.4.12, this implies that

$$\operatorname{Inv}(K_p, G_p\beta)_{\beta_{p'}} \le (\mathcal{X}_\alpha)_{G_p\beta}. \tag{4.4.12}$$

On the other hand, multiplication by $\beta_{p'}$ induces a bijection $f : G_p \to G_p\beta$. Obviously,

$$(\alpha_p)^f = \beta_{p'} \quad \text{and} \quad \operatorname{Inv}(GM_p, G_p)^f = \operatorname{Inv}(GM_p, G_p\beta).$$

Thus after applying f to the equality $(\mathcal{X}_p)_{\alpha_p} = \operatorname{Inv}(M_p, G_p)$ given by the hypothesis of the theorem, we get

$$\operatorname{Inv}(K_p, G_p\beta)_{\beta_{p'}} = \operatorname{Inv}(M_p, G_p\beta).$$

Together with (4.4.12), this proves the inclusion reversed to (4.4.11). □

To complete the proof of Theorem 4.4.10, set

$$\mathcal{X}' = (\mathcal{X}_\alpha)^{\pi_p(M)}.$$

Then from Lemma 4.4.13 and the obvious inclusion $M_p \le \pi_p(M)$, it follows that

$$\begin{aligned} \mathcal{X}'_{G_p\beta} &= ((\mathcal{X}_\alpha)^{\pi_p(M)})_{G_p\beta} \\ &= ((\mathcal{X}_\alpha)_{G_p\beta})^{\pi_p(M)} \\ &= \operatorname{Inv}(M_p, G_p\beta)^{\pi_p(M)} \\ &= \operatorname{Inv}(\pi_p(M), G_p\beta). \end{aligned} \tag{4.4.13}$$

Furthermore, let $\Delta \in \operatorname{Orb}(\pi_{p'}(M), G_{p'})$, where $\pi_{p'} = \pi_{G_{p'}}$. Then the set $G_p\Delta \subseteq G$ is the neighborhood of the point α in the relation

$$t = \{(a, b) \in G^2 : \ a^{-1}b \in G_p\Delta\}.$$

However, $t \in S^{\cup}$. Therefore, $G_p\Delta$ is a homogeneity set of $\mathcal{X}_\alpha$ (Lemma 3.3.5), and hence of $\mathcal{X}'$, because $(G_p\Delta)^{\pi_p(M)} = G_p\Delta$. So by the definition of the coherent configuration $\operatorname{pr}_p(\mathcal{X}')$, we have

$$\operatorname{pr}_p(\mathcal{X}') = \operatorname{pr}_p(\mathcal{X}'_{G_p\Delta}). \tag{4.4.14}$$

On the other hand, $\operatorname{Aut}(\mathcal{X}')$ obviously contains M and $\pi_p(M)$, and hence $\pi_{p'}(M)$. It follows that the set $\operatorname{pr}_p(\mathcal{X}'_{G_p\delta})$ does not depend on the choice of $\delta \in \Delta$. Thus formula (4.4.14) implies that

$$\operatorname{pr}_p(\mathcal{X}') = \operatorname{pr}_p(\mathcal{X}'_{G_p\delta}), \quad \delta \in G_{p'}.$$

Since the actions of M and $\pi_p(M)$ on G_p coincide, this equation together with equation (4.4.13) show that

$$\begin{aligned} \operatorname{pr}_p(\mathcal{X}_\alpha) = (\mathcal{X}_\alpha)_{G_p} = \mathcal{X}'_{G_p} = \operatorname{pr}_p(\mathcal{X}') = \operatorname{pr}_p(\mathcal{X}'_{G_pg}) \\ = \operatorname{pr}_p(\operatorname{Inv}(\pi_p(M), G_p\beta)) = \operatorname{Inv}(\pi_p(M), G_p), \end{aligned}$$

which completes the proof of the theorem. □

Proof of Theorem 4.4.7 Since a fission of partly regular scheme is partly regular, it suffices to prove Theorem 4.4.7 for $m = 1$ only. Let $\mathcal{X}$ be a normal circulant scheme with trivial radical. By Corollary 4.4.3, we may assume that $\mathcal{X} = \mathrm{Cyc}(M, G)$ for some $M \le \mathrm{Aut}(G)$.

Let us prove that the coherent configuration $\mathcal{X}_\alpha$ is partly regular; then by the transitivity of $\mathrm{Aut}(\mathcal{X})$, this is true for any one-point extension.

Claim. *Let p be a prime divisor of the order of G. Then the radical of $\mathcal{X}_p$ is trivial and $\mathcal{X}_p$ is a nontrivial scheme unless $|G_p| \le 3$.*

Proof. Since $M_p \le M$, we have

$$\mathrm{Cyc}(M_p, G) \ge \mathrm{Cyc}(M, G) = \mathcal{X}.$$

By statement (2) of Exercise 4.7.35, this implies that the radical of the scheme $\mathrm{Cyc}(M_p, G)$ is trivial. Since $M_p = \pi_p(M) \times 1_{G_{p'}}$, it follows that

$$\mathrm{Cyc}(M_p, G) = \mathrm{Cyc}(\pi_p(M), G_p) \otimes \mathrm{Cyc}(1_{G_{p'}}, G_{p'}). \tag{4.4.15}$$

Consequently, the radical of the scheme $\mathcal{X}_p = \mathrm{Cyc}(\pi_p(M), G_p)$ is also trivial. Finally, if $|G_p| > 3$, then this scheme cannot be trivial, for otherwise formula (4.4.15) implies that

$$\mathrm{Aut}(\mathcal{X}) \ge \mathrm{Aut}(\mathrm{Cyc}(M_p, G)) \ge \mathrm{Sym}(G_p) \times G_{p'},$$

which is impossible by the normality of $\mathcal{X}$. This proves the claim. □

Now if $|G_p| > 3$, then the scheme $\mathcal{X}_p$ satisfies the hypothesis of Theorem 4.4.9 and hence

$$(\mathcal{X}_p)_\alpha = \mathrm{Inv}(\pi_p(M), G_p) = \mathrm{Inv}(M_p, G_p).$$

Consequently by Theorem 4.4.10,

$$\mathrm{pr}_p(\mathcal{X}_\alpha) = \mathrm{Inv}(\pi_p(M), G_p).$$

Note that this equality obviously holds if $|G_p| = 2$ or 3. Thus,

$$\mathcal{X}_\alpha \ge \bigotimes_{p \in \mathcal{P}} \mathrm{pr}_p(\mathcal{X}_\alpha) = \bigotimes_{p \in \mathcal{P}} \mathrm{Inv}(\pi_p(M), G_p), \tag{4.4.16}$$

where $\mathcal{P}$ is the set of prime divisors of the order of G.

Each factor on the right-hand side of (4.4.16) is a partly regular coherent configuration. Therefore, the entire tensor product and hence the coherent configuration $\mathcal{X}_\alpha$ is also partly regular (Exercise 3.7.28). □

4.4.3 Separability and base numbers of a cyclotomic scheme

The following theorem represents the main result of this section. In particular, it gives an almost complete solution to the separability problem for cyclotomic schemes over a finite field. The proof is based on the results obtained in Subsections 4.4.1 and 4.4.2.

Theorem 4.4.14. *Let $\mathcal{X}$ be a cyclotomic scheme over a finite field. Then*

(1) *any m-point extension of $\mathcal{X}$ is schurian, $m \ge 1$;*
(2) *$b(\mathcal{X}) = b(K)$, where $K = \mathrm{Aut}(\mathcal{X})$;*
(3) *$s(\mathcal{X}) \le b(K)$; in particular, $s(\mathcal{X}) \le 3$.*

Proof. It suffices to verify statement (1). Indeed, we assume that it is true. Then in view of formula (3.3.4), for any points $\alpha, \beta, \ldots$ of the scheme $\mathcal{X}$, we have

$$\mathcal{X}_{\alpha,\beta,\ldots} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}_{\alpha,\beta,\ldots})) = \mathrm{Inv}(K_{\alpha,\beta,\ldots}).$$

Therefore, the coherent configuration $\mathcal{X}_{\alpha,\beta,\ldots}$ is discrete if and only if the group $K_{\alpha,\beta,\ldots}$ is trivial. This proves statement (2). Next, from statement (1) and Theorem 3.3.18, it follows that $\mathcal{X}$ admits a partly regular extension with respect to $b(K) - 1$ points. Thus statement (3) follows from Theorem 4.2.3.

To prove statement (1), we may assume that $\mathcal{X}$ is not trivial. Then the scheme $\mathcal{X}^*$ defined by formula (4.4.4) is a normal circulant scheme with trivial radical (Lemma 4.4.6). Consequently, it is cyclotomic (Corollary 4.4.3) and hence schurian. By statement (2) of Lemma 4.4.5, this implies that the coherent configuration $\mathcal{X}_\alpha$ is also schurian. By the transitivity of the group K this proves the required statement for $m = 1$.

Let $m \ge 2$. Since $\mathcal{X}^*$ is a normal circulant scheme with trivial radical, the extension of it with respect to any $m-1$ points $\beta, \gamma, \ldots$ is partly regular (Theorem 4.4.7). By Exercise 3.7.28, any fission of this extension is also partly regular. Therefore the coherent configuration

$$(\mathcal{X}_{\alpha,\beta,\gamma,\ldots})_{\mathbb{F}^\times} \ge (\mathcal{X}^*)_{\beta,\gamma,\ldots}$$

is partly regular.

By Theorem 3.3.19, the above inclusion shows that the coherent configuration on the left-hand side of the above inclusion is schurian. Consequently, so is the coherent configuration $\mathcal{X}_{\alpha,\beta,\gamma,\ldots}$. In view of the transitivity of the group K, the point α can be replaced for any other point of $\mathcal{X}$. Thus, any point extension of $\mathcal{X}$ is schurian. □

We complete the subsection by making some comments on the statements of the theorem. In what follows, $\mathbb{F}$ is the field associated with the scheme $\mathcal{X}$ and $A \le \mathrm{Sym}(\mathbb{F})$ is the additive group of $\mathbb{F}$. Then one of the following cases occurs:

- $\mathcal{X}$ is regular, i.e., $K = A$: $b(\mathcal{X}) = s(\mathcal{X}) = 1$;

- $\mathcal{X}$ is trivial, i.e., $K = \mathrm{Sym}(\mathbb{F})$: $b(\mathcal{X}) = q - 1$ and $s(\mathcal{X}) = 1$;
- $\mathcal{X}$ is proper and one of the two following statements holds:
  (a) $b(\mathcal{X}) = 2$ and $s(\mathcal{X}) = 1$ or 2; both cases are possible;
  (b) $b(\mathcal{X}) = 3$ and $s(\mathcal{X}) \leq 3$; no example for $s(\mathcal{X}) = 3$ is known.

One can also prove that the m-dimensional extension of the scheme $\mathcal{X}$ is schurian for all $m \geq 1$. However, the proof of this statement is beyond the scope of this text; the interested reader is referred to [**41**].

4.5 Schemes of prime degree

Let $\mathcal{X}$ be a scheme of prime degree p. From Corollary 2.1.23, it immediately follows that any parabolic of $\mathcal{X}$ is of valency 1 or p. Therefore, $\mathcal{X}$ must be primitive. In the schurian case, this is enough to classify such schemes completely.

Theorem 4.5.1. *Any scheme of prime degree p and transitive automorphism group is isomorphic to a cyclotomic scheme over a field of order p.*

Proof. Let $\mathcal{X}$ be a scheme of prime degree p and the group $K = \mathrm{Aut}(\mathcal{X})$ is transitive. The primality of p implies that any Sylow p-subgroup G of K is a (cyclic) group of order p. It follows that this group is regular. Therefore, $\mathcal{X}$ is isomorphic to a Cayley scheme over G. Being primitive, this scheme cannot be a nontrivial tensor or generalized wreath product. Thus the required statement follows from Theorem 4.4.1. □

Corollary 4.5.2. *Any schurian scheme of prime degree is isomorphic to a cyclotomic scheme over a prime field.*

By inspection of the Hanaki–Miyamoto list of association schemes [**56**], one can find that all schemes of prime degree p are schurian if $p \leq 13$ and there are non-schurian schemes for $p = 17, 19, 23$, and 29, and each of them is of rank 3. At present, the only known general construction of a non-schurian scheme of prime degree was suggested by D. Pasechnik in [**49**, p.75], and we describe it below.

Let $\mathcal{X}$ be an antisymmetric scheme of rank 3 on a set Ω. In the notation of Exercise 2.7.57, set $A_i = A_{s_i}$, $i = 0, 1, 2$, and define two $\{0,1\}$-matrices of size $(2n+1) \times (2n+1)$:

$$(4.5.1) \qquad A'_1 = \begin{pmatrix} 0 & 1_n & 0_n \\ 0'_n & A_1 & A_2 + I_n \\ 1'_n & A_2 & A_2 \end{pmatrix} \quad \text{and} \quad A'_2 = \begin{pmatrix} 0 & 0_n & 1_n \\ 1'_n & A_2 & A_1 \\ 0'_n & A_1 + I_n & A_1 \end{pmatrix},$$

where 0_n and 1_n (respectively, $0'_n$ and $1'_n$) are $1 \times n$ (respectively, $n \times 1$) matrices consisting of zeros and ones, respectively. Set Ω' to be the disjoint union of two copies of Ω and a singleton (corresponding to the first rows of the matrices A'_1 and A'_2), and

$$S' = \{s'_0, s'_1, s'_2\},$$

where $s'_0 = 1_{\Omega'}$ and s'_i is the relation on Ω' such that $A_{s'_i} = A'_i$, $i = 1, 2$.

Theorem 4.5.3. *Given an antisymmetric scheme $\mathcal{X}$ of degree n and rank 3, the pair $\mathcal{X}' = (\Omega', S')$ is an antisymmetric scheme of degree $2n+1$ and rank 3. Moreover, $\mathcal{X}'$ is not schurian if the group $\mathrm{Aut}(\mathcal{X})$ is intransitive.*

Proof. To prove the first statement, we make use of formulas (2.7.2) to obtain

$$\begin{aligned} A'_1 A'_1 &= \begin{pmatrix} 0 & m\,1_n & m'1_n \\ m'\,1'_n & A_1^2 + A_2^2 + A_2 & A_1A_2 + A_1 + A_2^2 + A_2 \\ m\,1'_n & J_n + A_2A_1 + A_2^2 & 2A_2^2 + A_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & m\,1_n & m'1_n \\ m'\,1'_n & mA_1 + m'A_2 & mI_n + m'A_1 + mA_2 \\ m\,1'_n & m'I_n + m'A_1 + mA_2 & m'A_1 + mA_2 \end{pmatrix} \\ &= mA'_1 + m'A'_2, \end{aligned}$$

where $m = (n-1)/2$ and $m' = m+1$. It easily follows that the matrices

$$A'_0 = I_{2n+1}, \quad A'_1, \quad A'_2$$

form the standard basis of a coherent algebra on Ω'. By Theorem 2.3.7, we conclude that $\mathcal{X}'$ is a coherent configuration.

To prove statement (2), denote by α' the index of the first row of the matrix A'_1 and identify the indices of the next n rows with the set Ω. Then

$$\Omega = \alpha' s'_1. \tag{4.5.2}$$

By statements (1) and (2) of Lemma 3.3.5, this implies that Ω is a homogeneity set of $\mathcal{Y}' = \mathcal{X}'_{\alpha'}$ and $s_1 = s'_1 \cap (\Omega \times \Omega)$ is a relation of $\mathcal{Y}'$. It follows that $(\mathcal{Y}')_\Omega$ is a fission of $\mathrm{WL}(s_1) = \mathcal{X}$. Therefore,

$$\mathrm{Aut}(\mathcal{Y}')^\Omega \le \mathrm{Aut}((\mathcal{Y}')_\Omega) \le \mathrm{Aut}(\mathcal{X}).$$

Now assuming that the group $\mathrm{Aut}(\mathcal{X})$ has an orbit $\Delta \subsetneq \Omega$, we conclude that $\mathrm{Aut}(\mathcal{Y}')$ has an orbit strictly contained in Δ and hence in Ω. By (4.5.2), this implies that $\alpha' s'_1$ is not an orbit of the group

$$\mathrm{Aut}(\mathcal{Y}') = \mathrm{Aut}(\mathcal{X}'_{\alpha'}) = \mathrm{Aut}(\mathcal{X}')_{\alpha'}.$$

Thus the scheme $\mathcal{X}'$ is not schurian by statement (3) of Proposition 2.2.5. □

Remark 4.5.4. *If the group* $\mathrm{Aut}(\mathcal{X})$ *is transitive, then the scheme* $\mathcal{X}'$ *in Theorem* 4.5.3 *can be schurian (e.g., if* $\mathcal{X}$ *is a regular scheme of degree* 3*) or not (e.g., if* $\mathcal{X}$ *is the Paley scheme of degree* 7*).*

Let $\mathcal{X}$ be a non-schurian antisymmetric scheme of degree $n \le 23$ and rank 3: there is one such scheme for $n = 15$, and 18 pairwise non-isomorphic schemes for $n = 23$. Then the group $\mathrm{Aut}(\mathcal{X})$ is intransitive: in the former case one can check this on computer, whereas in the latter case this follows from Theorem 4.5.1. By Theorem 4.5.3, this implies that the scheme $\mathcal{X}'$ of degree $2n+1 \in \{31, 47\}$ is not schurian.

One more general construction of antisymmetric schemes of prime degree and rank 3 comes from *skew Hadamard* matrices; a $\{-1,+1\}$-matrix H of order n is said to be skew Hadamard if

$$HH^T = H^TH = nI_n \quad \text{and} \quad (H - I_n)^T = I_n - H.$$

One can see that in this case, $n = 1, 2$ or a multiple of 4. Two such matrices are said to be equivalent if one can be transformed into the other by a series of row or column permutations and negations. Any skew Hadamard matrix of order n is always equivalent to the matrix

$$H = \begin{pmatrix} 1 & 1_{n-1} \\ -1'_{n-1} & C + I_{n-1} \end{pmatrix}, \tag{4.5.3}$$

where C is a skew-symmetric $\{0, -1, 1\}$-matrix. It can be represented in the form

$$C = A - A^T,$$

where the matrices I_{n-1}, A, and A^T form the standard basis of a coherent algebra, which is the adjacency algebra of an antisymmetric scheme of rank 3 if $n \geq 4$ (Exercise 4.7.39).

Conversely, any such scheme defines a skew Hadamard matrix of the form (4.5.3). Moreover, two such schemes are isomorphic if and only if the corresponding matrices are equivalent. By Corollary 4.5.2, this proves the following statement.

Theorem 4.5.5. *Given a prime p, a non-schurian (respectively, non-separable) antisymmetric scheme of degree p and rank 3 exists if and only if there are at least two nonequivalent skew Hadamard matrices of order $p+1$.*

In accordance with the list of known orders of skew Hadamard matrices [**87**], among the suitable primes p between 23 and 100, the only case where the existence of non-schurian antisymmetric scheme of degree p and rank 3 is not known, is $p = 79$. A long-standing conjecture [**121**] states that skew-Hadamard matrices exist for all dimensions divisible by 4.

It is interesting to find an infinite family of non-schurian schemes of prime degree: although there are many infinite sequences of naturals n for which there exist skew Hadamard matrices of order n, for none of them is known whether it contains an infinite subsequence for which $n-1$ is prime.

One more question here is to find a construction of non-schurian symmetric schemes of prime degree and rank 3 (such schemes are known for $p = 29$). Finally, we do not know any non-schurian scheme of prime degree and rank more than 3.

From Theorem 4.5.1, it follows that any schurian scheme of prime degree is Frobenius and hence pseudocyclic (Theorem 4.3.37). The following remarkable theorem proved by A. Hanaki and K. Uno in [**57**] generalizes this fact to all schemes of prime degree. To make the proof self-contained, we make use of Lemma 4.5.7 suggested and proved by A. Hanaki.

Theorem 4.5.6. *Every scheme of prime degree is commutative and pseudocyclic.*

Proof. Let $\mathcal{X}$ be a scheme of prime degree p. By Exercise 4.7.31, it suffices to verify that all non-principal irreducible characters of $\mathcal{X}$ are algebraically conjugate (then their multiplicities are constant).

Let $\xi \in \mathrm{Irr}(\mathcal{X})^{\#}$. Note that the algebraic conjugate of an irreducible character is again an irreducible character. Denote by Θ the sum of all algebraic conjugates of ξ, and by Θ' the sum of all nonprincipal irreducible characters which are not algebraically conjugate to ξ. Then the values of Θ and Θ' are algebraic integers and hence rational integers. If Θ' equals zero, then the assertion holds, so we assume that $\Theta' \neq 0$.

Lemma 4.5.7. *For any $s \in S$, the characteristic polynomial of the matrix A_s treated as a matrix with entries from $\mathbb{F} = \mathbb{F}_p$, is equal to $(x - n_s)^p$.*

Proof. Let $f(x)$ be the characteristic polynomial of $A_s \in \mathrm{Mat}(\mathbb{F})$. Then there exist an integer $m \geq 1$ and distinct elements $a_1, \ldots, a_k \in \mathbb{F}$ such that

$$f(x) = \prod_{i=1}^{k} (x - a_i)^{m_i}.$$

The great common divisor of the polynomials

$$g_i(x) = \frac{f(x)}{(x - a_i)^{m_i}}, \quad i = 1, \ldots, k,$$

is equal to 1. Therefore, there are polynomials $h_i(x) \in \mathbb{F}[x]$ such that

$$\sum_{i=1}^{k} g_i(x) h_i(x) = 1.$$

This implies that

$$\sum_{i=1}^{k} g_i(A_s) h_i(A_s) = I.$$

Since $f(A_s) = 0$, the matrices $E_i = g_i(A_s) h_i(A_s)$ are pairwise orthogonal. Thus the above decomposition implies that $E_i^2 = E_i$ for all i. Note also that $E_i \neq 0$.

Assume on the contrary that $k > 1$. Then the trace of E_1 is equal to the rank of E_1 and hence is nonzero in $\mathbb{F}$. Choose an algebraic number field $\mathbb{F}'$ and a prime ideal P of the ring of integers $O_{\mathbb{F}'}$ such that $\mathbb{F} \subseteq O_{\mathbb{F}'}/P$. Let

$$g_1'(x), h_1'(x) \in O_{\mathbb{F}'}[x]$$

be the lifts of $g_1(x), h_1(x)$, respectively.

The traces of the matrices $g_1'(A_s)$ and $h_1'(A_s)$ lie in $pO_{\mathbb{F}'}$, because the diagonal entries of them are constant. It follows that the trace of $E_1 = g_1(A_s) h_1(A_s)$ is equal to zero in $\mathbb{F}$, a contradiction. Thus, $k = 1$. Since n_s is an eigenvalue of A_s, we have $f(x) = (x - n_s)^p$, as required. □

Let $s \in S$. By Lemma 4.5.7, all eigenvalues of the matrix A_s are congruent to n_s modulo p. So there exist rational integers u_s and u_s' such that

$$\Theta(A_s) = \Theta(I) n_s - u_s p \quad \text{and} \quad \Theta'(A_s) = \Theta'(I) n_s - u_s' p,$$

where I is the identity matrix. Now applying the orthogonality relation (3.6.26) for $\eta = \xi_0$, we get

$$\begin{aligned} 0 &= \sum_{s\in S} \frac{1}{n_s} \xi_0(A_{s^*})\Theta(A_s) \\ &= \sum_{s\in S} \Theta(A_s) \\ &= \sum_{s\in S} (\Theta(I)n_s - u_s p) \\ &= p\big(\Theta(I) - \sum_{s\in S} u_s\big). \end{aligned}$$

It follows that

$$\sum_{s\in S} u_s = \Theta(I).$$

Similarly, one can check that

$$\sum_{s\in S} u'_s = \Theta'(I).$$

Using the orthogonality relation again, we obtain

$$\begin{aligned} 0 = \sum_{s\in S} \frac{1}{n_s}\Theta(A_{s^*})\Theta'(A_s) &= \sum_{s\in S} \frac{1}{n_s}(\Theta(I)n_{s^*} - u_{s^*}p)(\Theta'(I)n_s - u'_s p) \\ &= \sum_{s\in S}\Theta(I)\Theta'(I)n_s - \sum_{s\in S}\Theta(I)u'_s p - \sum_{s\in S}\Theta'(I)u_{s^*}p + \sum_{s\in S}\frac{1}{n_s}u_{s^*}u'_s p^2 \\ &= p\Theta(I)\Theta'(I) - p\Theta(I)\Theta'(I) - p\Theta(I)\Theta'(I) + \sum_{s\in S}\frac{1}{n_s}u_{s^*}u'_s p^2 \\ &= -p\Theta(I)\Theta'(I) + \sum_{s\in S}\frac{1}{n_s}u_{s^*}u'_s p^2. \end{aligned}$$

This implies that

$$\Theta(1)\Theta'(1) = \sum_{s\in S}\frac{1}{n_s}u_{s^*}u'_s p.$$

But $\Theta(1)\Theta'(1)$ is relatively prime to p, contrary to the fact that each summand in the above sum and hence the entire sum is a p-integer, because p and n_s are relatively prime for all s. □

Remark 4.5.8. *As shown in* [**57**, Theorem 5.3]*, every scheme* $\mathcal{X}$ *of prime degree* p *is algebraically isomorphic to a cyclotomic scheme over the field* $\mathbb{F}_p$ *whenever the minimal splitting field of* $\mathrm{Adj}(\mathcal{X})$ *is an abelian extension of* $\mathbb{Q}$.

By Theorem 4.5.6, any scheme of prime degree is pseudocyclic. By Theorem 4.3.41, this implies that at least asymptotically such a scheme is schurian and separable. More precisely, the following statement holds.

Theorem 4.5.9. *There exists a nonconstant function f such that for any prime p each scheme of degree p and valency less that $f(p)$ is schurian and separable.*

In accordance with [**27**], the function f in Theorem 4.5.9 can be taken to be of order $O(p^{1/3})$. It seems that this estimate is far from being tight.

4.6 The Weisfeiler–Leman method

In this section, we present the Weisfeiler–Leman method for testing isomorphism of graphs. The classical version of this method [**124**] (see Subsection 2.6.1) deals with binary relations.

A generalization to the m-ary relations with $m \geq 3$ was implicitly described in [**123**]. The modern version of the underlying algorithm known under the name the m-dimensional WL refinement (or the m-dim WL) was introduced by L. Babai in the end of 1970s; for detailed history we refer to [**21**].

In the first two subsections, we discuss connections between the Graph Isomorphism Problem and the theory of multidimensional extensions of coherent configurations, and present the algorithm m-dim WL.

The results of the last three subsections show that the output of the algorithm m-dim WL is very close to the multidimensional extension of a coherent configuration. More precisely, the information on symmetries of a coherent configuration $\mathcal{X}$ that is obtained by using the m-dim WL can also be taken from the m-dimensional extension of $\mathcal{X}$, and, conversely, the structure of the latter is completely determined from the output of the $3m$-dim WL applied to the colored graph of $\mathcal{X}$.

4.6.1 Graph isomorphism problem

The Graph Isomorphism Problem consists in finding a most efficient algorithm which given two graphs determines whether they are isomorphic or not. Informally, the term "most efficient" means that the maximal number $f(n)$ of elementary steps performed by the algorithm in processing of any two graphs with n vertices is as small as possible. The main question is whether there exists a polynomial-time algorithm for testing graph isomorphism, i.e., the algorithm for which the function $f(n)$ is a polynomial in n. At present, the best upper bound for the function $f(n)$ is quasipolynomial [**6**], i.e., of the form $n^{(\log n)^c}$ for a constant $c > 0$.

In 1968, B. Weisfeiler and A. Leman suggested a new approach to the Graph Isomorphism Problem that was based on the concept of coherent configuration [**124**]. The key point of this approach was a polynomial-time algorithm, the Weisfeiler–Leman algorithm, for constructing the coherent closure of the arc set of a graph, see Subsection 2.6.1. In view of the following statement (Exercise 2.7.54), this algorithm enables us to test isomorphism efficiently whenever the coherent configurations of the input graphs are schurian.

Proposition 4.6.1. *Let* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *be connected vertex-disjoint graphs. Assume that the coherent configuration* $\mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}')$ *is schurian. Then* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *are isomorphic if and only if*

$$\Delta \cap \Omega(\mathfrak{X}) \neq \varnothing \quad \text{and} \quad \Delta \cap \Omega(\mathfrak{X}') \neq \varnothing \tag{4.6.1}$$

for some (and hence for all) fibers Δ *of* $\mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}')$.

Proof. The graphs $\mathfrak{X}$ and $\mathfrak{X}'$ being connected, are isomorphic if and only if there exists $f \in \mathrm{Aut}(\mathfrak{X} \cup \mathfrak{X}')$ such that

$$\Omega(\mathfrak{X})^f = \Omega(\mathfrak{X}'). \tag{4.6.2}$$

On the other hand, by Corollary 2.6.6 we have

$$\mathrm{Aut}(\mathfrak{X} \cup \mathfrak{X}') = \mathrm{Aut}(\mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}')).$$

If the coherent configuration $\mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}')$ is schurian, then its fibers are the orbits of the group $\mathrm{Aut}(\mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}'))$ (statement (1) of Proposition 2.2.5). Therefore conditions (4.6.2) and (4.6.1) are equivalent, and we are done. □

The Weisfeiler–Leman algorithm can also be used to test graph isomorphism when the coherent configuration of one of the input graphs is separable. Indeed, let

$$\mathcal{Y} = \mathrm{WL}(\mathcal{X}) \quad \text{and} \quad \mathcal{Y}' = \mathrm{WL}(\mathcal{X}'),$$

where $\mathcal{X}$ and $\mathcal{X}'$ be the colored rainbows of the graphs $\mathfrak{X}$ and $\mathfrak{X}'$. Then in accordance with Exercise 2.7.52, these graphs are isomorphic if and only if the following conditions are satisfied:

(SC1) $|c_{\mathcal{Y}}| = |c_{\mathcal{Y}'}|$;
(SC2) $\psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}')$;
(SC3) $\mathrm{Iso}(\mathcal{Y}, \mathcal{Y}', \psi) \neq \varnothing$,

where $\psi : S(\mathcal{Y}) \to S(\mathcal{Y}')$ is the color preserving bijection. (Of course, if $\mathcal{Y}$ is separable, then (SC3) follows from (SC2).) This immediately implies the following analog of Proposition 4.6.1.

Proposition 4.6.2. *Let $\mathfrak{X}$ and $\mathfrak{X}'$ be graphs. Assume that the coherent configuration* $\mathrm{WL}(\mathfrak{X})$ *is separable. Then $\mathfrak{X}$ and $\mathfrak{X}'$ are isomorphic if and only if the conditions* (SC1) *and* (SC2) *are satisfied.*

In general, Propositions 4.6.1 and 4.6.2 are difficult to use in testing isomorphism of arbitrary graphs, because it is not clear how to verify the schurity or separability condition efficiently. However, these propositions are useful to solve the Graph Isomorphism Problem for every class of graphs the coherent configurations of which are known to be schurian or separable.

This idea was used in [**48**] to prove that the Weisfeiler–Leman algorithm solves the Graph Isomorphism Problem for several classes of graphs including, e.g., the class of forests, i.e., undirected graphs without cycles.

It may happen that, although the input graphs $\mathfrak{X}$ and $\mathfrak{X}'$ do not satisfy the conditions of Propositions 4.6.1 and 4.6.2, the schurity or separability numbers of their coherent configurations are enough small, say less than a certain integer $m \geq 1$. In this case, one can still use the Weisfeiler–Leman algorithm for testing graph isomorphism.

Proposition 4.6.3. *The isomorphism of n-vertex graphs $\mathfrak{X}$ and $\mathfrak{X}'$ can be tested in time $n^{O(m)}$, where*

$$m = t(\mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}')) \quad \textit{or} \quad m = \min\{s(\mathrm{WL}(\mathfrak{X})),\, s(\mathrm{WL}(\mathfrak{X}'))\}.$$

Proof. The algorithm in question starts with finding the m-closure of the coherent configuration $\mathcal{Y} = \mathrm{WL}(\mathfrak{X} \cup \mathfrak{X}')$ in the former case, and the m-closures of the coherent configurations $\mathcal{Y} = \mathrm{WL}(\mathfrak{X})$ and $\mathcal{Y}' = \mathrm{WL}(\mathfrak{X}')$ in the latter one. In each case, this can be done in time $n^{O(m)}$ with the help of the Weisfeiler–Leman algorithm. At the second step, one should verify condition (4.6.1) for $\Delta \in F(\mathcal{Y})$, or the conditions (SC1) and (SC2). The correctness of the algorithm follows from Exercise 4.7.40. □

We complete the subsection by remarking that in accordance with Theorem 4.2.4, the algorithm given in Proposition 4.6.3 is polynomial-time only if m is fixed.

4.6.2 Color refinement procedure

Let Ω be a finite set and $m \geq 1$ an integer. The key point of the m-dim Weisfeiler–Leman method is a procedure refining a coloring of the set Ω^m. To describe the refinement procedure, some technical notation are needed. For any point $\alpha \in \Omega$, any tuple $\tau \in \Omega^m$, and an index $i \in \{1, \ldots, m\}$, we define an m-tuple

$$\tau_{i,\alpha} = (\tau_1, \ldots, \tau_{i-1}, \alpha, \tau_{i+1}, \ldots, \tau_m), \tag{4.6.3}$$

the m-tuple of m-tuples

$$\tau/\alpha = (\tau_{1,\alpha}, \ldots, \tau_{m,\alpha}),$$

and given a coloring c of Ω^m, the m-tuple of colors

$$c(\tau/\alpha) = (c(\tau_{1,\alpha}), \ldots, c(\tau_{m,\alpha})).$$

In this notation, the following algorithm consistently refines a coloring c_0 of Ω^m to obtain a new coloring, which is no longer refined.

The m-dim WL color refinement

Step 1 Set $k = 0$.

Step 2 For each $\tau \in \Omega^m$, find a formal sum $S_k(\tau) = \sum_{\alpha \in \Omega} c_k(\tau/\alpha)$.

Step 3 Find a coloring c_{k+1} of Ω^m such that

$$c_{k+1}(\tau) = c_{k+1}(\tau') \quad \Leftrightarrow \quad c_k(\tau) = c_k(\tau') \text{ and } S_k(\tau) = S_k(\tau').$$

Step 4 If $|c_k| \neq |c_{k+1}|$, then $k := k+1$ and go to Step 2, else output $c = c_k$.

For brevity, we refer to the above procedure as the m-dim WL. It is easily seen that it does nothing, i.e., $c_0 = c$ whenever $m = 1$, or the initial coloring is trivial ($|c_0| = 1$) or discrete ($|c_0| = n^m$).

In general, the number of iterations of the m-dim WL is less than or equal to n^m, just because at each step the number $|c_k|$ strictly increases. At each iteration, the formal sum $S_k(\tau)$ at Step 2 can be constructed in time $O(nm)$ for each τ, and at Step 3, we have to sort n^m such sums. Thus Steps 2 and 3 can be implemented in time $O(m^2 n^{m+1} \log n)$ and the total complexity (after at most n^m iterations) is $O(m^2 n^{2m+1} \log n)$.

A more careful implementation of the m-dim WL described in [**78**] runs in time $O(m^2 n^{m+1} \log n)$. Concerning practical implementation of this algorithm, we refer to [**8**].

It should be noted that Step 3 can be implemented in different ways. For example, one can define a linear ordering on the set

$$\{S_k(\tau) : \ \tau \in \Omega^m\},$$

say, lexicographically, and then take $c_k(\tau)$ to be the position of $S_k(\tau)$ in this ordering. In what follows, we fix exactly this way. Then the m-dim WL respects color preserving bijections from Ω^m to another Cartesian m-power (Exercise 4.7.41). Let us show how this property is used for testing graph isomorphism.

Let $\mathcal{X}$ be a colored rainbow. For an integer $m \geq 2$, we define a coloring $c_0 = c_0(\mathcal{X})$ of the set Ω^m as follows.[4] Given a tuple $\tau \in \Omega^m$, put

$$A_m(\tau) = \begin{pmatrix} c_{\mathcal{X}}(\tau_1, \tau_1) & \cdots & c_{\mathcal{X}}(\tau_1, \tau_m) \\ \cdots & \cdots & \cdots \\ c_{\mathcal{X}}(\tau_m, \tau_1) & \cdots & c_{\mathcal{X}}(\tau_m, \tau_m) \end{pmatrix}, \tag{4.6.4}$$

where $c_{\mathcal{X}}$ is the standard coloring of $\mathcal{X}$. Then the color $c_0(\tau)$ is set to be the position of the matrix $A_m(\tau)$ in the lexicographical ordering of all $m \times m$ matrices with nonnegative entries less than or equal to $n^m = |\Omega^m|$.

Example 4.6.4. *Let $\mathfrak{X}$ be a loopless graph and $\mathcal{X}$ the colored rainbow of $\mathfrak{X}$ (with standard coloring). Then the coloring $c_0(\mathfrak{X}) = c_0(\mathcal{X})$ defined by condition (4.6.4) is such that two m-tuples τ and μ have the same color if and only if*

$$(\tau_i, \tau_j) \in D_k \quad \Leftrightarrow \quad (\mu_i, \mu_j) \in D_k \tag{4.6.5}$$

for all $i, j \in \{1, \ldots, m\}$ and $k \in \{0, 1, 2\}$, where $D_0 = 1_\Omega$, $D_1 = D$, and D_2 is the complement of $D_0 \cup D_1$ to Ω^2.

Let $c = c_m(\mathcal{X})$ be the output coloring obtained from c_0 by the m-dim WL. The color preserving property of isomorphisms implies that two rainbows $\mathcal{X}$ and $\mathcal{X}'$ on Ω are isomorphic only if for each color i,

$$|c^{-1}(i)| = |c'^{-1}(i)|, \tag{4.6.6}$$

where $c' = c_m(\mathcal{X}')$. This condition enables us to find an explicit isomorphism between $\mathcal{X}$ and $\mathcal{X}'$ if they are isomorphic and all (nonempty) color classes of c are singletons (Exercise 4.7.42).

Definition 4.6.5. *We say that the m-dim WL does not distinguish the rainbows $\mathcal{X}$ and $\mathcal{X}'$ if condition (4.6.6) is satisfied for all the colors i.*

Certainly, the m-dim WL does not distinguish isomorphic colored rainbows, but this may happen also for some non-isomorphic rainbows. Using the same terminology for graphs (considered as a special case of rainbows), we have the statement below, which immediately follows from the definition and the fact that m-dim WL runs in polynomial for a fixed m.

Proposition 4.6.6. *Let $\mathfrak{K}$ be a class of graphs. Assume that there exists $m \geq 1$ such that for any $\mathfrak{X}, \mathfrak{X}' \in \mathfrak{K}$,*

$$\text{the } m\text{-dim WL does not distinguish } \mathfrak{X} \text{ and } \mathfrak{X}' \quad \Rightarrow \quad \mathfrak{X} \cong \mathfrak{X}'.$$

Then the m-dim WL tests isomorphism of graphs belonging to $\mathfrak{K}$ in polynomial time.

[4] We do not consider here the case $m = 1$, in which the m-dim WL is defined to be the naive vertex classification (the color refinement algorithm in the sense of [**108**]).

The hypothesis of Proposition 4.6.6 is satisfied for many known classes of graphs, like trees ($m = 1$) or vertex-colored graphs with color class size at most 3 ($m = 3$, see [**78**]). Moreover, as $\mathfrak{K}$ one can take a class defined in [**7**] and consisting of almost all graphs.[5]

For each m, there are non-isomorphic graphs $\mathfrak{X}$ and $\mathfrak{X}'$ which are not distinguished by the m-dim WL: the direct construction was found in [**21**], but this result can also be obtained by combining Theorem 4.2.4 and Theorem 4.6.23 to be proved later. The fact that the m-dim WL does not distinguish some graphs has certain logical reasons. To see this, we recall some relevant concepts from mathematical logic.

In accordance with [**21**], the first-order language of graph theory is built up in the usual way from the variables $x, y, \ldots$, the relation symbols D and $=$, the logical connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, and the quantifiers $\forall$ and $\exists$. The quantifiers range over the vertices of the graph in question. When a formula φ of this language is true for a graph $\mathfrak{X}$, we write $\mathfrak{X} \models \varphi$.

Example 4.6.7. *The following formula establishes the property of a graph to be of diameter* 2*:* $\forall x\, \forall y\, \exists z\, [D(x, z) \wedge D(z, y)]$.

The first-order language of vertex-colored graphs is obtained by adding a countable set of unary relations $\{C_1, C_2, \ldots\}$ to the first-order language of graphs. It is assumed that finitely many of these relations are true at each vertex of a graph in question; they can be thought of as colorings of the vertices.

The first-order language of graph theory is too week to count how many vertices have the property expressed by a formula φ. In this sense, the counting quantifiers $\exists^k$ with positive integer k, are useful: the term $\exists^k x\, \varphi(x)$ is used in order to say that there are at least k vertices with property φ. The first order logic of vertex-colored graphs extended by counting quantifiers is called *counting logic.*

Example 4.6.8. *In counting logic, one can define the property of a graph to be* 3*-regular as follows:* $\forall x[\exists^3 y\, D(x, y) \;\wedge\; \neg\, \exists^4 y\, D(x, y)]$.

Denote by $\mathfrak{C}_m$ the set of all formulas in counting logic that contain at most m variables; although a variable can be used in a formula several times, it is counted only once (thus, the formula in Example 4.6.8 contains two variables). The vertex-colored graphs $\mathfrak{X}$ and $\mathfrak{X}'$ are said to be $\mathfrak{C}_m$-equivalent if for all formulas $\varphi \in \mathfrak{C}_m$,

$$\mathfrak{X} \models \varphi \quad \Leftrightarrow \quad \mathfrak{X}' \models \varphi.$$

The following theorem proved in [**78**] shows the relationship between the m-dim WL and the notion of $\mathfrak{C}_m$-equivalence; the proof of this theorem is out of scope of this text and we refer the reader to survey [**108**].

Theorem 4.6.9. *Let* $m \geq 2$ *be an integer. Then the* m*-dim WL does not distinguish graphs* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *if and only if* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *are* $\mathfrak{C}_{m+1}$*-equivalent.*

[5] For the classes $\mathfrak{K}$ mentioned in the paragraph, Proposition 4.6.6 holds even if $\mathfrak{X}' \notin \mathfrak{K}$.

We conclude the subsection by remarking that there are algorithms which are equivalent to the m-dim WL for each m in the sense that they distinguish the same graphs, e.g., the algorithm in [**3**] based on combinatorial optimization technique. More stronger algorithms distinguishing graphs in the spirit of the m-dim WL can be found in [**30, 32**].

4.6.3 WL-stable partitions

Throughout the subsection, we fix an integer $m \geq 1$ and a colored rainbow $\mathcal{X}$ on Ω. In what follows, $c_0 = c_0(\mathcal{X})$ and $c = c_m(\mathcal{X})$. Our goal is to analyze the partition $\mathcal{P}_c$ of the Cartesian power Ω^m that is constructed by the m-dim WL color refinement procedure. In view of Exercise 4.7.41, the partition $\mathcal{P}_c$ does not depend on the choice of colors of $c_{\mathcal{X}}$. Therefore it seems more natural to deal with $\mathcal{P}_c$ rather than c itself.

We start with the study of certain rather general properties of partitions $\mathcal{P}$ of the Cartesian product $\Omega^m = \Omega^M$, where $M = \{1, \ldots, m\}$. For any $L \subseteq M$, the natural projection operator from Ω^M to Ω^L is denoted by $\pi_L = \pi_L^M$, and we set

$$\pi_L(\mathcal{P}) = \{\pi_L(\Delta) : \ \Delta \in \mathcal{P}\}$$

and put $\pi_i = \pi_{\{1,\ldots,i\}}$ for $i \in M$.

Definition 4.6.10. *The partition $\mathcal{P}$ is said to be normal if*

$$\pi_L^{-1}(\mathrm{Diag}(\Omega^L)) \in \mathcal{P}^{\cup} \quad \text{for all } \ L \subseteq M. \tag{4.6.7}$$

Exercise 4.7.45 gives an equivalent definition of the normality. Using this definition, one can easily check that the partition $\mathcal{P}_{c_0}$ is normal. For $m = 2$, the normality implies the condition (CC1).

Given $g \in \mathrm{Sym}(M)$ and $\tau \in \Omega^M$, we set $\tau^g = (\tau_{1^{g^{-1}}}, \ldots, \tau_{m^{g^{-1}}})$. The induced action of the symmetric group $\mathrm{Sym}(M)$ permutes the partitions of Ω^M, namely, $\mathcal{P}^g = \{\Delta^g : \ \Delta \in \mathcal{P}\}$.

Definition 4.6.11. *The partition $\mathcal{P}$ is said to be invariant if*

$$\mathcal{P}^g = \mathcal{P} \quad \text{for all } \ g \in \mathrm{Sym}(M). \tag{4.6.8}$$

It is easily seen that the partition $\mathcal{P}_{c_0}$ is invariant. For $m = 2$, the invariance condition coincides with the condition (CC2).

In general, given $L \subseteq M$, the projection $\pi_L(\mathcal{P})$ of the partition $\mathcal{P}$ is not necessarily a partition. To control the projections, we introduce the regularity condition.

Definition 4.6.12. *The partition $\mathcal{P}$ is said to be regular if given $\Delta \in \mathcal{P}$, $L \subseteq M$, and $\Gamma \in \pi_L(\mathcal{P})$, the number*

$$c_{L,\Gamma}^{\Delta} = |\pi_L^{-1}(\gamma) \cap \Delta| \tag{4.6.9}$$

does not depend on $\gamma \in \Gamma$.

Note that the partition $\mathcal{P}_{c_0}$ is not necessarily regular. On the other hand, any projection of a regular partition is a partition. As the following example shows, the regularity condition is closely related with the condition (CC3).

Example 4.6.13. *Let $\mathcal{X}$ be a coherent configuration on Ω. Denote by $\mathcal{P}$ the partition of Ω^3 into the classes*

$$\Delta = \{\alpha \in \Omega^3 : \ r(\alpha_1, \alpha_3) = r, \ r(\alpha_3, \alpha_2) = s, \ r(\alpha_1, \alpha_2) = t\},$$

where r, s, t run over basis relations of $\mathcal{X}$. Then the condition (CC3) *implies that the partition $\mathcal{P}$ is regular and*

$$c_{rs}^{t} = c_{L,\Gamma}^{\Delta}$$

with $L = \{1, 2\}$ and $\Gamma = t$.

It seems that the most interesting partitions of Ω^M should be normal, invariant, and regular, see Example 4.7.46. This thesis is also justified by the theorem below showing that any WL-stable partition, i.e., the partition $\mathcal{P}_c$ with c as above, satisfies all these properties.

Theorem 4.6.14. *In the above notation, the partition $\mathcal{P}_m(\mathcal{X}) := \mathcal{P}_c$ is normal, invariant, and regular.*

Proof. In the process of refining the coloring c_0, the m-dim WL subsequently produces the colorings $c_1, \ldots, c_t = c$, where t is the number of iterations. It is easily seen that

$$\mathcal{P}_{c_0} \leq \mathcal{P}_{c_1} \leq \cdots \leq \mathcal{P}_{c_t} = \mathcal{P}_c, \tag{4.6.10}$$

i.e., for each $i = 0, 1, \ldots, t-1$ the partition $\mathcal{P}_{c_{i+1}}$ is a refinement of the partition of $\mathcal{P}_{c_i}$; in other words for each i,

$$\mathcal{P}_{c_i} \subseteq (\mathcal{P}_{c_{i+1}})^{\cup}.$$

As was mentioned, the partition $\mathcal{P}_{c_0}$ is normal and invariant. Using the induction on $i = 0, 1 \ldots$, one can easily check that the partition $\mathcal{P}_{c_{i+1}}$ is normal and invariant whenever so is $\mathcal{P}_{c_i}$. This proves the normality and invariance of $\mathcal{P}_c$.

Let us prove that the partition $\mathcal{P} = \mathcal{P}_c$ is regular. It suffices to verify that for $i = 1, \ldots, m-1$, $\Gamma \in \pi_{i+1}(\mathcal{P})$, and $\Lambda \in \pi_i(\mathcal{P})$, the number

$$|(\pi_i^{i+1})^{-1}(\lambda) \cap \Gamma| \tag{4.6.11}$$

does not depend on $\lambda \in \Lambda$, where π_i^{i+1} is the projection of Ω^{i+1} to the first i coordinates. In checking this statement, we will use the mapping

$$f_i : \Omega^m \to \Omega^m, \quad (\tau_1, \tau_2, \ldots, \tau_m) \mapsto (\tau_1, \tau_2, \ldots, \tau_i, \tau_1, \ldots, \tau_1)$$

defined for $i \in M$.

Lemma 4.6.15. *$\Delta \in \mathcal{P}$ implies $f_i(\Delta) \in \mathcal{P}$ for each i.*

Proof. Using induction on $i = m, m-1, \ldots$, we may assume without loss of generality that $\Delta = f_{i+1}(\Delta)$. Let $\tau \in \Delta$. Then by the normality of $\mathcal{P}$,

$$c(\tau_{i+1,\tau_1}) \neq c(\tau_{i+1,\alpha}) \quad \text{for all } \alpha \neq \tau_1,$$

see (4.6.3). By the termination condition at Step 3 of the m-dim WL color refinement algorithm, this implies that there exists a class $\Delta' \in \mathcal{P}$ such that

$$f_i(\Delta) \subseteq \Delta' \subseteq f_i(\Omega^m). \tag{4.6.12}$$

On the other hand, given $\tau' \in \Delta'$ the number of all $\alpha \in \Omega$ such that $\tau'_{i+1,\alpha} \in \Delta$ does not depend on τ', is positive since $f_i(\Delta) \neq \varnothing$. Thus,

$$\Delta' \subseteq f_i(\Delta).$$

In view of (4.6.12), this means that $f_i(\Delta) = \Delta' \in \mathcal{P}$. □

To verify that the number (4.6.11) does not depend on λ, let Γ and Λ be the projections of the classes $\Gamma' \in \mathcal{P}$ and $\Lambda' \in \mathcal{P}$ to the first $i+1$ and i coordinates, respectively. By Lemma 4.6.15, we may assume that

$$\Gamma' = f_{i+1}(\Gamma') \quad \text{and} \quad \Lambda' = f_i(\Lambda').$$

Then given $\lambda \in \Omega^i$ and $\tau \in f_i(\pi_i^{-1}(\lambda))$, we have

$$|(\pi_i^{i+1})^{-1}(\lambda) \cap \Gamma| = |\{\alpha \in \Omega : \ \tau_{i+1,\alpha} \in \Gamma'\}|.$$

Now if λ runs over Λ, then τ runs over Λ'. Since the right-hand side of the last equality does not depend on $\tau \in \Lambda'$, the left-hand side of it does not depend on $\lambda \in \Lambda$. □

Not every normal, invariant, and regular partition of Ω^M is WL-stable, see Exercise 4.7.47.[6] However, in fact, the only we need are the two following statements, which are deduced from Theorem 4.6.14.

Lemma 4.6.16. *Let $\mathcal{X}$ be a coherent configuration, $m \geq 1$, and $k \leq m$. Then the set $\pi_k(\mathcal{P}_m(\mathcal{X}))$ forms a normal and invariant partition of Ω^k.*

Proof. By Theorem 4.6.14, the partition $\mathcal{P} = \mathcal{P}_m(\mathcal{X})$ is normal, invariant, and regular. Let $k \leq m$. The regularity implies that $\pi_k(\mathcal{P})$ is a partition of Ω^k.

Furthermore, if $L \subseteq \{1, \ldots, k\} =: K$, then obviously,

$$(\pi_L^K)^{-1}(\mathrm{Diag}(\Omega^L)) = \pi_K(\pi_L^{-1}(\mathrm{Diag}(\Omega^L)).$$

Consequently, the partition $\pi_k(\mathcal{P})$ is normal. It is also invariant, because

$$\pi_k(\Delta)^g = \pi_k(\Delta^{g'}) \quad \text{for all } \ g \in \mathrm{Sym}(K),$$

where g' is the image of g with respect to the natural injection of $\mathrm{Sym}(K)$ into $\mathrm{Sym}(M)$. □

Under the conditions of Lemma 4.6.16, one can also prove that the partition $\pi_k(\mathcal{P}_m(\mathcal{X}))$ is regular, see Exercise 4.7.50. In what follows, the binary relations on Ω^m are identified with subsets of Ω^{2m} via the bijection

$$\Omega^m \times \Omega^m \to \Omega^{2m}, \ (\tau, \tau') \mapsto (\tau_1, \ldots, \tau_m, \tau'_1, \ldots, \tau'_m). \tag{4.6.13}$$

Lemma 4.6.17. *Let $\mathcal{X}$ be a coherent configuration, $m \geq 1$. Then the partition of $\Omega^m \times \Omega^m = \Omega^{2m}$ induced by $\pi_{2m}(\mathcal{P}_{3m}(\mathcal{X}))$ forms a coherent configuration on Ω^m.*

[6] A reason is that the regularity condition is not enough strong. A stronger condition is used in the definition of k-ary coherent configuration, see [**6**, Definition 2.3.5].

Proof. By Lemma 4.6.16, the partition $S = \pi_{2m}(\mathcal{P}_{3m}(\mathcal{X}))$ of Ω^{2m} is normal and invariant. The normality implies that the relation

$$1_{\Omega^m} = \bigcap_{i=1}^{m} \Big(\pi^{\{1,\ldots,2m\}}_{\{i,i+m\}}\Big)^{-1}(1_\Omega)$$

belongs to $S^\cup$. This proves that the condition (CC1) is satisfied. The invariance implies that given $s \in S$, the relation

$$s^* = s^g$$

belongs to S, where g is the permutation $(1, m+1)\cdots(m, 2m) \in \mathrm{Sym}(2m)$. This proves that the condition (CC2) is also satisfied.

To verify the condition (CC3), let $r, s, t \in S$, and $\alpha, \beta \in \Omega^m$ be such that $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$ belongs to t. Set

$$L = \{1, \ldots, 2m\} \quad \text{and} \quad \Gamma = t.$$

Denote by $\Delta_1, \ldots, \Delta_a$ the classes of the partition $\mathcal{P}_{3m}(\mathcal{X})$ such that the number

$$c^{\Delta_i}_{L,\Gamma} = |\pi_L^{-1}(\alpha \cdot \beta) \cap \Delta_i|$$

is positive, $1 \le i \le a$. By the regularity of $\mathcal{P}_{3m}(\mathcal{X})$, the set of these classes does not depend on $\alpha \cdot \beta \in t$ (Theorem 4.6.14). It follows that the set I of all i such that

$$\pi_{\{1,\ldots,m,2m+1,\ldots,3m\}}(\Delta_i) = r \quad \text{and} \quad \pi_{\{m+1,\ldots,2m,2m+1,\ldots,3m\}}(\Delta_i) = s$$

also does not depend on $\alpha \cdot \beta \in t$, and so is the number

$$|\alpha r \cap \beta s^*| = |\{\gamma \in \Omega^m :\ \alpha \cdot \gamma \in r,\ \gamma \cdot \beta \in s\}| = \sum_{i \in I} c^{\Delta_i}_{L,\Gamma},$$

as required. □

4.6.4 The WL-refinement of a coherent configuration

In this subsection, we establish a connection between the m-dim WL and the m-dimensional extensions and closures of coherent configurations. At this point it should be mentioned that in view of identification (4.6.13), the basis relations of the m-dimensional extension of a coherent configuration on Ω form a partition of Ω^{2m}.

Theorem 4.6.18. *Let $\mathcal{X}$ be a coherent configuration. Then for every $m \geq 1$,*

$$\mathcal{P}_m(\mathcal{X}) \leq F(\widehat{\mathcal{X}}^{(m)}) \quad \text{and} \quad S(\widehat{\mathcal{X}}^{(m)}) \leq \pi_{2m}(\mathcal{P}_{3m}(\mathcal{X})). \tag{4.6.14}$$

Proof. To prove the left-hand side inclusion, denote by c_k the coloring of Ω^m obtained on the kth iteration of m-dim WL applied to the initial coloring $c_0 = c_0(\mathcal{X})$. We make use of the induction on $k = 0, 1, \ldots$ to show that

$$\Gamma \in \mathcal{P}_{c_k} \quad \Rightarrow \quad \Gamma \in F(\widehat{\mathcal{X}})^{\cup}, \tag{4.6.15}$$

where $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$.

Let $k = 0$. By the definition of the coloring c_0, given $i, j \in \{1, \ldots, m\}$ the relation $r_{ij} = r(\tau_i, \tau_j)$ does not depend on the choice of $\tau \in \Gamma$. Therefore, $1_\Gamma = 1_{\Omega^m} \cap s$, where

$$s = \bigcap_{i,j=1}^{m} \mathrm{Cyl}_{r_{ij}}(i, j)$$

with $\mathrm{Cyl}_{r_{ij}}(i, j)$ defined by formula (3.5.5). By statement (1) of Theorem 3.5.7, s is a relation of $\widehat{\mathcal{X}}$. Thus implication (4.6.15) is true for $k = 0$.

Assume that implication (4.6.15) is true for all indices less than $k \geq 1$. It suffices to verify that given a fiber Δ of the coherent configuration $\widehat{\mathcal{X}}$, the formal sum $S_{k-1}(\tau)$ defined at Step 2 of the m-dim WL, does not depend on $\tau \in \Delta$. Note that this is true if given color classes $\Gamma_1, \ldots, \Gamma_m \in \mathcal{P}_{c_{k-1}}$, the number

$$a_\tau(\Gamma_1, \ldots, \Gamma_m) = |\{\alpha \in \Omega : \ \tau_{i,\alpha} \in \Gamma_i, \ i = 1, \ldots, m\}|$$

does not depend on $\tau \in \Delta$.

However, it is straightforward to check that $a_\tau(\Gamma_1, \ldots, \Gamma_m)$ equals the number $p_r(\tau, \tau; \widehat{s}_0, \ldots, \widehat{s}_m)$ defined in Exercise 2.7.25, where $r = 1_{\Omega^m}$ and

$$\widehat{s}_0 = (\Omega^m \times \Gamma_1) \cap \mathrm{Cyl}_1(2, 2) \cap \cdots \cap \mathrm{Cyl}_1(m, m),$$

$$\widehat{s}_i = (\Gamma_i \times \Gamma_{i+1}) \cap \mathrm{Cyl}_1(i, i+1) \cap \bigcap_{j \neq i, i+1} \mathrm{Cyl}_1(j, j) \quad (i = 1, \ldots, m-1),$$

$$\widehat{s}_m = (\Gamma_m \times \Omega^m) \cap \mathrm{Cyl}_1(1, 1) \cap \cdots \cap \mathrm{Cyl}_1(m-1, m-1).$$

By the induction hypothesis, $\Gamma_1, \ldots, \Gamma_m$ are homogeneity sets of $\widehat{\mathcal{X}}$. By statement (1) of Theorem 3.5.7, this implies that $\widehat{s}_0, \ldots, \widehat{s}_m$ are relations of $\widehat{\mathcal{X}}$. Thus the number $p_r(\tau, \tau; \widehat{s}_0, \ldots, \widehat{s}_m)$ and hence $a_\tau(\Gamma_1, \ldots, \Gamma_m)$ does not depend on $\tau \in \Delta$, as required.

Let us prove the right-hand side inclusion in (4.6.14). Denote by $\widehat{\mathcal{Y}}$ the coherent configuration on Ω^m defined in Lemma 4.6.17,

$$S(\widehat{\mathcal{Y}}) = \pi_{2m}(\mathcal{P}_{3m}(\mathcal{X})). \tag{4.6.16}$$

The normality of $\mathcal{P}_{3m}(\mathcal{X})$ implies that $\mathrm{Diag}(\Omega^m)$ is a homogeneity set of $\widehat{\mathcal{Y}}$. Furthermore from the definition of the coloring $c_0(\mathcal{X})$, it follows that

$$S(\mathcal{X}^m) \leq \pi_{2m}(\mathcal{P}_0(\mathcal{X})).$$

So, $\mathcal{X}^m \leq \widehat{\mathcal{Y}}$. Thus by the definition of the m-dimensional extension, we have

$$\widehat{\mathcal{X}}^{(m)} \leq \widehat{\mathcal{Y}},$$

and the required statement follows from (4.6.16). □

Let $\mathcal{X}$ be a rainbow on Ω. Choosing an arbitrary coloring of Ω^2 with color classes belonging to $S(\mathcal{X})$, we turn $\mathcal{X}$ to a colored rainbow. For any integer $m \geq 1$, set

$$S^{(m)} = \begin{cases} \pi_2(\mathcal{P}_m(\mathcal{X})), & \text{if } m \geq 2, \\ S(\mathcal{X}), & \text{if } m = 1. \end{cases}$$

By Lemma 4.6.16, $S^{(m)}$ forms a normal and invariant partition of Ω^2. This implies that the pair

$$\mathrm{WL}_m(\mathcal{X}) = (\Omega, S^{(m)}),$$

is a rainbow, and even a coherent configuration if $m \geq 2$ (Exercise 4.7.51). It should be noted that the partition $\mathcal{P}_m(\mathcal{X})$ does not depend on the colors of initial coloring (Exercise 4.7.41). Thus $\mathrm{WL}_m(\mathcal{X})$ is determined by the rainbow $\mathcal{X}$ only.

Theorem 4.6.19. *Let $\mathcal{X}$ be a coherent configuration, $m \geq 1$, and $\overline{\mathcal{X}}^{(m)}$ the m-closure of $\mathcal{X}$. Then*

$$\mathrm{WL}_m(\mathcal{X}) \leq \overline{\mathcal{X}}^{(m)} \leq \mathrm{WL}_{3m}(\mathcal{X}).$$

Proof. Let us prove the left-hand side inclusion. Without loss of generality, we may assume that $m \geq 2$. By the definition of $S^{(m)}$, for any basis relation s of the rainbow $\mathrm{WL}_m(\mathcal{X})$ there exist $k \geq 1$ classes $\Delta_1, \ldots, \Delta_k \in \mathcal{P}_m(\mathcal{X})$ such that

$$s = \pi_2(\Delta_1) \cup \cdots \cup \pi_2(\Delta_k).$$

By the left-hand side inclusion in (4.6.14), we may assume that each of these classes is a a fiber of the coherent configuration $\widehat{\mathcal{X}}^{(m)}$. Thus, s is a union of basis relations of $\overline{\mathcal{X}}^{(m)}$ by Theorem 3.5.16. This proves the required inclusion.

Let us prove the right-hand side inclusion. By the right-hand side inclusion in (4.6.14) and statement (1) of Exercise 3.7.46, we have

$$\begin{aligned} S(\mathrm{WL}_{3m}(\mathcal{X})) &= \pi_2(\mathcal{P}_{3m}(\mathcal{X})) \\ &= \pi_2(\pi_{2m}(\mathcal{P}_{3m}(\mathcal{X}))) \\ &\geq \pi_2(S(\widehat{\mathcal{X}}^{(m)})) \\ &= S(\overline{\mathcal{X}}^{(m)}), \end{aligned}$$

as required. □

4.6.5 The WL-refinement and algebraic isomorphisms

Let $\mathcal{X}$ and $\mathcal{X}'$ be colored rainbows on Ω and Ω', respectively. Assume that the m-dim WL does not distinguish $\mathcal{X}$ and $\mathcal{X}'$ for some $m \geq 1$. Then there exists a uniquely determined color preserving bijection

$$\psi : \mathcal{P}_m(\mathcal{X}) \to \mathcal{P}_m(\mathcal{X}'),\ \Delta \mapsto \Delta', \tag{4.6.17}$$

where under the color of a class we mean the color of any m-tuple belonging to it, such that for all $\Delta \in \mathcal{P}_m(\mathcal{X})$,

$$|\psi(\Delta)| = |\Delta|. \tag{4.6.18}$$

Now if

$$c_0 = c_0(\mathcal{X}), c_1, \ldots, c_t = c_m(\mathcal{X}) \quad \text{and} \quad c'_0 = c_0(\mathcal{X}'), c'_1, \ldots, c'_{t'} = c_m(\mathcal{X}')$$

are the colorings constructed by the m-dim WL, then using (4.6.18) and the induction on $k = t, \ldots, 0$, one can easily verify that $t = t'$ and

$$\Delta \in \mathcal{P}_{c_t},\ \Gamma \in \mathcal{P}_{c_k},\ \Delta \subseteq \Gamma \quad \Rightarrow \quad \psi(\Delta) \subseteq \psi(\Gamma), \tag{4.6.19}$$

(here $\psi(\Gamma)$ is defined to be the union of all $\psi(\Lambda)$ with $\Lambda \subseteq \Gamma$). The proof of the following statement is similar to that of Theorem 4.6.14.

Theorem 4.6.20. *Let $\mathcal{P} = \mathcal{P}_m(\mathcal{X})$ and $M = \{1, \ldots, m\}$. Then*

(1) $\psi(\pi_L^{-1}(\mathrm{Diag}(\Omega^L))) = \pi_L^{-1}(\mathrm{Diag}({\Omega'}^L))$ *for all* $L \subseteq M$;
(2) $\psi(\Delta^g) = \psi(\Delta)^g$ *for all* $g \in \mathrm{Sym}(M)$ *and* $\Delta \in \mathcal{P}$;
(3) $c^{\Delta}_{L,\pi_L(\Gamma)} = c^{\Delta'}_{L,\pi_L(\Gamma')}$ *for all* $\Gamma, \Delta \in \mathcal{P}$ *and* $L \subseteq M$.

Let $L = \{1, \ldots, l\}$, where $l \leq m$. The normality of $\mathcal{P} = \mathcal{P}_m(\mathcal{X})$ implies that $\mathrm{Diag}(\Omega^{\{l,\ldots,m\}})$ is the union of some classes of $\mathcal{P}$. By the regularity condition this implies that given $\Delta \in \pi_L(\mathcal{P})$, the set

$$\mathrm{Diag}(\Omega^{\{l,\ldots,m\}}) \cap \pi_L^{-1}(\Delta) = \{(\alpha_1, \ldots, \alpha_l, \ldots, \alpha_l) :\ (\alpha_1, \ldots, \alpha_l) \in \Delta\}$$

is a class of $\mathcal{P}$. Denote it by $\widehat{\Delta}$. Then the mapping

$$\pi_L(\mathcal{P}) \to \mathcal{P},\ \Delta \mapsto \widehat{\Delta}$$

is an injection, and hence the mapping

$$\psi_L : \pi_L(\mathcal{P}) \to \pi_L(\mathcal{P}'),\ \Delta \mapsto \pi_L(\psi(\widehat{\Delta})) \tag{4.6.20}$$

is a bijection, where $\mathcal{P}' = \mathcal{P}_m(\mathcal{X}')$. By statements (2) and (3) of Theorem 4.6.20, we have the following corollary.

Corollary 4.6.21. *Let $L \subseteq M$. Then given $\Delta \in \pi_L(\mathcal{P})$ and $\Gamma \in \mathcal{P}$,*

$$c^{\Gamma}_{L,\Delta} = c^{\psi(\Gamma)}_{L,\psi_L(\Delta)}.$$

To state the main result of the subsection, let $\mathcal{X} = (\Omega, S)$ be a rainbow. A function $c_{\mathcal{X}} : \Omega^2 \to \mathbb{N}$ is said to be a *coloring* of $\mathcal{X}$ if every nonempty $c_{\mathcal{X}}$-preimage belongs to S.

Now assume that $\mathcal{X}' = (\Omega', S')$ is a rainbow and

$$\varphi : S \to S', \;\; s \mapsto s' \tag{4.6.21}$$

is a bijection. Then for any coloring $c_{\mathcal{X}}$ of $\mathcal{X}$, there exists a uniquely determined coloring $c'_{\mathcal{X}}$ of $\mathcal{X}'$ such that

$$c_{\mathcal{X}}(s) = c_{\mathcal{X}'}(s') \quad \text{for all } \; s \in S. \tag{4.6.22}$$

As was mentioned earlier the partitions $\mathcal{P}_m(\mathcal{X})$ and $\mathcal{P}_m(\mathcal{X}')$ defined by the colored rainbows $(\mathcal{X}, c_{\mathcal{X}})$ and $(\mathcal{X}', c_{\mathcal{X}'})$, respectively, do not depend on the choice of the colors of $c_{\mathcal{X}}$ and $c_{\mathcal{X}'}$. This justifies the following definition.

Definition 4.6.22. *The bijection (4.6.21) is said to be an m-dim WL isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ if the m-dim WL does not distinguish the colored rainbows $(\mathcal{X}, c_{\mathcal{X}})$ and $(\mathcal{X}', c_{\mathcal{X}'})$ for some (and hence for all) coloring $c_{\mathcal{X}}$.*

The set of all m-dim WL isomorphisms between the rainbows $\mathcal{X}$ and $\mathcal{X}'$ is denoted by $\mathrm{Iso}_m^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}')$. It is easily seen that if $\mathcal{X}$ and $\mathcal{X}'$ are coherent configurations, then

$$\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}') \subseteq \mathrm{Iso}_2^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}').$$

The reverse inclusion is not true (Exercise 4.7.53).

Theorem 4.6.23. *For any coherent configurations $\mathcal{X}$, $\mathcal{X}'$, and $m \geq 1$,*

$$\mathrm{Iso}_m^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}') \supseteq \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}') \supseteq \mathrm{Iso}_{3m}^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}'). \tag{4.6.23}$$

Proof. Without loss of generality we may assume that $\mathcal{X}$, and $\mathcal{X}'$ are colored so that condition (4.6.22) is satisfied. To prove the left-hand side inclusion, let

$$\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}').$$

We have to check that the m-dim WL does not distinguish $\mathcal{X}$ and $\mathcal{X}'$, or equivalently, that given $\Delta \in \mathcal{P}_m(\mathcal{X})$ and $\Delta' \in \mathcal{P}_m(\mathcal{X}')$,

$$c(\Delta) = c'(\Delta) \quad \Rightarrow \quad |\Delta| = |\Delta'|,$$

where $c = c_m(\mathcal{X})$ and $c' = c_m(\mathcal{X}')$.

By Theorem 4.6.18, Δ and Δ' are homogeneity sets of the m-dimensional extensions of $\mathcal{X}$ and $\mathcal{X}'$, respectively. Thus by statement (2) of Proposition 2.3.22, it suffices to verify that

$$c(\Delta) = c'(\Delta) \quad \Rightarrow \quad \Delta^{\widehat{\varphi}} = \Delta', \tag{4.6.24}$$

where $\widehat{\varphi}$ the m-dimensional extension of φ. But this follows by induction on the number of the iterations of the m-dim WL applied to $c_0(\mathcal{X})$ (see the proof of the first part of Theorem 4.6.18).

To prove the right-hand side inclusion, let

$$\varphi \in \mathrm{Iso}_{3m}^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}') \tag{4.6.25}$$

and ψ a uniquely determined color preserving bijection from $\mathcal{P} = \mathcal{P}_{3m}(\mathcal{X})$ onto $\mathcal{P}' = \mathcal{P}_{3m}(\mathcal{X}')$ that satisfies condition (4.6.18) for all $\Delta \in \mathcal{P}$.

We claim that $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. Indeed, the partitions $\mathcal{P}$ and $\mathcal{P}'$ are normal, invariant, and regular (Theorem 4.6.14). Lemma 4.6.17 for $m = 1$ implies that

$$\mathcal{Y} = \pi_{\{1,2\}}(\mathcal{P}) \quad \text{and} \quad \mathcal{Y}' = \pi_{\{1,2\}}(\mathcal{P}')$$

are coherent configurations on Ω and Ω', respectively. The formulas for their intersection numbers obtained in the proof of Lemma 4.6.17 together with Corollary 4.6.21 imply that

$$\psi_{\{1,2\}} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}'). \tag{4.6.26}$$

The definition of the colorings $c_0(\mathcal{X})$ and $c_0(\mathcal{X}')$, and formula (4.6.19) for $k = 0$, show that $\psi_{\{1,2\}}$ extends φ, i.e.,

$$\psi_{\{1,2\}}(s) = \varphi(s) \quad \text{for all } \ s \in S(\mathcal{X}).$$

Together with (4.6.26), this proves the claim.

The formulas for the intersection numbers of the coherent configurations $\pi_{2m}(\mathcal{P})$ and $\pi_{2m}(\mathcal{P}')$, found in the proof of Lemma 4.6.17, together with Corollary 4.6.21, show that

$$\psi_{2m} \in \mathrm{Iso}_{\mathrm{alg}}(\pi_{2m}(\mathcal{P}), \pi_{2m}(\mathcal{P}')),$$

where $\psi_{2m} = \psi_{\{1,\ldots,2m\}}$. Moreover, from statement (1) of Theorem 4.6.20, it follows that

$$\mathrm{Diag}(\Omega^m)^{\psi_{2m}} = (\pi_{2m}(\mathrm{Diag}(\Omega^{3m})))^{\psi_{2m}} = \pi_{2m}(\mathrm{Diag}({\Omega'}^{3m})) = \mathrm{Diag}({\Omega'}^m).$$

Furthermore, the definition of the initial colorings $c_0(\mathcal{X})$ and $c_0(\mathcal{X}')$ implies that

$$\varphi^m(s) = \psi_{2m}(s) \quad \text{for all } \ s \in S(\mathcal{X}^m).$$

Thus the algebraic isomorphism ψ_{2m} induces the m-dimensional extension of φ. It follows that $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}')$, as required. □

Theorem 4.6.23 enables us to compare the powers of the m-dim WL and m-dimensional extension to test graph isomorphism. Namely, the statement below is an immediate consequence of the first inclusion in (4.6.23).

Corollary 4.6.24. *Let $\mathfrak{X}$ be a graph such that the coherent configuration* $\mathrm{WL}(\mathfrak{X})$ *is m-separable, $m \ge 1$. Then the $3m$-dim WL does not distinguish $\mathfrak{X}$ from a graph $\mathfrak{X}'$ only if $\mathfrak{X} \cong \mathfrak{X}'$.*

As was mentioned earlier, two vertex-colored graphs $\mathfrak{X}$ and $\mathfrak{X}'$ with color class size at most 3 are $\mathfrak{C}_3$-equivalent if and only if they are isomorphic [**78**]. Now this result is a direct consequence of Theorem 4.6.9, Corollary 4.6.24, and Exercise 3.7.20.

4.7 Exercises

In what follows, $\mathcal{X}$ is a coherent configuration on Ω and $S = S(\mathcal{X})$, $F = F(\mathcal{X})$, and $E = E(\mathcal{X})$.

4.7.1 [72] Let $\mathfrak{G}$ be a family of finite simple groups. Then $\mathfrak{G}$-configuration is the direct sum of semiregular coherent configurations. In particular, any quasiregular coherent configuration whose homogeneous components are the schemes of simple groups is schurian and separable.

4.7.2 [72] Let $\mathfrak{G}$ be a family of groups with distributive lattices of normal subgroups. Then any $\mathfrak{G}$-configuration is schurian and separable.

4.7.3 Let $\mathcal{X}$ be a $\mathfrak{G}$-configuration. Then for any $i, j \in I$, any basis relation $r \in S_{ij}$, and any pair $(\alpha, \beta) \in r$, there exists $t \in G_i$ such that

$$r = \bigcup_{s \in G_i} \alpha\, G_{ij} s \,\times\, \beta\, f_{ij}(G_{ij} ts).$$

4.7.4 Let $\mathcal{X}$ be a non-semiregular Klein configuration and $K \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$. Suppose that K acts regularly on F. Then

(1) the thin residue of $\mathcal{X}^K$ is of order 2 or a Klein group;
(2) if $|F|$ is a 2-power and K is abelian, then $\mathrm{Aut}(\mathcal{X}^K)$ is a metabelian 2-group.

4.7.5 Let $\mathcal{X}$ be a Klein configuration and $\sim$ the equivalence relation defined by (4.1.12). Then given systems J and J' of distinct representatives for $I/\sim$, we have:

(1) there is a unique bijection $J \to J'$, $j \mapsto j'$, such that $j \sim j'$;
(2) given $j \in J$, the set $S_{jj'}$ consists of thin relations; fix one of them, say s_j;
(3) the mapping $f : \Omega_J \to \Omega_{J'}$ such that $f^{\Omega_j} = f_{s_j}$ for all $j \in J$, is a bijection;
(4) $f \in \mathrm{Iso}(\mathcal{X}_{\Omega_J}, \mathcal{X}_{\Omega_{J'}})$.

4.7.6 Let $\mathcal{X}$ be a Klein configuration and $\mathcal{Y}$ a reduction of $\mathcal{X}$. Assume that the group $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$ acts transitively on $F(\mathcal{X})$. Then the group $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{Y})$ acts transitively on $F(\mathcal{Y})$.

4.7.7 Let $\mathfrak{G} = \{G_i\}_{i \in I}$ and $\mathfrak{S} = \{G_{ij}\}_{i,j \in I}$ be families as in the conditions (F1) and (F2). Assume that for all i,

(1) $G_i = G$ is a Klein group;
(2) $|G_{ij}| = |G_{ji}| \ge 2$ for all $j \ne i$;
(3) G_{ij} satisfies condition (4.1.14) for all j.

Then there exists a (unique) Klein configuration $\mathcal{X}$ such that

$$\mathcal{T}(\mathcal{X}) = (\mathfrak{G}, \mathfrak{S}, \mathfrak{F})$$

for a certain family $\mathfrak{F}$.

4.7.8 Two cubic Klein configurations with isomorphic associated graphs are algebraically isomorphic.

4.7.9 A cubic Klein configuration is a nontrivial direct sum if and only if the associated graph is disconnected.

4.7.10 Let $\mathcal{X}$ be a cubic Klein configuration. Assume that the graph associated with $\mathcal{X}$ is acyclic. Then $\mathcal{X}$ is schurian.

4.7.11 Let $\mathcal{X}$ be a primitive scheme of degree n. Then

$$t(\mathcal{X}) < \lceil 4\sqrt{n}\log n\rceil + 1 \quad \text{and} \quad s(\mathcal{X}) < \lceil 4\sqrt{n}\log n\rceil + 1.$$

4.7.12 Let $\mathcal{X}$ be the scheme of a distance-regular graph $\mathfrak{X}$. Then

(1) $\mathfrak{X}$ is distance-transitive if and only if $t(\mathcal{X}) = 1$;
(2) $\mathfrak{X}$ is uniquely determined by parameters if and only if $s(\mathcal{X}) = 1$.

4.7.13 [**40**, Theorem 4.6] The following inequalities hold:

(1) $s(\mathcal{X}) \le s(\mathcal{X}_\alpha) + 1$ for all $\alpha \in \Omega$;
(2) $t(\mathcal{X}) \le t(\mathcal{X}_\alpha) + 1$ if $\mathcal{X}_\alpha$ is $t(\mathcal{X}_\alpha)$-separable for some $\alpha \in \Omega$;
(3) $s(\mathcal{X}) \le m\, s(\widehat{\mathcal{X}}^{(m)})$, $t(\mathcal{X}) \le m\, t(\widehat{\mathcal{X}}^{(m)})$ for all $m \ge 1$.

4.7.14 Let $\mathcal{X}$ be an imprimitive equivalenced scheme. Then

$$t(\mathcal{X}) \le 2 \quad \text{and} \quad s(\mathcal{X}) \le 2.$$

4.7.15 [**35**, Theorem 3.29] The coherent configurations in Theorem 4.2.4 can be chosen homogeneous, and for the first inequality in (4.2.2) even schurian.

4.7.16 In the notation of Theorem 4.2.6, suppose that φ_1 and φ_2 are algebraic automorphisms of $\mathcal{X}$ leaving each fiber of $\mathcal{X}$ fixed and such that

$$(\varphi_1)_{\Omega_i} = (\varphi_2)_{\Omega_i} \quad \text{for all } i \in I.$$

Then the algebraic automorphism $\varphi_1\varphi_2^{-1}$ is induced by an isomorphism if and only if $t(\varphi_1) = t(\varphi_2) \pmod 2$.

4.7.17 Any colored graph, which is $(\mathfrak{Y}, \mathfrak{Z})$-regular of degree d with respect to each of relations r_1 and r_2, is also $(\mathfrak{Y}, \mathfrak{Z})$-regular of degree d with respect to $r_1 \cup r_2$.

4.7.18 Let $\mathfrak{X} = \mathfrak{X}(\mathcal{X})$ be a colored graph. Then given $s \in S$,

$$\mathrm{Cyl}_s(i, j) = s_{\mathfrak{X}}(\mathfrak{Y}, \mathfrak{Z}, d)$$

for suitable colored graphs $\mathfrak{Y}$, $\mathfrak{Z}$, and a positive integer d.

4.7.19 A generating set of a projective plane $\mathcal{P}$ is a base of the coherent configuration associated with $\mathcal{P}$.

4.7.20 The Doob graphs are pairwise nonisomorphic and can be distinguished each from other with the help of the 4-vertex condition.

4.7.21 Let $\mathcal{X}$ be a scheme and $k \ge 2$ an integer. Then any algebraic isomorphism $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ induces an isomorphism of the graphs $\mathfrak{X}_k$ and $\mathfrak{X}'_k$ (see p. 259). In particular, $\mathcal{X}$ is k-saturated if and only if $\mathcal{X}'$ is k-saturated.

4.7.22 Prove that statement (3) of Theorem 4.3.6 remains true if condition (4.3.10) is replaced by a weaker one: for all $x, y, z \in S_k$ such that

$x \sim y \sim z \sim x$, there exist $a, b \in S$ for which

$$S'_{xz} \cdot S'_{zy} \subseteq S_{xy},$$

where $S'_{xz} = S_{xz} \setminus \{a\}$ and $S'_{zy} = S_{zy} \setminus \{b\}$.

4.7.23 Prove that statement (3) of Theorem 4.3.6 remains true without condition (4.3.10) and with the saturation condition replaced by a weaker one: the graph $\mathfrak{X}_k$ is connected.

4.7.24 In the notation of Exercise 4.7.21, the elements r and s are linked with respect to (x, y, z) if and only if the elements $\varphi(r)$ and $\varphi(s)$ are linked with respect to $(\varphi(x), \varphi(y), \varphi(z))$. In particular, $\mathcal{X}$ is Desarguesian if and only if so is $\mathcal{X}'$.

4.7.25 Let $\mathcal{X}$ be a two-valenced weakly Desarguesian scheme. Assume that x, y, z and r, s are basis relations of $\mathcal{X}$ that form an initial configuration. Then

$$|rs \cap x^*y| = 1.$$

4.7.26 Let $\mathcal{X}$ be the scheme of an affine plane $\mathcal{A}$. Then $\mathcal{X}$ is Desarguesian if and only if $\mathcal{A}$ is a Desarguesian plane.

4.7.27 Let $\mathcal{X}$ be a quasi-thin scheme. Then

(1) every symmetric basis relation of $\mathcal{X}$ is the disjoint union of undirected cycles of the same length;
(2) every homogeneous fission of $\mathcal{X}$ is quasi-thin;
(3) $\mathcal{X} \otimes \mathcal{Y}$ is quasi-thin if and only if $\mathcal{Y}$ regular;
(4) if $\mathcal{X}$ is primitive, then $\mathcal{X}$ is schurian and separable.

4.7.28 [**70**, Lemma 4.1] Let $\mathcal{X}$ be a quasi-thin scheme. Then for any $s \in S$, there exists $t \in S^{\#}$ such that $s\,s^* = \{1, t\}$.

4.7.29 [**103**, Lemma 5.1] Let u and v be are non-thin basis relations of a quasi-thin scheme $\mathcal{X}$. Then

(1) $u^\perp = v^\perp$ and $u^\perp \in S_1$ if and only if either $A_{u^*}A_v = 2A_a + 2A_b$ with $a, b \in S_1$, or $A_{u^*}A_v = A_a$ with $a \in S_2$;
(2) $u^\perp = v^\perp$ and $u^\perp \notin S_1$ if and only if $A_{u^*}A_v = 2A_a + A_b$ with $a \in S_1$ and $b \in S_2$;
(3) $u^\perp \neq v^\perp$ if and only if $A_{u^*}A_v = A_a + A_b$ with $a, b \in S_2$.

4.7.30 [**103**, Lemma 5.4] Assume that $\mathcal{X}$ is a commutative Kleinian scheme. Then $|S^\perp| = 3$.

4.7.31 Let $\mathcal{X}$ be a scheme such that m_ξ does not depend on $\xi \in \mathrm{Irr}(\mathcal{X})^{\#}$. Then $\mathcal{X}$ is commutative and pseudocyclic.

4.7.32 Any cyclotomic scheme over a finite field is pseudocyclic.

4.7.33 [**102**, Theorem 4.3] Let q be the order of an affine plane. Then given a divisor m of $q + 1$, there exists an amorphic pseudocyclic scheme of degree q^2, valency $(q^2 - 1)/m$ and rank $m + 1$.

4.7.34 [**102**, Theorem 3.4] Let $\mathcal{X}$ be a commutative pseudocyclic scheme of valency k. Assume that a group $G \leq \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$ acts semiregularly on $S^{\#}$. Then $\mathcal{X}^G$ is a commutative pseudocyclic scheme of valency $k|G|$.

4.7.35 Let $\mathcal{X}$ be a Cayley scheme over a cyclic group G. Then

(1) if $H \leq \mathrm{rad}(\mathcal{X})$ and $H^\rho \in E$, then $\mathrm{rad}(\mathcal{X}_{G/H}) = \mathrm{rad}(\mathcal{X})/H$;
(2) if $\mathcal{Y} \leq \mathcal{X}$, $E(\mathcal{Y}) = E(\mathcal{X})$, and $\mathrm{rad}(\mathcal{Y}) = 1$, then $\mathrm{rad}(\mathcal{X}) = 1$.

4.7.36 Find an example of a Cayley scheme $\mathcal{X}$ over a cyclic group G and a group $H \leq G$ such that $\mathrm{rad}(\mathcal{X}) = 1_G$, $H^\rho \in E$, and $\mathrm{rad}(\mathcal{X}_{G/H}) \neq 1_{G/H}$.

4.7.37 [**41**, Theorem 6.1] Let $\mathcal{X}$ be a Cayley scheme over a cyclic group G. Then $\mathcal{X}$ is normal if and only if the following conditions are satisfied:

(1) $\mathcal{X}$ is cyclotomic over G;
(2) $|\,\mathrm{rad}(\mathcal{X})| \leq 2$;
(3) if G_p is a Sylow p-subgroup of G, $|G_p| = p$, and $\mathrm{Aut}(\mathcal{X})^{G_p}$ contains $\mathrm{Aut}(G_p)$, then $p = 2$ or 3.

4.7.38 [**41**, Theorem 6.6] The class of normal circulant schemes is separable with respect to the class of all circulant schemes.

4.7.39 Let H be a skew Hadamard matrix (4.5.3) of order $n \geq 4$, A the $\{0,1\}$-matrix such that $C = A - A^T$, and s the relation such that $A_s = A$. Then $\{1, s, s^*\}$ is the set of basis relations of an antisymmetric scheme of rank 3.

4.7.40 Propositions 4.6.1 and 4.6.2 remain true if the graphs $\mathfrak{X}$ and $\mathfrak{X}'$ are assumed to be colored.

4.7.41 Let c and c' be the output colorings obtained by the m-dim WL applied to the colorings c_0 and c'_0 of the sets Ω^m and Ω'^m, respectively. Then for any bijection $f : \Omega^m \to \Omega'^m$ induced by a bijection from Ω to Ω',

$$c_0(\tau) = c'_0(\tau^f) \text{ for all } \tau \in \Omega^m \quad \Rightarrow \quad c(\tau) = c'(\tau^f) \text{ for all } \tau \in \Omega^m.$$

4.7.42 Let $\mathcal{X}$ and $\mathcal{X}'$ be two isomorphic colored rainbows on Ω and Ω', respectively. Assume that

$$|c^{-1}(i)| = |c'^{-1}(i)| \leq 1$$

for all colors i, where $c_0 = c_0(\mathcal{X})$, $c'_0 = c_0(\mathcal{X}')$, $c = c_m(\mathcal{X})$, $c' = c_m(\mathcal{X}')$, and $m \geq 2$. Then the mapping

$$f : \Omega \to \Omega',\ \alpha \mapsto \alpha',$$

where α' is the unique point of Ω' for which $c'(\alpha', \ldots, \alpha') = c(\alpha, \ldots, \alpha)$, is a well-defined bijection. Moreover, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$.

4.7.43 The property of an undirected graph to be strongly regular is expressible in the counting logic language.

4.7.44 Any two strongly regular graphs with the same parameters are $\mathfrak{C}_2$-equivalent.

4.7.45 A partition $\mathcal{P}$ of Ω^m is normal if and only if for any $\Delta \in \mathcal{P}$ and $1 \le i, j \le m$, we have

$$\tau, \tau' \in \Delta \quad \text{and} \quad \tau_i = \tau_j \qquad \Rightarrow \qquad \tau'_i = \tau'_j.$$

4.7.46 For any group $K \le \mathrm{Sym}(\Omega)$, the partition $\mathrm{Orb}(K, \Omega^m)$ of the set Ω^m is normal, invariant, and regular.

4.7.47 The set of basis relations of a coherent configuration on Ω forms a normal, invariant, and regular partition of Ω^2. Find an example showing that not every such partition forms a coherent configuration.

4.7.48 Let $\mathcal{X}$ and $\mathcal{X}'$ be rainbows, $\mathcal{Y} = \mathrm{WL}(\mathcal{X})$ and $\mathcal{Y}' = \mathrm{WL}(\mathcal{X}')$, and let c and c' be standard colorings of $\mathcal{X}$ and $\mathcal{X}'$, respectively. Then at least one of the following statements holds:

(1) there is $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}')$ such that $c(s) = c'(\varphi(s))$ for all $s \in S(\mathcal{Y})$;
(2) there is no $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ such that $c(s) = c'(s^f)$ for all $s \in S(\mathcal{X})$.

4.7.49 For any $m \ge 1$, the mapping $\mathcal{X} \mapsto \mathrm{WL}_m(\mathcal{X})$ is a closure operator.

4.7.50 [**38**, Lemma 6.3] Let $\mathcal{P}$ be a normal, invariant, and regular partition of Ω^m, $m \ge 1$. Then given $k \le m$, the partition $\pi_k(\mathcal{P})$ is also normal, invariant, and regular.

4.7.51 Let $m \ge 2$. Then the partition of Ω^2 induced by $\pi_2(\mathcal{P}_m(\mathcal{X}))$ forms a coherent configuration on Ω.

4.7.52 Prove Theorem 4.6.20.

4.7.53 Find an example of 2-dim WL isomorphism between two coherent configurations, which is not an algebraic isomorphism.

4.7.54 For every $l \le m$,

$$\mathrm{Iso}_l^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}') \supseteq \mathrm{Iso}_m^{\mathrm{WL}}(\mathcal{X}, \mathcal{X}').$$

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List of Notation