World-sheet Duality for Superspace $\sigma$-Models

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Symposium on Theoretical and Mathematical Physics
Euler International Mathematical Institute, St. Petersburg

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[This research receives funding from an Intra-European Marie-Curie Fellowship]
**Motivation**

World-sheet

- 2D Riemann surface (w/wo boundaries)

Target space

- Riemannian manifold (plus extra structure)

**Appearance of superspace $\sigma$-models**

- **String theory**
  - Quantization of strings in flux backgrounds
  - AdS/CFT correspondence
  - Moduli stabilization in string phenomenology

- **Disordered systems**
  - Quantum Hall systems
  - Self avoiding random walks, polymer physics, ...
  - Efetov’s supersymmetry trick
### Ingredients

- Superspace $\sigma$-model encoding geometry and fluxes
- Pure spinors: Curved ghost system
- BRST procedure

[Berkovits et al] [Grassi et al] [Hogeveen, Skenderis] [...]

### Features

- Manifest target space supersymmetry
- Manifest world-sheet conformal symmetry
- Geometry encodes physical properties
- Action quantizable, but quantization hard in practice
Step 1: Trading disorder for supersymmetry

\[
\langle \mathcal{O} \rangle = \left( \frac{\int D F \mathcal{O} e^{-S(F)}}{\int D F e^{-S(F)}} \right) = \left( \int D F D B \mathcal{O} e^{-\left[ S(F) + S(B) \right]} \right)
\]

\[
= \int D F D B \mathcal{O} e^{-S_{\text{eff}}(F, B)} \quad \leftarrow \text{supersymmetric}
\]

Typically:

\[
S_{\text{eff}}(F, B) = S_{\text{free}} + g \int d^2x (B \bar{B} + F \bar{F})^2
\]

Step 2: Hubbard-Stratonovich transformation

- Remove BF interaction by introducing auxiliary field
- Integrate out B and F

\[ \Rightarrow \text{Supersymmetric } \sigma\text{-model} \]
The structure of this talk

Outline

1. Supercoset $\sigma$-models
   - Occurrence in string theory and condensed matter theory
   - Ricci flatness and conformal invariance

2. Some particular examples
   - Three-dimensional Anti-de Sitter space
   - Superspheres
   - Projective superspaces

3. Quasi-abelian perturbation theory
   - Exact boundary spectra
   - World-sheet duality for supersphere $\sigma$-models
Appearance of supercosets

### String backgrounds as supercosets...

<table>
<thead>
<tr>
<th>Minkowski</th>
<th>$\text{AdS}_5 \times S^5$</th>
<th>$\text{AdS}_4 \times \mathbb{CP}^3$</th>
<th>$\text{AdS}_2 \times S^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>super-Poincaré Lorentz</td>
<td>PSU(2,2</td>
<td>4) ÷ SO(1,4) × SO(5)</td>
<td>OSP(6</td>
</tr>
</tbody>
</table>

[Metsaev, Tseytlin] [Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach] [Arutyunov, Frolov]

### Supercosets in statistical physics...

<table>
<thead>
<tr>
<th>IQHE</th>
<th>Dilute polymers (SAW)</th>
<th>Dense polymers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1,1</td>
<td>2)$ ÷ $U(1</td>
<td>1) \times U(1</td>
</tr>
</tbody>
</table>

[Weidenmüller] [Read, Saleur]
Symmetric and generalized symmetric spaces

Definition of the cosets

\[ G/H = \{ g \in G \, | \, gh \sim g, \, h \in H \} \]

Some additional requirements

- \( H \subset G \) is invariant subgroup under an automorphism
- Ricci flatness \( \Leftrightarrow \) super Calabi-Yau \( \Leftrightarrow \) vanishing Killing form

\[ f f = 0 \]

Examples: Cosets of \( PSU(N|N) \), \( OSP(2S + 2|2S) \), \( D(2, 1; \alpha) \).

Remark: To describe supergroups choose \( K = \frac{K \times K}{K_{\text{diag}}} \)
## Properties of supercoset models

### Properties in a nutshell

- Conformal invariance
- Family of CFTs with continuously varying exponents
- Geometric realization of supersymmetry: $g \mapsto hg$
- Completely new type of 2D conformal field theory
  - **Standard methods do not apply!**
- Integrability

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Kagan, Young [Babichenko] [Pohlmeyer] Lüscher ... Bena, Polchinski, Roiban [Young]
Sketch of conformal invariance

The $\beta$-function vanishes identically...

$$\beta = \sum \text{certain } G\text{-invariants} = 0$$

**Ingredients:**
- Invariant form: $\kappa^{\mu\nu}$
- Structure constants: $f^{\mu\nu\lambda}$
Sketch of conformal invariance

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\[ f \bullet f = 0 \]

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[Bershadsky, Zhukov, Vaintrob'99] [Babichenko'06]
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[ Bershadsky, Zhukov, Vaintrob'99 ] [ Babichenko'06 ]
AdS$_3 \times S^3$ alias PSU$(1, 1|2)$
The moduli space of AdS$_3 \times S^3$

Deformed WZW models: mixed fluxes

WZW models: pure NS flux

Principal chiral model: pure RR flux

Action & physical interpretation of the PSU(1,1|2) $\sigma$-model

\[
S = fS_{\text{kin}} + kS_{\text{top}}
\]

\[
\begin{align*}
  k &= Q_5^{\text{NS}} \\
  f &= \sqrt{(Q_5^{\text{NS}})^2 + g_5^2(Q_5^{\text{RR}})^2}
\end{align*}
\]

String theory
[Berkovits,Vafa,Witten]
[Bershadsky,Vaintrob,Zhukov]

Condensed matter theory (IHQE)
[Zirnbauer]
[Bhaseen,Kog.,Sol.,Tan.,Tsvelik]
[Tsvelik]
[Obuse,Sub.,Fur.,Gruz.,Ludwig]
Recent progress

Enhanced symmetry?

1. Solve the PSU(1, 1|2) WZW model $\rightarrow$ LCFT
2. Study marginal deformations by $S_{\text{def}} = \int \text{str}(J \cdot \text{Ad}_g(\bar{J}))$
3. Quasi-abelian deformation theory $\rightarrow$ anomalous dimensions

$Z(f, k) \rightarrow$ Degeneracies for certain values of $f$ and $k$?
OSP(4|2) Gross-Neveu model
The OSP(4|2) Gross-Neveu model

Field content
- Fundamental OSP(4|2)-multiplet \((\psi_1, \psi_2, \psi_3, \psi_4, \beta, \gamma)\)
- All these fields have conformal weight \(h = 1/2\)

Formulation as a Gross-Neveu model

\[
S_{GN} = S_{\text{free}} + g^2 S_{\text{int}}
\]

\[
S_{\text{free}} = \int \left[ \bar{\psi} \partial \psi + 2 \bar{\beta} \partial \gamma + \text{h.c.} \right]
\]

\[
S_{\text{int}} = \int \left[ \bar{\psi} \psi + \bar{\beta} \gamma - \gamma \bar{\beta} \right]^2
\]

Formulation as a deformed OSP(4|2) WZW model

\[
S_{GN} = S_{\text{WZW}} + g^2 S_{\text{def}}
\]

with \(S_{\text{def}} = \int \text{str}(J\bar{J})\)
More on the reformulation

- At $g = 0$ there is an affine $\hat{OSP}(4|2)_{-1/2}$ symmetry.
- It has a “bosonic” realization as an orbifold:

$$\hat{OSP}(4|2)_{-1/2} \cong \left[ \hat{SU}(2)_{-1/2} \times \hat{SU}(2)_1 \times \hat{SU}(2)_1 \right] / \mathbb{Z}_2$$

Towards a boundary spectrum for $g = 0$

- Employ twisted gluing conditions.
- The spectrum can be calculated using standard techniques.
A boundary spectrum

Strong coupling \( g^2 \) \hspace{1cm} Weak coupling

Free ghosts / WZW model

The main result: The full partition function

\[
Z_{GN}(g^2 = 0) = \sum \psi_{[j_1,j_2,j_3]}^{WZW}(q) \chi_{[j_1,j_2,j_3]}(z)
\]

Energy levels

OSP(4\mid2) content

\[
\psi_{[j_1,j_2,j_3]}^{WZW}(q) = \frac{1}{\eta(q)^4} \sum_{n,m=0}^{\infty} (-1)^{n+m} q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+n^2-\frac{1}{8}}
\]

\[
\times (q^{j_2-\frac{n}{2}} - q^{j_2+\frac{n}{2}+1})(q^{j_3-\frac{n}{2}} - q^{j_3+\frac{n}{2}+1})^2
\]
**A boundary spectrum**

![Diagram](image)

**The main result:** The full partition function

\[
Z_{GN}(g^2) = \sum q^{-\frac{1}{2}} \frac{g^2}{1+g^2} C_\Lambda \psi_{WZW}^{[j_1,j_2,j_3]}(q) \chi_{[j_1,j_2,j_3]}(z)
\]

\[
\psi_{WZW}^{[j_1,j_2,j_3]}(q) = \frac{1}{\eta(q)^4} \sum_{n,m=0}^{\infty} (-1)^{n+m} q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+n+\frac{1}{8}} \times (q^{j_2-\frac{n}{2}} - q^{j_2+\frac{n}{2}+1})(q^{j_3-\frac{n}{2}} - q^{j_3+\frac{n}{2}+1})
\]
Interpolation of the spectrum

At the two extremal values of $g^2$, the spectrum has the form...

\[ WZW \text{ model } (g = 0) \quad \text{Strong deformation } (g = \infty) \]
Interpolation of the spectrum

At the two extremal values of $g^2$, the spectrum has the form...

**WZW model ($g = 0$)**

**Strong deformation ($g = \infty$)**

**Supersphere $\sigma$-model at $R \to \infty$**
A world-sheet duality for superspheres
The supersphere $S^{3|2}$

Realization of $S^{3|2}$ as a submanifold of flat superspace $\mathbb{R}^{4|2}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \eta_1 \\ \eta_2 \end{pmatrix}$$

with

$$\vec{X}^2 = \vec{x}^2 + 2\eta_1\eta_2 = R^2$$

Realization as a symmetric space

$$S^{3|2} = \frac{\text{OSP}(4|2)}{\text{OSP}(3|2)}$$
The supersphere $\sigma$-model

### Action functional

$$S_\sigma = \int \partial \vec{X} \cdot \bar{\partial} \vec{X}$$

with

$$\vec{X}^2 = R^2$$

### Properties of this $\sigma$-model

- There is no topological term
- Conformal invariance for each value of $R$
- Central charge: $c = 1$
- Non-unitarity

[Read, Saleur] [Polchinski, Mann] [Candu, Saleur]$^2$ [Mitev, TQ, Schomerus]

### The space of states on a space-filling brane

$$\prod X^{a_i} \prod \partial X^{b_j} \prod \partial^2 X^{c_k} \ldots$$

and

$$\vec{X}^2 = R^2$$

$\Rightarrow$ Products of coordinate fields and their derivatives
A world-sheet duality for superspheres?

**Supersphere $\sigma$-model**

Large volume

Strong coupling

$R^2 = 1 + g^2$

Strong coupling

Weak coupling

$1/R$

geometric

$Z_{\sigma}(q, z, R)$

non-geometric

$Z_{GN}(q, z, g^2)$

**OSP(4|2) Gross-Neveu model**

$[\text{Candu, Saleur}]^2 [\text{Mitev, TQ, Schomerus}]$
Quasi-abelian deformations
Radius deformation of the free boson

A Neumann brane on a circle of radius $R$...

\[ Z_N(R) = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} q^{\frac{w^2}{2R^2}} = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} q^{\frac{w^2}{2R_0^2(1+\gamma)}} \]

Interpret this as a deformation...

\[ R = R_0 \sqrt{1 + \gamma} \]

Anomalous dimensions

\[ \delta \gamma h_w = \frac{w^2}{2R_0^2} \left[ \frac{1}{1 + \gamma} - 1 \right] = -\frac{\gamma}{1 + \gamma} \frac{w^2}{2R_0^2} = -\frac{\gamma}{1 + \gamma} C_2(w) \]
Quasi-abelianness of supergroup WZW theories

The effective deformation for conformal dimensions

- Vanishing Killing form $\Rightarrow$ the perturbation is quasi-abelian (for the purposes of calculating anomalous dimensions)
  
  \[ [\text{Bershadsky,Zhukov,Vaintrob}, \text{TQ,Schomerus,Crutzig}] \]

- The currents behave as if they were abelian
  
  \[ J^\mu(z) J^\nu(w) = \frac{k\kappa^{\mu\nu}}{(z-w)^2} + \frac{if^{\mu\nu\lambda} J^\lambda(w)}{z-w} \sim \frac{k\kappa^{\mu\nu}}{(z-w)^2} \]

- For $\widehat{\text{OSP}(4|2)}^{-\frac{1}{2}}$ a representation $\Lambda$ is shifted according to
  
  \[ \delta h_\Lambda(g^2) = -\frac{1}{2} \frac{g^2 C_\Lambda}{1 + g^2} = -\frac{1}{2} \left( 1 - \frac{1}{R^2} \right) C_\Lambda \]
Projective superspaces
New features

- Non-trivial topology
  \[ \Rightarrow \text{ Monopoles and } \theta\text{-angle} \]
- Symplectic fermions as a subsector
  \[ \theta\text{-angle } \Rightarrow \text{ twists} \]
- \(\sigma\)-model brane spectrum can be argued to be
  \[
  Z_{R,\theta}(q, z) = q^{\frac{1}{2} \lambda(R, \theta) \left[ 1 - \lambda(R, \theta) \right]} \sum_{\Lambda} q^{f(R, \theta) C_{\Lambda}} \psi_{\Lambda}^\infty(q) \chi_{\Lambda}(z)
  \]
  \[ \text{twist} \quad \text{Casimir} \quad \text{result for } R \to \infty \]
- Currently no free field theory point is known...

Remark: The family contains the supertwistor space \( \mathbb{CP}^{3|4} \)
Conclusions
Conclusions

- Using supersymmetry we determined the full spectrum of anomalous dimensions for certain boundary spectra in various models as a function of the radius.
- The result provided strong evidence for a duality between supersphere $\sigma$-models and Gross-Neveu models.

World-sheet methods appear to be more powerful than expected!
Several open issues remain...
- More points with enhanced symmetry?
- Deformation of the bulk spectrum
- Correlation functions
- Interplay with integrability ("S-matrix approach")
- Path integral derivation?

Outlook
- Other spaces: AdS-spaces, conifold, nil-manifolds, ...
- Applications to condensed matter physics, ...