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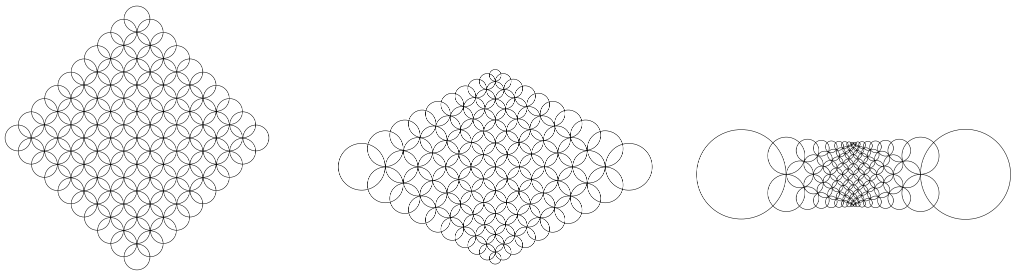
Variational principles for circle patterns and polyhedra; Koebe's theorem and its generalizations

A circle packing is a configuration of disjoint discs which may touch but not intersect. In 1936, Koebe published the following theorem about circle packings in the sphere.

Theorem For every triangulation of the sphere there is a packing of circles in the sphere such that circles correspond to vertices and two circles touch if and only if the corresponding vertices are adjacent. This circle pattern is unique up to Möbius transformations of the sphere.

A generalization of this theorem yields a powerful version of Steinitz' theorem: For every polytopal cell decomposition of the sphere, there is a combinatorially equivalent polyhedron with edges tangent to a sphere. This polyhedron is unique up to projective transformations which fix the sphere. There is a simultaneous realization of the dual polyhedron, such that corresponding edges of the dual and the original polyhedron touch the sphere in the same points and intersect orthogonally.



In the talk based on the paper [1] these theorems are generalized in two directions. First we consider patterns of circles intersecting at arbitrary angles. Second, we consider not only circle patterns in the sphere but also in other surfaces of constant curvature. We present new variational principles for the circle patterns in Euclidean and hyperbolic surfaces. The functionals are given explicitly in terms of the dilogarithm function Li_2 of the radii of the circles. The convexity of the functional allows us to prove the existence and uniqueness results for the corresponding circle patterns using the technique developed by Colin de Verdière [2].

Our functional is an effective tool for constructing of circle patterns. Three circle patterns with square grid combinatorics and intersection angles $\pi/2$ are presented in the Figure. They are uniquely determined by their boundary data (boundary angles) and were obtained using our variational principle.

REFERENCES

[1] A.I. Bobenko, B. Springborn, *Variational principles for circle patterns and Koebe's theorem*, Preprint (2002) http://xxx.lanl.gov/abs/math.GT/0203250

[2] Y. Colin de Verdière *Un principe variationnel pour les empilements de cercles*, Invent. Math. 104, 655-669, 1991