Lengths and volumes in Riemannian manifolds

Christopher B. Croke, University of Pennsylvania, USA;
email: ccroke@math.upenn.edu
Nurlan S. Dairbekov, Sobolev Institute of Mathematics, Russia;
email: dair@math.nsc.ru

We consider the question of inequalities on volumes implied by inequalities on lengths of geodesics. This is in fact a collection of related problems.

For a compact manifold N without boundary, the marked length spectrum for a Riemannian metric g on N is a function, $MLS_g : \mathcal{C} \to \mathbb{R}^+$, from the set $\mathcal{C}$ of free homotopy classes of the fundamental group $\pi_1(N)$ to the nonnegative reals. For each $\langle\gamma\rangle \in \mathcal{C}$, $MLS_g(\langle\gamma\rangle)$ is the length of the shortest curve in $\langle\gamma\rangle$ (always a geodesic). Given two Riemannian metrics g_0 and g_1 on N, we say that $MLS_{g_1} \geq MLS_{g_0}$ if $MLS_{g_1}(\langle\gamma\rangle) \geq MLS_{g_0}(\langle\gamma\rangle)$ for all free homotopy classes $\langle\gamma\rangle$. It was conjectured that if g_0 and g_1 are negatively curved metrics on N then $MLS_{g_1} \geq MLS_{g_0}$ would imply $Vol(g_1) \geq Vol(g_0)$. It was further conjectured that the equality of volumes would hold if and only if g_0 and g_1 are isometric.

Our most general result proves the first part for surfaces (two dimensional manifolds). In higher dimensions we can prove the result, assuming the metrics g_0 and g_1 conformal. In this case we also get rigidity in the equality case.

The problem for a compact manifold M with boundary ∂M would be to compare the volumes, $Vol(g_0)$ and $Vol(g_1)$, of two Riemannian metrics g_0 and g_1 on M if we know that for every pair of points $x, y \in \partial M$ we have $d_{g_0}(x, y) \leq d_{g_1}(x, y)$. In the above $d_{g_i}(x, y)$ represents the distance in M with respect to the the metric g_i, i.e. the length of the g_i-shortest path in M.

The most general result one could hope for would be of the form: If g_0 is an SGM metric on $(M, \partial M)$ and g_1 is another metric with $d_{g_0}(x, y) \leq d_{g_1}(x, y)$ then $Vol(g_0) \leq Vol(g_1)$ with equality of volumes implying isometry of the metrics. This is still very much an open question, which as stated includes the boundary rigidity problem. The SGM condition of a metric g (which is given in terms of $d_g : \partial M \times \partial M \to \mathbb{R}$ only) means, loosely speaking, that each geodesic segment which lies in the interior of M except possibly for the endpoints is a unique path whose length is the distance between the endpoints.

Recently Ivanov considered the case of compact metrics g_0 and g_1 on a disk, assuming g_0 convex in the sense that every pair of interior points can be joined by a unique geodesic. Using completely different methods, we establish a similar though somewhat different result, assuming both the metrics to be SGM, and prove the higher dimensional version for conformal metrics.