Illuminating a convex lateral surface by horizontal directions

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We deal with a modification of the following well-known Illumination Problem.

Let C be a convex body in $E^d, d \geq 2$. Let x be a point in ∂C and u be a direction. Consider the axis ℓ passing through x and having the direction u. One says that u **illuminates** x if ℓ contains a point $y \in \operatorname{int} C$ which succeeds x. If each point of a part of ∂C is illuminated by at least one direction from a set U, this part is said to be illuminated by U. When the whole ∂C is illuminated by U, we say that C is illuminated by U.

The Hadwiger Conjecture, still unproved, states that each convex body in E^d can be illuminated by 2^d directions. There are numerous partial results in this area.

If a restriction is applied to the directions, one naturally might need for illumination more than 2^d such directions. Fix a pair P, Q of parallel ("horizontal") supporting hyperplanes and call the set $L(C) \stackrel{\text{def}}{=} \partial C \backslash [(C \cap P) \cup (C \cap Q)]$ the **lateral surface of** C. If P and Q are chosen orthogonal to a diameter of C, then $C \cap P$ and $C \cap Q$ are points, and two directions will illuminate them. Try now to illuminate the lateral surface $L(C)$ by horizontal directions only. It turns out that, for $d \geq 3$, no finite number of horizontal directions might be sufficient for this. Moreover, a body C is constructed such that, if a set of horizontal directions illuminates $L(C)$, it must be dense in the sphere S^{d-2} of all horizontal directions. This is a part of a more general theorem here.