SEMIGROUPS OF NORMAL SURFACES FOR $\mathbf{L_{p,q}}$, $\mathbf{S^3/Q_{4n}}$ AND STALLINGS MANIFOLDS

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Haken's theory of normal surfaces is one of the main tools in 3-manifold topology [1]. The set of normal surfaces in a 3-manifold M equipped with a triangulation, maybe singular, possesses a natural structure of an abelian partial semigroup generated by a finite set of fundamental surfaces. We present a complete explicit description of the semigroup for pseudo-minimal singular triangulations of three infinite sets of 3-manifolds.

Let M be either a lens space $L_{p,q}$ ($p \geq 4$), or a generalized quaternion space S^3/Q_{4n} ($n \geq 2$), or a once-punctured torus bundle over S^1 with a hyperbolic monodromy map. Instead of triangulations we use dual objects called *special spines*. Let P be a symmetric chain spine of M. Note that all manifolds listed above possess such spines [3]. Denote by $\mathcal{N}(P)$ the set of surfaces in M normal with respect to P.

Let $\mathcal{C}(P) = \{P_1, \dots, P_n\}$ be the set of all fake subsurfaces of P. All of them are connected. A formal linear combination $x_1 P_1 + \ldots + x_n P_n$ with non-negative integer coefficients is called *admissible*, if for any $x_i \neq 0, x_j \neq 0$ the polyhedron $P_i \cap P_j$ lies in $\mathcal{C}(P)$. Denote by $\mathcal{L}(P)$ the set of all admissible combinations, which is an abelian partial semigroup with respect to the natural addition. Construct a linear map $\Psi\colon \mathcal{L}(P) \to \mathcal{N}(P)$ as follows. If $P_i \in \mathcal{C}(P)$ is a subsurface of P, then we set $\Psi(P_i) = P_i$. If $P_i \in \mathcal{C}(P)$ is a fake subsurface of P, then $\Psi(P_i)$ is the boundary of a small regular neighborhood of P_i.

Lemma 1. *The map Ψ is a well-defined homomorphism of partial semigroups.*

It is known [2,3] that the set $\Psi(P_1), \dots, \Psi(P_n)$ coincides with the set of all fundamental surfaces. Hence, Ψ is a surjection. Unfortunately, Ψ is not an injection.

A formal linear combination $x_1 P_1 + \ldots + x_n P_n$ is called *monotone*, if there exist a permutation π and a number $k > 0$ such that $P_{\pi(1)} \subset \ldots \subset P_{\pi(k)}$ and $x_{\pi(k+1)} = \ldots = x_{\pi(n)} = 0$. Denote by $\mathcal{L}_m(P)$ the set of all monotone combinations. It is obvious that $\mathcal{L}_m(P) \subset \mathcal{L}(P)$.

Lemma 2. *The restriction of Ψ onto the set $\mathcal{L}_m(P)$ is a bijection.*

Note that the set $\mathcal{L}_m(P)$ is not closed with respect to the summation of combinations. On the other hand, it follows from Lemma 2 that there exists a retraction $r\colon \mathcal{L}(P) \to \mathcal{L}_m(P)$ such that for any $L \in \mathcal{L}(P)$ the condition $\Psi(L) = \Psi(r(L))$ holds. Define a partial binary operation $\oplus$ on $\mathcal{L}_m(P)$ as follows: if $L_1, L_2 \in \mathcal{L}_m(P)$, then $L_1 \oplus L_2 = r(L_1 + L_2)$. It means that the sum $L_1 \oplus L_2$ is defined if and only if the sum $L_1 + L_2$ is defined. Thus, we have constructed an abelian partial semigroup $\mathcal{L}_m(P)$. The following theorem can be considered as a complete description of the semigroup $\mathcal{N}(P)$ for symmetric chain spines.

Theorem. *The abelian partial semigroups $\mathcal{L}_m(P)$ and $\mathcal{N}(P)$ are isomorphic.*

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References

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