Displacement energy for open bounded subsets of $\mathbb{R}^{2n}$

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Consider the standard $2n-$dimensional euclidean space $\mathbb{R}^{2n}$ equipped with the euclidean symplectic form $\omega_0 = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$. A *relative capacity* associates to every open subset Ω of $\mathbb{R}^{2n}$ a number $c(\Omega) \in [0, \infty]$ which measures the symplectic size of Ω.

The *cylindrical capacity* associates to $\Omega \in \mathcal{O}$ the infimum $c(\Omega)$ of all numbers $r > 0$ for which there is a symplectomorphism of $\mathbb{R}^{2n}$ which maps Ω into the cylinder $B^2(r) \times \mathbb{R}^{2n-2}$. The cylindrical capacity is the biggest relative capacity.

A compactly supported smooth time dependent function $H(t,x)$ on $[0,1] \times \mathbb{R}^{2n}$ induces a time-dependent Hamiltonian flow on $\mathbb{R}^{2n}$. Its time-one map φ is a symplectomorphism of $\mathbb{R}^{2n}$ whose *Hofer-norm* equals

$$\|\varphi\| = \inf_H \big(\sup_{t\in[0,1]} (\sup_{x\in\mathbb{R}^{2n}} H(t,x) - \inf_{x\in\mathbb{R}^{2n}} H(t,x)) \big).$$

Here H ranges over the set of all compactly supported time dependent functions whose Hamiltonian flows induce φ as their time-one map.

For a bounded set $A \subset \mathbb{R}^{2n}$ the *displacement energy* $d(A)$ is defined to be the infimum of the Hofer norms $\|\varphi\|$ of all those $\varphi \in \mathcal{D}$ which *displace* A, i.e. for which we have $\varphi(A) \cap A = \emptyset$.

We discuss the following two results.

Theorem: *Let $\Omega \subset \mathbb{R}^{2n}$ be an open bounded connected set; then* $c(\Omega) = \inf\{d(K) \mid K \text{ compact}, K \supset \Omega, K \text{ simply connected}\}$.

Corollary : *Let $\Omega \in \mathcal{O}$ be an open bounded connected set. If $H^1(\overline{\Omega}, \mathbb{R}) = 0$ then $c(\Omega) = d(\overline{\Omega})$.*

We give an example that the assumptions in the statement of our theorem and in the corollary can not be omitted.