Acute triangulations

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Triangulations are a basic geometric tool. Restricting ourselves here to 2-dimensional compact surfaces, triangulations are finite sets of triangles satisfying certain natural conditions: the intersection of any two of the triangles is either empty or consists of a vertex or of an edge. The edges of any triangle must be shortest paths.

Now, we raise the question of existence and minimality (least number of triangles used) of *acute triangulations*, for various types of surfaces. This means that the involved triangles must be acute, i.e. all their angles must be less than $\frac{\pi}{2}$.

If the considered surface is homeomorphic to the sphere and smooth, then every vertex of any acute triangulation must obviously have degree at least 5. This immediately implies that any acute triangulation of such a surface contains at least 20 triangles.

Any flat torus admits an acute triangulation made up by 18 triangles. However, in some cases, 14 are enough!

All these things suggest a few questions.

Problem 1. Does there exist a natural number N such that every convex surface admits an acute triangulation consisting of at most N triangles? If yes, find the minimal such N.

This question does not seem to be easy. A solution to this problem restricted to smooth surfaces would also be of interest.

Problem 2. Find the analogous numbers in the nontrivial cases of Platonic solids, i.e. for the surface of the cube, of the dodecahedron, and of the icosahedron.

H. Maehara independently discussed on acute triangulations of polygons.

In this talk, we shall announce the following results with respect to Problem 2 and more.

Theorem 1 (Hangan, Zamfirescu & I.). *The cube admits several acute triangulations with 24 triangles, and no acute triangulation with fewer triangles.*

Theorem 2 (Zamfirescu & I.). *The icosahedron admits an acute triangulation with 12 triangles and no acute triangulation with less than 12 triangles.*

Theorem 3 (Zamfirescu & I.). *The dodecahedron admits an acute triangulation with 14 triangles and no acute triangulation with less than 12 triangles.*

Theorem 4. (Zamfirescu & I.). *Every rotated ellipse is triangulable with at most 20 acute triangles. Every double of ellipse is triangulable with at most 72 acute triangles.*

A *triangle* on the unit sphere is called *proper*, if all angles are less than π.

Theorem 5 (Zamfirescu & I.). *Each proper triangle is triangulable with at most 10 acute triangles, and this is the best possible estimate.*