Two-generator hyperbolic orbifolds

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Let $\mathbf{H}^3$ be the hyperbolic space and $\mathrm{Isom}(\mathbf{H}^3)$ be its isometry group. It is well-known that $\mathrm{Isom}(\mathbf{H}^3)$ is isomorphic to $\mathrm{PSL}(2,\mathbf{C})$. A hyperbolic 3-orbifold Q is a quotient space $\mathbf{H}^3/\Gamma$, where Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{C})$, possibly with torsion. Hence, the classification problem for hyperbolic orbifolds is equivalent to the classification problem for discrete subgroups of $\mathrm{PSL}(2,\mathbf{C})$. However, both questions are still open in general.

Recently, Gehring, Gilman, and Martin suggested the investigation of an important class of two-generator groups with real parameters:

$$\mathcal{RP} = \{\Gamma = \langle f, g\rangle \mid f, g \in \mathrm{PSL}(2,\mathbf{C});\ \beta,\ \beta', \gamma \in \mathbf{R}\},$$

where $\beta = \mathrm{tr}^2 f - 4$, $\beta' = \mathrm{tr}^2 g - 4$, $\gamma = \mathrm{tr}[f,g] - 2$.

Parameters β, β', and γ have a definite geometric meaning: β and β' define the type of generators and γ is responsible for the relative position of the axes or invariant planes of f and g. Clearly, the generators of an $\mathcal{RP}$ group can be only elliptic, parabolic, hyperbolic, and π-loxodromic (geometrically a π-loxodromic element is a glide-reflection in each plane passing through its axis).

Elementary discrete groups and discrete two-generator groups with invariant plane have been completely described by numerous authors. We are interested in those $\mathcal{RP}$ groups that are non-elementary and have no invariant plane.

Reality of parameters allows us to break the study of $\mathcal{RP}$ groups into a finite number of cases. Each case is defined by the types of the generators and relative position of their axes or invariant planes. Most of these cases have been investigated by E. Klimenko.

We consider $\mathcal{RP}$ groups generated by elliptic and hyperbolic elements with intersecting axes and present a list of orbifolds corresponding to the discrete ones. It is interesting to see how the underlying spaces and the singular sets of the orbifolds are changing together with the change of the parameters.