E. Kudryavtseva

# Simple curves on surfaces and an analog of a theorem of Magnus for surface groups

(by joint work [1] with O. Bogopolski and H. Zieschang)

In [2] Magnus proved that *if elements $u, v$ of a free group $\mathcal{F}$ have the same normal closures, then $u$ is conjugate to $v^{\pm 1}$*. We know the following two generalizations of this theorem.

1) In 1961 Greendlinger proved that if two subsets $\mathcal{U}$ and $\mathcal{V}$ of a free group $\mathcal{F}$ satisfy some small cancellation conditions and have the same normal closures then there is a bijection $\psi\colon \mathcal{U} \to \mathcal{V}$ such that $u$ is conjugate to $\psi(u)^{\pm 1}$.

2) A group is said to be locally indicable if each of its non-trivial, finitely generated subgroups admits an epimorphism onto the infinite cyclic group. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-trivial locally indicable groups. In 1989 Edjvet proved that if $u, v \in \mathcal{A} * \mathcal{B}$ are cyclically reduced words each of length at least two, and if the normal closures of $u$ and $v$ coincide, then $u$ is a conjugate of $v^{\pm 1}$.

**Theorem 1.** [1] *Let $S$ be a closed surface and $g$, $h$ non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves $\gamma$ and $\chi$, resp. If $h$ belongs to the normal closure of $g$ then $h$ is conjugate to $g^\varepsilon$ or to $(gug^\eta u^{-1})^\varepsilon$, $\varepsilon, \eta \in \{1, -1\}$; here $u$ is a homotopy class containing a simple closed curve $\mu$ which properly intersects $\gamma$ exactly once.*

*Moreover, if $h$ is not conjugate to $g^\varepsilon$ then $\eta = 1$ if $\mu$ is one-sided and $\eta = -1$ elsewise, and $\chi$ is homotopic to the boundary of a regular neighbourhood of $\gamma \cup \mu$.*

A direct consequence is the following analog of the Magnus' theorem [2].

**Corollary.** [1] *Let $S$ be a closed surface and $g$, $h$ be non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves. If the normal closures of $g$ and $h$ coincide then $h$ is conjugate to $g$ or $g^{-1}$.*

The proof of Theorem 1 is geometrical and uses coverings, intersection numbers of curves, and Brouwer's fixed-point theorem. As a corollary we obtain the following Theorem 2 concerning *normal* automorphisms (an automorphism of a group $\mathcal{G}$ is called normal if it maps each normal subgroup of $\mathcal{G}$ into itself).

**Theorem 2.** [1] *If $S$ is a closed surface different from the torus and the Klein bottle, then every normal automorphism of $\pi_1(S)$ is an inner automorphism.*

The proof of this theorem uses Theorem 1 and purely algebraic technique (amalgamated free products, HNN-extensions, Magnus' solution of the word problem for one-relator groups), as well as the Bass-Serre theory of groups acting on trees.

Earlier Lubotsky (1980) and Lue (1980) proved that every normal automorphism of a free group of rank at least 2 is an inner automorphism. In 1996 Neshchadim proved that any normal automorphism of the free product of two non-trivial groups is an inner automorphism.

Remark that Theorem 1 and Corollary admit some analogs for non-simple curves satisfying additional assumptions.

## References


[1] Bogopolski, O.; Kudryavtseva, E.; Zieschang, H.: *Simple curves on surfaces and an analog of a theorem of Magnus for surface groups.* Submitted.

[2] Magnus, W.: *Über diskontinuerliche Gruppen mit einer definierenden Relation (Der Freiheitssatz).* J. reine angew. Math. **163**, 141–165 (1930).