A sphere theorem for non-reversible Finsler metrics

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The classical sphere theorem in Riemannian geometry states, that a simply-connected and compact Riemannian manifold with sectional curvature $1/4 < K \leq 1$ is homeomorphic to the n-sphere. In [Kl] W. Klingenberg showed that using Morse theory for the free loop space one can prove this result using only the injectivity radius estimate and a Rauch comparison argument. P. Dazord noticed in [Da] that this proof carries over to the case of a *reversible* Finsler metric F, where the sectional curvature is replaced by the *flag curvature.* The Finsler metric F is called reversible, if for all tangent vectors X we have $F(-X) = F(X)$.
In this talk we consider also non–reversible Finsler metrics and we define the *reversibility* $\lambda = \lambda(F) := \sup\{F(-X) \,|\, F(X) = 1\} \geq 1$. Then we obtain a lower bound for the length of a closed geodesic:

Theorem 1 *Let (M^n, F) be a simply-connected compact Finsler manifold with reversibilty λ and with flag curvature $0 < K \leq 1$. If n is odd we assume in addition that $K > \left(1 - \frac{1}{1+\lambda}\right)^2$. Then every non-constant closed geodesic c has length $L(c) \geq \pi \left(1 + \frac{1}{\lambda}\right)$.*

We notice that the examples of irreversible Finsler metrics on the 2-sphere with only two closed geodesics due to A. Katok define particular Finsler metrics of Randers type, for which Z. Shen proves in [Sh1] that they have constant flag curvature. These examples show that the above estimate is sharp.
The length estimate implies that the Morse index of a closed geodesic is at least $n-1$, hence we conclude:

Theorem 2 *A simply-connected and compact Finsler manifold (M^n, F) with reversibilty λ and flag curvature $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$ is homotopy-equivalent to the n-sphere.*

References

[Da] P. Dazord: *Variétés finslériennes en forme des sphères.* C. R. Acad. Sc. Paris, **267** (1968) 353-355

[Kl] W. Klingenberg: *Manifolds with restricted conjugate locus.* Ann. Math. **78** (1963) 527-547

[Sh1] Z. Shen: *Two-dimensional finsler metrics with constant curvature.* Preprint 2001; `arXiv:math.DG/0109097`