One-dimensional sectional curvature of Riemannian manifolds.

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At research of Riemannian manifolds the important role is played by a tensor which is defined with the help of the formula

$$A_{ij} = 3D\frac{1}{n-2}\left(R_{ij} - \frac{Rg_{ij}}{2(n-1)}\right),$$

where R_{ij} denotes the Ricci tensor, R denotes the scalar curvature and g_{ij} denotes the metric tensor on manifold M^n.

Definition. *The one-dimensional sectional curvature in the tangent direction ξ is defined as the value*

$$A(\xi) = 3D\frac{A_{ij}\xi^i\xi^j}{g_{ij}\xi^i\xi^j},$$

where ξ^i is an arbitrary tangent vector generating the direction ξ.

In these notations we have the following results:

Theorem. *Let $(G/H, ds^2)$ be a homogeneous Riemannian manifold and let o denotes the origin of G/H. Then the following statements are true:*

(i) *If the one-dimensional sectional curvature $A(\xi)$ is nonnegative for any vector $\xi \in T_oG/H$, then the Ricci curvature* $\mathrm{Ric}(\xi)$ *is nonnegative for any vector $\xi \in T_oG/H$. Moreover, if there is a vector $\eta \in T_oG/H$ such that $\mathrm{Ric}(\eta) = 3D\sum R_{ij}\eta^i\eta^j = 3D0$, then $(G/H, ds^2)$ is isometric to the direct Riemannian product of a flat torus and Euclidean space.*

(ii) *There are homogeneous Riemannian manifolds of non-negative Ricci curvature and sign-changing one-dimensional sectional curvature.*

Theorem. *Let $(G/H, ds^2)$ be a homogeneous Riemannian manifold of nonnegative one-dimensional sectional curvature wich is not isometric to the direct Riemannian product of a flat torus and Euclidean space. Then the following statements are true:*

(i) *The Lie group G is compact and the Levi subgroup LG of G (i.e. maximal connected semisimple subgroup of G) acts transitively on M.*

(ii) *The fundamental group $\pi_1(M)$ is finite.*

Theorem. *A connected Lie group G admits a left-invariant Riemannian metric of positive one-dimensional sectional curvature if and only* *if G is compact and its fundamental group $\pi_1(G)$ is finite. In such a case, G also admits a standard metric with the above property.*

Besides, we investigate the behaviour of one-dimensional sectional curvature of Moebius manifolds.

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