SMALL LIMIT CYCLES BIFURCATIONS OF POLYNOMIAL VECTOR FIELDS

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Consider a family of two-dimensional real polynomial vector fields

$$\dot{x} = P_n(x,y), \dot{y} = Q_n(x,y) \tag{1}$$

where $P_n(x,y), Q_n(x,y)$ are polynomials of degree n, and suppose that the coefficients of the polynomials P_n, Q_n are from a parameter space $\mathcal{E}$. We say that the singular point (x_0, y_0) of the vector field $E \in \mathcal{E}$ *has cyclicity* k with respect to $\mathcal{E}$ if and only if any perturbation of E in $\mathcal{E}$ has at most k limit cycles in a neighborhood of (x_0, y_0) and k is the minimal number with this property. In fact, the problem of cyclicity is a part of 16th Hilbert problem and it is often called *the local 16th Hilbert problem*. For the first time the cyclicity of singular points of center or focus type was investigated for quadratic system by Bautin [1]. An approach to the problem is as follows. For $w = x + iy$, if the origin is a fine focus or a center, then we can rewrite the system (1) in the form

$$i\frac{dw}{dt} = w - \sum_{j+l=2}^{n} a_{lj} w^l \overline{w}^j, \tag{2}$$

with complex coefficients a_{lj}. According to the Poincaré-Lyapunov theorem there exists a formal series $\Phi(w, \bar{w}; a_{20}, \bar{a}_{20}, \ldots) = w\bar{w} + \ldots$, with the property $\dot{\Phi} = \tilde{g}_3 |w|^4 + \cdots$, where $g_3 := i\tilde{g}_3, g_5 := i\tilde{g}_5, \ldots$ are polynomials in $a_{ij}, \bar{a}_{ij}$ (called *focus quantities*). They generate an ideal $J = \langle g_3, g_5, \ldots \rangle$, the *Bautin ideal.* It is well known and follows, e.g., from [1,3] that, if J is generated by the m first focus quantities $J = \langle g_3, g_5, \ldots, g_{2m+1} \rangle$, then the cyclicity of the elementary center or focus $w = 0$ of any system $E \in \mathcal{E}$ is less or equal to $m - 1$. In fact, the first step in the investigation of the cyclicity problem is the solution of *the problem of distinguishing between a focus and a center*, for the first time considered by Dulac in 1908. The latter problem is to separate in $\mathcal{E}$ the set of points, which correspond to systems having a center at the origin from those with a focus. The set of systems with a center is described by the variety $V(J) \subset \mathcal{E}$ of zeroes of the Bautin ideal J.

We apply methods based on the theory of Gröbner bases to the investigation of the cyclicity and center-focus problems for polynomial systems. The idea of our investigation of the cyclicity is as follows. First, using methods from the theory of plane polynomial vector fields, we find the variety of the Bautin ideal, i.e. we get an ideal $J_m = \langle g_3, g_5, \ldots, g_{2m+1} \rangle$, such that $\mathbf{V}(J_m) = \mathbf{V}(J)$. Then we compute the radical of the ideal J_m. If the ideal J_m is a radical one, then this yields $J = J_m$ and, hence, we get m as an upper bound for the cyclicity of the singular point. However the more common situation is that J_m is not a radical ideal. We demonstrate that using our methods it is possible to find a basis of Bautin ideal (and, therefore, to resolve the cyclicity problem) also in some cases when the ideal is not radical one.

[1] Bautin, N. N. (1952): Mat. Sb. **30**, 396-413 (in Russian); (1954): Trans Amer. Math. Soc. **100**.
[2] Jarrah, A., Laubenbacher, R. and Romanovski, V. (2001): The center variety of polynomial differential systems. Preprint.
[3] Romanovski, V. G., Rauh, A. (1998): Dynamic Systems and Applications, **7**, Issue 4, 529-552.