Magnetic flows on Riemannian manifolds

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A magnetic field is given by a closed 2–form Ω on a Riemannian manifold, and magnetic geodesics are solutions of

$$\nabla_{\dot\gamma}\dot\gamma = Y_\gamma(\dot\gamma)$$

where the so–called Lorentz force is defined by $\Omega_x(v,w) = g_x(Y_x(v), w)$.

The aim of our work is to understand the dynamics of magnetic flows, in particular its relation to the underlying geodesic flow. We concentrate on the dynamical Lagrangian approach and study minimal magnetic geodesics. It is known from Mañé's work that there exists a critical energy level $c = c(\Omega)$ at which a significant change of behaviour in the dynamics takes place. Our main results are the following:

Magnetic geodesics are Riemannian quasi–geodesics For every energy value $k_0 > c$, there exists a constant $A > 1$ such that minimal magnetic geodesics of energy $k \geq k_0$ are Riemannian $(A, 0)$–quasi–geodesics, with $A \to 1$ as $k_0 \to \infty$. The example of a constant magnetic field on the hyperbolic plane shows that the boundary value c is sharp.

The Morse–Lemma for magnetic geodesics Assume that the manifold M has negative curvature. Then the classical Morse–Lemma states that (A, α)–quasi–geodesics lie in tubes around genuine geodesics. We prove here that the optimal width of these tubes goes to zero when (A, α) tends to $(1, 0)$. Although this might be intuitively clear we have not been able to locate this fact in the literature.

Then the fact that magnetic geodesics are quasi–geodesics implies the Morse–Lemma for magnetic geodesics, where the tubes can be chosen arbitrarily small if the magnetic field is weak enough (or, equivalently, the energy of the magnetic geodesic is sufficiently high).

The magnetic length is the Riemannian length Using his so–called action potential, Mañé defined for each energy value k a pseudo–metric on the universal cover $\widetilde{M}$ which becomes a genuine metric when $k > c$. This metric should encode properties of the magnetic field. Somewhat surprisingly, however, it turns out that the corresponding *length* does not contain any information about the magnetic field anymore—it is just the original Riemannian length. This follows from a careful estimate for the difference between Mañé's and the Riemannian metric.