Alexandrov-type and Minkowski-type theorems for polyhedral herissons

Victor Alexandrov

A closed strictly convex surface F in Euclidean space $\mathbb{R}^3$ may be considered as the envelope of its support planes $\langle x,p\rangle = h(p)$, $(p \in S^2)$. R. Langevin, G. Levitt, and H. Rosenberg [Banach Cent. Publ. 20, 245-253 (1988)] began to study such envelopes F_h for the case when thier "support" functions $h : S^2 \to \mathbb{R}$ are arbitrary functions of the class C^∞ and the corresponding Gauss maps are bijective. Note that, from this point of view, F_h need not be neither convex nor smooth. They call such objects herissons that means hedgehogs in French.

It turns out that herissons appear in the study of wave front singularities, minimal surfaces, and convex surfaces (including the theory of mixed volumes). See, for example, a paper by by Y. Martinez-Maure in [Adv. Math. 158, No.1, 1-17 (2001)] for more details.

The notion of a polyhedral herisson was introduced by L. Rodriguez and H. Rosenberg in the paper [Rigidity of certain polyhedra in $\mathbb{R}^3$. Comment. Math. Helv. 75, No.3, 478-503 (2000)] where they proved several rigidity theorems similar to the classical Cauchy–Alexandrov rigidity theorem for convex polyhedra.

¿From our point of view it is very natural to ask whether it is possible to develop a theory of herissons which will be similar to the theory of convex bodies? In the present talk we discuss three theorems on the polyhedral herissons which are similar to the following three well-known theorems for convex polyhedra in the Euclidean 3-space:

Uniqueness theorem (A. D. Alexandrov). If two convex polyhedra have mutually parallel faces and there is no a pair of parallel faces such that a face can be located inside the other face of the pair by a translation then the polyhedra are congruent and may be obtained one from another by a translation.

Uniqueness theorem (H. Minkowski). If two convex polyhedra have mutually parallel faces of pair-wise equal area then the polyhedra are congruent and may be obtained one from another by a translation.

Existence theorem (H. Minkowski). Let $n_1, \dots, n_m$ be non-coplanar unit vectors and $f_1, \dots, f_m$ be positive reals such that $\sum_{j=1}^m f_j n_j = 0$. Then there exists a convex closed polyhedron for which vectors n_j are unit normal vectors to faces and reals f_j are areas of its corresponding faces $(j = 1, \dots, m)$.