Integrability and Algebraic curve in AdS/CFT

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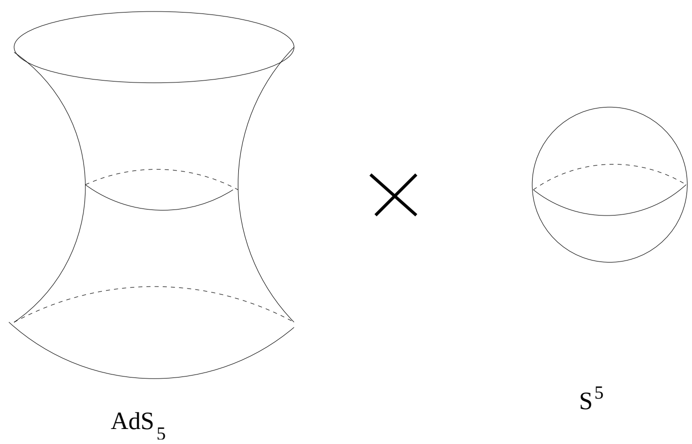


A. Marshakov, J. Minahan, K. Zarembo, V. K., hep-th/0402207
K. Zarembo, V. K., hep-th/0410105
N. Beisert, K. Sakai, V.K., hep-th/0410253
N. Beisert, K. Sakai, K. Zarembo, V.K., hep-th/0502226, hep-th/0503200

• **Motivation:** quest for integrable structure of superconformal $\mathcal{N} = 4$ Yang-Mills (SYM) theory (CFT) in $4D$ at $N \to \infty$, and its dual, string on $AdS_5 \times S^5$.

• **Main result:** full *algebraic curves* (=solutions) for classical string on $AdS_5 \times S^5$ and for long operators in perturbative SYM.
AdS/CFT correspondence: Curves coincide (at 1-loop)

Plan:

• *Algebraic curve* of classical superstring (sigma-model) on $AdS_5 \times S^5$ by the finite gap method.

• CFT (SYM): perturbative (1-loop) matrix of anomalous dimensions for ALL operators

$$\Downarrow$$

Hamiltonian of integrable spin chain on the superconformal group $\mathrm{PSU}(2,2|4)$

$$\Downarrow$$

algebraic Bethe ansatz

$$\Downarrow$$

Algebraic curve (=solution) in the "thermodynamical" limit of long operators.

History and References

- AdS:

[Maldacena'98]
[Gubser,Klebanov,Polyakov '98,'02]
[Berenstein,Maldacena Nastase '02]
[Metsaev,Tseytlin '02]
[Frolov,Tseytlin '02]
[Bena,Polchinski,Roiban,'02]...

- CFT:

[Lipatov,'94,'97]
[Faddeev,Korchemsky,'95]
[Minahan,Zarembo '02]
[Beisert,Kristjansen,Staudacher '02]
[Beisert, '03, Beisert,Staudacher '03]
[Beisert,Minahan,Staudacher,Zarembo '03]
[Serban,Staudacher '04] ...

- Integrability:

[H.Bethe'31,"Zur Theorie der Metalle", Z.Phys.,71,p.205]
[Zakharov,Shabat,Mikhailov, 70's-80's]
[Reshetikhin,Smirnov'83],
[Its,Matveew'81], [Krichever'81], [Kulish'85]...

CFT: $\mathcal{N}=4$ super Yang-Mills (SYM) theory,
$10D \Rightarrow 4D$ reduction of $\mathcal{N}=1$ SYM

$$S=\frac{N\mathrm{Tr}}{\lambda}\int_{\mathbf{R}_4}\mathcal{F}_{\mu\nu}^2+(\mathcal{D}_\mu\Phi_a)^2-2i\bar{\Psi}[\not{\mathcal{D}},\Psi]+[\Phi_a,\Phi_b]^2-2i\bar{\Psi}[\Gamma^a\Phi_a,\Psi]$$

- *Field content*: $\mathcal{D}_\mu=\partial_\mu+iA_\mu$, fermions Ψ,

Scalars: $X=\Phi_1+i\Phi_2$, $Y=\Phi_3+i\Phi_4$, $Z=\Phi_5+i\Phi_6$.

- Local single trace operators form superconf. algebra $psu(2,2|4)$: $\mathcal{O}=\mathrm{Tr}\ \mathcal{W}_{A_1}\mathcal{W}_{A_2}\mathcal{W}_{A_3}\dots\mathcal{W}_{A_L}$ with $\mathcal{W}_A\in\{\mathcal{D}^m\Phi,\mathcal{D}^m\Psi,\mathcal{D}^m\mathcal{F}\}$

Dual to:

AdS: Green-Schwarz-Metsaev-Tseytlin $10D$ superstring on $AdS_5\times S^5$ of the radius $R^2=\sqrt{\lambda}\alpha'$.

- Point-like string on $AdS_5\times S^5$ with ang. mom. $J \Leftrightarrow$ "vacuum" BPS state of SYM: $\mathcal{O}_V=|0;J\rangle=\mathrm{Tr}\ \ Z^J$.

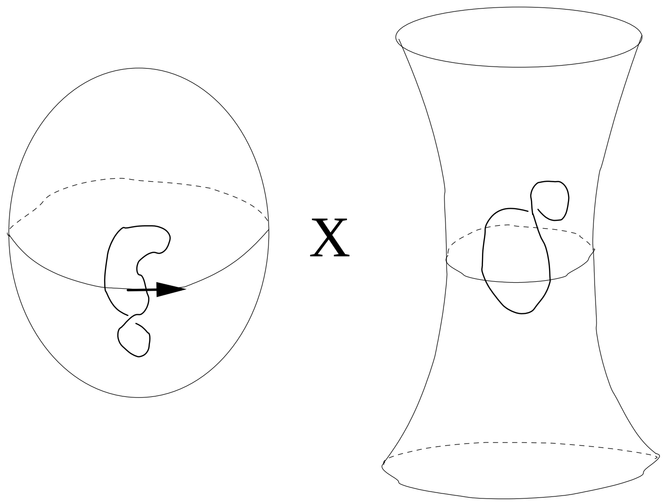
- Energy of classical string $\Leftrightarrow$ Dimension of a long SYM operator $J\sim J_1\sim\cdots\sim\infty$.

Frolov-Tseytlin'02

Motion of superstring on $AdS_5 \times S^5$

$\mathcal{N} = 4$ SYM is dual to:

AdS: Green-Schwarz-Metsaev-Tseytlin $10D$ superstring on $AdS_5 \times S^5$ of the radius $R^2 = \sqrt{\lambda}$:



$$\mathrm{AdS}_5 : \quad -X_{-1}^2 - X_0^2 + X_7^2 + X_8^2 + X_9^2 + X_{10}^2 = -R^2$$

$$S^5 : \quad X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2$$

• The radial coordinate z and the Lorentzian space-time x_μ of AdS are recovered from $X_{-1} + X_{10} = R/z$, $(X_0, X_7, X_8, X_9) = R\frac{x_\mu}{z}$, giving $ds^2 = R^2\,\frac{dx^2 + dz^2}{z^2}$.

Metsaev-Tseytlin superstring

• MT superstring: sigma model on the coset
$\mathrm{AdS}_5 \times \mathrm{S}^5 \sim \mathrm{PSU}(2,2|4)/\left(\mathrm{Sp}(2,2)\times \mathrm{Sp}(4)\right)$.
Supergroup element g is a $(4|4)\times(4|4)$ supermatrix

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \quad \in \mathrm{PSU}(2,2|4)$$

• Decompose: $J = -g^{-1}dg = H + Q_1 + P + Q_2$
$P \in AdS_5 \times S^5$, $H \in \mathrm{sp}(2,2)\times\mathrm{sp}(4)$, $Q_{1,2}$ fermions.

• $P = \frac{1}{4}\left(J + CJ^{st}C^{-1} + \eta J \eta + \eta C J^{st} C^{-1}\eta\right)$, etc.

where: $\eta = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1)$,

$$C = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & -iE \end{array}\right), \quad E = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)_{4\times 4}$$

• Natural Z_4 grading w.r.t. a const. matrix C
$C^{-1}(H, Q_1, P, Q_2)C = -(H^{st}, i\ Q_1^{st}, i^2\ P^{st}, i^3\ Q_2^{st})$

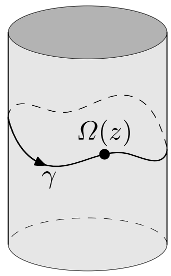
• Action: $\qquad S_{MT} = \frac{\sqrt{\lambda}}{4\pi}\,\mathrm{str}\int_{\mathcal{M}_2}\left[\mathrm{P}\wedge *\mathrm{P} - \mathrm{Q}_1\wedge\mathrm{Q}_2\right]$

Monodromy matrix for MT superstring

- All Bianchi identities and eqs. of motion (current conserv.) are packed into a Lax eq. [Bena et al.'02]:

$$(d+A(z))\wedge(d+A(z))=0,$$

$$A(z)=H+\frac{1}{2}\left(z^{-2}+z^{2}\right)P+\frac{1}{2}\left(z^{-2}-z^{2}\right)(*P)+z^{-1}Q_1+z\,Q_2$$



- Monodromy matrix:

$$\Omega(z)=P\exp\int_0^{2\pi}d\sigma\,A_\sigma(z)\simeq P\exp\oint A(z)$$

- Conserved quantities: eigenvalues of $\Omega(z)$

$$\mathrm{diag}\{e^{i\tilde{p}_1(z)},e^{i\tilde{p}_2(z)},e^{i\tilde{p}_3(z)},e^{i\tilde{p}_4(z)}||e^{i\hat{p}_1(z)},e^{i\hat{p}_2(z)},e^{i\hat{p}_3(z)},e^{i\hat{p}_4(z)}\}$$

- Super-unimodularity $\mathrm{sdet}\Omega(\mathrm{z})=\mathbf{1}$:
$\tilde{p}_1+\tilde{p}_2+\tilde{p}_3+\tilde{p}_4-\hat{p}_1-\hat{p}_2-\hat{p}_3-\hat{p}_4\in 2\pi\mathbb{Z}$

Algebraic curve of quasimomentum

- $y_k(z) = -izp'_k(z)$: good variables, having only:

- branch cuts at $\tilde{z}_i, \hat{z}_j$ where same grading e.v.'s cross;

- poles at z_j^* where opposite grading e.v.'s cross. Corresponding $(1|1) \times (1|1)$ sub-supermatrix of $\Omega(z)$

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) = u(z) \left(\begin{array}{c|c} \frac{bc}{a-d} + a & 0 \\ \hline 0 & \frac{bc}{a-d} + d \end{array}\right) u^{-1}(z)$$

- Finite gap solutions, a large, may be exhaustive class.

$\mathrm{diag}\{\tilde{y}_1(z), \tilde{y}_2(z), \tilde{y}_3(z), \tilde{y}_4(z) || \hat{y}_1(z), \hat{y}_2(z), \hat{y}_3(z), \hat{y}_4(z)\}$ are sheets of 8-sheeted Riemann surface of alg. curve $\mathcal{D}(y,z) = \mathrm{sdet}\left(y - \left[u(z)\left(-izp'(z)\right) u^{-1}(z)\right]\right) = \{\frac{0}{0}\}$

$$\mathcal{D}(y,z) \sim \frac{\tilde{F}_4(x)y^4 + \tilde{F}_3(x)y^3 + \tilde{F}_2(x)y^2 + \tilde{F}_1(x)y + \tilde{F}_0(x)}{\hat{F}_4(x)y^4 + \hat{F}_3(x)y^3 + \hat{F}_2(x)y^2 + \hat{F}_1(x)y + \hat{F}_0(x)} = \{\frac{0}{0}\}$$

where $x = (1+z^2)/(1-z^2)$.

- All singularities are encoded in

$$\tilde{F}_4(x) = x^4 \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^+) \prod_{a=1}^{2\tilde{A}} (x - \tilde{x}_a^-) \prod_{a=1}^{2A^*} (x - x_a^*)^2$$
$$\hat{F}_4(x) = x^4 \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^+) \prod_{a=1}^{2\hat{A}} (x - \hat{x}_a^-) \prod_{a=1}^{2A^*} (x - x_a^*)^2$$

Fixing moduli of the curve

- Symmetry $\Omega(1/x) = C\Omega^{-ST}(x)C^{-1}$: monodromy

$\tilde{y}_k, \hat{y}_k(1/x) = -\tilde{y}_{k'}, \hat{y}_{k'}(x), \qquad (1,2,3,4) \leftrightarrow (2,1,4,3)$

- Global charges: ang. momenta J_1, J_2, J_3,energy E, spins S_1, S_2, fix $x \to \infty$ (or $x \to 0$) asymptotics:

$$S^5: \quad \tilde{p}_1(x) = -\frac{2\pi}{\sqrt{\lambda}}(J_1 + J_2 - J_3)\frac{1}{x} + \ldots, \quad etc.$$
$$AdS^5: \quad \hat{p}_1(x) = \frac{2\pi}{\sqrt{\lambda}}(E + S_1 - S_2)\frac{1}{x} + \ldots, \quad etc.$$

- Energy E of state $\Leftrightarrow$ dimension of operator in SYM.

- Local charges: fixed by asymptotics $\delta = (x \pm 1) \to 0$:

$A(\epsilon^{\pm 1}) \to \frac{1}{2}\delta^{-2}(P \pm *P)$

and the Virasoro conditions: $\mathrm{str}(\mathrm{P} \pm *\mathrm{P})^2 = 0$

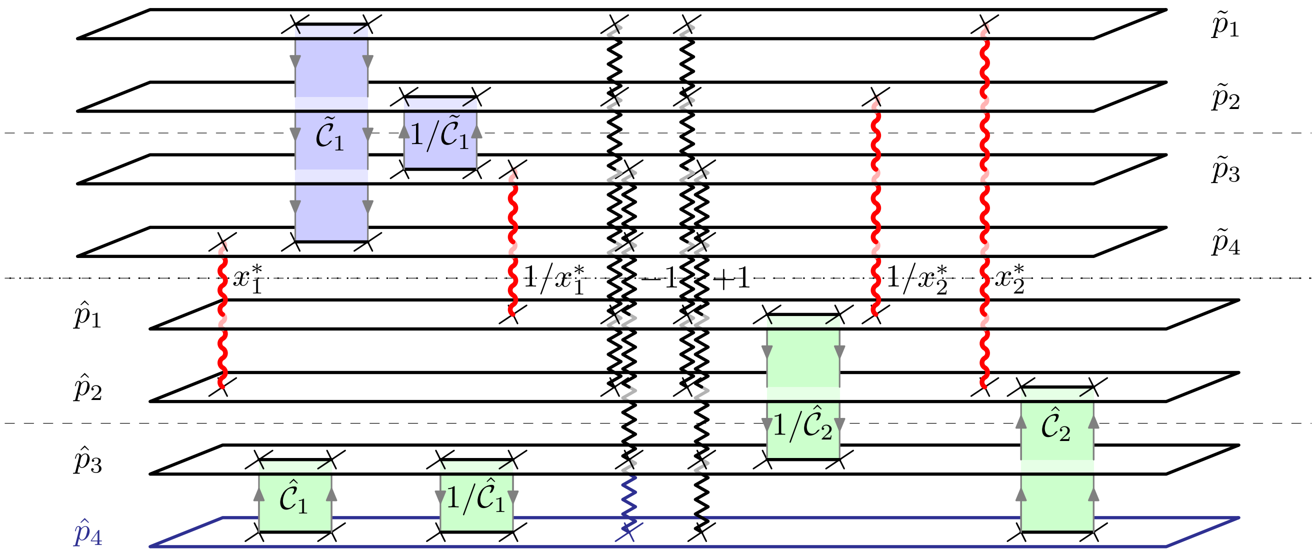
- On a cut C_{ij}: $p_i(x) - p_j(x) = p(x_+) + p(x_-) = 2\pi n$.

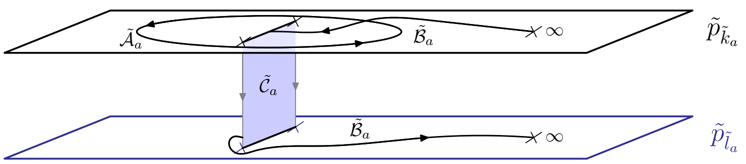
- From $\Omega(\infty) = \Omega(0) = I$ and the $x \to 1/x$ symmetry, for the total momentum on S^5:

$$\tilde{p}_{1,2}(0) = -\tilde{p}_{3,4}(0) = \int_\infty^0 d\tilde{p}_{1,2} = -\int_\infty^0 d\tilde{p}_{3,4} = 2\pi m$$

$\hat{p}_k(0) = \int_\infty^0 d\hat{p}_k = 0$ - no time windings on AdS_5.

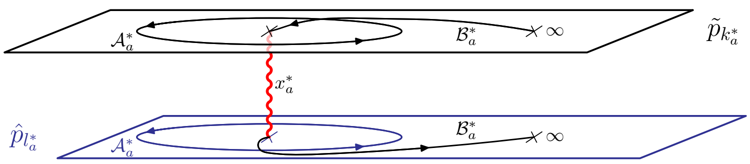
Riemann surface





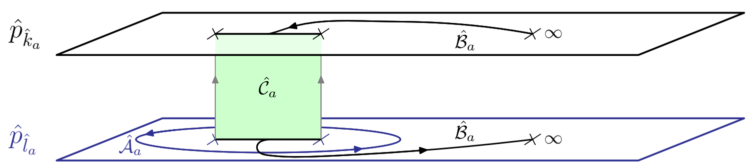
S^5 cuts

$$\int_{\tilde{B}_a} d\tilde{p} = 2\pi \tilde{n}_a$$



Fermionic poles

$$\int_{B_a^*} dp = 2\pi n_a^*$$



AdS_5 cuts

$$\int_{\hat{B}_a} d\hat{p} = 2\pi \hat{n}_a$$

Choice of A-cycles: $\oint_{\tilde{A}_a} d\tilde{p} = \oint_{\hat{A}_a} d\hat{p} = \oint_{A_a^*} dp = 0$

SYM: operators as spin chains

• Dilatation operator $\hat{D} = \hat{D}^{(0)} + \lambda \hat{D}^{(2)} + \lambda^2 \hat{D}^{(4)} + \ldots$ ($\lambda = N g_{YM}^2$ - 't Hooft coupling):

$$\mathcal{O}(x/\Lambda) = \Lambda^{\hat{D}} \mathcal{O}(x) = \Lambda^{\hat{D}^{(0)}} \left(1 + \lambda \log \Lambda \hat{D}^{(2)} + \ldots\right) \mathcal{O}(x)$$

• Conf. dimensions $\Delta = \Delta^{(0)} + \lambda \Delta^{(2)} + \lambda^2 \Delta^{(4)} + \ldots$ are eigenvalues of "hamiltonian" $\hat{D}$ ("time" $\sim \log \Lambda$).

• $\hat{D}$ computed by by perturbation theory (point splitting and renormalization) for some sectors [Lipatov'97, Minahan,Zarembo '02] and for superconformal group $PSU(2,2|4)$ (all operators of SYM) [Beisert,Staudacher'03].

Bethe Ansatz and "thermodynamical" limit of long operators

- Example: Bethe eq. for XXX chain:

$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{q=1}^{J_1}{}' \frac{u_p - u_q + \frac{i}{2}}{u_p - u_q - \frac{i}{2}}$$

- Thermodynamical limit: many, but rearely changing fields under trace: $L \to \infty$; roots are big: $u_p^{(j)} \sim L$:

$$|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\nearrow\nearrow\nearrow\nearrow\rightarrow\rightarrow\rightarrow\rightarrow\searrow\searrow\searrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\searrow\searrow\searrow\searrow\rangle$$

- Take $\log$ of both parts and expand in $1/u \sim 1/L$

$$2\pi n + \frac{1}{u_p} = \frac{1}{L}\sum_{q=1}^{J_1}{}' \frac{1}{u_p - u_q} + O\left(\frac{1}{L}\sum_{q=1}^{J_1}{}' \frac{1}{(u_p - u_q)^3}\right)$$

or
$$2\pi n_k + \frac{1}{x} = \int_{C_k} \frac{dy\, \rho(y)}{x - y}$$

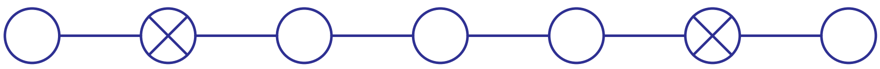
Riemann-Hilbert problem for the density of roots $\rho(x)$! Solved by hyperelliptic curve [Reshetikhin,Smirnov'83].

Bethe ansatz for 1-loop SYM

$$\left(\frac{u_p^{(j)}+\frac{i}{2}V_j}{u_p^{(j)}-\frac{i}{2}V_j}\right)^L=\prod_{j'=1}^{7}\prod_{\substack{q=1\\(j,p)\neq(j',q)}}^{K_{j'}}\frac{u_p^{(j)}-u_q^{(j')}+\frac{i}{2}M_{j,j'}}{u_p^{(j)}-u_q^{(j')}-\frac{i}{2}M_{j,j'}}$$

$$M_{j,j'}=\left(\begin{array}{c|c|ccc|c|c}
-2 & +1 & & & & & \\ \hline
+1 & 0 & -1 & & & & \\ \hline
 & -1 & +2 & -1 & & & \\
 & & -1 & +2 & -1 & & \\
 & & & -1 & +2 & -1 & \\ \hline
 & & & & -1 & 0 & +1 \\ \hline
 & & & & & +1 & -2
\end{array}\right),\quad V_j=\left(\begin{array}{c}
0\\ \hline 0\\ \hline 0\\ 1\\ 0\\ \hline 0\\ \hline 0
\end{array}\right)$$

- Global charges: determined by excitation numbers of 7 flavors (nodes of the $psu(2,2|4)$ Dynkin diagram):



$$0\le K_1\le K_2\le K_3\le K_4\ge K_5\ge K_6\ge K_7\ge 0,$$

- Vacuum with $K_j=0$ represents the state $\mathrm{tr}\ Z^L$.
- Local conserved charges

$$Q_r=\frac{i}{r-1}\sum_{p=1}^{K_4}\left(\frac{1}{(u_p^{(4)}-\frac{i}{2})^{r-1}}-\frac{1}{(u_p^{(4)}-\frac{i}{2})^{r-1}}\right)$$

- cyclicity of trace (zero total momentum): $Q_1=2\pi m$
- 1-loop anomalous dimension: $\delta E=(\lambda/8\pi^2)Q_2$

Stacks (bound states of roots of different flavors)

• "Standard" procedure for the thermodynamical limit does not give the complete result.

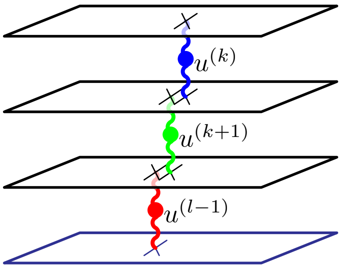
• Stacks are bound states of close roots of a few adjacent flavors: $u_p^{(j)} - u_p^{(j+1)} \sim O(L^0)$.

• Electrostatic analogy:
- bosonic roots $j = 1, 3, 4, 5, 7$ of same flavor repulse
- fermionic roots $j = 2, 6$ do not feel each other
- any roots of adjacent flavors attract, the reason for formation of stacks
- roots of the middle node $u_q^{(j=4)}$ are locked by the overall "potential" $V'(u) \sim 1/u + 2\pi n_4$.

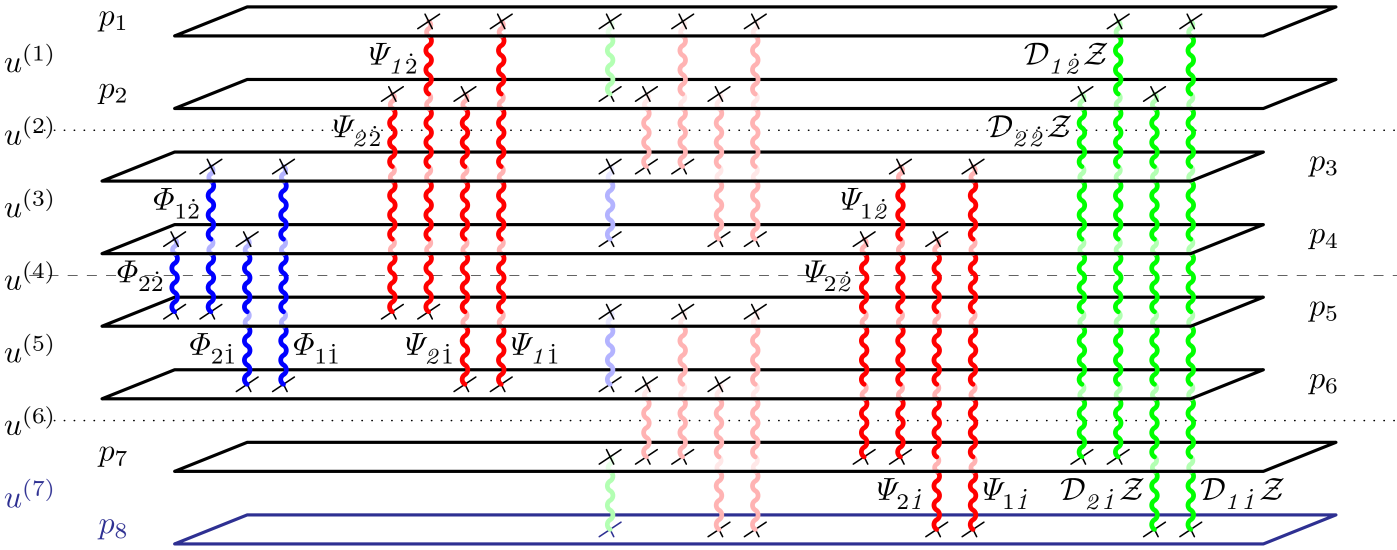
• Stacks containing $u_q^{(j=4)}$ form momentum and "energy" carrying states. Other states are "auxiliary".

• Riemann surface with 8 sheets (alge. curve): roots of j'th flavor form cuts connecting the $(j-1)$'s and the j's sheets. Stacks connect non-adjacent sheets.

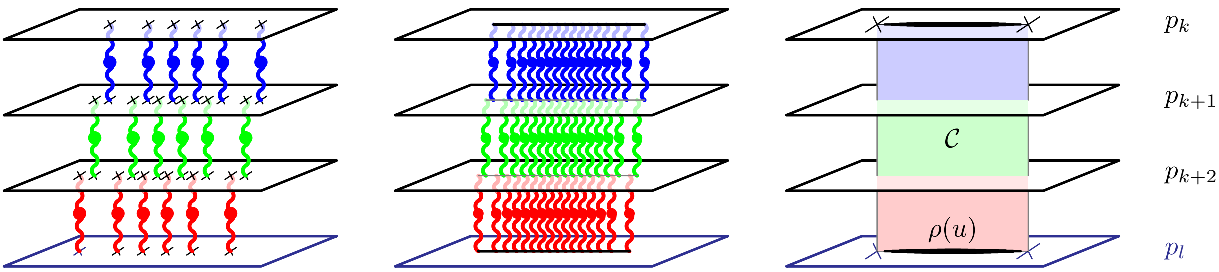
- A stack of three flavors:



- Dictionary between stacks and fields of SYM:



- Formation of cuts from strings of stacks:



Strings of stacks

• Macroscopic strings of stacks form linear supports (cuts $\mathcal{C}_{kl}$ on a Riemann surface) with density $\rho_{kl}(u)$. Resolvent a stack centered at $u^{(kl)} \sim L$

$$G_{kl}(u) = \sum_{p=1}^{K_{kl}} \frac{1}{u_p^{(kl)} - uL} = \int_{\mathcal{C}_{kl}} \frac{dv\, \rho_{kl}(v)}{v-u} \qquad 1 \le k < l \le 8$$

• 8 quasi-momenta: $p_k(u) = \frac{\epsilon_k}{2u} + \eta_k \sum_{l=1}^{8} G_{kl}(u)$,

where $G_{lk}(u) = -G_{kl}(u)$, $\eta_{1,2,7,8} = -1$, $\eta_{3,4,5,6} = 1$; $\epsilon_k = (+1, +1, +1, +1, -1, -1, -1, -1)$.

• Bethe equations: a stack makes a tour around the chain, its scattering phase after interaction with all other stacks is $2\pi n$, due to the periodicity:

$$\not{p}_l(u) - \not{p}_k(u) = 2\pi n_{kl,a}, \qquad \text{for } u \in \mathcal{C}_{kl,a}.$$

Equivalent to eqs. for the string: $\oint_{B_{kl}} dp = 2\pi n_{kl,a}$

Local and Global Charges

- Define the sum of momentum carrying resolvents
$$G_{\mathrm{mom}}(u) = \sum_{k=1}^{4} \sum_{l=5}^{8} G_{kl}(u)$$
- Trace cyclicity: $G_{\mathrm{mom}}(0) = 2\pi m$
- 1-loop anomalous dimension: $\delta E = \frac{\lambda}{8\pi^2 L} G'_{\mathrm{mom}}(0)$
- Local conserved charges:
$$G_{\mathrm{mom}}(u) = \sum_{r=1}^{\infty} (Lu)^{r-1} Q_r$$
- Global charges from asymptotics $\tilde{p}_k(u)$ at $u = \infty$:

$\mathfrak{so}(6)$: $\quad \tilde{p}_1(u) = \frac{1}{2Lu}(+J_1 + J_2 - J_3) + \ldots$

$\mathfrak{so}(2,4)$: $\quad \hat{p}_1(u) = -\frac{1}{2Lu}(+E + S_1 - S_2) + \ldots$

Curve of SYM: Comparison to String

• The curve for $y(u) = u^2 p'(u)$ is similar to the one for $AdS_5 \times S_5$, but there are some differences.

• By analogy with SYM define for string:

"filling fractions": $\tilde{K}_a = -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\tilde{\mathcal{A}}_a} dx \left(1 - \frac{1}{x^2}\right) \tilde{p}_{\tilde{k}_a}(x)$
"length": $mL = \sum_{a=1}^{\tilde{A}} \tilde{n}_a \tilde{K}_a$

• Frolov-Tseytlin (FT) limit for the string sigma model: $\sqrt{\lambda}/L \to 0$ and rescale $u = \frac{\sqrt{\lambda}}{4\pi L} x$. Then the string pole term is same as in SYM:

$$\frac{u}{u^2 - \lambda/(16\pi^2 L^2)} \simeq \frac{1}{u}$$

• Half of the cuts stay finite and their $x \leftrightarrow 1/x$-mirrors move to $u = 0$ and can be neglected.
• Energy, zero mom. condition, definition of filling fractions and of length coincide with SYM in FT limit.

The algebraic curves of string on $AdS_5 \times S_5$ and of the $\mathcal{N} = 4$ SYM coincide in this limit by the appropriate identification of parameters, i.e. they probably describe the same state (AdS/CFT correspondence at 1-loop).

Conclusions

• Full explicit solution of classical string on $AdS_5 \times S^5$ and for operators in 1-loop $\mathcal{N} = 4$ SYM in terms of their algebraic curves. Fermions included.

• Equivalence of AdS/CFT curves in 1-loop/FT limit.

• Probably, equivalence at 2 loops as well (known in $SO(6)$, $SL(2)$ and $SU(2|1)$ sectors).

• Discrepancy at ≥ 3 loops. *Possible reason*: non-commutativity of $1 << \lambda << L^2$ and $\lambda << 1 << L^2$ limits in the full theory?

Prospects

• Find an integrable model incorporating the full quantum theory of the string sigma model. The $\lambda \to 0$ limit should reproduce one-loop SYM results. Hopefully it amounts to the exact solution of $\mathcal{N} = 4$ SYM at large N.