Correlation functions of the XXZ spin chain, $\Delta = \frac{1}{2}$ case.

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The XXZ spin-1/2 Heisenberg chain

1. Hamiltonian

$$H = \sum_{m=1}^{M} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) \right) - \frac{h}{2} \sum_{m=1}^{M} \sigma_m^z \quad \Delta \text{ - anisotropy}$$

h- external magnetic field.

Periodic boundary conditions: $\sigma_{M+1} = \sigma_1$

Special point: $\Delta = \frac{1}{2}$, $h = 0$ (massless regime)

2. Correlation functions

- Two-point functions

$$f_{aa}(m) = \langle \psi_g | \sigma_{m+1}^a \sigma_1^a | \psi_g \rangle, \quad a = x, y, z$$

- Generating function

$$Q_m(e^\beta) = \langle \psi_g | \exp \left(\beta \sum_{j=1}^{m} \frac{1}{2} (1 - \sigma_j^z) \right) | \psi_g \rangle,$$

$$f_{zz}(m) = \left(2\mathcal{D}_m^2 \frac{\partial^2}{\partial \beta^2} - 4\mathcal{D}_m \frac{\partial}{\partial \beta} + 1\right) Q_m(e^\beta)\Big|_{\beta=0}$$

We denote $\kappa = e^\beta$

- Emptiness formation probability

$$\tau(m) = \lim_{\kappa \to \infty} \kappa^{-m} Q_m(\kappa)$$

Multiple integral representations for the elementary blocks (massless case):

$\star$ 1996 Jimbo and Miwa $\longrightarrow$ from qKZ equation

$\star$ 1999 N.K., J.M. Maillet, V. Terras $\longrightarrow$ from Algebraic Bethe Ansatz

Algebraic Bethe Ansatz

L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan (1979):

monodromy matrix:

$$T(\lambda) = L_{aM}(\lambda - \xi_M) \ldots L_{a1}(\lambda - \xi_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[a]}$$

with $$L_{an}(\lambda) = \begin{pmatrix} \sinh(\lambda + i\frac{\zeta}{2}\sigma_n^z) & i\sin\zeta\,\sigma_n^- \\ i\sin\zeta\,\sigma_n^+ & \sinh(\lambda + i\frac{\zeta}{2}\sigma_n^z) \end{pmatrix}_{[a]}, \quad \Delta = \cos\zeta.$$

$\hookrightarrow$ **Yang-Baxter algebra**: ∘ generators $A,\ B,\ C,\ D$

∘ commutation relations given by the R-matrix of the model

$$R_{ab}(\lambda,\mu)\,T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)\,R_{ab}(\lambda,\mu)$$

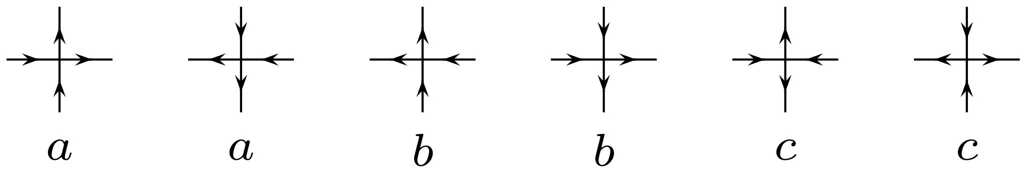
$\longrightarrow$ commuting conserved charges: $\mathcal{T}(\lambda) = A(\lambda) + D(\lambda)$

$\longrightarrow$ construction of the space of states by action of B (creation) and C (annihilation) on a reference state $|0\rangle \equiv |\uparrow\uparrow \ldots \uparrow\rangle$
eigenstates : $|\psi\rangle = \prod_k B(\lambda_k)|0\rangle$ with $\{\lambda_k\}$ solution of the Bethe equations.

$$\prod_{a=1}^{M} \frac{\sinh(\lambda_j - \xi_a - i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_a + i\frac{\zeta}{2})} \prod_{k=1}^{N} \frac{\sinh(\lambda_j - \lambda_k + i\zeta)}{\sinh(\lambda_k - \lambda_j - i\zeta)} = -1,$$

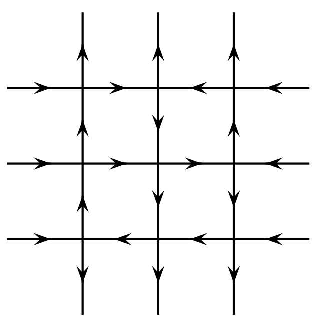
Six vertex model

1. Rectangular lattice $N \times M$, six possible arrow configurations with their statistical weights:



2. Boundary conditions:

1. Periodic boundary conditions,
2. Domain wall boundary condition (DWBC), (**square lattice** $M \times M$)



3. Partition function

$$Z_M = \sum_{configurations} a^{n_a} b^{n_b} c^{n_c}$$

4. Inhomogeneous model

$$a_{jk} = \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2}), \quad b_{jk} = \sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}), \quad c_{jk} = i\sin\zeta$$

5. Ice model

$$a = b = c = 1$$

$\longrightarrow$ corresponds to the point $\Delta = \frac{1}{2}$, $\zeta = \frac{1}{2}$

$Z_M =$ number of configurations

Partition function

Partition function, inhomogeneous case:

$$Z_M(\{\lambda_j\},\{\xi_k\}) = \langle 0'|B(\lambda_1)B(\lambda_2)\dots B(\lambda_M)|0\rangle$$

$|0\rangle$ ($|0'\rangle$)- ferromagnetic states with all spins up (down).

This function is defined by a set of recursion relations (Korepin 1982)

Determinant formula (Izergin 1987)

$$Z_M(\{\lambda_j\},\{\xi_k\}) = \frac{\prod\limits_{j=1}^{M}\prod\limits_{k=1}^{M}\sinh(\lambda_j-\xi_k+i\frac{\zeta}{2})\sinh(\lambda_j-\xi_k-i\frac{\zeta}{2})}{\prod\limits_{j>k}\sinh(\lambda_j-\lambda_k)\sinh(\xi_k-\xi_j)}\det_M \mathcal{M}$$

$$\mathcal{M}_{jk} = \frac{i\sin\zeta}{\sinh(\lambda_j-\xi_k+i\frac{\zeta}{2})\sinh(\lambda_j-\xi_k-i\frac{\zeta}{2})}$$

Alternating sign matrices

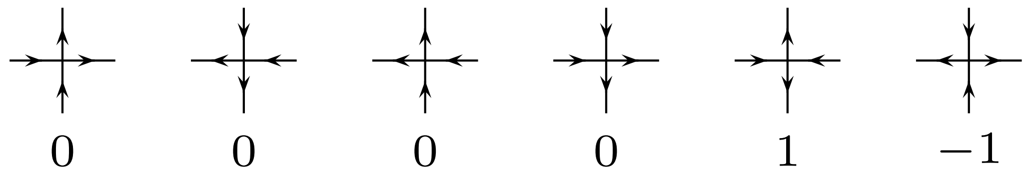
An *alternating sign matrix* (ASM) is a matrix of 0's, 1's, and -1's such that the non-zero elements in each row and column alternate between 1 and -1 and begin and end with 1, for example:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$A(n)$ - number of ASM $n \times n$
Theorem (Zeilberger 1996):

$$A(n) = \frac{1!4!7!\dots(3n-2)!}{n!(n+1)!(n+2)!\dots(2n-1)!}$$

Six vertex correspondence (Kuperberg 1997)



Square ice model with domain wall boundary conditions:

$$A(n) = Z_n$$

Calculation of the Izergin determinant in the ice point $\zeta = \frac{\pi}{3}$ and in the homogeneous limit: $\lambda_j \to i\frac{\pi}{2}$, $\xi_k \to 0$.

Razumov-Stroganov Conjecture

Emptiness formation probability in the thermodynamic limit: $M \to \infty$:

$$\tau(m) = \langle \psi_g | \prod_{j=1}^{m} \left(\frac{1}{2}(1 - \sigma_j^z) \right) | \psi_g \rangle$$

Conjecture (Razumov, Stroganov 2000)

$$\tau(m) = \left(\frac{1}{2} \right)^{m^2} A(m)$$

Asymptotic behavior $m \to \infty$:

$$\tau(m) \to c \left(\frac{\sqrt{3}}{2} \right)^{3m^2} m^{-\frac{5}{36}}, \quad c = \exp \left[\int_0^\infty \left(\frac{5e^{-t}}{36} - \frac{\sinh \frac{5t}{12} \sinh \frac{t}{12}}{\sinh^2 \frac{t}{2}} \right) \frac{dt}{t} \right].$$

Method: ABA + solution of the inverse problem

Compute $\langle\psi_g|\,\prod_j \sigma_j^{\alpha_j}\,|\psi_g\rangle$?

1. Diagonalise the Hamiltonian using ABA

$\longrightarrow$ key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$

$\longrightarrow$ eigenstates: $B(\lambda_1)\ldots B(\lambda_n)|0\rangle$

2. Act with local operators on eigenstates

$\longrightarrow$ problem: relation between B (creation) and σ_j^α a priori very complicated !

$\longrightarrow$ solve the quantum inverse problem:

$$\sigma_j^{\alpha_j} = f_j^{\alpha_j}(A, B, C, D) = \prod(A, B, C, D)$$

$\longrightarrow$ use Yang-Baxter commutation relations

3. Compute the resulting scalar products

4. Thermodynamic limit

$\longrightarrow$ elementary building blocks of correlation functions as multiple integrals

5. Further resummations. . .

Emptiness formation probability

$$\tau(m) \equiv \langle\psi_g| \prod_{k=1}^{m} \frac{1-\sigma_k^z}{2} |\psi_g\rangle$$

$$= (-1)^m \Big(-\frac{\pi}{\zeta}\Big)^{\frac{m(m-1)}{2}} \int_{-\infty}^{\infty} \frac{d^m\lambda}{2\pi} \prod_{a>b}^{m} \frac{\sinh\frac{\pi}{\zeta}(\lambda_a-\lambda_b)}{\sinh(\lambda_a-\lambda_b-i\zeta)}$$

$$\times \prod_{j=1}^{m} \frac{\sinh^{j-1}(\lambda_j-i\zeta/2)\sinh^{m-j}(\lambda_j+i\zeta/2)}{\cosh^m\frac{\pi}{\zeta}\lambda_j}$$

Integral representation as a single elementary block but previous expression not symmetric

$\longrightarrow$ **symmetrisation** of the integrand:

$$\tau(m) = \lim_{\xi_1,\ldots\xi_m \to -\frac{i\zeta}{2}} \frac{1}{m!} \int_{-\infty}^{\infty} d^m\lambda \prod_{a,b=1}^{m} \frac{1}{\sinh(\lambda_a - \lambda_b - i\zeta)}$$
$$\times \prod_{a<b}^{m} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(\xi_a - \xi_b)} \cdot Z_m(\{\lambda\},\{\xi\}) \cdot \det{}_m(\rho(\lambda_j - \xi_k))$$

where $Z_m(\{\lambda\},\{\xi\})$ is the partition function of the 6-vertex model with domain wall boundary conditions and $\rho(\lambda,\xi) = [-2i\zeta \sinh \frac{\pi}{\zeta}(\lambda_j - \xi_k)]^{-1}$ is the inhomogeneous version of the density for the ground state

Exact computation for $\Delta = 1/2$

$$\tau_{inh}(m,\{\xi_j\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_b-\xi_a)}{\sinh(\xi_b-\xi_a)} \prod_{\substack{a,b=1\\ a\neq b}}^{m} \frac{1}{\sinh(\xi_a-\xi_b)} \cdot \det_m \left(\frac{3\sinh\frac{\xi_j-\xi_k}{2}}{\sinh\frac{3(\xi_j-\xi_k)}{2}} \right).$$

Exact result:

$$\tau(m) = \left(\frac{1}{2}\right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!} = \left(\frac{1}{2}\right)^{m^2} A_m$$

$\longrightarrow$ A_m - number of alternating sign matrices

$\longrightarrow$ first exact result for $\Delta \neq 0$

Generating function

$$Q_m(\kappa) = \langle\psi_g| \prod_{k=1}^{m} \left(\frac{1-\kappa}{2} + \frac{1+\kappa}{2}\sigma_k^z \right) |\psi_g\rangle$$

Polynom of κ

$$Q_m(\kappa) = \sum_{s=0}^{m} \kappa^s G_s(m).$$

$$G_s(m) = \frac{1}{s!(m-s)!\sin^m \zeta} \prod_{j<k} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \dots \int_{-\infty}^{\infty} d\lambda_m \times$$

$$\times\, Z_m(\{\lambda\}|\{\xi\})\ \det{}_m(\rho(\{\lambda\},\{\xi\})) \times \Theta_s(\{\lambda\}).$$

where $Z_m(\{\lambda\},\{\xi\})$ is the partition function of the 6-vertex model with domain wall boundary conditions

$$\Theta_m^s(\lambda_1,\dots,\lambda_m)=\prod_{k=1}^{s}\prod_{j=s+1}^{m}\frac{1}{\sinh(\lambda_j-\lambda_k)}\prod_{m\geqslant j>k>s}\frac{\sinh(\lambda_j-\lambda_k)}{\sinh(\lambda_j-\lambda_k+i\zeta)\sinh(\lambda_j-\lambda_k-i\zeta)}$$
$$\times\prod_{s\geqslant j>k\geqslant 1}\frac{\sinh(\lambda_j-\lambda_k)}{\sinh(\lambda_j-\lambda_k+i\zeta)\sinh(\lambda_j-\lambda_k-i\zeta)}$$

Generating function. $\Delta = \frac{1}{2}$.

Inhomogeneous case:

$$\langle Q_\kappa(m)\rangle = \frac{3^m}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_a-\xi_b)}{\sinh^3(\xi_a-\xi_b)} \sum_{n=0}^{m} \kappa^{m-n} \sum_{\substack{\{\xi\}=\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\}\\ |\gamma_+|=n}} \det{}_m \hat{\Phi}^{(n)}$$

$$\times \prod_{a\in\gamma_+}\prod_{b\in\gamma_-} \frac{\sinh(\xi_b-\xi_a-\frac{i\pi}{3})\sinh(\xi_a-\xi_b)}{\sinh^2(\xi_b-\xi_a+\frac{i\pi}{3})},$$

with

$$\hat{\Phi}^{(n)}(\{\xi_{\gamma_+}\},\{\xi_{\gamma_-}\}) = \left(\begin{array}{c|c} \Phi(\xi_j-\xi_k) & \Phi(\xi_j-\xi_k-\frac{i\pi}{3}) \\ \hline \Phi(\xi_j-\xi_k+\frac{i\pi}{3}) & \Phi(\xi_j-\xi_k) \end{array}\right), \quad \Phi(x) = \frac{\sinh\frac{x}{2}}{\sinh\frac{3x}{2}}.$$

Homogeneous limit

$$\langle Q_\kappa(m)\rangle = \frac{(-1)^{\frac{m^2-m}{2}}3^m}{2^{m^2}m!}\prod_{a>b}^{m}\frac{\sinh 3(\xi_a-\xi_b)}{\sinh(\xi_a-\xi_b)}\sum_{n=0}^{m}\kappa^{m-n}C_m^n\oint\limits_{\Gamma\{\xi-\frac{i\pi}{6}\}}\frac{d^n z}{(2\pi i)^n}\oint\limits_{\Gamma\{\xi+\frac{i\pi}{6}\}}\frac{d^{m-n}z}{(2\pi i)^{m-n}}$$

$$\times\prod_{b=1}^{m}\left\{\prod_{j=1}^{n}\frac{1}{\sinh(z_j-\xi_b+\frac{i\pi}{6})}\prod_{j=n+1}^{m}\frac{1}{\sinh(z_j-\xi_b-\frac{i\pi}{6})}\right\}$$

$$\times\prod_{a=1}^{n}\prod_{b=n+1}^{m}\frac{\sinh(z_a-z_b-\frac{i\pi}{3})\sinh(z_a-z_b+\frac{i\pi}{3})}{\sinh^2(z_a-z_b)}\cdot\det_m\Phi(z_j-z_k). \quad (1)$$

Here the integration contours $\Gamma\{\xi\mp\frac{i\pi}{6}\}$ surround the points $\{\xi-\frac{i\pi}{6}\}$ for $z_1,\ldots,z_n$ and $\{\xi+\frac{i\pi}{6}\}$ for $z_{n+1},\ldots,z_m$ respectively.

Numerical results

If the lattice distance m is not too large, the representations can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely.
First results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 6$:

$$P_1(\kappa) = 1 + \kappa,$$

$$P_2(\kappa) = 2 + 12\kappa + 2\kappa^2,$$

$$P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,$$

$$P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4$$

$$P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,$$

$$P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 + 96289380\kappa^5 + 7436\kappa^6,$$

Numerical results, two-point functions

$$\begin{aligned}
\langle\sigma_1^z\sigma_2^z\rangle =& -2^{-1},\\
\langle\sigma_1^z\sigma_3^z\rangle =& 7\cdot 2^{-6},\\
\langle\sigma_1^z\sigma_4^z\rangle =& -401\cdot 2^{-12},\\
\langle\sigma_1^z\sigma_5^z\rangle =& 184453\cdot 2^{-22},\\
\langle\sigma_1^z\sigma_6^z\rangle =& -95214949\cdot 2^{-31},\\
\langle\sigma_1^z\sigma_7^z\rangle =& 1758750082939\cdot 2^{-46},\\
\langle\sigma_1^z\sigma_8^z\rangle =& -30283610739677093\cdot 2^{-60},\\
\langle\sigma_1^z\sigma_9^z\rangle =& 5020218849740515343761\cdot 2^{-78}.
\end{aligned}$$