Gaston Darboux and comte de Sparre: retrospective.

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Introduction

Darboux-Verdier-Treibich model, (first discovered by Darboux in 1882 and rediscovered by Treibich and Verdier in 1990), represents the remarkable particular example of the finite-gap Schrödinger operator with elliptic potential labeled by 4 integers. Sparre differential equation found in 1883 represents the integrable second order linear differential equation with elliptic coefficients labeled by by 4 integers and 3 arbitrary real parameters. Setting these 3 parameters to be equal to zero we recover the Darboux-Verdier-Treibich operator. Here we show that, in a sense, Sparre equation considered before as a generalization of Darboux equation, in fact, is naturally isomorphic to the later: Darboux-Verdier-Treibich equation represents a canonical form of the Sparre model.

COMTE de SPARRE EQUATION

$$\frac{d^2y}{dx^2} + \left[2\nu \frac{k^2 \operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} + 2\nu_1 \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x} - 2\nu_2 \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x}\right] \frac{dy}{dx} =$$
$$= \Big[\frac{1}{\operatorname{sn}^2 x}(n_3 - \nu_2)(n_3 + \nu_2 + 1) +$$
$$\frac{\operatorname{dn}^2 x}{\operatorname{cn}^2 x}(n_2 - \nu_1)(n_2 + nu_1 + 1) +$$
$$+ \frac{k^2 \operatorname{cn}^2 x}{\operatorname{dn}^2 x}(n_1 - \nu)(n_1 + \nu + 1) +$$
$$k^2 \operatorname{sn}^2 x(n + \nu + \nu_1 + \nu_2)(n - \nu - \nu_1 - \nu_2 + 1) + h\Big]\, y,$$

Here n, n_1, n_2, n_3 are four integers and $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ are the standard Jacobi elliptic functions. The parameters ν, ν_1, ν_2 are the complex numbers choosen in arbitrary way.

Particular cases of the Sparre equation were solved respectively by Lamé, Hermite , Picard and Darboux.

LAMÉ equation:

$$y'' = [n(n+1)k^2 \,\mathrm{sn}^2 x + h]y.$$

PICARD equation:

$$y'' + n\frac{k^2 \,\mathrm{sn}\, x \,\mathrm{cn}\, x}{\mathrm{dn}\, x}y' + \alpha y = 0,$$

HERMITE equations:

$$y'' + 2(\nu+1)\frac{k^2 \,\mathrm{sn}\, x \,\mathrm{cn}\, x}{\mathrm{dn}\, x}y' = [(n-\nu)(n+\nu+1)k^2 \,\mathrm{sn}^2 x + h]y,$$

$$y'' + 2(\nu+1)\frac{\mathrm{sn}\, x \,\mathrm{dn}\, x}{\mathrm{cn}\, x}y' = [(n-\nu)(n+\nu+1)k^2 \,\mathrm{sn}^2 x + h]y,$$

$$y'' + 2(\nu+1)\frac{\mathrm{cn}\, x \,\mathrm{dn}\, x}{\mathrm{sn}\, x}y' = [(n-\nu)(n+\nu+1)k^2 \,\mathrm{sn}^2 x + h]y,$$

In 1882 DARBOUX obtained a 4-integer dependent extension of the Lamé equation:

$$\frac{d^2y}{dx^2} = \Big[\frac{\mu(\mu+1)}{\mathrm{sn}^2 x} + \frac{\mu'(\mu'+1)}{\mathrm{cn}^2 x}\,\mathrm{dn}^2 x + \frac{\mu''(\mu''+1)}{\mathrm{dn}^2 x}k^2 \,\mathrm{cn}^2 x + n(n+1)k^2 \,\mathrm{sn}^2 x + h\Big]\, y.$$

It is obvious that all equations listed before may be obtained as a trivial reductions of the SPARRE equation simply setting some parameters to be equal to zero.

SOLUTION TO THE SPARRE EQUATION

It is enough to consider the case $n, n_1, n_2, n_3 \geq 0$. In this case (Sparre 1883) the general solution reads

$$y(x) =$$

$$= \mathrm{dn}^{\nu}\, x\, \mathrm{cn}^{\nu_1}\, x\, \mathrm{sn}^{\nu_2}\, x \left[A \frac{H(x-a_1)H(x-a_2)\ldots H(x-a_\beta)}{H^{n_3}(x)H_1^{n_2}(x)\Theta_1^{n_1}(x)\Theta^{n}(x)} e^{Cx/2} \right.$$

$$\left. +B \frac{H(x+a_1)H(x+a_2)\ldots H(x+a_\beta)}{H^{n_3}(x)H_1^{n_2}(x)\Theta_1^{n_1}(x)\Theta^{n}(x)} e^{-Cx/2}, \right.$$

$\beta := n + n_1 + n_2 + n_3,$

Here A B — are arbitrary constants, $H(x)$, $H_1(x)$. $\Theta(x)$, $\Theta_1(x)$ — are elliptic Jacobi Theta functions.Constant C is defined by the formula

$$\frac{C}{2} = \sum_{i=1}^{\beta} \frac{H'(a_i)}{H(a_i)} = \sum_{i=1}^{\beta} \frac{H_1'(a_i)}{H_1(a_i)} = \sum_{i=1}^{\beta} \frac{\Theta'(a_i)}{\Theta(a_i)} = \sum_{i=1}^{\beta} \frac{\Theta_1'(a_i)}{\Theta_1(a_i)}.$$

a_i — are the solutions of the system of transcendental equations written below:*

*Indeed, all of the a_i are the functions depending on the modular parameter k

$$\sum_{i=1}^{\beta} (\mathrm{sn}\, a_i)^{2j-1}\, \mathrm{cn}\, a_i\, \mathrm{dn}\, a_i = 0, \qquad -n_3+2 \le j \le n-1;$$

$$\sum_{i=1}^{\beta} \frac{(\mathrm{cn}\, a_i)^{2j-1}}{(\mathrm{dn}\, a_i)^{2j+1}}\, \mathrm{sn}\, a_i = 0, \qquad 1 \le j \le n_1 - 1;$$

$$\sum_{i=1}^{\beta} \frac{(\mathrm{dn}\, a_i)^{2j-1}}{(\mathrm{cn}\, a_i)^{2j+1}}\, \mathrm{sn}\, a_i = 0, \qquad 1 \le j \le n_2 - 1;$$

The above system * contains $\beta-1$ equations, it should be completed by the equation relating the spectral parameter h with a_i which can be written in 4 equivalent forms given on the next slide.

*including also the equations on the previous slide

Spectral parameter h can be expressed by means of the modular parameter k using one of the following, (equivalent), formulas:

$$(2n-1)\sum_{i=1}^{\beta} k^2 \,\mathrm{sn}^2\, a_i - (n+n_1+\nu_1+\nu_2)(n+n_1-\nu_1-\nu_2) - \\ - k^2(n+n_2+\nu+\nu_2)(n+n_2-\nu-\nu_2) = h;$$

$$(2n_3-1)\sum_{i=1}^{\beta} k^2 \,\mathrm{sn}^2(a_i+iK') - (n_3+n_2+\nu_1+\nu_2)(n_3+n_2-\nu_1-\nu_2) - \\ - k^2(n_3+n_1+\nu+\nu_2)(n_3+n_1-\nu-\nu_2) = h;$$

$$(2n_2-1)\sum_{i=1}^{\beta} k^2 \,\mathrm{sn}^2(a_i+K+iK') - \\ - (n_2+n_3+\nu_1+\nu_2)(n_2+n_3-\nu_1-\nu_2) - \\ - k^2(n_2+n+\nu+\nu_2)(n_2+n-\nu-\nu_2) = h;$$

$$(2n_1-1)\sum_{i=1}^{\beta} k^2 \,\mathrm{sn}^2(a_i+K) - (n_1+n+\nu_1+\nu_2)(n_1+n-\nu_1-\nu_2) - \\ - k^2(n_1+n_3+\nu+\nu_2)(n_1+n_3-\nu-\nu_2) = h;$$

SIMPLEST EXAMPLES

a. $n = n_1 = n_2 = n_3 = 0$,

$$y(x) = \mathrm{dn}^{\nu} x\, \mathrm{cn}^{\nu_1} x\, \mathrm{sn}^{\nu_2} x \left[Ae^{C_1 x} + Be^{-C_1 x} \right],$$

where

$$C_1^2 = h - (\nu_1 + \nu_2)^2 - k^2(\nu + \nu_2)^2.$$

b. $n = 1$, $n_1 = n_2 = n_3 = 0$:

$$y(x) = \mathrm{dn}^{\nu} x\, \mathrm{cn}^{\nu_1} x\, \mathrm{sn}^{\nu_2} x \left[A\frac{H(x-a)}{\Theta(x)} e^{\frac{\Theta'(a)}{\Theta(a)}x} + B\frac{H(x+a)}{\Theta(x)} e^{-\frac{\Theta'(a)}{\Theta(a)}x} \right],$$

where a is a solution of the following equation:

$$\mathrm{sn}^2(a) = \frac{h - (\nu_1+\nu_2+1)(\nu_1+\nu_2-1) - k^2(\nu+\nu_2+1)(\nu+\nu_2-1)}{k^2}.$$

c. $n_1 = 1$, $n = n_2 = n_3 = 0$:

$$y(x) = \mathrm{dn}^{\nu} x\, \mathrm{cn}^{\nu_1} x\, \mathrm{sn}^{\nu_2} x \left[A\frac{H(x-a)}{\Theta_1(x)} e^{\frac{\Theta_1'(a)}{\Theta_1(a)}x} + B\frac{H(x+a)}{\Theta_1(x)} e^{-\frac{\Theta_1'(a)}{\Theta_1(a)}x} \right],$$

where a is defined by the formula:

$$\mathrm{sn}^2(a+K) = \frac{h - (\nu_1+\nu_2+1)(\nu_1+\nu_2-1) - k^2(\nu+\nu_2+1)(\nu+\nu_2-1)}{k^2}.$$

The solutions to the Sparre equation in the case where $n_3 = 1$, $n = n_1 = n_2 = 0$ (or $n_2 = 1$, $n = n_1 = n_3 = 0$) are described by the formulae similar to the case b. where all the functions except the factor near the square bracket , are transformed following the rule

$$x \to x + iK', \quad a \to a + iK', \quad (x \to x + K + iK', \quad a \to a + K + iK').$$

GENERAL CASE :HOW TO FIND a_i?

Theorem 1. *(Comte de Sparre) The solutions a_i of the transcendental equations written above can be found from the relation:**

$$f(\mathrm{sn}^2 a_i) = 0,$$

where $f(t)$ is a polynomial described by the next transparency. The sign of a_1 can be chosen in arbitrary way. The signs of other a_i are chosen in a way to satisfy the following consistency conditions

$$\frac{f'(\mathrm{sn}^2 a_1)}{\mathrm{sn}^{2n_3-1} a_1 \,\mathrm{cn}^{2n_2-1} a_1 \,\mathrm{dn}^{2n_1-1} a_1} = \ldots =$$
$$= \frac{f'(\mathrm{sn}^2 a_\beta)}{\mathrm{sn}^{2n_3-1} a_\beta \,\mathrm{cn}^{2n_2-1} a_\beta \,\mathrm{dn}^{2n_1-1} a_\beta} = \ldots =$$
$$= \frac{f'(\mathrm{sn}^2 a_\beta)}{\mathrm{sn}^{2n_3-1} a_\beta \,\mathrm{cn}^{2n_2-1} a_\beta \,\mathrm{dn}^{2n_1-1} a_\beta}.$$

*Here and below we assume that the modular parameter k satisfies the relation $0 < k^2 < 1$. The roots a_i are defined by the conditions above modulo periods of $\mathrm{sn}(x, k)$.

COEFFICIENTS of the POLYNOMIAL $f(t)$

$$f(t) := t^\beta + \alpha_0 t^{\beta-1} + \alpha_1 t^{\beta-2} + \ldots + \alpha_{\beta-1},$$

are defined as follows:

$$\begin{gathered}\alpha_0\,(2n-1)k^2 + h := \\ = (\nu_1+\nu_2+n+n_1)(\nu_1+\nu_2-n-n_1) + \qquad (1) \\ + k^2(\nu+\nu_2+n+n_2)(\nu+\nu_2-n-n_1).\end{gathered}$$

Next coefficients of $f(t)$ are defined by two recursive relations: ($\alpha_{-3} := \alpha_{-2} = 0$, $\alpha_{-1} := 1$, $\mu_i := 0$ for $i < 0$ or $i > \beta - 2$):

$$\begin{gathered}k^2\alpha_i[\beta(\beta-2i-2)-(i+1)(2n-2i-1)] + \\ + \alpha_{i-1}\{-\beta(\beta-2i)(1+k^2) + \\ + 2i[n+n_1-i+k^2(n+n_2-i)] + (2n-1)k^2\alpha_0\} + \\ + \alpha_{i-2}(\beta-i+1)(2\beta-2i-2n_3+1) - \\ - \mu_{i-1} + (1+k^2)\mu_i - k^2\mu_{i+1} = 0, \qquad (2)\end{gathered}$$

$$\begin{gathered}2(n_1+n_2)\mu_{i-1} - \\ - 2(n_1+n_2k^2)\mu_i - k^2(i+1)(2n-2i-1)(2n-i)\alpha_i + \\ + 2\{(2n-1)(n-i)k^2\alpha_0 + \beta(\beta-2i)(n_1+n_2k^2) + \\ + i(2n-i)[(n-i)(1+k^2)+n_1+n_2k^2]\}\alpha_{i-1} - \qquad (3) \\ - (\beta-2n+i)(\beta-i+1)(2\beta-2n_3-2i+1)\alpha_{i-2} = 0.\end{gathered}$$

Setting $i = -1$ and $i = 0$ we find from (2) the values of μ_0 and μ_1. Next, setting $i = 1$, from equation (3) we find α_1 as a function of α_0, (α_0 was defined by (1)).

In general case, knowing the quantities $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $\mu_1, \mu_2, \ldots, \mu_s$, it is possible to find from (2) the value of μ_{s+1}, and later from (3) the value of α_{s+1}. Somehow, for $\beta > 2n + 1$ this procedure does not allow to determine α_{2n}. In the later case , following the same procedure till $i = \beta - 1$, we find that $\mu_{2n+1}, \ldots, \mu_{\beta-2}$, and $\alpha_{2n+1}, \ldots, \alpha_{\beta-1}$ as a functions of α_0 and and α_{2n}. The obtained expressions are linear in α_{2n}. Substituting $i = \beta$ in (2) or (3) we obtain the relation

$$\alpha_{\beta-1}\{\beta[(n - n_3)(1 + k^2) + (n_1 - n_2)(k')^2] + (2n - 1)k^2\alpha_0\} - (2n_3 - 1)\alpha_{\beta-2} = 0,$$

from which, due to the linear dependence of $\alpha_{\beta-1}$ and $\alpha_{\beta-2}$ from α_{2n}, we find the last unknown coefficient of the polynomial $f(t)$.

ALTERNATIVE FORM OF THE SOLUTION.

Let $\beta = 2m$ or $\beta = 2m-1$. Consider the polynomials

$$\varphi(t) = A_0 + A_1 t + \ldots + A_m t^m,$$
$$\theta(t) = B_0 + B_1 t + \ldots + B_{m-1} t^{m-1},$$
$$\psi(t) = R_0 + R_1 t + \ldots + R_{\beta-m-1} t^{\beta-m-1},$$

satisfying the relation

$$t^{n_3}(1-t)^{n_2}(1-k^2 t)^{n_1}\psi(t) = \varphi(t)f'(t) - f(t)\theta(t).$$

Let us introduce the following notations

$$\omega = \sum_{i=1}^{\beta} a_i + (\beta+1)iK', \qquad \lambda = -\frac{\mathrm{sn}\,\omega\,\mathrm{cn}\,\omega\,\mathrm{dn}\,\omega\psi(\mathrm{sn}^2\,\omega)}{\varphi(\mathrm{sn}^2\,\omega)}.$$

Theorem 2. *(Comte de Sparre) In this notations the solution to the Sparre equation takes the form:*

$$y(x) = \mathrm{dn}^{\nu-n_1} x\,\mathrm{cn}^{\nu_1-n_2} x\,\mathrm{sn}^{\nu_2-n_3} x \Bigg[A[\lambda\varphi(\mathrm{sn}^2 x) - $$
$$- \mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\psi(\mathrm{sn}^2 x)] \times$$
$$\times \frac{\Theta(x)}{H(x+\omega)} e^{\left\{\frac{H'(\omega)}{H(\omega)} + \frac{R_0}{\lambda A_0}\right\} x} +$$
$$+ B[\lambda\varphi(\mathrm{sn}^2 x) +$$
$$\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\psi(\mathrm{sn}^2 x)] \frac{\Theta(x)}{H(x-\omega)} e^{-\left\{\frac{H'(\omega)}{H(\omega)} + \frac{R_0}{\lambda A_0}\right\} x} \Bigg].$$

Here A and B — are arbitrary constants , $H(x)$, $\Theta(x)$ — are Jacobi elliptic theta functions.

LAMÉ EQUATION

Consider the particular case $n_1 = n_2 = n_3 = \nu = \nu_1 = \nu_2 = 0$ of Sparre equation which is nothing but Lamé equation. In this case it was shown in that the coefficients of the polynomial

$$f(t) = t^n + \alpha_0 t^{n-1} + \alpha_1 t^{n-2} + \ldots + \alpha_{n-1}$$

are defined by the three terms recursive relation

$$\begin{gathered}\alpha_i k^2(2n-2i-1)(i+1)(2n-i) = \\ = [i(2n-i)(1+k^2) + (2n-1)k^2\alpha_0]2(n-i)a_{i-1} + \\ + (n-i)(n-i+1)(2n-2i+1)\alpha_{i-2},\end{gathered}$$

where $\alpha_{-1} = 1$,

$$\alpha_0 = -\frac{h + n^2(1+k^2)}{(2n-1)k^2}.$$

Similarly to the general case the quantities a_i, can be found from the solution of the equation $f(\operatorname{sn}^2 a_i) = 0$. The signs of a_i are chosen in a way to satisfy the relation

$$\begin{gathered}\operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1 f'(\operatorname{sn}^2 a_1) = \ldots \\ = \operatorname{sn} a_i \operatorname{cn} a_i \operatorname{dn} a_i f'(\operatorname{sn}^2 a_i) = \ldots = \\ \operatorname{sn} a_n \operatorname{cn} a_n \operatorname{dn} a_n f'(\operatorname{sn}^2 a_n).\end{gathered}$$

The solutions to the Sparre equation for this special case are obtained by simply setting

$n_1 = n_2 = n_3 = \nu = \nu_1 = \nu_2 = 0$, $\beta = n$ in general solution.

EQUIVALENCE OF THE SPARRE AND DARBOUX EQUATIONS

The coefficient of $y'(x)$ in Sparre equation can be written as follows

$$2\nu\frac{k^2\,\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x}+2\nu_1\frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x}-2\nu_2\frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x}=$$
$$=-2\frac{d}{dx}\ln\left(\mathrm{dn}^{\nu}\,x\,\mathrm{cn}^{\nu_1}\,x\,\mathrm{sn}^{\nu_2}\,x\right).$$

As everybody knows the equation

$$y''-2\frac{d\ln f(x)}{dx}y'=Q(x)y,$$

can be reduced to a normal (Schrödinger) form :

$$z''=\left(Q(x)-\frac{d^2\ln f(x)}{dx^2}+\left(\frac{d\ln f(x)}{dx}\right)^2\right)z$$

by the change of variables $y=f(x)z$.

This means that Sparre equation may be reduced to a normal form $z'' = \widetilde{Q}(x)z,$ by the change of variables

$$y = \mathrm{dn}^{\nu} x \, \mathrm{cn}^{\nu_1} x \, \mathrm{sn}^{\nu_2} x \cdot z$$

. The related potential $\widetilde{Q}(x)$ is given by the formula

$$\begin{aligned}
\widetilde{Q}(x) = h + \frac{1}{\mathrm{sn}^2 x}(n_3 - \nu_2)(n_3 + \nu_2 + 1) + \\
+ \frac{\mathrm{dn}^2 x}{\mathrm{cn}^2 x}(n_2 - \nu_1)(n_2 + nu_1 + 1) + \\
+ \frac{k^2 \mathrm{cn}^2 x}{\mathrm{dn}^2 x}(n_1 - \nu)(n_1 + \nu + 1) + \\
+ k^2 \mathrm{sn}^2 x(n + \nu + \nu_1 + \nu_2)(n - \nu - \nu_1 - \nu_2 + 1) + \\
+ \frac{d}{dx}\left[\nu \frac{k^2 \mathrm{sn}\, x \,\mathrm{cn}\, x}{\mathrm{dn}\, x} + \nu_1 \frac{\mathrm{sn}\, x \,\mathrm{dn}\, x}{\mathrm{cn}\, x} - \nu_2 \frac{\mathrm{cn}\, x \,\mathrm{dn}\, x}{\mathrm{sn}\, x}\right] + \\
+ \left[\nu \frac{k^2 \mathrm{sn}\, x \,\mathrm{cn}\, x}{\mathrm{dn}\, x} + \nu_1 \frac{\mathrm{sn}\, x \,\mathrm{dn}\, x}{\mathrm{cn}\, x} - \nu_2 \frac{\mathrm{cn}\, x \,\mathrm{dn}\, x}{\mathrm{sn}\, x}\right]^2 .
\end{aligned}$$

Simplifying the previous expression by use of the differential equations for Jacobi elliptic functions we get

$$\widetilde{Q}(x) = \frac{n_3(n_3+1)}{\operatorname{sn}^2 x} + \frac{n_2(n_2+1)}{\operatorname{cn}^2 x}\operatorname{dn}^2 x + \\ + \frac{n_1(n_1+1)}{\operatorname{dn}^2 x} k^2 \operatorname{cn}^2 x + n(n+1)k^2 \operatorname{sn}^2 x + \widetilde{h},$$

where

$$\widetilde{h} = h - (\nu_1+\nu_2)^2 - k^2(\nu+\nu_2)^2$$

, i.e $\widetilde{Q}(x)$ is exactly the Darboux potential.

From thus it turns out that Darboux-Verdier-Treibich equation — is not only the special case of the Sparre equation but also its canonical form.

Another obvious conclusion is that the general solution of the Darboux equation can be written in two equivalent forms:

$$y(x) = \left[A \frac{H(x-a_1)H(x-a_2)\dots H(x-a_\beta)}{H^\mu(x)H_1^{\mu'}(x)\Theta_1^{\mu''}(x)\Theta^n(x)} e^{Cx/2} + \right. \\ \left. + B \frac{H(x+a_1)H(x+a_2)\dots H(x+a_\beta)}{H^\mu(x)H_1^{\mu'}(x)\Theta_1^{\mu''}(x)\Theta^n(x)} e^{-Cx/2} \right],$$

or

$$y(x) = \operatorname{dn}^{-n_1} x \operatorname{cn}^{-n_2} x \operatorname{sn}^{-n_3} x \times \\ \times \left[A[\lambda\varphi(\operatorname{sn}^2 x) - \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \psi(\operatorname{sn}^2 x)] \frac{\Theta(x)}{H(x+\omega)} e^{\left\{\frac{H'(\omega)}{H(\omega)} + \frac{R_0}{\lambda A_0}\right\}x} + \right. \\ \left. + B[\lambda\varphi(\operatorname{sn}^2 x) + \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \psi(\operatorname{sn}^2 x)] \frac{\Theta(x)}{H(x-\omega)} e^{-\left\{\frac{H'(\omega)}{H(\omega)} + \frac{R_0}{\lambda A_0}\right\}x} \right].$$

Concluding remarks

Sparre and Darboux equations are linear second order differential equations with periodic coefficients. Hence, for certain values h_j of the spectral parameter h,for which one of the linearly independent solutions is a periodic function of x ($y(x+2K) = y(x)$) or antiperiodic ($y(x+2K) = -y(x)$) function. As a matter of fact, the solutions, of the Sparre equation displayed above are generic for all values of h, except the eigenvalues h_j of periodic or antiperiodic Sturm-Liouville problems. It easy to recognize that h_j are the values of h, for which at least one of a_i satisfies the condition $2a_i \equiv 0$ modulo periods of the lattice.

Replacing in the Darboux equation the Jacobi elliptic functions by Wierstrass elliptic functions with help of the well known relations we get its Weierstrass form in which this equation was discovered independently from Darboux by Trebich and Verdier:

$$\frac{d^2y}{d\xi^2} = [\mu(\mu+1)\wp(\xi) + \mu'(\mu'+1)\wp(\xi+\omega_1) + \\ + \mu''(\mu''+1)\wp(\xi+\omega_2) + \\ + n(n+1)\wp(\xi+\omega_3) + \widehat{h}]y.$$

Considering the related Schrödinger operator,

Verdier and Treibich proved that it belongs to the class of the finite gap Schrödinger operators. They also obtained the formula evaluating a number of gaps in the spectrum by means of μ, μ', μ'' and n. Sparre's articles have an advantage to provide an efficient formula in term of elliptic functions for the solutions.

An appropriate change of variables allows to reduce Darboux-Verdier-Treibich equation to the Heun equation (i.e to the standard Fuchs equation with 4 regular singular points)

$$\frac{d^2y}{dz^2} + \frac{1}{2}\left(\frac{1-2\mu'}{z} + \frac{1-2\mu''}{z-1} + \frac{1-2n}{z-a}\right)\frac{dy}{dz} + \frac{\beta(\beta-2\mu-1)z+q}{4z(z-1)(z-a)}y = 0,$$

Here $\beta = \mu + \mu' + \mu'' + n$. In Smirnov's recent work the general solution of this special Heun equation and its dependence on the accessory parameter q was determined. The relation between Sparre equation and Heun equation found here provides a new way to calculate the monodromy data for the special Heun equation based on the formula for the solution of the Darboux equation mentioned above.

ELLIPTIC FUNCTIONS

$$\frac{d}{dx}\,\mathrm{sn}\,x = \mathrm{cn}\,x \cdot \mathrm{dn}\,x, \qquad \mathrm{sn}(0,k) = 0,$$
$$\frac{d}{dx}\,\mathrm{cn}\,x = -\,\mathrm{sn}\,x \cdot \mathrm{dn}\,x, \qquad \mathrm{cn}(0,k) = 1,$$
$$\frac{d}{dx}\,\mathrm{dn}\,x = -k^2\,\mathrm{sn}\,x \cdot \mathrm{cn}\,x; \qquad \mathrm{dn}(0,k) = 1$$

$$\mathrm{cn}^2\,x = 1 - \mathrm{sn}^2\,x, \qquad \mathrm{dn}^2\,x = 1 - k^2\,\mathrm{sn}^2\,x.$$

Translation properties with respect to the shifts on the periods and half-periods of the lattice

$$\mathrm{sn}(x + 2K) = -\,\mathrm{sn}\,x \qquad \mathrm{sn}(x + 2iK') = \mathrm{sn}\,x,$$
$$\mathrm{cn}(x + 2K) = -\,\mathrm{cn}\,x \qquad \mathrm{cn}(x + 2iK') = -\,\mathrm{cn}\,x,$$
$$\mathrm{dn}(x + 2K) = \mathrm{dn}\,x \qquad \mathrm{dn}(x + 2iK') = -\,\mathrm{dn}\,x,$$

$$\mathrm{sn}(x + K) = \frac{\mathrm{cn}\,x}{\mathrm{dn}\,x} \qquad \mathrm{sn}(x + iK') = \frac{1}{k\,\mathrm{sn}\,x},$$
$$\mathrm{cn}(x + K) = -k'\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x} \qquad \mathrm{cn}(x + iK') = -i\frac{\mathrm{dn}\,x}{k\,\mathrm{sn}\,x},$$
$$\mathrm{dn}(x + K) = \frac{k'}{\mathrm{dn}\,x} \qquad \mathrm{dn}(x + iK') = -i\frac{\mathrm{cn}\,x}{\mathrm{sn}\,x},$$

Here $i^2 = -1$, $k^2 + (k')^2 = 1$.

JACOBI THETA FUNCTIONS

$$\vartheta_3(x,\tau) := \sum_{m=-\infty}^{\infty} e^{i\pi\tau m^2 + 2\pi i m x}, \qquad \Im\tau > 0,$$

$$\vartheta_0(x) := \vartheta_3\left(x + \frac{1}{2}\right),$$

$$\vartheta_1(x) := i e^{-i\pi\left(x - \frac{\tau}{4}\right)} \vartheta_3\left(x + \frac{1-\tau}{2}\right),$$

$$\vartheta_2(x) := e^{-i\pi\left(x - \frac{\tau}{4}\right)} \vartheta_3\left(x - \frac{\tau}{2}\right),$$

$$\theta_3(x) := \vartheta_3\left(\frac{x}{2K}\right) \equiv \Theta_1(x),$$

$$\theta_0(x) := \vartheta_0\left(\frac{x}{2K}\right) \equiv \Theta(x),$$

$$\theta_1(x) := \vartheta_1\left(\frac{x}{2K}\right) \equiv H(x)$$

$$\theta_2(x) := \vartheta_2\left(\frac{x}{2K}\right) \equiv H_1(x).$$

$$K := \frac{\pi}{2}\vartheta_3^2(0,\tau) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

CONNECTION OF JACOBI ELLIPTIC FUNCTIONS WITH THETA FUNCTIONS

$$\operatorname{sn} x = \frac{1}{\sqrt{k}}\frac{H(x)}{\Theta(x)}, \qquad \operatorname{cn} x = \sqrt{\frac{k'}{k}}\frac{H_1(x)}{\Theta(x)}, \qquad \operatorname{dn} x = \sqrt{k'}\frac{\Theta_1(x)}{\Theta(x)};$$

$$\sqrt{k} = \frac{H_1(0)}{\Theta_1(0)}, \qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)}.$$

TRANSFORMATION PROPERTIES of the JACOBI THETA FUNCTIONS.

$$\begin{array}{ll} H(x+K) = H_1(x) & H(x+iK') = i\lambda\Theta(x), \\ \Theta(x+K) = \Theta_1(x) & \Theta(x+iK') = i\lambda H(x), \\ H_1(x+K) = -H(x) & H_1(x+iK') = \lambda\Theta_1(x), \\ \Theta_1(x+K) = \Theta(x) & \Theta_1(x+iK') = \lambda H_1(x), \end{array}$$

where $\lambda = e^{-\dfrac{\pi i}{4K}(2x+iK')}$ and $iK' := \tau K$ or

equivalently $K' = \int_0^1 \dfrac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$

CONNECTION BETWEEN JACOBI ELLIPTIC FUNCTIONS AND WEIERSTRASS ELLIPTIC FUNCTION $\wp(x)$

$$\wp(\xi) = e_3 + (e_1 - e_3)\frac{1}{\mathrm{sn}^2 x}, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad (k')^2 = \frac{e_1 - e_2}{e_1 - e_3},$$

$$\xi = \frac{x}{\sqrt{e_1 - e_3}}, \quad \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_3 = \frac{iK'}{\sqrt{e_1 - e_3}},$$

$$\wp(\xi + 2\omega_1) = \wp(\xi + 2\omega_2) = \wp(\xi + 2\omega_3) = \wp(\xi),$$
$$\omega_2 = \omega_1 + \omega_3, \qquad e_j = \wp(\omega_j), \quad j = 1, 2, 3.$$

$$\wp(\xi + \omega_1) = e_3 + (e_1 - e_3)\frac{\mathrm{dn}^2 x}{\mathrm{cn}^2 x},$$
$$\wp(\xi + \omega_2) = e_3 + (e_1 - e_3)k^2\frac{\mathrm{cn}^2 x}{\mathrm{dn}^2 x},$$
$$\wp(\xi + \omega_3) = e_3 + (e_1 - e_3)k^2\,\mathrm{sn}^2 x.$$

$$[\wp'(x)]^2 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3).$$

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