Rate optimality of wavelet series expansions of Fractional Brownian Motion

Antoine AYACHE

Laboratoire Paul Painlevé, UMR CNRS 8524,
Université Lille 1,
59655 Villeneuve d'Ascq Cedex, FRANCE
&
CLAREE, UMR CNRS 8020,
IAE, Université Lille 1,
104, Avenue du peuple Belge,
59043 Lille Cedex, FRANCE
E-Mail: **Antoine.Ayache@math.univ-lille1.fr**

1 Introduction

1.1 Small ball probabilities and l-approximation numbers

A random variable X with values in a separable Banach space E is *Gaussian centered* when $< X, x^* >$ is *a real-valued normal random variable* for any $x^* \in E^*$.

Example 1 *Any real-valued, continuous and centered Gaussian process defined on a compact metric space K, can be viewed as a centered Gaussian random variable with values in $\mathcal{C}(K)$, the Banach space of continuous functions over K equipped with the uniform norm.*

Theorem 2 *X can be represented as an a.s. convergent random series of the form:*

$$X = \sum_{k=1}^{+\infty} \epsilon_k x_k, \qquad (1)$$

where the ϵ_k's are independent $\mathcal{N}(0,1)$ real-valued Gaussian random variables *and the x_k's are* some deterministic vectors of *E.*

See the book of Lifshits or that of Ledoux and Talagrand for a proof.

It is natural to look for *optimal* representations of the type (1) i.e. those where the tail of the series $\Sigma_{k=n}^{+\infty} \epsilon_k x_k$ tends to zero as fast as possible.

This leads to the study of the quantity

$$l_n(X) = \inf \left\{ \left(E\| \sum_{k=n}^{+\infty} \epsilon_k f_k \|^2 \right)^{1/2} ;\ X = \sum_{k=0}^{+\infty} \epsilon_k f_k \right\}. \qquad (2)$$

$l_n(X)$ is called the *nth l-approximation number of* X.

Remark 3 *The value of* $l_n(X)$ *remains the same even if the random variables* ϵ_k *are allowed to be dependent.*

Clearly, $\lim_{n\to+\infty} l_n(X) = 0$. The speed of convergence is closely connected to the small ball behaviour of X:

Theorem 4 *(Li and Linde 1999) Let* $\alpha > 0$ *and* $\beta \in I\!R$ *be fixed.*

(a) If for some constant $c_1 > 0$ *and every integer* $n \geq 1$*,*

$$l_n(X) \leq c_1 n^{-1/\alpha}(1 + \log n)^{\beta}. \qquad (3)$$

Then, there is a constant $c_2 > 0$ *such that for any* $\epsilon > 0$ *small enough,*

$$\log(P(\|X\| \leq \epsilon) \geq -c_2 \epsilon^{-\alpha}(\log 1/\epsilon)^{\alpha\beta}. \qquad (4)$$

(b) Conversely, if (4) holds then there is a constant $c_3 > 0$ *such that for each integer* $n \geq 1$*,*

$$l_n(X) \leq c_3 n^{-1/\alpha}(1 + \log n)^{\beta+1}. \qquad (5)$$

Remark 5 *(Li and Linde 1999) When* E *is* K*-convex (e.g.* L^p*,* $1 < p < \infty$*) then (3) and (4) are equivalent.*

1.2 l-approximation numbers and small ball behaviour of Fractional Brownian Sheet

The *Fractional Brownian Motion* (FBM) with Hurst parameter H, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$, is the continuous centered Gaussian process with the covariance kernel, for all $s,t\in\mathbb{R}$,

$$E\left(B_H(s)B_H(t)\right) = K_H(s,t) = \frac{1}{2}\left(|s|^{2H}+|t|^{2H}-|s-t|^{2H}\right).(6)$$

There are two natural extensions to $\mathbb{R}^N$ of the FBM:

→ The *Lévy FBM* with Hurst parameter $H\in(0,1)$, denoted by $\{X_H(t)\}_{t\in\mathbb{R}^N}$, is the continuous, centered and *isotropic* Gaussian field with the covariance kernel, for all $s,t\in\mathbb{R}^N$,

$$E\left(X_H(s)X_H(t)\right) = \frac{1}{2}\left(\|s\|^{2H}+\|t\|^{2H}-\|s-t\|^{2H}\right), \qquad (7)$$

where $\|\cdot\|$ denotes the Euclidian norm.

→ The *Fractional Brownian Sheet* with Hurst multiparameter $\alpha=(\alpha_1,\dots,\alpha_N)\in(0,1)^N$, denoted by $\{Y_\alpha(t)\}_{t\in\mathbb{R}^N}$, is the continuous, centered and *anisotropic* Gaussian field, with the covariance kernel, for all $s=(s_1,\dots,s_n)$ and $t=(t_1,\dots,t_n)$,

$$E\left(Y_\alpha(s)Y_\alpha(t)\right) = \prod_{l=1}^{N} K_{\alpha_l}(s_l,t_l). \qquad (8)$$

Sharp bounds of the small ball probabilities of Lévy FBM, under the uniform norm, were determined by Monrad, Rootzen, Shao, Talagrand and Wang, in the middle of 90's:

$$-c_4\epsilon^{-N/H} \le \log P\left(\sup_{t\in[0,1]^N} |X_H(t)| \le \epsilon\right) \le c_5\epsilon^{-N/H}. \qquad (9)$$

$\longrightarrow$ The lower bound was obtained by using the following inequality: For all $s, t \in \mathbb{R}^N$,

$$E\left(|X_H(s) - X_H(t)|^2\right) \le c_6|s-t|^{2H}. \qquad (10)$$

$\longrightarrow$ The upper bound was obtained by using the *strong local non determinism* property of $\{X_H(t)\}_{t\in\mathbb{R}^N}$: There is a constant $c_7 > 0$ such that for all $t \in (0,1)^N$ and $\eta > 0$ small enough,

$$\mathrm{Var}\left(X_H(t)/X(s);\ s \in [0,1]^N,\ |s-t| \ge \eta\right) \ge c_7\eta^{2H}. \qquad (11)$$

FBS has non-stationary increments and a complex covariance structure. It is therefore difficult to prove that it satisfies certain nice properties (strong local non-determinism,...).

$\Rightarrow$ The problem of finding sharp bounds of the small ball probabilities of this field is challenging:

- It has been completely solved only in the particular case of the Brownian Sheet on $\mathbb{R}^2$ (denoted by $\{B(t)\}_{t\in\mathbb{R}^2}$): For all $0 < \epsilon \leq 1$,

  $$-c_8\epsilon^{-2}\log^3(1/\epsilon) \leq \log P\left(\sup_{t\in[0,1]^2} |B(t)| \leq \epsilon\right) \leq -c_9\epsilon^{-2}\log^3(1/\epsilon).$$

  The lower bound is due to Lifshits (1986) and Bass (1988) and the upper bound to Talagrand (1994).

- The combinatorial arguments used for $N = 2$ fail for $N \geq 3$. $\Rightarrow$ The problem becomes much more tricky and there is still a gap between the upper and lower bounds.

From now on, the parameters α_i of the FBS $\{Y_\alpha(t)\}_{t\in\mathbb{R}^N}$ will be ordered in an increasing way:

$$0 < \alpha_1 = \ldots = \alpha_\nu \leq \alpha_{\nu+1} \leq \ldots \leq \alpha_N < 1.$$

Kühn and Linde (2002) obtained sharp bounds of the l-approximation numbers of FBS: There are $0 < c_{10} \leq c_{11}$ two constants such that for all $n \geq 1$,

$$\begin{aligned} c_{10} n^{-\alpha_1}(1 &+ \log n)^{(\nu-1)\alpha_1+\nu/2} \\ &\leq l_n(Y_\alpha) \leq c_{11} n^{-\alpha_1}(1+\log n)^{(\nu-1)\alpha_1+\nu/2}. \quad (12) \end{aligned}$$

Before that (12) was only known in the particular case of Brownian Motion (Maiorov and Wasilkowski 1996).

To obtain (12) Kühn and Linde have studied the approximations properties of some fractional integral operators related to FBS.

For FBM (12) becomes

$$c_{10} n^{-H}(1+\log n)^{1/2} \leq l_n(B_H) \leq c_{11} n^{-H}(1+\log n)^{1/2}. \quad (13)$$

Relation (12) is an important result at least for the following two reasons:

1) It allows to bound the small ball probabilities of FBS (Kühn and Linde): There are $c_{12} > 0$ and $c_{13} > 0$ two constants such that for all $0 < \epsilon \leq 1$,

$$
\begin{aligned}
&-c_{12}\epsilon^{-1/\alpha_1}(\log 1/\epsilon)^{(\nu-1)\left(1+\frac{1}{2\alpha_1}\right)+\frac{1}{2\alpha_1}} \\
&\leq \log P\left(\sup_{t\in[0,1]^N} |Y_\alpha(t)| \leq \epsilon\right) \leq -c_{13}\epsilon^{-1/\alpha_1}(\log 1/\epsilon)^{(\nu-1)\left(1+\frac{1}{2\alpha_1}\right)}.
\end{aligned}
$$

2) It gives a universal lower bound for the rate of convergence of random series of the type:

$$\sum_{k=1}^{+\infty} x_k(t)\epsilon_k,$$

to FBS.

2 Optimal series representations of Brownian Motion

It seems natural to look for optimal random series representations of FBS i.e. the speed of convergence is

$$cn^{-\alpha_1}(1+\log n)^{(\nu-1)\alpha_1+\nu/2}.$$

To find good candidates let us first examine the case of Brownian Motion on $[0,1]$, denoted by $\{B(t)\}_{t\in[0,1]}$.

Expansion of $\{B(t)\}_{t\in[0,1]}$ in the *trigonometric system*:

$$B(t) = \epsilon_0 t + \sqrt{2}\sum_{k=1}^{+\infty} \epsilon_k \frac{\sin(\pi k t)}{\pi k}, \tag{14}$$

where the ϵ_k's are independent $\mathcal{N}(0,1)$ Gaussian random variables and the series is a.s. uniformly convergent in t.

The following proposition entails that (14) is an optimal series representation of $\{B(t)\}_{t\in[0,1]}$.

Proposition 6 *(Kühn and Linde 2002) Let*

$$R_H(t) = \int_0^t (t-s)^{H-1/2}\, dB(s), \tag{15}$$

be the Riemann-Liouville FBM with Hurst parameter H. Then for any $H \in (0,1/2]$,

$$\begin{aligned} &R_H(t) \qquad (16)\\ &= \epsilon_0 \int_0^t (t-s)^{H-1/2}\, ds + \sqrt{2}\sum_{k=1}^{+\infty} \epsilon_k \int_0^t (t-s)^{H-1/2} \cos(\pi k s)\, ds, \end{aligned}$$

is an optimal series representation of $\{R_H(t)\}_{t\in[0,1]}$.

Expansion of $\{B(t)\}_{t\in[0,1]}$ in the *Faber-Schauder system* (Lévy 1948):

$$B(t) = \epsilon_0 t + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \epsilon_{j,k} 2^{-j/2} \tau(2^j t - k), \tag{17}$$

where:

- ϵ_0 and $\epsilon_{j,k}$ are independent $\mathcal{N}(0,1)$ Gaussian random variables,
- τ is the triangle function based on $[0,1]$,
- the series is a.s. uniformly convergent in t.

The expansion (17) has been used to study some fine properties of the trajectories of $\{B(t)\}_{t\in[0,1]}$, as for instance to prove that they do not satisfy a uniform Hölder condition of order 1/2.

Proposition 7 *The series expansion of Brownian Motion in the Faber-Schauder system is optimal in the sense of Kühn and Linde.*

For proving Proposition 7 we need the following lemmas.

Lemma 8 *There is a constant* $c_{14} > 0$ *such that for any* $N \geq 1$ *and any centered Gaussian sequence* $Z_1, \ldots, Z_N$ *one has,*

$$E\left(\sup_{1\leq k\leq N} |Z_k|\right) \leq c_{14}(1+\log N)^{1/2} \sup_{1\leq k\leq N} \left(E|Z_k|^2\right)^{1/2}. \quad (18)$$

Lemma 9 *For any* $t \in [0,1]$ *and any integer* $j \geq 0$*, there is at most one integer* $0 \leq k < 2^j$ *such that* $\tau(2^j t - k) \neq 0$*.*

Proof of Proposition 7: Lemma 9 implies that for all $j \geq 0$ and $t \in [0,1]$,

$$\sum_{k=0}^{2^j-1} |\epsilon_{j,k}||\tau(2^j t - k)| \leq \left(\sup_{0 \leq k < 2^j} |\epsilon_{j,k}| \right) \|\tau\|_\infty. \tag{19}$$

By using (19), Lemma 8 and the fact that the $\epsilon_{j,k}$'s have the same standard Gaussian distribution, one has for any integer $m \geq 1$,

$$\begin{aligned}
Q_m &= E\left(\sup_{t \in [0,1]} \left| B(t) - \epsilon_0 t - \sum_{j=0}^{m} \sum_{k=0}^{2^j-1} 2^{-j/2} \epsilon_{j,k} \tau(2^j t - k) \right| \right) \\
&\leq c_{15} \sum_{j=m+1}^{+\infty} 2^{-j/2} E\left(\sup_{0 \leq k < 2^j} |\epsilon_{j,k}| \right) \\
&\leq c_{16} \sum_{j=m+1}^{+\infty} 2^{-j/2} (1+j)^{1/2} \left(\sup_{0 \leq k < 2^j} E|\epsilon_{j,k}|^2 \right)^{1/2} \\
&= c_{16} \sum_{j=m+1}^{+\infty} 2^{-j/2} (1+j)^{1/2} \\
&\leq c_{17} \int_{m+1}^{+\infty} 2^{-x/2} (1+x)^{1/2} \, dx \\
&\leq c_{18} 2^{-m/2} (1+m)^{1/2},
\end{aligned}$$

which means that the expansion of $\{B(t)\}_{t \in [0,1]}$ in the Faber-Schauder system is optimal in the sense of Kühn and Linde.

3 Wavelet series representations of FBM

They consist in expressing FBM as a series of approximations with successive scale refinements.

$\Rightarrow$ They are the counterpart of the representation of Brownian Motion in the Faber-Schauder system.

There are mainly two kinds of wavelet series representations of FBM:

- The representations without scaling functions,
- The representations with scaling functions.

3.1 Wavelets and scaling functions

Definition 10 *A* wavelet *is a Borel function ψ satisfying the following two properties:*

- *ψ is a* well-localized *function: There are $\alpha > 1$ and $c_{19} > 0$ such that, for almost all x,*
$$|\psi(x)| \leq c_{19}(1 + |x|)^{-\alpha}. \qquad (20)$$
The derivatives of ψ of any order are also well-localized, when they exist.

- *ψ is an* oscillating *function: There is an integer $p \geq 0$ such that,*
$$\int_{I\!R} \psi(x)\, dx = \int_{I\!R} x\psi(x)\, dx = \ldots = \int_{I\!R} x^p \psi(x)\, dx = 0.$$

Example 11 *The Haar function $\chi_{[0,1/2)} - \chi_{[1/2,1]}$, where χ_I denotes the indicator of an interval I.*

Definition 12 *A scaling function* ϕ *is the unique non trivial solution of a two-scale equation i.e. an equation of the form:*

$$\phi(t) = \sum_{k\in\mathbb{Z}} a_k \phi(2t-k), \tag{21}$$

where the coefficients a_k *satisfy* $\Sigma_{k\in\mathbb{Z}} a_k = 1$*.*

Example 13 *The Haar scaling function i.e. the indicator of the interval* $[0,1]$*.*

Two-scale equations were introduced by De Rham in the 50's for constructing continuous and nowhere differentiable functions. It later turned out that such kind of functional equations can also have very smooth solutions.

Definition 14 *An orthonormal wavelet basis of* $L^2(\mathbb{R})$ *is an orthonormal basis of the form:*

$$\{2^{j/2}\psi(2^j t - k);\ j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\},$$

where ψ *is called a* mother wavelet*.*

Thus any function $f \in L^2(\mathbb{R})$ can be expressed as:

$$f(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} c_{j,k}\psi(2^j t - k), \tag{22}$$

where the series converges in $L^2(\mathbb{R})$ and for every $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(t)\psi(2^j t - k)\, dt. \tag{23}$$

Theorem 15 *(Mallat and Meyer 1986) Let ϕ be a scaling function i.e.*

$$\phi(t) = \sum_{k\in\mathbb{Z}} a_k \phi(2t - k), \tag{24}$$

and let ψ be a function of the form

$$\psi(t) = \sum_{k\in\mathbb{Z}} b_k \phi(2t - k). \tag{25}$$

Then, under some conditions on the coefficients a_k and b_k the series (24) and (25) converge in $L^2(\mathbb{R})$ and ψ is a mother wavelet i.e. it generates an orthonormal wavelet basis of $L^2(\mathbb{R})$.

Corollary 16 *For any $m \in \mathbb{Z}$, the functions*

$$\{2^{m/2}\phi(2^m t - l),\, l \in \mathbb{Z},\, 2^{j/2}\psi(2^j t - k),\, j \geq m,\, k \in \mathbb{Z}\},$$

form an orthonormal basis of $L^2(\mathbb{R})$.

Thus each function $f \in L^2(\mathbb{R})$ can be expressed as:

$$f(t) = \sum_{l \in \mathbb{Z}} d_{m,l}\phi(2^m t - l) + \sum_{j=m}^{+\infty} \sum_{k \in \mathbb{Z}} c_{j,k}\psi(2^j t - k), \qquad (26)$$

where the series converges in $L^2(\mathbb{R})$ and for every $l \in \mathbb{Z}$, $j \geq m$ and $k \in \mathbb{Z}$,

$d_{m,l} = 2^m \int_{\mathbb{R}} f(t)\phi(2^m t - l)\,dt$ and $c_{j,k} = 2^j \int_{\mathbb{R}} f(t)\psi(2^j t - k)\,dt$.

Usually one approximates the function $f(t)$ by

$$\sum_{l \in \mathbb{Z}} d_{m,l}\phi(2^m t - l)$$

and the higher is the integer m the better is the approximation.

One of the main advantages of the decomposition (26) with respect to the decomposition without scaling function is that *Mallat pyramidal algorithm* allows to compute by induction the coefficient $d_{m,l}$, $l \in \mathbb{Z}$, for any $m \geq 1$, starting from the coefficients $d_{0,l}$, $l \in \mathbb{Z}$ and $c_{j,k}$, $0 \leq j \leq m-1$, $k \in \mathbb{Z}$.

Later we will see that *biorthogonal wavelet bases* are more adapted to the analysis of FBM than orthonormal wavelet bases.

Definition 17 $\{e_n\}_{n\in I\!N}$ *and* $\{\tilde{e}_n\}_{n\in I\!N}$ *are two biorthogonal bases of a Hilbert space* H *when they satisfy the following conditions:*

(a) For all n *and* n',

$$< e_n, \tilde{e}_{n'} >= \begin{cases} 1 \text{ when } n = n', \\ \\ 0 \text{ else.} \end{cases}$$

(b) Any $x \in H$ *can be expressed as:*

$$x = \sum_{n=0}^{+\infty} < x, \tilde{e}_n > e_n = \sum_{n=0}^{+\infty} < x, e_n > \tilde{e}_n. \tag{27}$$

3.2 Wavelet representations of FBM without scaling functions

We suppose that:

- ψ is a smooth and well-localized mother wavelet,
- $\{2^{j/2}\psi(2^j t-k);\ j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$,
- $\widehat{\psi}$ is a smooth and well-localized function.

$\Rightarrow$ ψ is typically a Lemarié-Meyer wavelet or a compactly supported Daubechies wavelet with at least 6 vanishing moments.

Under these assumptions, for any $H \in (0,1)$,

$$\Psi_H(t) = \int_{\mathbb{R}} e^{it\xi} \frac{\widehat{\psi}(\xi)}{(i\xi)^{H+1/2}}\, d\xi \text{ and } \Psi_{-H}(t) = \int_{\mathbb{R}} e^{it\xi} (i\xi)^{H+1/2} \widehat{\psi}(\xi)\, d\xi,$$

the *fractional primitive* of ψ of order $H+1/2$ and its *fractional derivative* of order $H+1/2$ are well-defined, smooth and well-localized functions.

Theorem 18 *(Meyer, Sellan and Taqqu 1999) The functions*

$$\{2^{j/2}\Psi_H(2^j t - k);\ j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$$

and

$$\{2^{j/2}\Psi_{-H}(2^j t - k);\ j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$$

are two biorthogonal bases of $L^2(\mathbb{R})$*.*

These bases are well-adapted to the analysis of FBM:

Theorem 19 *(Meyer, Sellan and Taqqu 1999) The FBM* $\{B_H(t)\}_{t\in\mathbb{R}}$ *can be expressed as:*

$$B_H(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \Big(\Psi_H(2^j t - k) - \Psi_H(-k)\Big)\, \epsilon_{j,k}, \quad (28)$$

where the $\epsilon_{j,k}$*'s are independent* $\mathcal{N}(0,1)$ *Gaussian random variables and the series is a.s. uniformly convergent in* t *on compact intervals. Moreover,*

$$\epsilon_{j,k} = 2^{j(H+1)} \int_{\mathbb{R}} B_H(t)\Psi_{-H}(2^j t - k)\, dt. \quad (29)$$

$\rightarrow$ (28) is almost a Karhunen-Loeve expansion of $\{B_H(t)\}_{t\in\mathbb{R}}$.

$\rightarrow$ It allows to obtain some local and asymptotic properties of $\{B_H(t)\}_{t\in\mathbb{R}}$ (nowhere differentiability, behaviour as $|t| \rightarrow +\infty$,...).

Proof of Theorem 19: Let us start from the *harmonizable representation* of FBM:

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}}\, d\widehat{B}(\xi), \tag{30}$$

where $d\widehat{B}$ is the Fourier transform of the White Noise. By expanding the function $\xi \mapsto \frac{e^{it\xi}-1}{(i\xi)^{N+1/2}}$ in the orthonormal basis

$$\left\{2^{-j/2} e^{ik\xi/2^j}\overline{\widehat{\psi}(2^{-j}\xi)};\ j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\right\}$$

and by using the isometry property of the integral $\int_{\mathbb{R}}(\cdot)d\widehat{B}$, it follows that

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}}\, d\widehat{B}(\xi) = \sum_{j,k\in\mathbb{Z}} \alpha_{j,k}(t)\epsilon_{j,k}, \tag{31}$$

where the

$$\epsilon_{j,k} = 2^{-j/2} \int_{\mathbb{R}} e^{ik\xi/2^j}\overline{\widehat{\psi}(2^{-j}\xi)}\, d\widehat{B}(\xi)$$

are independent $\mathcal{N}(0,1)$ Gaussian random variables and

$$\alpha_{j,k}(t) = 2^{-j/2} \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}} e^{-ik\xi/2^j}\widehat{\psi}(2^{-j}\xi)\, d\xi. \tag{32}$$

The series (31) is a.s. uniformly convergent in t on compact intervals (Itô-Nisio Theorem).

Let us now prove that

$$\alpha_{j,k}(t) = 2^{-jH}\left(\Psi_H(2^j t - k) - \Psi_H(-k)\right). \tag{33}$$

As the first two moments of the wavelet ψ vanish one has

$$\widehat{\psi}(\xi) = O(\xi^2), \tag{34}$$

which implies that

$$\alpha_{j,k}(t) \tag{35}$$
$$= 2^{-j/2}\int_{\mathbb{R}} e^{i(t-k/2^j)\xi}\frac{\widehat{\psi}(2^{-j}\xi)}{(i\xi)^{H+1/2}}\,d\xi - 2^{-j/2}\int_{\mathbb{R}} e^{-ik\xi/2^j}\frac{\widehat{\psi}(2^{-j}\xi)}{(i\xi)^{H+1/2}}\,d\xi.$$

Finally, setting $\eta = 2^{-j}\xi$ one obtains (33).

3.3 Wavelet representations of FBM with well-localized scaling functions

The mother wavelet ψ satisfies the same conditions as previously and ϕ is a corresponding scaling function.

As $\widehat{\phi}(0) = 1 \neq 0$, the fractional primitive of ϕ of order $H + 1/2$ exists only when $H \in (0, 1/2)$. Moreover, it is irregular and bad localized.

$\Rightarrow$ The problem of finding a wavelet expansion of FBM with a well-localized scaling function is tricky. However, we need to have this well-localization for the expansion to be optimal in the sense of Kühn and Linde.

To overcome this difficulty Meyer, Sellan and Taqqu have used the function Φ_H defined as:

$$\widehat{\Phi}_H(\xi) = \left(\frac{1+e^{-i\xi}}{i\xi}\right)^{H+1/2} \widehat{\Phi}(\xi). \tag{36}$$

Theorem 20 *(Meyer, Sellan and Taqqu 1999) Let Ψ_H and Ψ_{-H} be the fractional primitive of order $H+1/2$ and the fractional derivative of order $H+1/2$ of the mother wavelet ψ. Let Φ_H be as in (36) and let Φ_{-H} be defined as:*

$$\widehat{\Phi}_{-H}(\xi) = \left(\frac{1+e^{-i\xi}}{i\xi}\right)^{-H-1/2} \widehat{\Phi}(\xi). \tag{37}$$

Then for any $m \in \mathbb{Z}$,

$$\{2^{m/2}\Phi_H(2^m t - l),\, l \in \mathbb{Z}, 2^{j/2}\Psi_H(2^j t - k),\, j \geq m,\, k \in \mathbb{Z}\}$$

and

$$\{2^{m/2}\Phi_{-H}(2^m t - l),\, l \in \mathbb{Z}, 2^{j/2}\Psi_{-H}(2^j t - k),\, j \geq m,\, k \in \mathbb{Z}\}$$

are two biorthogonal bases of $L^2(\mathbb{R})$.

Theorem 21 *(Meyer, Sellan and Taqqu 1999) For any* $m \in \mathbb{Z}$*, the FBM* $\{B_H(t)\}_{t\in\mathbb{R}}$ *can be expressed as:*

$$\begin{aligned} B_H(t) &= 2^{-mH} \sum_{l=-\infty}^{+\infty} \Phi_H(2^m t - l) S_{m,l}^{(H)} \\ &\quad + \sum_{j=m}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \Psi_H(2^j t - k) - b_0, \end{aligned} \tag{38}$$

where $\{S_{m,l}^{(H)}\}_{l\in\mathbb{Z}}$ *is a FARIMA(0,H-1/2,0) random walk and* $\{\epsilon_{j,k}\}_{j\geq m, k\in\mathbb{Z}}$ *is a sequence of independent* $\mathcal{N}(0,1)$ *Gaussian random variables and where the series is a.s. uniformly convergent in* t *on compact intervals. Moreover,*

$$S_{m,l}^{(H)} = 2^{m(H+1)} \int_{\mathbb{R}} B_H(t) \Phi_{-H}(2^m t - l)\, dt$$

and

$$\epsilon_{j,k} = 2^{j(H+1)} \int_{\mathbb{R}} B_H(t) \Psi_{-H}(2^j t - k)\, dt.$$

The main advantages of the representation (38) with respect to the representation without scaling function are the following:

- The first term isolates the low frequencies and gives the tendency while the second term involves fluctuations around it.
- FBM can be approximated by the first term and Mallat pyramidal algorithm allows to compute by induction the coefficients $S_{m,l}$, $l \in \mathbb{Z}$ for any $m \geq 1$.

4 Optimality of the wavelet representations of FBM

The representations without scaling functions are optimal in the sense of Kühn and Linde:

Theorem 22 *(Taqqu and Ayache 2002) For every integer $J \geq 0$, let*

$$F_J = \{(j,k) \in \mathbb{Z}^2;\ 0 \leq j \leq J \text{ and } |k| \leq (J-j+1)^{-2} 2^{J+4}\},$$

and

$$P_J = \{(j,k) \in \mathbb{Z}^2;\ -J \leq j \leq -1 \text{ and } |k| \leq 2^{J/2}\}.$$

For every integer $n \geq 2\beta$ with $\beta = 64 \sum_{l=1}^{+\infty} \frac{1}{l^2}$, let $I_n \subset \mathbb{Z}^2$ be a set satisfying the following properties:

- *I_n contains, at most, n indices (j,k),*
- *for every $n \geq 2\beta$, $F_{J(n)} \cup P_{J(n)} \subset I_n$, where $J(n)$ is the unique integer such that*

$$(2\beta)2^{J(n)} \leq n < (2\beta)2^{J(n)+1}.$$

At last let

$$B_{H,n}(t) = \sum_{(j,k)\in I_n} 2^{-jH} \Big(\Psi_H(2^j t - k) - \Psi_H(-k)\Big)\, \epsilon_{j,k}. \quad (39)$$

Then there is a random variable $C > 0$ of finite moment of any order such that a.s. for all $n \geq 1$,

$$\sup_{t\in[0,1]} |B_H(t) - B_{H,n}(t)| \leq C n^{-H} (1 + \log n)^{1/2}. \quad (40)$$

The representations with scaling functions are optimal as well:

Theorem 23 *(Taqqu and Ayache 2002) We suppose that β, F_J and $J(n)$ are the same as in Theorem 22. For every integer $J \geq 0$ let*

$$Q_J = \{l \in \mathbb{Z};\ |l| \leq 2^J\}. \tag{41}$$

For every integer $n \geq 2\beta$, let $I'_n \subset \mathbb{Z}$ and $I''_n \subset \mathbb{Z}^2$ be two sets satisfying the following properties:

- *I'_n contains at most $n/2$ indices l and I''_n contains at most $n/2$ indices (j,k),*
- *for every $n \geq 2\beta$, $Q_{J(n)} \subset I'_n$ and $F_{J(n)} \subset I''_n$.*

At last let

$$\begin{aligned} B_{H,n}(t) &= \sum_{l \in I'_n} \Phi_H(t-l) S_{0,l}^{(H)} \\ &\quad + \sum_{(j,k) \in I''_n} 2^{-jH} \left(\Psi_H(2^j t - k) - \Psi_H(-k)\right) \epsilon_{j,k}. \end{aligned} \tag{42}$$

Then there is a random variable $C > 0$ of finite moment of any order such that a.s. for all $n \geq 1$,

$$\sup_{t \in [0,1]} |B_H(t) - B_{H,n}(t)| \leq C n^{-H} (1 + \log n)^{1/2}. \tag{43}$$

5 Concluding remarks

$\rightarrow$ "The tensor products" of the wavelet series representations of Fractional Brownian Motion lead to optimal series representations of Fractional Brownian Sheet.

$\rightarrow$ Another optimal series representation of Fractional Brownian Sheet has been introduced in 2003 by Dzhaparidze and Van Zanten. It has some similarities with the expansion of Brownian Motion in the trigonometric system.

$\rightarrow$ Lifshits and Simon have recently used some wavelet techniques for obtaining a sharp lower bound of the small ball probability of the symmetric α-stable Riemann Liouville process with Hurst parameter $H > 0$, under a quite general norm.